Dynamics of Elastic Knots

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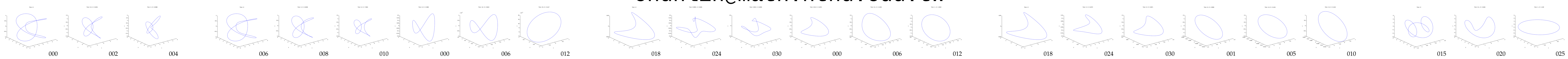


Motivations

Finding an analytic proof of **Smale conjecture**: the topological space consisted of unknotted loops is homotopic to the topological space of round circles.
However,

- A topological proof of Smale conjecture was given by Hatcher in 1983 [3].
- It's more natural and intuitive to have an analytic proof of Smale conjecture instead of an abstract topological proof (see [2]).

*** This poster is based on the joint work in [6].**

Introduction

Assume $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is sufficiently smooth. Let $\gamma = |\partial_x f|$, $ds = \gamma\, dx$ the arclength element, and $\partial_s = \gamma^{-1}\partial_x$ the arclength differentiation. Denote by $T = \partial_s f$ the unit tangent vector, by I the set of arclength parameter of f, and by $\kappa = \partial_s^2 f$ the curvature vector of f. Define the **bending energy** of f, $\mathcal{K}[f]$, by

$$\int_I |\kappa|^2\, ds, \tag{1}$$

and the so-called **Möbius energy** (which is a kind of **electrostatic energy**) of f, $\mathcal{E}_M[f]$, by

$$\iint_{S^1\times S^1} \left[\frac{1}{|f(\sigma)-f(s)|^2} - \frac{1}{D(f(\sigma),f(s))^2}\right] ds d\sigma, \tag{2}$$

where $D(f(\sigma), f(s))$ denotes the minimum length of subarcs of f with end points $f(\sigma)$ and $f(s)$, σ and s both denote the arclength parameter of f. Let the total energy of f be

$$\mathcal{E}_{\alpha,\beta,\lambda}[f] := \alpha\cdot\mathcal{K}[f] + \gamma\cdot\mathcal{E}_M[f] + \lambda\cdot\mathcal{L}[f], \tag{3}$$

where $\mathcal{L}[f]$ is the total length of curve f and α, γ, λ are non-negative constants. Note that the Möbius energy $\mathcal{E}_M$ is induced from a renormalized electrostatic energy. Also, as $\alpha > 0, \gamma = 0$ in Eq.(3), the total energy functional $\mathcal{E}_{\alpha,\beta,\lambda}$ corresponds to the **Euler-Bernoulli** model of **elastic curves**.

Below we investigate the L^2 gradient flows of $\mathcal{E}_{\alpha,\gamma,\lambda}[f]$. The energy decreasing evolution equation induced from $\mathcal{E}_{\alpha,\beta,\lambda}$ can be written as,

$$\partial_t f = -2\alpha\cdot\nabla_s^2\kappa - \frac{|\kappa|^2}{2}\kappa + \lambda\cdot\kappa - \gamma\cdot\mathcal{H}_f, \tag{4}$$

where the covariant derivative $\nabla_s\eta$ denotes the normal component of $\partial_s\eta$, and $\mathcal{H}_f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ is induced from the so-called Gateaux differential of $\mathcal{E}_M$. Notice that the leading term of this parabolic equation (4), $\nabla_s^2\kappa$, is fourth-order, $\mathcal{H}_f$ is a pseudo differential operator of third-order (see [4]), and keeps the curve f_t away from self-intersection (embedding) all the time.
The short time existence of (4) is a standard matter. Notice that $\mathcal{H}_f$ is non-local, and whose differential-order is less than the highest term $\nabla_s^2\kappa$ in the linearized equation of Eq.(4). Hence $\mathcal{H}_f$ remains a compact operator between the relevant parabolic Hölder spaces. Therefore, the short time existence can be argued the same as the case of curve-straightening flow (e.g., see [7], in which $\mathcal{H}_f$ doesn't appear).
Thus we just need to focus on the long time existence and the asymptotics (see the Main Theorem below).

Main Theorem

Let f_0 be a given smooth initial loop in the Euclidean 3-space. Assume f_t is the solution of

$$\partial_t f = -\nabla\mathcal{E}_{\alpha,\gamma,\lambda}[f],$$

where α, γ, λ are non-negative constants. Then,

1. the solution of evolution equation f_t remains smooth for all $t > 0$;
2. the asymptotic solution, f_∞, is an equilibrium configuration of $\mathcal{E}_{\alpha,\gamma,\lambda}$, i.e., f_∞ satisfies
$$\delta\mathcal{E}_{\alpha,\gamma,\lambda}[f_\infty] = 0.$$

In the **proof of the main theorem**, the mathematical analysis follows [7], namely it is based on L^2 *curvature estimates* and *Gagliardo-Nirenberg interpolation inequalities*.

Numerical Simulations

The algorithm here is an extension of that in [7], where we exploited the divergence form of the main part in the evolution equation and the partition into a second-order parabolic-elliptic system for the position vector f and the curvature vector κ. We choose $\alpha = \frac{1}{2}$, and write

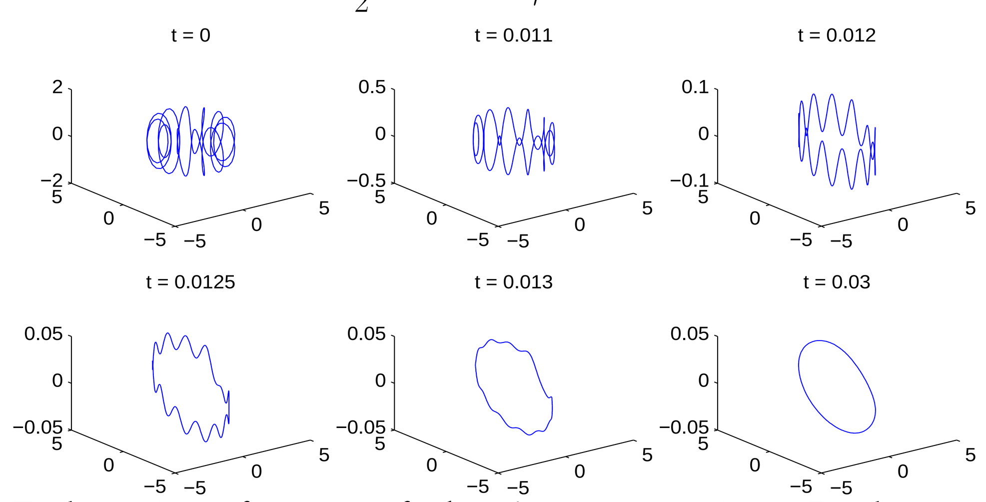
$$\partial_t f + \partial_s\left(\partial_s\kappa + \tfrac{3}{2}|\kappa|^2 T\right) + \gamma\mathcal{H}_f = \lambda\kappa, \tag{5}$$
$$\partial_s^2 f = \kappa \tag{6}$$

and discretize the problem using an semi-implicit scheme in time and piecewise-affine finite elements for the space dependence.

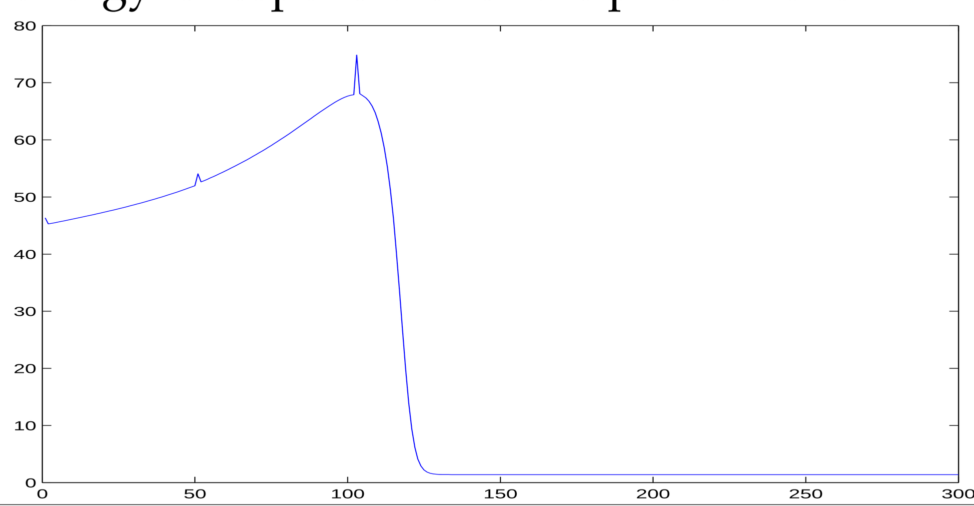
Computational Experiments

Below we show examples in exhibit interesting dynamical behavior for the gradient flow equation in Eqs.(5) and (6).

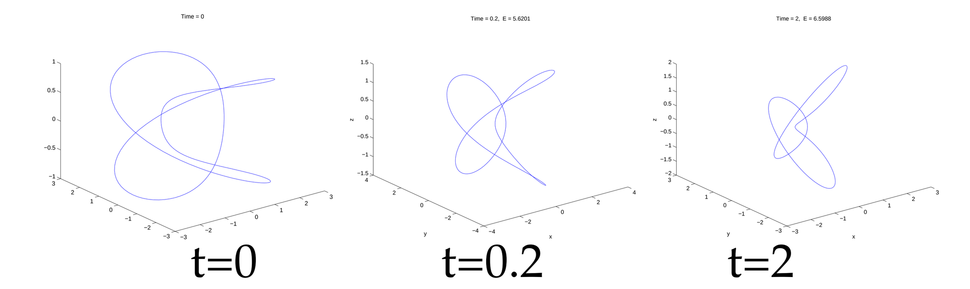
A (1,12)-knot initial curve Below shows an example of competition between the elastic energy and the Möbius energy during untangling a unknotted loop of (1,12)-knot type into a round circle. Notice that the shape changes rapidly during $t = 0.011$ and $t = 0.013$. Here we choose $\frac{1}{2} = \alpha = \gamma = \lambda$.



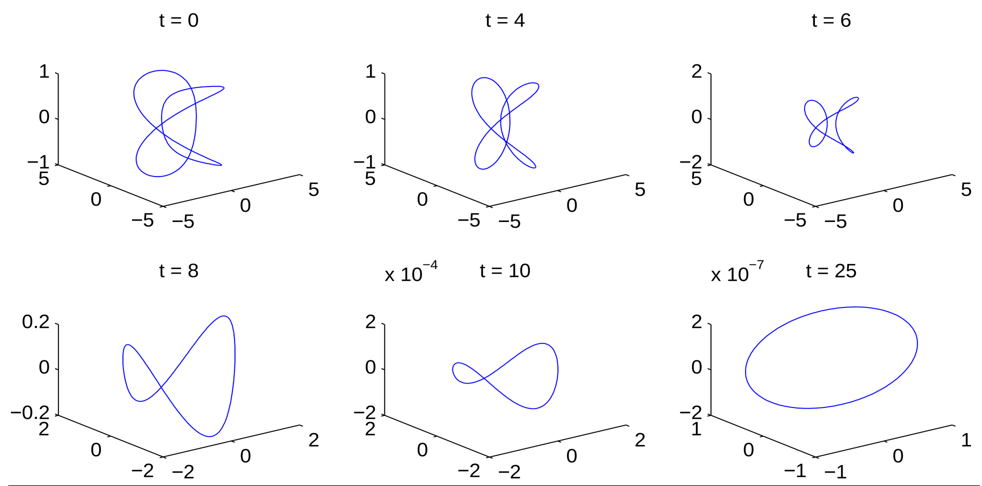
Below is a figure of *elastic energy* via t. It shows that the elastic energy was forced to increase before $t = 0.01$ and then decreased rapidly after $t = 0.01$. The decreasing of the Möbius energy is responsible for the phenomenon.



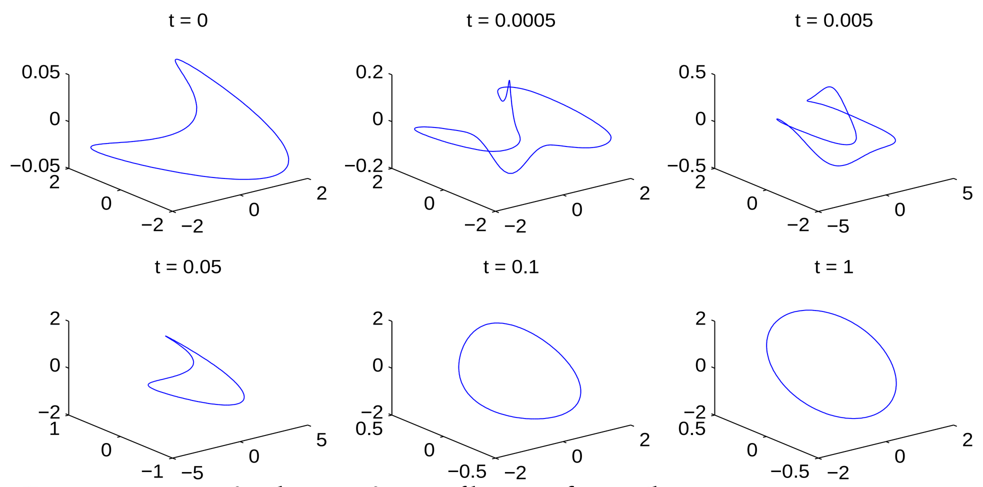
A (2,3)-knot initial curve The figures below show that a nearly flat trefoil remains in the same knot type during the flow. Here we choose $\frac{1}{2} = \alpha = \gamma = \lambda$.



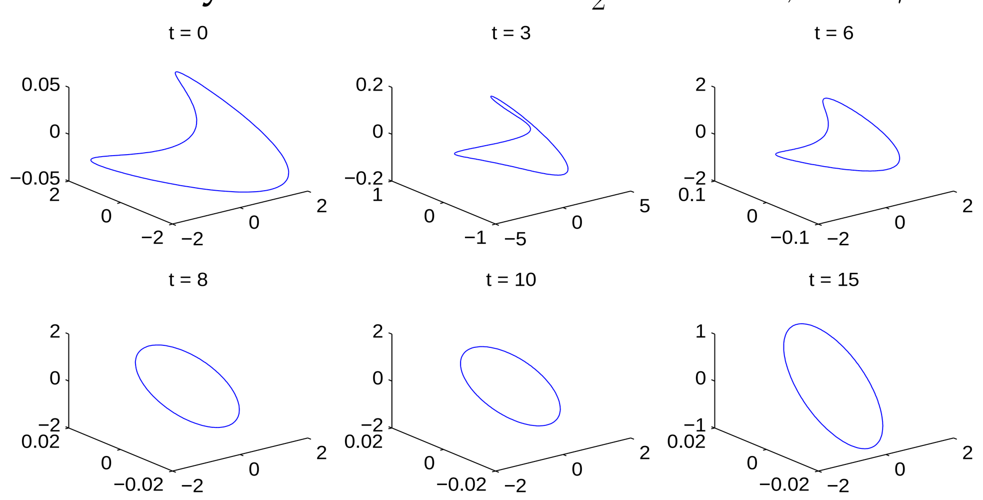
Curve straightening flow for the same (2,3)-knot initial curve The figures below show that the curve-straightening flow for the same initial curve of nearly flat trefoil doesn't prevent from self-intersection. Here we choose $\frac{1}{2} = \alpha = \lambda; 0 = \gamma$.



A figure-eight type initial curve The figures below show that the part of two nearby arcs of an almost flat figure-eight was initially pulled apart. Here we choose $\frac{1}{2} = \alpha = \gamma = \lambda$.



Curve straightening flow for the same figure-eight type initial curve The figures below show that the corresponding curve-straightening flow (i.e., the case of $\gamma = 0$ with the same initial curve as above behaves differently. Here we choose $\frac{1}{2} = \alpha = \lambda; 0 = \gamma$.



A Remained Problem

- Are round circles the only equilibrium configurations of the elastic knot energy, $\mathcal{E}_{\alpha,\gamma,\lambda}$, in the class of unknotted loops?

Notice that one already know that round circles are equilibrium configurations of $\mathcal{E}_{\alpha,\gamma,\lambda}$ for all non-negative constants α, γ, λ. Thus if the answer is positive, then combining with our result in [6] gives an analytic proof of Smale conjecture.

Other Applications

There are two aspects in applications. We hope to explore them in the future.

1. Constructing approximate solutions of the **N-body problem** (see [1]).
2. Modeling **over-damped dynamics of elastic rods** (e.g., **bio-polymers**, see [7] and [5]).

References

[1] G. Buck, *Most smooth closed space curves contain approximate solutions of the n-body problem*, Nature,Vol. 395, 3, Sep. 1998.

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