# A Note on Complex Representations of $\mathrm{GL}(2, \mathbb{F}_q)$


## Hua-Chieh Li

Department of Mathematics, National Tsing Hua University, Hsin Chu, Taiwan, R.O.C.

*E-mail address*: `li@math.nthu.edu.tw`

# Contents

# General Properties on Linear Representations of Finite Groups

All groups we consider in this chapter are finite group.

## 1.1. Basic Definitions

Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and let $\mathrm{Aut}(V)$ be the group of automorphisms of $V$ onto itself. A *linear representation* of a finite group $G$ on $V$ is a homomorphism $\rho : G \to \mathrm{Aut}(V)$ from $G$ to the group $\mathrm{Aut}(V)$. In this way we have the equalities

$$\rho(s \cdot t) = \rho(s) \circ \rho(t)\ \forall s,\, t \in G, \quad \rho(1) = 1 \;\text{ and }\; \rho(s^{-1}) = \rho(s)^{-1}.$$

We will also frequently write $\rho_s$ instead of $\rho(s)$.

When $V$ is given, we say that $V$ is a *representation space* of $G$, denoted $V_\rho$ and also say that $G$ *acts on* $V$ through $\rho$. The dimension of $V$ is called the *dimension* of $\rho$, denoted $\dim(\rho)$. If we have $\rho(s)$ equals to the identity map for all $s \in G$, the representation is called the *trivial representation*.

**Example 1.1.1.** Let $g$ be the order of $G$ and let $V$ be the vector space of dimension $g$ with a basis $(v_t)_{t\in G}$ indexed by the elements $t$ of $G$. For $s,\, t \in G$, let $\rho_s$ be the linear map of $V$ into $V$ such that $\rho_s(v_t) = v_{st}$; this defines a linear representation, which is called the *regular representation* of $G$. Note that if $e$ is the identity of $G$, the orbit of $v_e$ form a basis of $V$.

Let $\rho$ and $\rho'$ be two representations of the same group $G$ in $V$ and $V'$, respectively. These representations are said to be *isomorphic* if there exists a linear isomorphism $\tau : V \to V'$ such that $\tau \circ \rho(s) = \rho'(s) \circ \tau$ for all $s \in G$. We shall usually identify isomorphic representations.

## 1.2. Subrepresentations and Irreducible Representations

Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation and let $W$ be a subspace of $V$. Suppose that $w \in W$ implies $\rho_s(w) \in W$ for all $s \in G$. The restriction $\rho_s|_W$ of $\rho_s$ to $W$ is then an automorphism of $W$ and we have $\rho_{st}|_W = \rho_s|_W \circ \rho_t|_W$. Thus $W$ is *stable* under the action of $G$ and $\rho|_W : G \to \mathrm{Aut}(W)$ is a linear representation of $G$ in $W$; $W$ is said to be a *subrepresentation* of $V$.

There are some important subrepresentations. Let $\rho$ and $\rho'$ be representations of $G$ into $V$ and $W$ respectively. A *G-linear map* from $V$ to $W$ is a linear map $\phi : V \to W$ such that $\phi(\rho_s(v)) = \rho'_s(\phi(v))$

for all $s \in G$ and $v \in V$. We denote the space of all $G$-linear maps from $V$ to $W$ by $\mathrm{Hom}_G(V,W)$. It is easy to check that for a given $\phi \in \mathrm{Hom}_G(V,W)$, the space $\mathrm{Ker}(\phi) = \{v \in V \mid \phi(v) = 0\}$ gives a subrepresentation of $G$ in $V$ and the space $\mathrm{Im}(\phi) = \{w \in W \mid w = \phi(v) \text{ for some } v \in V\}$ gives a subrepresentation of $G$ in $W$.

A representation of $G$ in $V$ is called *irreducible* if there is no proper nonzero subrepresentation of $V$.

**Lemma 1.2.1.** *Let $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(W)$ be two representations of $G$. Suppose that $\phi \in \mathrm{Hom}_G(V,W)$ is not the zero map. Then we have the following:*

(1) *If $V$ is an irreducible representation of $G$, then $\phi$ is injective.*

(2) *If $W$ is an irreducible representation of $G$, then $\phi$ is surjective.*

*In particular, if both $V$ and $W$ are irreducible representations of $G$, then $V$ and $W$ are isomorphic.*

**Proof.** Since $\phi$ is not zero, we have $\mathrm{Ker}(\phi) \neq V$ and $\mathrm{Im}(\phi) \neq \{0\}$. Therefore, $V$ is irreducible implies $\mathrm{Ker}(\phi) = \{0\}$ and $W$ is irreducible implies $\mathrm{Im}(\phi) = W$. ☐

**Corollary 1.2.2.** *Let $V$ and $W$ be two representations of $G$ where $V$ is irreducible and let $\phi_1$, $\phi_2 \in \mathrm{Hom}_G(V,W)$. Suppose that there exist $v \neq 0$ in $V$ such that $\phi_1(v) = \phi_2(v)$. Then $\phi_1 = \phi_2$.*

**Proof.** The assumption says that $\phi_1 - \phi_2$ is not injective. Since $\phi_1 - \phi_2 \in \mathrm{Hom}_G(V,W)$, it implies that $\phi_1 - \phi_2$ is the zero mapping by Lemma 1.2.1. ☐

## 1.3. Schur's Lemma and Its Applications

For each $n \times n$ matrix $A$, since it is over $\mathbb{C}$ which is *algebraically closed*, there exist eigenvalues of $A$. By this, we can derive that there exists a *unitary matrix* $U$ (*i.e.* $\overline{U}^T \cdot U = I$) such that $\overline{U}^T \cdot A \cdot U$ is a upper triangular matrix. This is what called *Schur's Theorem* in Linear Algebra [**1**, Section 6.5]. Here, by using similar argument, we have the following:

**Proposition 1.3.1** (Schur's Lemma). *Let $\rho : G \to \mathrm{Aut}(V)$ be an irreducible representation of $G$ and let $f$ be a linear mapping of $V$ into $V$ such that $\rho_s \circ f = f \circ \rho_s$ for all $s \in G$. Then $f$ is a homothety (i.e. $f = \lambda I$ for some $\lambda \in \mathbb{C}$ where $I$ is the identity map of $V$).*

**Proof.** Because $f$ is an endomorphism of $V$, there exists an eigenvalue $\lambda$ with eigenvector $v \in V$. Thus $f(v) = \lambda I(v)$. By Corollary 1.2.2, $f$ is equal to $\lambda I$. ☐

Let $G$ be a finite abelian group and let $\rho : G \to \mathrm{Aut}(V)$ be a representation of $G$. It is easy to show that for every $s \in G$, $\rho_s$ is a $G$-linear mapping of $V$ into $V$. Hence by Schur's Lemma, we have the following:

**Corollary 1.3.2.** *Let $G$ be a finite abelian group and let $\rho : G \to \mathrm{Aut}(V)$ be an irreducible representation of $G$. Then we have that $\dim(V) = 1$.*

We will see latter that there are many applications for Schur's Lemma. Here we give some important ones which are very useful for developing *character theory.*

**Corollary 1.3.3.** *Let $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(W)$ be two irreducible representations of $G$ and let $g$ be the order of $G$. Let $h$ be a linear mapping of $V$ into $W$ (note: $h$ may not be a $G$-linear mapping). Put*

$$h^0 = \frac{1}{g} \sum_{t \in G} (\rho'_t)^{-1} \circ h \circ \rho_t.$$

*Then:*

(1) *If $\rho$ and $\rho'$ are not isomorphic, then we have $h^0 = 0$.*

(2) *If $V = W$ and $\rho = \rho'$, then $h^0$ is a homothety of ratio $(1/n)\mathrm{Tr}(h)$, where $n = \dim(V)$.*

**Proof.** We have $\rho'_s h^0 = h^0 \rho_s$ for all $s \in G$. Applying Lemma 1.2.1 and Schur's Lemma with $f = h^0$, we see in case (1) that $h^0 = 0$ and in case (2) that $h^0 = \lambda I$ for some $\lambda \in \mathbb{C}$. For the value of $\lambda$, we have $n\lambda = \mathrm{Tr}(\lambda I) = (1/g) \sum_{t \in G} \mathrm{Tr}((\rho_t)^{-1} h \rho_t) = \mathrm{Tr}(h)$. □

Now we rewrite Corollary 1.3.3 in matrix form. Suppose that $\dim(W) = m$ and the linear mapping $h$ is defined by an $m \times n$ matrix $(h_{kj})$ and likewise $h^0$ is defined by $(h^0_{kj})$. Assume $\rho$ and $\rho'$ are given in matrix form $\rho_t = (r_{ij}(t))$, $1 \le i, j \le n$ and $\rho'_t = (r'_{kl}(t))$, $1 \le k, l \le m$ respectively. We have by the definition of $h^0$:

$$h^0_{kj} = \frac{1}{g} \sum_{t \in G, 1 \le l \le m, 1 \le i \le n} r'_{kl}(t^{-1}) \cdot h_{li} \cdot r_{ij}(t).$$

Since $h$ is any linear mapping, we choose $h$ with matrix form $E_{li}$, the matrix which is 1 in the $(l, i)$-place and 0 everywhere else. Notice that $\mathrm{Tr}(E_{li}) = \delta_{li}$ ($\delta_{ij}$ denotes the *Kronecker symbol*, equal to 1 if $i = j$ and 0 otherwise). Whence:

**Corollary 1.3.4.** *Keeping the hypothesis and notation of Corollary* 1.3.3*, we have:*

(1) *If $\rho$ and $\rho'$ are not isomorphic, then*

$$\frac{1}{g} \sum_{t \in G} r'_{kl}(t^{-1}) r_{ij}(t) = 0, \ \forall\, 1 \le k, l \le m,\ 1 \le i, j \le n.$$

(2) *If $V = W$ and $\rho = \rho'$, then*

$$\frac{1}{g} \sum_{t \in G} r_{kl}(t^{-1}) r_{ij}(t) = \begin{cases} 1/n & \textit{if } i{=}l \textit{ and } k{=}j, \\ 0 & \textit{otherwise.} \end{cases}$$

## 1.4. Direct Sum and Tensor Product

There are many ways to construct new representations from old ones. Here we introduce direct sum and tensor product. Let $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(W)$ be linear representations of $G$ in $V$ and $W$, respectively. Define a linear representation $\rho \oplus \rho'$ of $G$ in $V_1 \oplus V_2$ by setting $(\rho \oplus \rho')_s(v \oplus w) = \rho_s(v) \oplus \rho'_s(w)$, for all $s \in G$, $v \in V$ and $w \in W$. $\rho \oplus \rho'$ is called *direct sum representation* of the given $\rho$ and $\rho'$. The direct sum of an arbitrary finite number of representations is defined similarly.

The *tensor product representation* $\rho \otimes \rho'$ of $G$ in $V \otimes W$ of the given representations $\rho$ of $G$ in $V$ and $\rho'$ in $W$ is defined by the condition $(\rho \otimes \rho')_s(v \otimes w) = \rho_s(v) \otimes \rho'_s(w)$, for all $s \in G$, $v \in V$ and $w \in W$. The tensor product of an arbitrary finite number of representations is defined similarly.

We can easily see that

$$\dim(\rho \oplus \rho') = \dim(\rho) + \dim(\rho') \quad \text{and} \quad \dim(\rho \otimes \rho') = \dim(\rho) \cdot \dim(\rho')\ .$$

## 1.5. Complete Reducibility

As in any study, before we begin our attempt to classify the representations of a finite group in earnest we should try to simplify life by restricting our search somewhat. The key to all this is

**Proposition 1.5.1.** *Let $\rho$ be a linear representation of $G$ in $V$ and let $W$ be a subrepresentation of $G$ in $V$. Then there exists a complement $W^0$ of $W$ in $V$ which is stable under $G$.*

**Proof.** Choose $W'$ an arbitrary complement of $W$ in $V$, and let $p : V \to W$ be the corresponding *projection* of $V$ onto $W$ (*i.e.* writing $v \in V$ uniquely as $v = w + w'$ with $w \in W$ and $w' \in W'$, $p(v) = w$). Define

$$p^0 = \frac{1}{g} \sum_{t \in G} (\rho_t)^{-1} \circ p \circ \rho_t,$$

where $g$ is the order of $G$. Since $p$ maps $V$ into $W$ and $\rho_t$ preserves $W$ for all $t \in G$, we see that $p^0$ maps $V$ into $W$. Furthermore, because $p(w) = w$ and $\rho_t^{-1}(w) = \rho_{t^{-1}}(w) \in W$ for all $w \in W$, it implies that $p^0(w) = w$ for all $w \in W$. Thus $p^0$ is a projection of $V$ onto $W$, corresponding to some complement $W^0 = \mathrm{Ker}(p^0)$ of $W$. We have moreover $\rho_s \circ p^0 = p^0 \circ \rho_s$ for all $s \in G$. Hence $p^0 \circ \rho_s(w^0) = \rho_s \circ p^0(w^0) = 0$ for $w^0 \in W^0$ and $s \in G$, which shows that $W^0$ is stable under $G$ and complete the proof. □

This proposition says that for any subrepresentation $W$ of $G$ in $V$, there exists another subrepresentation $W^0$ of $G$ in $V$ such that $V = W \oplus W^0$ is a direct sum representation of $W$ and $W^0$. Therefore, an irreducible representation is equivalent to saying that it is not the direct sum of two representations. We have the following *complete reducibility* property.

**Theorem 1.5.2.** *Every representation is a direct sum of irreducible representations.*

**Proof.** We proceed by induction on the dimension of representation. If the representation is irreducible, there is nothing to prove. Otherwise, because of Proposition 1.5.1, it can be decomposed into a direct sum of subrepresentations with smaller dimensions. By the induction hypothesis, these subrepresentations are direct sum of irreducible representations and so is our original representation. □

**Remark .** This property is not always true for representations of infinite group or over a field other than $\mathbb{C}$. For example, the additive group $\mathbb{R}$ does not have this property. Note also that the argument of Proposition 1.5.1 would fail if the vector space was over a field of *finite characteristic.*

We can ask if this decomposition of $V$ is unique. The case where all the $\rho_s$ are equal to identity shows that this is not true in general (in this case the irreducible representations are lines, and we have an infinity of ways to decompose a vector space into a direct sum of lines). Nevertheless, we have a decomposition of $V$ which is "coarser" than the decomposition into irreducible representations, but which has the advantage of being *unique.* It is obtained as follows. First decompose $V$ into direct sum of irreducible representations $V = W_1 \oplus \cdots \oplus W_k$ and then collect together the isomorphic representations. A representation is said to be *isotypic* if it is a direct sum of isomorphic irreducible representation. Thus, we have $V = V_1 \oplus \cdots \oplus V_h$ where every $V_i$ is isotypic. This will be the *canonical decomposition* we have in mind.

There is another concept for the proof of Proposition 1.5.1 which is very useful.

Let $T$ be a linear mapping of $V$ into $V$, where $V$ is endowed with an inner product $\langle\ ,\ \rangle$. Suppose that $\langle T(v), T(w)\rangle = \langle v, w\rangle$ for all $v, w \in V$ and suppose further that $U$ is the matrix representation of $T$ with respect to an orthonormal basis of $V$. Then $U$ is *unitary* (*i.e.* $\overline{U}^T \cdot U = U \cdot \overline{U}^T = I$). We say that an $n \times n$ matrix $A$ is *normal* if $\overline{A}^T \cdot A = A \cdot \overline{A}^T$ (so a unitary matrix is normal). Using Schur's theorem we can prove the *spectral theorem* which says that if $A$ is normal, then there exists a unitary matrix $U$ such that $\overline{U}^T \cdot A \cdot U$ is a diagonal matrix. This amounts to saying that $A$ is normal if and only if $A$ possesses a orthonormal basis which are eigenvectors.

Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation where $V$ is endowed with an inner product $\langle\ ,\ \rangle$. Consider the product $\langle\langle u, v\rangle\rangle := \sum_{t\in G}\langle\rho_t(u), \rho_t(v)\rangle$. Then $\langle\langle u, v\rangle\rangle$ is an inner product with the property $\langle\langle\rho_s(u), \rho_s(v)\rangle\rangle = \langle\langle u, v\rangle\rangle$ for all $s \in G$. We can deduce from this that there exists a basis of $V$ such that the matrix form of $\rho_s$ with respect to this basis is a unitary matrix for every $s \in G$. Now, if $W$ is a subrepresentation of $G$ in $V$, then with respect to the inner product $\langle\langle u, v\rangle\rangle$, the orthogonal complement $W^\perp$ of $W$ in $V$ is stable under $G$; another proof of Proposition 1.5.1 is thus obtained.

## 1.6. Characters for Representations

Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation of $G$ in $V$. Since the trace of the linear mapping $\rho_s$ does not depend on the choice of basis of $V$, we put:

$$\chi_\rho(s) = \mathrm{Tr}(\rho_s) \quad \text{for each } s \in G.$$

The complex valued function $\chi_\rho$ on $G$ thus obtained is called the *character* of the representation $\rho$. We remark that if two representations $\rho$ and $\rho'$ are isomorphic, then $\chi_\rho = \chi_{\rho'}$.

Suppose that $\dim(\rho) = n$. We have $\mathrm{Tr}(I) = n$, and so $\chi_\rho(e) = n$ where $e$ is the identity of $G$. Recall that from 1.5, the matrix form of $\rho_s$ is normal, and hence diagonalizable. Thus for $s \in G$, a basis $(v_1, \ldots, v_n)$ of $V$ can be chosen such that $\rho_s(v_i) = \lambda_i v_i$ with $\lambda_i \in \mathbb{C}^*$, and so $\chi_\rho(s) = \sum_{i=1}^n \lambda_i$. Also note that $s \in G$ has finite order, the values $\lambda_i$ are roots of unity; in particular we have $\overline{\lambda_i} = \lambda_i^{-1}$. Because $\rho_{s^{-1}} = \rho_s^{-1}$, we have $\chi_\rho(s^{-1}) = \sum_{i=1}^n \lambda_i^{-1} = \sum_{i=1}^n \overline{\lambda_i}$ and $\mathrm{Tr}(\rho_{tst^{-1}}) = \mathrm{Tr}(\rho_t \circ \rho_s \circ \rho_t^{-1}) = \mathrm{Tr}(\rho_s)$.

We can summarize what we have shown so far in

**Proposition 1.6.1.** *If $\chi_\rho$ is the character of a dimension $n$ representation $\rho : G \to \mathrm{Aut}(V)$ of $G$ in $V$, we have:*

(1) $\chi_\rho(e) = n$.

(2) $\chi_\rho(s^{-1}) = \overline{\chi_\rho(s)}$ *for* $s \in G$.

(3) $\chi_\rho(tst^{-1}) = \chi_\rho(s)$ *for* $s, t \in G$.

**Proposition 1.6.2.** *Let $\rho$ and $\rho'$ be two linear representations of $G$ in $V$ and $W$, and let $\chi_\rho$ and $\chi_{\rho'}$ be their characters, respectively. Then:*

(1) *The character of the direct sum representation $\rho \oplus \rho'$ is equal to $\chi_\rho + \chi_{\rho'}$.*

(2) *The character of the tensor product representation $\rho \otimes \rho'$ is equal to $\chi_\rho \cdot \chi_{\rho'}$.*

**Proof.** This is a consequence of followings. Suppose that $\{v_i\}$ and $\{w_j\}$ are bases of $V$ and $W$ which are eigenvectors of $\rho_s$ and $\rho'_s$ with eigenvalues $\{\lambda_i\}$ and $\{\lambda'_j\}$, respectively. Then $\{v_i \oplus 0_W, 0_V \oplus w_j\}$ and $\{v_i \otimes w_j\}$ are eigenvectors of $(\rho \oplus \rho')_s$ and $(\rho \otimes \rho')_s$ with eigenvalues $\{\lambda_i, \lambda'_j\}$ and $\{\lambda_i \cdot \lambda'_j\}$, respectively. □

## 1.7. Orthogonality Relations for Characters

Let $G$ be a group of order $g$. If $\phi$ and $\psi$ are two complex valued functions on $G$, we put:

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{s \in G} \phi(s) \overline{\psi(s)}.$$

This is an inner product.

**Theorem 1.7.1.** *Let $\rho$ and $\rho'$ be two irreducible representations of $G$ with characters $\chi_\rho$ and $\chi_{\rho'}$, respectively.*

(1) *If $\rho$ and $\rho'$ are not isomorphic, then we have $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$.*

(2) *If $\rho$ and $\rho'$ are isomorphic, then we have $\langle \chi_\rho, \chi_{\rho'} \rangle = 1$.*

**Proof.** Because the character dose not depend on the choices of basis, without lose of generality by suitable choice of basis, we suppose that the matrix form $(r_{ij}(s))$ of $\rho_s$ and $(r'_{kl}(s))$ of $\rho'_s$ are unitary matrices. Thus $(r_{ij}(s))^{-1} = (\overline{r_{ij}(s)})^T$ and $(r'_{kl}(s))^{-1} = (\overline{r'_{kl}(s)})^T$. We have then $r_{ij}(s^{-1}) = \overline{r_{ji}(s)}$ and $r'_{kl}(s^{-1}) = \overline{r'_{lk}(s)}$. Suppose $\dim(\rho) = n$ and $\dim(\rho') = m$. By definition, $\chi_\rho(s) = \sum_{i=1}^n r_{ii}(s)$ and $\chi_{\rho'}(s) = \sum_{k=1}^m r'_{kk}(s)$, and hence

$$\langle \chi_\rho, \chi_{\rho'} \rangle = \sum_{k=1}^{m} \sum_{i=1}^{n} \langle r_{ii}, r'_{kk} \rangle \quad \text{and} \quad \langle r_{ii}, r'_{kk} \rangle = \frac{1}{g} \sum_{s \in G} r_{ii}(s) \overline{r'_{kk}(s)} = \frac{1}{g} \sum_{s \in G} r_{ii}(s) r'_{kk}(s^{-1}).$$

If $\rho$ is not isomorphic to $\rho'$, then by Corollary 1.3.4, we have $\langle r_{ii}, r'_{kk} \rangle = 0$, and hence $\langle \chi_\rho, \chi_{\rho'} \rangle = 0$. If $\rho$ is isomorphic to $\rho'$, then $n = m$ and $\chi_\rho = \chi_{\rho'}$. By Corollary 1.3.4, we have $\langle r_{ii}, r_{kk} \rangle = \delta_{ik}/n$, and hence $\langle \chi_\rho, \chi_{\rho'} \rangle = \langle \chi_\rho, \chi_\rho \rangle = \sum_{i,k=1}^n \delta_{ik}/n = 1$. □

Theorem 1.7.1 says that in terms of the inner product defined above, the characters of irreducible representations of $G$ are orthonormal. There are many applications of these orthogonality relations.

**Corollary 1.7.2.** *Let $\rho$ be a representation of $G$ in $V$ with character $\chi_\rho$ and suppose $V$ decomposes into a direct sum of irreducible representations: $V = W_1 \oplus \cdots \oplus W_k$. Let $\theta$ be an irreducible representation of $G$ in $W$ with character $\chi_\theta$. Then the number of $W_i$ which is isomorphic to $W$ is equal to $\langle \chi_\rho, \chi_\theta \rangle$.*

**Proof.** Let $\chi_i$ be the character of the irreducible representation of $G$ in $W_i$. By Proposition 1.6.2, we have $\chi_\rho = \chi_1 + \cdots + \chi_k$. Thus $\langle \chi_\rho, \chi_\theta \rangle = \langle \chi_1, \chi_\theta \rangle + \cdots + \langle \chi_k, \chi_\theta \rangle$. By Theorem 1.7.1, $\langle \chi_i, \chi_\theta \rangle$ is equal to 1 (*resp.* 0) if $W_i$ is (*resp.* is not) isomorphic to $W$. The result follows. □

Since $\langle \chi_\rho, \chi_\theta \rangle$ does not depend on the decomposition of $V$, this result says that the number of irreducible representations in any decomposition of $V$ which are isomorphic to $W$ is the same. This shows the fact that the canonical decomposition of $V$ is unique (*cf.* Section 1.5). This number is called the *multiplicity* of $W$ occurs in $V$. If $W_1, \ldots, W_h$ are the distinct non-isomorphic irreducible representations occur in $W$ with multiplicities $m_1, \ldots, m_h$ respectively, and $\chi_1, \ldots, \chi_h$ denote corresponding characters, then $V$ is isomorphic to $m_1 W_1 \oplus \cdots \oplus m_h W_h$ and the character $\chi_\rho$ of $V$ is equal to $m_1\chi_1 + \cdots + m_h\chi_h$ with $m_i = \langle \chi_\rho, \chi_i \rangle$. Whence:

**Corollary 1.7.3.** *Two representations have the same character if and only if they are isomorphic.*

The above results reduce the study of representations to that of their characters. In particular, we have:

**Corollary 1.7.4.** *If $\chi_\rho$ is the character of a representation $\rho$ of $G$ in $V$, then $\langle \chi_\rho, \chi_\rho \rangle$ is a positive integer. Furthermore, we have $\langle \chi_\rho, \chi_\rho \rangle = 1$ if and only if $V$ is irreducible.*

**Proof.** Suppose that $\chi_\rho = m_1\chi_1 + \cdots + m_h\chi_h$ where $\chi_i$ are irreducible characters of $G$. The orthogonality relations among the $\chi_i$ imply $\langle \chi_\rho, \chi_\rho \rangle = \sum_{i=1}^h m_i^2$. Furthermore, $\sum_{i=1}^h m_i^2 = 1$ if only one of the $m_i$ is equal to 1. Our result follows. □

## 1.8. The Space of Class Functions on $G$

A Complex valued function $f$ on $G$ is called a *class function* if $f(tst^{-1}) = f(s)$ for all $s$, $t \in G$. By Proposition 1.6.1, all characters of a representation of $G$ are class functions. Recall that two elements $s$ and $s'$ in $G$ are said to be *conjugate* if there exists $t \in G$ such that $s' = tst^{-1}$; this is an equivalence relation, which partitions $G$ into *conjugacy classes*. Let $C_1, \ldots, C_h$ be the distinct conjugacy classes of $G$. To say that a function $f$ on $G$ is a class function is equivalent to saying that $f$ is constant on each of $C_1, \ldots, C_h$.

We introduce now the space $\mathcal{H}$ of class functions on $G$. This is an inner product space endowed with the inner product defined in 1.7. The dimension of $\mathcal{H}$ is equal to the number of conjugacy classes of $G$.

Given a linear representation $\rho : G \to \mathrm{Aut}(V)$ of $G$ in $V$, for $f \in \mathcal{H}$, we define a linear mapping $\rho_f : V \to V$ by:

$$\rho_f(v) = \sum_{t \in G} f(t)\rho_t(v), \quad \text{for } v \in V.$$

Because $f$ is a class function on $G$, we have

$$\rho_s^{-1} \circ \rho_f \circ \rho_s = \sum_{t \in G} f(t)\rho_{s^{-1}ts} = \sum_{u \in G} f(sus^{-1})\rho_u = \sum_{u \in G} f(u)\rho_u = \rho_f.$$

Hence, $\rho_f$ is a $G$-linear mapping of $V$ into $V$.

**Lemma 1.8.1.** *Let $G$ be a group of order $g$ and let $f$ be a class function on $G$. Suppose that $\rho : G \to \mathrm{Aut}(V)$ is an irreducible linear representation of $G$ of dimension $n$ and character $\chi$. Then $\rho_f = \sum_{t \in G} f(t)\rho_t$ is a homothety of ratio $\lambda$ given by:*

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{g}{n} \langle f, \overline{\chi} \rangle.$$

**Proof.** Since $\rho_f \in \mathrm{Hom}_G(V,V)$ and $V$ is irreducible, by Schur's lemma (Proposition 1.3.1), $\rho_f = \lambda I$. Because $\dim(V) = n$, we have

$$\lambda\, n = \mathrm{Tr}(\lambda I) = \mathrm{Tr}(\rho_f) = \sum_{t\in G} f(t)\mathrm{Tr}(\rho_t) = \sum_{t\in G} f(t)\chi(t).$$

The proof is complete. □

Theorem 1.7.1 show that the characters of the irreducible representations of $G$ are orthonormal in $\mathcal{H}$. Therefore, they are linearly independent over $\mathbb{C}$. This amounts to saying that the number of the irreducible representations of $G$ is less than or equal to the number of conjugacy classes of $G$. In fact, they generate $\mathcal{H}$.

**Theorem 1.8.2.** *The characters of irreducible representations of $G$ form an orthonormal basis of the space of class functions on $G$.*

**Proof.** Suppose that $\chi_1, \dots, \chi_h$ are the distinct characters of the irreducible representations of $G$. We know that $\overline{\chi}_1, \dots, \overline{\chi}_h$ are also characters of $G$, and since $\langle \overline{\chi}_i, \overline{\chi}_i\rangle = \langle \chi_i, \chi_i\rangle = 1$, they are also irreducible. Therefore, we only have to show that the orthogonal complement of $\mathcal{W}$ =span$(\overline{\chi}_1, \dots, \overline{\chi}_h)$ in $\mathcal{H}$ is $\{0\}$. Let $f \in \mathcal{W}^\perp$ and for any representation $\rho$ of $G$, put $\rho_f = \sum_{t\in G} f(t)\rho_t$. Since $\langle f, \overline{\chi}_i\rangle = 0$, Lemma 1.8.1 above shows that $\rho_f$ is the zero mapping so long as $\rho$ is irreducible. However, by Theorem 1.5.2, every representation is a direct sum of irreducible representations. We conclude that for any representation $\rho$, $\rho_f$ is always the zero mapping.

Now let $\rho$ be the regular representation of $G$ (*cf.* Example 1.1.1) in the vector space of dimension $g$ with a basis $(v_t)_{t\in G}$. Let $e$ be the identity of $G$. Computing the image of $v_e$ under $\rho_f$, we have $\rho_f(v_e) = \sum_{t\in G} f(t)\rho_t(v_e) = \sum_{t\in G} f(t)v_t = 0$. Since $(v_t)_{t\in G}$ is linearly independent, $f(t) = 0$ for all $t \in G$ and the proof is complete. □

This theorem says that the number of irreducible representations of $G$ (up to isomorphic) is equal to the number of conjugacy classes of $G$. We have another consequence of Theorem 1.8.2:

**Proposition 1.8.3.** *Let $\chi_1, \dots, \chi_h$ be the distinct characters of irreducibles representations of $G$. Let $g$ be the order of $G$ and for $s \in G$, let $c(s)$ be the number of elements in the conjugacy class of $s$. Then we have:*

$$\sum_{i=1}^{h} \overline{\chi_i(s)}\chi_i(t) = \begin{cases} \frac{g}{c(s)} & \text{if } t \text{ is conjugate to } s\ , \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let $f_s : G \to \mathbb{C}$ be the function on $G$ such that $f_s(t) = 1$ if $t$ is conjugate to $s$ and $f_s(t) = 0$ otherwise. Since $f_s \in \mathcal{H}$, by Theorem 1.8.2, it can be written as $f_s = \sum_{i=1}^h \lambda_i\, \chi_i$. Because $\chi_1, \dots, \chi_h$ are orthonormal,

$$\lambda_i = \langle f_s, \chi_i\rangle = \frac{1}{g}\sum_{t\in G} f_s(t)\overline{\chi_i(t)} = \frac{c(s)}{g}\overline{\chi_i(s)}.$$

We then have for each $t \in G$,

$$f_s(t) = \frac{c(s)}{g}\sum_{i=1}^{h} \overline{\chi_i(s)}\chi_i(t).$$

Our proof is complete by evaluating $f_s$. □

Let $e$ be the identity of $G$. Then $c(e) = 1$ and $\chi_i(e)$ equals to the dimension of the corresponding irreducible representation of $\chi_i$. Hence, we have the following:

**Corollary 1.8.4.** *Let $G$ be a group of order $g$. Let $\chi_1, \dots, \chi_h$ be all the distinct characters of the irreducible representations of $G$ and let $n_1, \dots, n_h$ be the dimensions of their corresponding representations. Then $\sum_{i=1}^h n_i^2 = g$ and if $s \neq e$ then $\sum_{i=1}^h n_i\, \chi_i(s) = 0$.*

In Corollary 1.3.2, we know that every irreducible representation of an abelian group has dimension 1. In fact, the converse is also true.

**Corollary 1.8.5.** *$G$ is abelian if and only if all the irreducible representations of $G$ have dimension 1.*

**Proof.** Suppose that $W_1, \dots, W_h$ are distinct irreducible representations of $G$ of dimension $n_1, \dots, n_h$ respectively, where $h$ is the number of conjugacy classes of $G$. Suppose that $g$ is the order of $G$. By Corollary 1.8.4, $n_1^2 + \cdots + n_h^2 = g$. Since $G$ is abelian if and only if $h = g$, which is equivalent to all the $n_i$ are equal to 1, our claim follows. □

## 1.9. Characters of a Group

A representation of $G$ of dimension 1 is a homomorphism of $G$ into the multiplicative group $\mathbb{C}^*$ and is called a *character* of $G$. In particular, we call the trivial 1-dimensional representation of $G$, the *unit character* of $G$.

Let $\rho$ be a representation of $G$. Suppose that $\mu$ is a character of $G$ for which there exists a non-zero $v \in V_\rho$ such that $\rho_s(v) = \mu(s)v$ for every $s \in G$. Then $\mu$ is said to be an *eigenvalue* of $G$ with respective to $\rho$ and $v$ is said to be an *eigenvector* of $G$ that belongs to $\mu$.

Let $A$ be a finite abelian group. Then Proposition 1.8.3 says that the irreducible representation of $A$ are of dimension 1 and that their number is equal to $|A|$. Hence, in this case, the number of characters of $A$ is equal to the number of $A$. Furthermore, the set of characters of $A$ forms a multiplicative group $\hat{A}$ which is isomorphic to $A$.

For arbitrary group, the subgroup of $G$ generated by the set $\{sts^{-1}t^{-1} \mid s, t \in G\}$ is called the *commutator subgroup* of $G$ and denoted $G'$. $G'$ is the smallest normal subgroup of $G$ such that $G/G'$ is abelian. We can deduce that, $G$ has $[G : G']$ characters. The following properties for characters are useful.

**Lemma 1.9.1** (Orthogonality). *If $\chi$ is not the unit character of $G$, then $\sum_{s \in G} \chi(s) = 0$.*

**Proof.** Since $\chi$ is not the unit character, there exists $t \in G$ such that $\chi(t) \neq 1$. We have $\sum_{s\in G} \chi(s) = \sum_{s\in G} \chi(t)\chi(s)$. Subtracting both side by $\sum_{s\in G} \chi(s)$, we obtain $(\chi(t) - 1) \sum_{s\in G} \chi(s) = 0$. Since $\chi(t) \neq 1$, our proof is complete. □

**Lemma 1.9.2** (Artin's Lemma). *If $\chi_1, \dots, \chi_n$ are distinct characters of $G$, then the only elements $a_1, \dots, a_n$ in $\mathbb{C}$ such that $\sum_{i=1}^n a_i\chi_i(s) = 0$ for all $s \in G$ are $a_1 = \cdots = a_n = 0$.*

**Proof.** We prove the result by induction. We may assume that every $a_i \neq 0$. Since $\chi_1 \neq \chi_2$, there exists $t \in G$ such that $\chi_1(t) \neq \chi_2(t)$. We have $\sum_{i=1}^n a_i\chi_i(t)\chi_i(s) = 0$ and $\sum_{i=1}^n a_i\chi_1(t)\chi_i(s) = 0$. Subtracting these two relations we obtain $\sum_{i=2}^n a_i(\chi_1(t) - \chi_i(1))\chi_i(s) = 0$ for all $s \in G$. Since $a_2(\chi_1(t) - \chi_2(t)) \neq 0$, this contradicts the validity of the result for $n - 1$ and complete the proof. □

**Remark .** Suppose $G$ is abelian. Then $G$ is canonically isomorphic to the dual $\hat{\hat{G}}$ of $\hat{G}$. Hence the dual of these two lemmas is also true.

## 1.10. Restricted Representation

If $H \subseteq G$ is a subgroup, any representation $\rho$ of $G$ in $V$ restricts a representation of $H$ in $V$, denoted $\rho_H$ (or $\mathrm{Res}_H^G(V)$).

Suppose that $W$ is a subrepresentation of $\rho_H$, that is, a vector subspace of $V$ stable under $\rho_t$, for $t \in H$. Let $s \in G$; the vector space $\rho_s W$ depends only on the left coset $sH$ of $s$; indeed, if $t \in H$, we have $\rho_{st}(W) = \rho_s\rho_t(W) = \rho_s(W)$ because $\rho_t(W) = W$. Hence, if $\tau$ is a left coset of $H$ in $G$, we can thus define a subspace $W_\tau$ of $V$ to be $\rho_s W$ for any $s \in \tau$. Because the set of left cosets of $H$ are permuted among themselves by multiplying an element $s \in G$ on the left, it is clear that the $W_\tau$ are permuted among themselves by the $\rho_s$, $s \in G$. Their sum $\sum_{\tau \in G/H} W_\tau$ is thus a subrepresentation of $V$.

We are interested in the case that $G$ has an abelian subgroup.

**Proposition 1.10.1.** *Let $G$ be a group of order $g$ and let $A$ be an abelian subgroup of $G$ of order $a$. Then each irreducible representation of $G$ has dimension $\leq g/a$.*

**Proof.** Let $\rho$ be an irreducible representation of $G$ in $V$ and $\rho_A$ be the restriction to $A$. Suppose that $W \subseteq V$ is an irreducible subrepresentation of $\rho_A$. By Corollary 1.8.5, we have $\dim(W) = 1$. Since $V' = \sum_{\tau \in G/A} W_\tau$ is thus a subrepresentation of $V$ and $V$ is irreducible, we have that $V = V'$, and hence $\dim(V) \leq g/a$. □

## 1.11. Induced Representations

Let $H$ be a subgroup of $G$ and let $W$ be a subspace of $V$ which is stable under $H$. We say that the representation $\rho$ of $G$ in $V$ is *induced* by the representation $\theta$ of $H$ in $W$, if $V$ is equal to the direct sum of the $W_\tau$, $\tau \in G/H$ (thus, if $V = \bigoplus_{\tau \in G/H} W_\tau$). Recall that if $\tau$ is a left coset of $H$ in $G$, $W_\tau$ of $V$ is $\rho_s W$ for any $s \in \tau$. Therefore, we have $\dim(V) = \sum_{\tau \in G/H} \dim(W_\tau) = [G : H] \cdot \dim(W)$, where $[G : H]$ is the number of left cosets of $H$ in $G$, *i.e.* the index of $H$ in $G$. Later (Theorem 1.11.4) we will see that given a linear representation $\theta : H \to \mathrm{Aut}(W)$, there exists a unique (up to isomorphic) representation $\rho : G \to \mathrm{Aut}(V)$ such that $\rho$ in $V$ is induced by $\theta$ in $W$. In this case we write $V = \mathrm{Ind}_H^G(W)$ and $\rho = \mathrm{Ind}_H^G(\theta)$.

From the definition, it is easy to see that $\mathrm{Ind}_H^G(W \oplus W') = \mathrm{Ind}_H^G(W) \oplus \mathrm{Ind}_H^G(W')$.

**Example 1.11.1.** Take for $\rho$ the regular representation of $G$ in $V$; $V$ has a basis $(v_t)_{t \in G}$ such that $\rho_s(v_t) = v_{st}$. Let $W$ be the subspace of $V$ with basis $(v_t)_{t \in H}$. The representation $\theta$ of $H$ in $W$ is the regular representation of $H$ and it is clear that $\rho$ is induced by $\theta$.

Now we show the existence and uniqueness of induced representations.

**Lemma 1.11.2.** *If the representation $\rho : G \to \mathrm{Aut}(V)$ is induced by $\theta : H \to \mathrm{Aut}(W)$, and if $W'$ is a subspace of $W$ which is stable under $H$, then the subspace $V' = \sum_{\tau \in G/H} W'_\tau$ of $V$ is stable under $G$ and the representation of $G$ in $V'$ is induced by the representation of $H$ in $W'$.*

**Proof.** Let $\tau \in G/H$ and $t \in \tau$. Then we have $W'_\tau = \rho_t(W') \subseteq \rho_t(W) = W_\tau$. Since $V = \bigoplus_{\tau \in G/H} W_\tau$, it implies that $V' = \bigoplus_{\tau \in G/H} W'_\tau$. □

By using the lemma above, we can prove the existence of induced representation of $\theta : H \to \mathrm{Aut}(W)$. Because $\mathrm{Ind}_H^G(W \oplus W') = \mathrm{Ind}_H^G(W) \oplus \mathrm{Ind}_H^G(W')$, we may assume the $\theta$ is irreducible. In this case, (using Corollary 1.7.2) $\theta$ is isomorphic to a subrepresentation of the regular representation of $H$ and the regular representation of $H$ induces the regular representation of $G$ (*cf.* the example above). Applying Lemma 1.11.2, there exists a subrepresentation of the regular representation of $G$ which is induced by $\theta$.

In next section, we will give a concrete construction for the induced representation.

**Lemma 1.11.3.** *Suppose that the representation $\rho : G \to \mathrm{Aut}(V)$ is induced by $\theta : H \to \mathrm{Aut}(W)$. Let $\rho' : G \to \mathrm{Aut}(V')$ be a linear representation of $G$ and let $f : W \to V'$ be a $H$-linear map (i.e. $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$). Then there exists a unique linear map $F : V \to V'$ which extends $f$ (i.e. $F(w) = f(w)$ for all $w \in W$) and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.*

**Proof.** Let $\tau \in G/H$. If $F$ satisfies these conditions, for $s \in \tau$ and $w \in W$, we have $F(\rho_s(w)) = \rho'_s(F(w)) = \rho'_s(f(w))$. This determines $F$ on $\rho_s(W) = W_\tau$ and hence on $V$ because $V = \bigoplus_{\tau \in G/H} W_\tau$. This proves the uniqueness of $F$.

For the existence of $F$; if $v = \rho_s(w) \in W_\tau$, we define $F(v) = \rho'_s(f(w))$. This definition does not depend on the choice of $s$ in $\tau$ and $w$ in $W$. If $\rho_{st}(w') = \rho_s(w)$ for some $t \in H$ and $w' \in W$, then we have $\rho_t(w') = \theta_t(w') = w$. Hence, $\rho'_{st}(f(w')) = \rho'_s(\rho'_t(f(w'))) = \rho'_s(f(\theta_t(w'))) = \rho'_s(f(w))$. Again, since $V = \bigoplus_{\tau \in G/H} W_\tau$, by linearity, there exists a unique linear map $F : V \to V'$ which extends the partial mappings thus defined on every $W_\tau$. One easily checks that $F \circ \rho_{s'} = \rho'_{s'} \circ F$ for all $s' \in G$. In fact, if $v = \rho_s(w) \in W_\tau$, then $F \circ \rho_{s'}(\rho_s(w)) = F(\rho_{s's}(w)) = \rho'_{s's}(f(w)) = \rho'_{s'}(\rho'_s(f(w))) = \rho'_{s'} \circ F(\rho_s(w))$. □

**Theorem 1.11.4.** *Let $H$ be a subgroup of $G$ and let $\theta : H \to \mathrm{Aut}(W)$ be a linear representation of $H$ in $W$. Then there exists a unique (up to isomorphic) representation $\rho : G \to \mathrm{Aut}(V)$ such that $\rho$ in $V$ is induced by $\theta$ in $W$.*

**Proof.** Because we have proved the existence, we only have to prove the uniqueness. Suppose that $\rho : G \to \mathrm{Aut}(V)$ and $\rho' : G \to \mathrm{Aut}(V')$ are two representations of $G$ induced by $\theta : H \to \mathrm{Aut}(W)$. Considering $\iota : W \hookrightarrow V'$ the injection of $W$ into $V'$, by Lemma 1.11.3 there exists a unique linear map $F : V \to V'$ which is identity on $W$ and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. For every $\rho'_s(w) \in \rho'_s(W)$, we have $F(\rho_s(w)) = \rho'_s(F(w)) = \rho'_s(w)$. Hence the image of $F$ contains all the $\rho'_s(W)$ and thus $F$ is onto. Since $V$ and $V'$ have the same dimension $[G : H] \dim(W)$, we see that $F$ is an isomorphism which proves the uniqueness. □

## 1.12. A Concrete Construction for Induced Representation

Let $G$ be a finite group and let $H$ be a subgroup of $G$. Let $\theta : H \to \mathrm{Aut}(W)$ be a linear representation of $H$. Define a vector space $V$ to be the set of all functions $f : G \to W$ that satisfy

$$f(ts) = \theta_t(f(s)) \ \forall\, t \in H,\ s \in G.$$

Thus, an element $f \in V$ is uniquely decided by its values on a system of representatives $H \backslash G$ of the right cosets of $H$ in $G$. Define an action of $G$ on $V$ by

$$\rho_s(f)(r) = f(r \cdot s) \ \forall\, r,\ s \in G \text{ and } f \in V.$$

It is easy to check that $\rho$ gives a linear representation of $G$ with representation space $V$.

We embed $W$ into $V$ by mapping each $w \in W$ onto the function $f_w : G \to W$ defined by

$$f_w(s) = \begin{cases} \theta_s(w) & \text{if } s \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly we have that $\rho_t(f_w) = f_{\theta_t(w)}$ for all $t \in H$ and $W$ is isomorphic onto the subspace of $V$ consisting of functions which vanish off $H$.

Let now $R$ be a system of representatives of the left cosets $G/H$. For every $f \in V$ and $r \in R$, we define a function $f_r \in V$ by

$$f_r(s) = \begin{cases} f(s) & \text{if } s \in Hr^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = \sum_{r \in R} \rho_r(\rho_r^{-1}(f))$ and $\rho_r^{-1}(f_r) = \rho_{r^{-1}}(f_r)$ belongs to $W$ (after identifying $W$ with its image in $V$). Thus $V = \bigoplus_{\tau \in G/H} W_\tau$ and hence $V = \mathrm{Ind}_H^G(W)$.

There is another point of view of induced representation. Let $\rho$ be a linear representation of $G$. Then $V_\rho$ can be also considered as a module over the group-ring $\mathbb{C}[G]$. Using this form, if $\rho'$ is another representation of $G$, then we write $(\rho, \rho') = \dim \mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'})$. The form $(\rho, \rho')$ is clearly symmetric and bilinear. In fact, decomposing $V_\rho$ and $V_{\rho'}$ into direct sum of irreducible representations, by Theorem 1.7.1 we have that

$$\langle \chi_\rho, \chi_{\rho'} \rangle = (\rho, \rho')\ .$$

From this point of view, for induced representation, we obtain also a canonical isomorphism

$$\mathrm{Ind}_H^G(W) \cong \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W\ .$$

This characterization of induced representation makes it obvious that the induced representation exists and is unique. On the other hand, given a $\mathbb{C}[G]$-module $V$ which is a direct sum $V = \oplus_{i\in I} W_i$ of vector space permuted transitively by $G$. Choose $i_0 \in I$ and $W = W_{i_0}$ and let $H$ be the subgroup $H = \{\ s \in G\ |sW = W\}$. Then it is clear that the $\mathbb{C}[G]$-module $V$ is induced by the $\mathbb{C}[H]$-module $W$.

This form of induced representation is convenient to prove the following fundamental properties, by using elementary property of tensor product.

**Proposition 1.12.1.** *Let $J$ be a subgroup of $H$ and $H$ be a subgroup of $G$.*

(1) (Lemma 1.11.3) *Let $W$ be a $\mathbb{C}[H]$-module and let $E$ be a $\mathbb{C}[G]$-module. Then we have*

$$\mathrm{Hom}_{\mathbb{C}[H]}(W, E) \cong \mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G(W), E)\ .$$

(2) *Let $U$ be a $\mathbb{C}[J]$-module. Then*

$$\mathrm{Ind}_J^G(U) = \mathrm{Ind}_H^G(\mathrm{Ind}_J^H(U))\ .$$

## 1.13. Characters of Induced Representations

Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation of $G$ which is induced by the representation $\theta : H \to \mathrm{Aut}(W)$ and let $\chi_\rho$ and $\chi_\theta$ be the corresponding characters. Since by the uniqueness, $\theta$ determines $\rho$ up to isomorphic, we ought to be able to compute $\chi_\rho$ from $\chi_\theta$.

**Theorem 1.13.1.** *Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation of $G$ which is induced by the representation $\theta : H \to \mathrm{Aut}(W)$ and let $\chi_\rho$ and $\chi_\theta$ be the corresponding characters. Let $h$ be the order of $H$. For each $s \in G$, we have*

$$\chi_\rho(s) = \frac{1}{h} \sum_{\substack{r\in G \\ r^{-1}sr\in H}} \chi_\theta(r^{-1}sr).$$

**Proof.** Choose $R$ being a system of representatives of $G/H$, so $V = \bigoplus_{r\in R} \rho_r(W)$. For $s \in G$ and $r \in R$, we have that $sr = r't$ with $r' \in R$ and $t \in H$. We see that $\rho_s$ sends $\rho_r(W)$ into $\rho_{r'}(W)$. We choose a basis of $V$ which is the union of bases of $\rho_r(W)$, $r \in R$. The indices $r$ such that $r \neq r'$ give zero diagonal terms, and for the indices $r$ such that $r = r'$, $\rho_r(W)$ is stable under $\rho_s$ (because $W$ is stable under $\rho_t = \theta_t$, for $t \in H$). Observe that $r = r'$ if and only if $r^{-1}sr = t \in H$. We thus only have to compute the trace of the restriction of $\rho_s$ on $\rho_r(W)$ for those $r \in R$ such that $r^{-1}sr \in H$. Note that in this case $\rho_s \circ \rho_r = \rho_r \circ \rho_t = \rho_r \circ \theta_t$ and $\rho_r$ defines an isomorphism of $W$ into $\rho_r(W)$. Hence the restriction of $\rho_s$ on $\rho_r(W)$ is equal to $\rho_r\theta_t\rho_r^{-1}$ and thus its trace is equal to that of $\theta_t$, that is, to $\chi_\theta(t) = \chi_\theta(r^{-1}sr)$. Our formula follows from the fact that if $r^{-1}sr \in H$ , then every element $u \in rH$ has the property $u^{-1}su \in H$ and $\chi_\theta(u^{-1}su) = \chi_\theta(r^{-1}sr)$. □

Let $H$ be a subgroup of $G$. For a linear representation of $\rho : G \to \mathrm{Aut}(V)$ with character $\chi_\rho$, we denote by $\mathrm{Res}_H^G(\chi_\rho)$ the character of the restricted representation $\rho_H$ of $G$ on $H$. For a linear representation of $\theta : H \to \mathrm{Aut}(W)$ with character $\chi_\theta$, we denote by $\mathrm{Ind}_H^G(\chi_\theta)$ the character of the representation of $G$ induced by $\theta$.

**Theorem 1.13.2** (Frobenius Reciprocity)**.** *Let $H$ be a subgroup of $G$. Let $\rho : G \to \mathrm{Aut}(V)$ be a linear representation of $G$ with character $\chi_\rho$ and let $\theta : H \to \mathrm{Aut}(W)$ be a linear representation of $H$ with character $\chi_\theta$. Then we have*

$$\langle\chi_\rho, \mathrm{Ind}_H^G(\chi_\theta)\rangle_G = \langle\mathrm{Res}_H^G(\chi_\rho), \chi_\theta\rangle_H,$$

*where $\langle\ ,\ \rangle_G$ and $\langle\ ,\ \rangle_H$ denote the inner products of the spaces of class functions on $G$ and $H$ defined in* 1.7.

**Proof.** Observe first that if $\rho$ and $\rho'$ are linear representations of $G$ in $V$ and $V'$ with characters $\chi$ and $\chi'$, respectively, then $\langle\chi, \chi'\rangle_G$ is equal to $\dim(\mathrm{Hom}_G(V, V'))$. Lemma 1.11.3 shows that every $H$-linear mapping from $W$ into $\mathrm{Res}^G_H V$ can be uniquely extended to a $G$-linear mapping from $\mathrm{Ind}^G_H(W)$ into $V$. Therefore,

$$\mathrm{Hom}_G(\mathrm{Ind}^G_H(W), V) \simeq \mathrm{Hom}_H(W, \mathrm{Res}^G_H V)$$

and our theorem follows. $\square$

**Remark .** Let $\rho$ be an irreducible representation of $G$ in $V$ and let $\theta$ be an irreducible representation of $H$ in $W$. Frobenius reciprocity says that the number of times that $W$ occurs in $\mathrm{Res}^G_H V$ is equal to the number of times that $V$ occurs in $\mathrm{Ind}^G_H(W)$.

## 1.14. Restrictions of Induced Representations

Let $H$ and $J$ be two subgroups of $G$, and let $\theta : H \to \mathrm{Aut}(W)$ be a linear representation of $H$, and let $V = \mathrm{Ind}^G_H(W)$. We shall determine the restriction $\mathrm{Res}^G_J(V)$ of $V$ to $J$.

First choose a set of representatives $S$ for the double cosets $J\backslash G/H$; this means that $G$ is the disjoint union of the $JsH$ for $s \in S$. Given $s \in S$, let $H_s = sHs^{-1} \cap J$, which is a subgroup of $J$. Define a homomorphism $\theta^s : H_s \to \mathrm{Aut}(W)$ by setting $\theta^s(x) = \theta(s^{-1}xs)$, for $x \in H_s$. This is a linear representation of $H_s$. Though, the representation space for $\theta^s$ is also $W$, to distinguish it with the representation $\theta$ we denote it by $W_s$. Since $H_s$ is a subgroup of $J$, the induced representation $\mathrm{Ind}^J_{H_s}(W_s)$ is defined.

**Proposition 1.14.1.** *Let $H$ and $J$ be two subgroups of $G$ and $S$ be a representatives for the double cosets $J\backslash G/H$. The representation $\mathrm{Res}^G_J(\mathrm{Ind}^G_H(W))$ is isomorphic to the direct sum of the representations $\mathrm{Ind}^J_{H_s}(W_s)$, for $s \in S$.*

**Proof.** Let $\rho = \mathrm{Ind}^G_H(\theta)$ and for $s \in S$ let $V(s)$ be the subspace of $V = \mathrm{Ind}^G_H(V)$ generated by $\rho_x(W)$, for all $x \in JsH$. $V(s)$ is a $\mathbb{C}[J]$-module and the space is a direct sum of the $V(s)$. It remains to claim that $V(s)$ is $\mathbb{C}[J]$-isomorphic to $\mathrm{Ind}^J_{H_s}(W_s)$. $V(s)$ is the direct sum of $\rho_x(\rho_s(W))$, $x \in J/H_s$ and the subgroup of $J$ consisting of the elements $x$ such that $\rho_x(\rho_s(W)) = \rho_s(W)$ is equal to $H_s$. Therefore, $V(s) = \mathrm{Ind}^J_{H_s}(\rho_s(W))$. Consider the map $f : W_s \to \rho_s(W)$ given by $f(w) = \rho_s(w)$. This is a $\mathbb{C}[H_s]$-isomorphism, because $f((\theta^s)_x(w)) = \rho_s(\theta_{s^{-1}xs}(w)) = \rho_s(\rho_{s^{-1}xs}(w)) = \rho_x(\rho_s(w))$, for $x \in H_s$. Our claim follows. $\square$

We apply Proposition 1.14.1 to the case $J = H$; the representation $\theta$ of $H$ defines a representation $\mathrm{Res}^H_{H_s}(\theta)$ of $H_s$ which should not be confused with the representation $\theta^s$ defined above.

**Proposition 1.14.2** (Mackey's irreducibility criterion)**.** *The induced representation $\rho = \mathrm{Ind}^G_H(\theta)$ is irreducible if and only the following two conditions are satisfied:*

(1) *$\theta$ is irreducible.*
(2) *For each $s \in G - H$, $(\mathrm{Res}^H_{H_s}(\theta), \theta^s) = 0$, as representations for $H_s$.*

**Proof.** From Proposition 1.14.1, we have $\mathrm{Res}^G_H(\rho) = \oplus_{s\in H\backslash G/H}\mathrm{Ind}^H_{H_s}(\theta^s)$. Applying Frobenius reciprocity (Theorem 1.13.2), we obtain

$$(\rho,\rho) = (\theta, \mathrm{Res}^G_H(\rho)) = \sum_{s\in H\backslash G/H} (\theta, \mathrm{Ind}^H_{H_s}(\theta^s)) = \sum_{s\in H\backslash G/H} (\mathrm{Res}^H_{H_s}(\theta), \theta^s)\ .$$

Since $\rho$ is irreducible if and only if $(\rho, \rho) = 1$, our proof is complete. $\square$

**Corollary 1.14.3.** *Suppose that $H$ is a normal subgroup of $G$. In order that $\mathrm{Ind}^G_H(\theta)$ is irreducible, it is necessary and sufficient that $\theta$ is irreducible and not isomorphic to any of its conjugates $\theta^s$, for $s \notin H$.*

**Proof.** Indeed, we have then $H_s = H$ and $\mathrm{Res}^H_{H_s}(\theta) = \theta$. $\square$

## 1.15. Method of Little Group

The principle of the method of little group is to show that the irreducible presentations of $G$ can be constructed from those of certain subgroups of $G$.

**Proposition 1.15.1.** *Let $A$ be a normal subgroup of $G$, and let $\rho$ be an irreducible presentation of $G$. Then:*

(1) *either there exists a subgroup $H$, $A \subseteq H \subsetneq G$, and an irreducible representation $\theta$ of $H$ such that $\rho$ is induced by $\theta$;*

(2) *or else the restriction of $\rho$ to $A$ is isotypic.*

**Proof.** Let $V_\rho = \oplus V_i$ be the canonical decomposition of $\mathrm{Res}^G_A(\rho)$ into a direct sum of isotypic representations. Because $A$ is normal in $G$, for $s \in G$ we see that $\rho(s)$ permutes the $V_i$ and since $V$ is irreducible, $G$ permutes them transitively. Let $V_{i_0}$ be one of these. If $V_{i_0} = V$, we have case (2). Otherwise, let $H$ be the subgroup consisting of those $s \in G$ such that $\rho(s)(V_{i_0}) = V_{i_0}$. we have $A \subseteq H \subsetneq G$ and $\rho$ is induced by the nature representation of $H$ in $V_{i_0}$, which is irreducible by (1) of Proposition 1.14.2. □

Let $J$ and $H$ be two subgroup of $G$, with $J$ normal. We say that $G$ is the *semidirect product* of $H$ by $J$, if $G = J \cdot H$ and $H \cap J = \{1\}$. Suppose that $J$ is abelian and $G$ is the semidirect product of $H$ by $J$. We are going to show that the irreducible representations of $G$ can be constructed from those of certain subgroups of $H$ (this is the method of "little group" of Wigner and Mackey).

Since $J$ is abelian, its irreducible representations are of dimension 1 and form the character group $\hat{J}$ of $J$. The group $G$ act on $\hat{J}$ by

$$(s * \psi)(j) = \psi(s^{-1}js), \quad \text{for } s \in G,\ \psi \in \hat{J},\ j \in J.$$

Given $\psi \in \hat{J}$, the subset of $\hat{J}$ consisting of all elements $t * \psi$ with $t \in H$ is denoted by $H\psi$ and is called the *orbit* of $\psi$ under $H$.

Let $X$ be a system of representatives for the orbits of $\hat{J}$ under $H$. For each $\psi \in X$, let $H_\psi$ be a subgroup of $H$ consisting of those elements $h$ such that $h * \psi = \psi$, and let $G_\psi = J \cdot H_\psi$ be the corresponding subgroup of $G$. We can extend $\psi$ to a function of $G_\psi$ by setting

$$\psi(jh) = \psi(j) \ \text{ for } j \in J \text{ and } h \in H_\psi.$$

Because $J$ is normal and $H_\psi$ fixes $\psi$, we have that $\psi((jh)(j'h')) = \psi(j(hj'h^{-1})hh') = \psi(j(hj'h^{-1})) = \psi(j)\psi(j') = \psi(jh)\psi(j'h')$ for $j$, $j' \in J$ and $h$, $h' \in H_\psi$. Hence, $\psi$ is a character of $G_\psi$. Now let $\theta$ be an irreducible representation of $H_\psi$. By composing with the canonical projection $G_\psi \to H_\psi$, we obtain an irreducible representation $\tilde{\theta}$ of $G_\psi$. The tensor product $\psi \otimes \tilde{\theta}$ is also an irreducible representation of $G_\psi$. Let $\rho_{\psi,\theta} = \mathrm{Ind}^G_{G_\psi}(\psi \otimes \tilde{\theta})$.

**Proposition 1.15.2.** *Let $X$ be a system representatives of the orbits of $\hat{J}$ under $H$. For $\psi \in X$ and let $\theta$ be an irreducible representation of $G_\psi$. Then $\rho_{\psi,\theta}$ is an irreducible representation of $G$. Furthermore, given an irreducible representation $\rho$ of $G$, there exist $\psi \in X$ and $\theta$ such that $\rho$ is isomorphic to $\rho_{\psi,\theta}$.*

**Proof.** We prove the irreducibility of $\rho_{\psi,\theta}$ by using Mackey's criterion (Proposition 1.14.2). For $s \notin G_\psi$, Let $G_s = G_\psi \cap sG_\psi s^{-1}$. We only have to claim that

$$((\psi \otimes \tilde{\theta})^s, \mathrm{Res}^{G_\psi}_{G_s}(\psi \otimes \tilde{\theta})) = 0.$$

Since $A \subseteq G_s$, it is enough to check for the restrictions of theses representations to $J$. The restriction of $(\psi \otimes \tilde{\theta})^s$ to $J$ is $s * \psi$ and the restriction of $(\psi \otimes \tilde{\theta})$ to $J$ is $\psi$. Since $s \notin G_\psi = J \cdot H_i$, we have $s * \psi \neq \psi$. Our claim follows.

Finally, let $\rho$ be an irreducible representation of $G$. Let $V_\rho = \oplus_{\psi \in \hat{J}} W_\psi$ be the canonical decomposition of $\mathrm{Res}^G_J(\rho)$ ($W_\psi$ is the space of vectors $w$ in $V_\rho$ such that $\rho_j(w) = \psi(j)w$ for $j \in J$). If $s \in G$

and $w \in W_\psi$, we have $\rho_j(\rho_s(w)) = \rho_s(\rho_{s^{-1}js}(w)) = \rho_s(\psi(s^{-1}js)w) = (s * \psi)(j)(\rho_s(w))$. Hence, $\rho_s$ transforms $W_\psi$ into $W_{s*\psi}$. Thus, if $W_\psi$ is nonzero, then $W_{\psi'}$ is nonzero for every $\psi'$ in the orbit of $\psi$ under $H$. Suppose $\psi \in X$ such that $W_\psi$ is nonzero. $H_\psi$ maps $W_\psi$ into itself and hence $W_\psi$ is a $\mathbb{C}[H_\psi]$-module. Choose an irreducible sub-$\mathbb{C}[H_\psi]$-module $U_\psi$ of $W_\psi$ and let $\theta$ be the corresponding representation of $H_\psi$. It is clear that the corresponding representation of $\mathrm{Res}^G_{G_\psi}(\rho)$ on $U_\psi$ is isomorphic to $\psi \otimes \tilde{\theta}$. Hence $(\mathrm{Res}^G_{G_\psi}(\rho), \psi \otimes \tilde{\theta}) \geq 1$. By Proposition 1.13.2, we have $(\rho, \mathrm{Ind}^G_{G_\psi}(\psi \otimes \tilde{\theta})) \geq 1$. Since both $\rho$ and $\mathrm{Ind}^G_{G_\psi}(\psi \otimes \tilde{\theta}) = \rho_{\psi,\theta}$ are irreducible, this implies they are isomorphic. □

**Remark .** Let $X$ be a system representatives of the orbits of $\hat{J}$ under $H$. Let $\psi$, $\psi' \in X$ and suppose that $\rho_{\psi,\theta}$ is isomorphic to $\rho_{\psi',\theta'}$. From the proof above, we know that the restriction of $\rho_{\psi,\theta}$ involves only characters belonging to the orbit of $\psi$ under $H$. Hence, we have $\psi = \psi'$. Further, the space $W_\psi$ is stable under $H_\psi$ and one checks immediately that $\theta$ is isomorphic to $\theta'$. This says that given an irreducible presentation $\rho$ of $G$, the $\psi$ and $\theta$ we find in (2) of Proposition 1.15.2 is in fact uniquely determined by $\rho$.

## 1.16. The Schur Algebra

Given a representation $\rho$ of $G$, $\mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_\rho)$ is an algebra over $\mathbb{C}$ called the *Schur algebra*.

If $\rho$ is irreducible, then $\mathrm{Hom}_{\mathbb{C}[G]}(V_{n\rho}, V_{n\rho})$ is isomorphic to $M_n(\mathbb{C})$, the algebra of all $n \times n$ matrices over $\mathbb{C}$. If $\rho = \bigoplus n_i\rho_i$ is the canonical decomposition of $\rho$, then, by Schur's lemma, $\mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_\rho) = \bigoplus M_{n_i}(\mathbb{C})$. It follows that $\rho$ has no multiple components, *i.e.* $n_i = 1$ for all $i$, if and only if $\mathrm{Hom}_{\mathbb{C}[G]}(V_\rho, V_\rho)$ is commutative.

**Proposition 1.16.1.** *Let $H$ and $J$ be subgroup of $G$ and let $\theta$ and $\sigma$ be representations of $H$ and $J$, respectively. Then $\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}^G_H(V_\theta), \mathrm{Ind}^G_J(V_\sigma))$ is isomorphic to the vector space of all functions $F : G \to \mathrm{Hom}_{\mathbb{C}}(V_\theta, V_\sigma)$ satisfying*

$$(\star) \qquad F(jsh) = \sigma_j \circ F(s) \circ \theta_h \text{ for all } j \in J,\ s \in G \text{ and } h \in H.$$

**Proof.** Let $\hat{\theta} = \mathrm{Ind}^G_H(\theta)$, $\hat{\sigma} = \mathrm{Ind}^G_J(\sigma)$ and $n = [G : H]$. Let $V_{\hat{\theta}}$ and $V_{\hat{\sigma}}$ be the representation spaces we constructed in 1.12 for $\hat{\theta}$ and $\hat{\sigma}$, respectively. Denote by $\mathcal{F}$ the space of all functions $F : G \to \mathrm{Hom}_{\mathbb{C}}(V_\theta, V_\sigma)$ satisfying $(\star)$. For every $F \in \mathcal{F}$, denote by $T_F$ the element of $\mathrm{Hom}_{\mathbb{C}}(V_{\hat{\theta}}, V_{\hat{\sigma}})$ defined by

$$T_F(f)(s) = \frac{1}{n}\sum_{r\in G} F(sr^{-1})(f(r)), \quad \text{for } f \in V_{\hat{\theta}} \text{ and } s \in G.$$

We remark that $T_F(f) \in V_{\hat{\sigma}}$ for $f \in V_{\hat{\theta}}$ because $F(js) = \sigma_j \circ F(s)$ for $j \in J$. Moreover, the map $F \to T_F$ is a homomorphism $\mathcal{F} \to \mathrm{Hom}_{\mathbb{C}[G]}(V_{\hat{\theta}}, V_{\hat{\sigma}})$ since clearly

$$T_F(\hat{\theta}_t(f))(s) = \frac{1}{n}\sum_{r\in G} F(sr^{-1})(f(rt)) = \frac{1}{n}\sum_{r\in G} F(str^{-1})(f(r)) = \hat{\sigma}_t(T_F(f))(s).$$

We have to show that this is in fact an isomorphism.

It is injective. Indeed, suppose that $T_F = 0$. Given $t \in G$ and $v \in V_\theta$, we define a function $f_{t,v} \in V_{\hat{\theta}}$ by

$$f_{t,v}(s) = \begin{cases} \theta_h(v) & \text{if } s = ht \text{ for some } h \in H, \\ 0 & \text{if } s \notin Ht. \end{cases}$$

Then we have that $0 = T_F(f_{t,v})(s) = \frac{1}{n}\sum_{h\in H} F(st^{-1})(v) = F(st^{-1})(v)$ (here, we use the fact that $F(sh) = F(s) \circ \theta_h$). Hence $F(st^{-1}) = 0$ for all $s, t \in G$, *i.e.* $F = 0$.

To show the map is surjective, we first remark that $\dim(\mathcal{F})$ is not equal to $|J\backslash G/H| \dim(\theta) \dim(\sigma)$. In fact, though it suffices to give values of $F \in \mathcal{F}$ on a system of representatives of $J\backslash G/H$, we need an extra condition. Because it is possible that there exist $j \neq j'$ in $J$ and $h \neq h'$ in $H$ such that

$jsh = j'sh'$. Hence for $s \in J\backslash G/H$, if $jsh = j'sh'$ for $j, j' \in J$ and $h, h' \in H$, in order to have $F \in \mathcal{F}$ (*i.e.* satisfies the relation ($\star$)) we need:

$$(\star') \qquad F(s) = \sigma_{j^{-1}j'} \circ F(s) \circ \theta_{h'h^{-1}} \ .$$

Let $x = j^{-1}j' \in sHs^{-1} \cap J$. Recall in section 1.14, we define a representation of $H_s = sHs^{-1} \cap J$ by setting $\theta^s_x = \theta_{s^{-1}xs}$ for $x \in H_s$. The relation ($\star'$) is just saying that $F(s) \in \mathrm{Hom}_{\mathbb{C}[H_s]}(\theta^s, \mathrm{Res}^J_{H_s}(\sigma))$. Observe that $\dim(\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}^G_H(V_\theta), \mathrm{Ind}^G_J(V_\sigma))) = (\hat{\theta}, \hat{\sigma})$. According to Frobenius reciprocity, we have $(\hat{\theta}, \hat{\sigma}) = (\mathrm{Res}^G_J(\hat{\theta}), \sigma)$. However, from Proposition 1.14.1 we have $\mathrm{Res}^G_J(\hat{\theta}) = \oplus_{s\in J\backslash G/H} \mathrm{Ind}^J_{H_s}(\theta^s)$. Once more applying the Frobenius reciprocity, we obtain:

$$(\hat{\theta}, \hat{\sigma}) = \sum_{s\in J\backslash G/H} (\theta^s, \mathrm{Res}^J_{H_s}(\sigma)) \ .$$

This proves that the dimension of $\mathcal{F}$ is equal to the dimension of $\mathrm{Hom}_{\mathbb{C}[G]}(V_{\hat{\theta}}, V_{\hat{\sigma}})$ and hence proves the surjectivity. □

**Corollary 1.16.2.** *Let $H$ and $J$ be subgroup of $G$ and let $\theta$ and $\sigma$ be representations of $H$ and $J$, respectively. We have* $(\mathrm{Ind}^G_H(\theta), \mathrm{Ind}^G_J(\sigma)) \leq |J\backslash G/H| \dim(\theta)\dim(\sigma)$.

The most interesting conclusion of Proposition 1.16.1 arises in the special case where $H = J$ and $\theta = \sigma$. In this case, Proposition 1.16.1 turns $\mathcal{F}$ into the Schur algebra of $\mathrm{Ind}^G_H(\theta)$ and the product between two elements $F_1$ and $F_2$ of $\mathcal{F}$ is given by

$$(F_1 \cdot F_2)(s) = \frac{1}{[G:H]} \sum_{r\in G} F_1(sr^{-1})F_2(r)$$

which can be easily verified from the basic relation $T_{F_1} \circ T_{F_2} = T_{F_1 \cdot F_2}$ .

# The Group $\mathrm{GL}(2, \mathbb{F}_q)$ and Its Subgroups

In this chapter $\mathbb{F}_q$ is a finite field with $q$ elements, where $q > 2$.

## 2.1. Notational Conventions

We denote by $G$ the group $\mathrm{GL}(2, \mathbb{F}_q)$ of all $2 \times 2$ invertible matrices with entries in $\mathbb{F}_q$. We further reserve some letters for distinguished subgroups of $G$ that will concern us in the sequel.

The letter $B$ stands for the *Borel* subgroup of $G$ consisting of all upper triangular matrices

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbb{F}_q^\times;\ \beta \in \mathbb{F}_q \right\} .$$

$B$ contains the normal abelian subgroup

$$U = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \mid \beta \in \mathbb{F}_q \right\} .$$

The quotient group $B/U$ is isomorphic to the *Cartan group*

$$D = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \mid \alpha, \delta \in \mathbb{F}_q^\times \right\} .$$

Another important subgroup of $B$ is

$$P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_q^\times,\ \beta \in \mathbb{F}_q \right\} .$$

A complement of $U$ in $P$ is the group

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \mid \alpha \in \mathbb{F}_q^\times \right\} .$$

The center of $G$ is

$$Z = \left\{ \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} \mid \delta \in \mathbb{F}_q^\times \right\} .$$

The idempotent matrix

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

will play an important role in the sequel.

## 2.2. The Subgroups $U$ and $P$

$U$ is a normal abelian subgroup of $B$ which contains all unipotent upper triangular matrices. This group is isomorphic to the additive group $\mathbb{F}_q^+$ of the field $\mathbb{F}_q$. Indeed

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta + \beta' \\ 0 & 1 \end{pmatrix} .$$

We shall sometimes identify an element $\beta$ of $\mathbb{F}_q$ with the corresponding matrix of $U$.

$P$ is another important subgroup of $B$. It is of order $(q-1)q$. Note that $U$ is contained in $P$. In fact, $U$ is also the commutator subgroup of $P$.

$A$ is a complement of $U$ in $P$ and is isomorphic to $\mathbb{F}_q^\times$. Thus $P$ is the semidirect product of $A$ by $U$. The action of $A$ on $U$ by conjugation corresponds to the action of $\mathbb{F}_q^\times$ on $\mathbb{F}_q^+$ by multiplication

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha\beta \\ 0 & 1 \end{pmatrix} .$$

We use the method of little groups of Wigner and Mackey (*cf.* Section 1.15) in order to determine the representations of $P$.

We consider first all the irreducible representations of $U$. Fix a non-unit character $\psi$ of $U$ (we consider it also as a character of $\mathbb{F}_q^+$). For every $s \in G$, we define an action $s * \psi(u) = \psi(s^{-1}us)$. Consider the orbit of $\psi$ under $A$. For $a, a' \in A$, we have that $a * \psi = a' * \psi$ if and only if $a = a'$. Indeed, if

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad a' = \begin{pmatrix} \alpha' & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix},$$

then $a * \psi(u) = \psi(\alpha^{-1}\beta)$, and $a * \psi = a' * \psi$ implies that $\psi((\alpha - \alpha')\beta) = 1$ for all $\beta \in \mathbb{F}_q$. Since $\psi$ is non-unit, this implies that $\alpha = \alpha'$ and hence $a = a'$. We thus get $q-1$ distinct representations of $U$. These together with the unit representation of $U$, are all the irreducible representations of $U$, since $\mid U \mid = \mid \hat{U} \mid = q$.

From the discussion above, we know that the non-unit character $\psi$ and the unit character 1 is a system of representatives of the orbits of $\hat{U}$ under $A$. We also know that $A_\psi = \{a \in A \mid a * \psi = \psi\} = \{1\}$. The only representation of $A_\psi$ is the trivial representation and $P_\psi = U \cdot A_\psi = U$. Hence, by Proposition 1.15.2 we have $\rho_{\psi,1} = \mathrm{Ind}_U^P(\psi)$ is an irreducible representation of $P$. For the unit character 1 of $U$, it is clear that $A_1 = A$ and $P_1 = U \cdot A_1 = P$. Every irreducible representation $\theta$ of $A_1$, *i.e.* every character of $A$, can be lifted to a character $\tilde{\theta}$ of $P$ defined by $\tilde{\theta}(ua) = \theta(a)$, for $u \in U$ and $a \in A$ (this is $\rho_{1,\theta}$ in Proposition 1.15.2). Notice that the $q-1$ characters $\tilde{\theta}$ of $P$ obtained in this way are all the characters of $P$ since $[P : P'] = [P : U] = q-1$. Using Proposition 1.15.2 again, we obtain all the irreducible representations of $P$. We have therefore proved the following:

**Theorem 2.2.1.** *The group $P$ has $q$ irreducible representations:*

(1) $q-1$ *of them are* 1*-dimensional representations which are the lifting of the characters of $A$; and*

(2) *one* $(q-1)$*-dimensional representation which is* $\mathrm{Ind}_U^P(\psi)$*, where $\psi$ is any non-unit character of $U$.*

**Remark .** The $(q-1)$-dimensional irreducible representation of $P$ is independent of the choice of non-unit character $\psi$ of $U$. We fix for the rest of this note a non-unit character $\psi$ of $U$ and let

$$\pi = \mathrm{Ind}_U^P(\psi) .$$

We would like to show more directly that $\pi$ is independent of the choice of $\psi$. Consider first

$$\mathrm{Res}_U^P \mathrm{Ind}_U^P(\psi) = \bigoplus_{a \in A} a * \psi .$$

This can be proved by applying Proposition 1.14.1 to the case $G = P$, $J = H = U$ and $U$ is normal in $P$ (so that $U_s = U$ and $\psi^s = s * \psi$). We can also construct this direct sum concretely. For every $a \in A$, we define a function $f_a \in \mathrm{Ind}_U^P V_\psi$ by

$$f_a(a') = \begin{cases} 1 & \text{if } a' = a^{-1}, \\ 0 & \text{if } a' \neq a^{-1} \end{cases}, \quad \text{where } a' \in A.$$

Then we can see that $f_a$ is an eigenvector of $U$ that belongs to the eigenvalue $a * \psi$, *i.e.* $\pi_u(f_a)(p) = f_a(pu) = (a*\psi)(u)f_a(p)$ for every $p \in P$ and $u \in U$. Thus, the vector $f_a$ generates the one-dimensional space $V_{a*\psi}$. If we let $a$ vary, we get $q-1$ linearly independent vector $f_a$ of the $(q-1)$-dimensional vector space $\mathrm{Ind}_U^P(V_\psi)$. Hence we have the direct sum decomposition. From this we can show again that $\mathrm{Ind}_U^P(\psi)$ is an irreducible representation of $P$ (this can also be seen by using Corollary 1.14.3). Indeed, by Frobenius reciprocity

$$(\mathrm{Ind}_U^P(\psi), \mathrm{Ind}_U^P(\psi)) = (\psi, \bigoplus_{a\in A} a * \psi) = \sum_{a \in A} (\psi, a * \psi) = 1 \ .$$

Now for any non-unit character $\psi'$ of $U$, we have $\psi' = a * \psi$ for some $a' \in A$. Hence

$$\mathrm{Res}_U^P \mathrm{Ind}_U^P(\psi') = \bigoplus_{a\in A} a * (a' * \psi) = \bigoplus_{a\in A} a * \psi \ .$$

and as before $\mathrm{Ind}_U^P(\psi')$ is irreducible. Therefore, by Frobenius reciprocity again, $(\mathrm{Ind}_U^P(\psi), \mathrm{Ind}_U^P(\psi')) = 1$. Thus, $\pi = \mathrm{Ind}_U^P(\psi) = \mathrm{Ind}_U^P(\psi')$.

Once we know that $\pi$ is an irreducible representation of $P$, combining with the $q-1$ characters of $P$, Theorem 2.2.1 can also be proved by using Corollary 1.8.4 to show that there is no additional representation of $P$. In fact, we have that $(q-1)\cdot 1^2 + (q-1)^2 = q(q-1) =\mid P \mid$.

We can also show that there is no additional representation of $P$ without using the counting method (Corollary 1.8.4). Let $\sigma$ be an arbitrary irreducible representation of $P$. If there exists a non-unit character $\psi$ of $U$ such that $(\mathrm{Res}_U^P(\sigma), \psi) > 0$, then we have $(\sigma, \pi) = (\mathrm{Res}_U^P(\sigma), \psi) > 0$. Hence $\sigma = \pi$. Otherwise, $\mathrm{Res}_U^P(\sigma)$ is a multiple of the unit character of $U$, *i.e.*, $\sigma_u(v) = v$ for every $v \in V_\sigma$. Since $A$ is abelian, there exists $0 \neq v \in V_\sigma$ and a character $\theta$ of $A$ such that $\sigma_a(v) = \theta(a)v$ for every $a \in A$. Hence, if $u \in U$, then $\sigma_{au}(v) = \sigma_a(\sigma_u(v)) = \theta(a)v$. It follows that $\sigma = \tilde{\theta}$.

## 2.3. The Borel Subgroup $B$

The Borel subgroup $B$ consists of all upper triangular matrices in $\mathrm{GL}(2, \mathbb{F}_q)$. Clearly, $\mid B \mid= (q-1)^2 q$.

$B$ is a *solvable group.* (One says that a group $G$ is solvable if there exists a sequence $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ with $G_{i-1}$ normal in $G_i$ and $G_i/G_{i-1}$ abelian.) Indeed, $B$ contains the normal abelian subgroup $U$. The quotient group $B/U$ is isomorphic to the Cartan group $D$ of all invertible diagonal matrices. Clearly, $U \cap D = \{1\}$ and $B = U \cdot D$. Hence $B$ is the semidirect product of $D$ by $U$. Direct computation shows that $U$ is the commutator subgroup of $B$, if $\mid \mathbb{F}_q \mid \neq 2$. In particular, it follows that $B$ has exactly $(q-1)^2$ characters.

$P$ is another important normal subgroup of $B$ and is of index $q-1$. The center $Z$ is also contained in $B$. Clearly, $Z \cap P = \{1\}$ and $B = Z \cdot P$. Hence $B$ is also the semidirect product of $P$ by $Z$.

We use the method of little groups of Wigner and Mackey again in order to determine the representations of $P$.

First, consider $B$ is the semidirect product of $D$ by $U$. In last section, we know that for a nonunit character $\psi$ of $U$, the orbit of $\psi$ under $A$ is the set of all nonunit character of $U$. Since $A \subset D$, we have that $\psi$ and 1 is a representatives of the orbit of $\hat{U}$ under $D$.

For the unit character 1 of $U$, we know that $D_1 = \{d \in D \mid d * 1 = 1\} = D$ and hence $B_1 = U \cdot D_1 = B$. The extension of 1 to $B_1 = B$ is also a unit character of $B$. Now let $\theta$ be an irreducible representation of $D_1 = D$. It is a character of $D$ because $D$ is abelian. In fact, every pair $(\mu_1, \mu_2)$ of characters of $\mathbb{F}_q^\times$ defines a unique character $\theta$ of $D$ by the formula

$$\theta \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \mu_1(\alpha)\mu_2(\delta), \quad \text{for } \alpha, \delta \in \mathbb{F}_q^\times \ .$$

By composing with the canonical projection $B_1 = B \to D_1 = D$, we obtain a character $\tilde{\theta}$ of $B$, *i.e.*

$$\tilde{\theta} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \mu_1(\alpha)\mu_2(\delta), \quad \text{for } \alpha, \delta \in \mathbb{F}_q^\times \ .$$

From this, by the notation of Section 1.15, $\rho_{1,\theta} = \mathrm{Ind}_{B_1}^B(1 \otimes \tilde{\theta}) \simeq \tilde{\theta}$. Recall that the commutator of $B$ is $U$, so $B$ has exactly $(q-1)^2$ characters. Thus the $(q-1)^2$ characters of $B$ given here are all the characters of $B$.

By Proposition 1.15.2, another kind of irreducible representation comes from $\psi$. It is easy to see that $D_\psi = \{d \in D \mid d * \psi = \psi\} = Z$. We know that the method of little groups will involve the group $B_\psi = U \cdot D_\psi = U \cdot Z$ which we have not studied yet. Therefore, we move to another form of semidirect product, $B = Z \cdot P$.

Recall that the abelian group $Z$ is isomorphic to $\mathbb{F}_q^\times$ and has $q-1$ characters. For each of the character $\chi$ of $Z$, since $Z$ is the center of $B$, $B$ acts trivially on $\chi$, *i.e.*, $b * \chi(z) = \chi(b^{-1}zb) = \chi(z)$ for every $b \in B$ and $z \in Z$. Hence, we have the the orbit of $\chi$ under $P$ is $\chi$ itself and $P_\chi = \{p \in P \mid p * \chi = \chi\} = P$. We can extend $\chi$ to a character of $B_\chi = Z \cdot P_\chi = B$ by setting $\tilde{\chi}(zp) = \chi(z)$ for $z \in Z$ and $p \in P$. Now let $\theta$ be an irreducible representation of $P_\chi = P$. By composing with the canonical projection $B_\chi = B \to P_\chi = P$, we obtain an irreducible representation $\tilde{\theta}$ of $B$. By Proposition 1.15.2, $\tilde{\chi} \otimes \tilde{\theta}$ is an irreducible representation of $B$ and every irreducible representation of $B$ is isomorphic to $\tilde{\chi} \otimes \tilde{\theta}$ for some $\chi \in \hat{Z}$ and irreducible representation $\theta$ of $P$. From Theorem 2.2.1, we know that $\theta$ is either a character of $P$ or $\theta = \pi$. If $\theta$ is a character of $P$, then $\tilde{\chi} \otimes \tilde{\theta}$ is a character of $B$ (which we have already found above). If $\theta = \pi$, then $\tilde{\chi} \otimes \tilde{\pi}$ is an irreducible $(q-1)$-dimensional representation of $B$. Varying $\chi$ on all the characters of $Z$, we get $q-1$ of $(q-1)$-dimensional irreducible representations of $B$. These, together with the $(q-1)^2$ characters of $B$, are all the irreducible representations of $B$ (this can also be seen by computing $(q-1)^2 \cdot 1^2 + (q-1) \cdot (a-1)^2 = q(q-1)^2 =\mid B \mid$).

We sum up our results in the following:

**Theorem 2.3.1.** *The group $B$ has $q(q-1)$ irreducible representations:*

(1) $(q-1)^2$ *of them are* 1*-dimensional representations which are lifting of the characters of $D$; and*

(2) $q-1$ *of them are* $(q-1)$*-dimensional irreducible representation isomorphic to $\tilde{\chi} \otimes \tilde{\pi}$ for some $\chi \in \hat{Z}$.*

## 2.4. The Group $\mathrm{GL}(2, \mathbb{F}_q)$

We denote by $G$ the group $\mathrm{GL}(2, \mathbb{F}_q)$. Straightforward calculations show that the matrices

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \ \gamma \in \mathbb{F}_q$$

form a system of representatives for the left classes of $G$ modulo $B$. Hence $[G : B] = q + 1$ and thus $\mid G \mid= (q-1)^2 q(q+1)$.

On the other hand, we have the *Bruhat's decomposition* of $G$, namely $G = B \cup BwU$. Notice that $B \cap BwU = \emptyset$. Indeed, if $\gamma \neq 0$, then

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \beta - \alpha\gamma^{-1}\delta & \alpha \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \gamma^{-1}\delta \\ 0 & 1 \end{pmatrix} \ .$$

We need a description of $\mathrm{GL}(2,\mathbb{F}_q)$ by generators and relations for an explicit presentation of the representations of $\mathrm{GL}(2,\mathbb{F}_q)$.

Let

$$w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then we have the following relations between $w'$ and the elements of $B$:

$$w' \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} w'^{-1} = \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}, \quad w'^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and } (w'u)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

**Proposition 2.4.1.** $\mathrm{GL}(2,\mathbb{F}_q)$ *is the free group generated by* $B$ *and* $\tilde{w}$ *with the following as the defining relations.*

(1)
$$\tilde{w} \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \tilde{w}^{-1} = \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix},$$

(2)
$$\tilde{w}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

(3)
$$(\tilde{w}u)^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

**Proof.** Let $\tilde{G}$ be the free group generated by $B$ and $\tilde{w}$ with the above defining relations. Then there exists a unique epimorphism $\theta : \tilde{G} \to G$ which is the identity on $B$ and maps $\tilde{w}$ onto $w'$. We have to prove that its kernel consists of 1.

We first show that in $\tilde{G}$, for every $b \in B - D$, there exist $b_1, b_2 \in B$ such that $\tilde{w}b\tilde{w} = b_1\tilde{w}b_2$. Indeed, if $\beta \neq 0$, then

$$b = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \delta\beta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} = d'ud'';$$

also by(3), $\tilde{w}u\tilde{w} = u^{-1}\tilde{w}^{-1}u^{-1}$; hence

$$\tilde{w}u\tilde{w} = u^{-1} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{w}u^{-1}$$

by (2). It follows that

$$\tilde{w}b\tilde{w} = (\tilde{w}d'\tilde{w}^{-1})\tilde{w}u\tilde{w}(\tilde{w}^{-1}d''\tilde{w}) = b_1\tilde{w}b_2,$$

where

$$b_1 = (\tilde{w}d'\tilde{w}^{-1})u^{-1} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad b_2 = u^{-1}(\tilde{w}^{-1}d''\tilde{w})$$

are in $B$ because of (1).

Next note that if $d \in D$, then $\tilde{w}d\tilde{w} = (\tilde{w}d\tilde{w}^{-1})\tilde{w}^2 \in B$ by (1) and (2).

Now let $g \neq 1$ be in the kernel of $\theta$. Then $g \neq B$; hence by using

$$\tilde{w}b\tilde{w} = \begin{cases} b & \text{if } b \in D, \\ b_1\tilde{w}b_2 & \text{if } b \in B - D, \end{cases}$$

$g$ can be written as

$$g = \begin{pmatrix} \alpha' & \beta' \\ 0 & \delta' \end{pmatrix} \tilde{w} \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \quad \text{where } \alpha, \alpha', \delta, \delta' \neq 0.$$

The right-hand side is mapped by $\theta$ to the element $\begin{pmatrix} -\alpha'\alpha & \alpha'\delta - \beta'\beta \\ -\delta'\alpha & -\delta'\beta \end{pmatrix}$ of $\mathrm{GL}(2,\mathbb{F}_q)$. But $g$ is mapped to 1. Hence $\delta'\alpha = 0$, which is a contradiction. $\square$

## 2.5. Inducing Characters from $B$ to $G$

As a first step toward the determination of the irreducible representations of $G$, we investigate those that appear as components of $\mathrm{Ind}_B^G(\mu)$ where $\mu$ is a character of $B$. In order to shorten the notation, we make the convention

$$\hat{\mu} = \mathrm{Ind}_B^G(\mu)$$

and stick to it for the rest of this note. The dimension of $\hat{\mu}$ is $[G : B] = q+1$. Our task in this section is to determine the connection between $\mu$ and $\hat{\mu}$.

**Proposition 2.5.1.** *Let $\mu$ be a character of $B$ and let $\hat{\mu} = \mathrm{Ind}_B^G(\mu)$. Then either $\hat{\mu}$ is irreducible, or $\hat{\mu}$ decomposes into a direct sum of two non-isomorphic irreducible representations.*

**Proof.** Bruhat's decomposition of $G$ implies that $\mid B\backslash G/B \mid= 2$. Hence, by Corollary 1.16.2 $(\hat{\mu}, \hat{\mu}) \leq 2$. Thus, we have either $(\hat{\mu}, \hat{\mu}) = 1$ or $(\hat{\mu}, \hat{\mu}) = 2$. Our proof is complete. □

Now we shall determine the restriction $\mathrm{Res}_B^G(\hat{\mu})$. Bruhat's decomposition of $G$ also tells us that $\{1, w\}$ is a representatives for the double cosets $B\backslash G/B$. We have that $B_1 = B$, $\mu^1 = \mu$ and $B_w = wBw^{-1} \cap B = D$, $\mu^w(x) = \mu(w^{-1}xw)$ for $x \in D$. By Proposition 1.14.1,

$$\mathrm{Res}_B^G(\hat{\mu}) = \mu \oplus \mathrm{Ind}_D^B(\mu^w).$$

Moreover, Mackey's irreducible criterion (Proposition 1.14.2) tell us that $\hat{\mu}$ is irreducible if and only if $(\mathrm{Res}_D^B(\mu), \mu^w) = 0$ as representations for $D$. Notice that since both $\mathrm{Res}_D^B(\mu)$ and $\mu^w$ are characters of $D$, we have that $(\mathrm{Res}_D^B(\mu), \mu^w) = 0$ if and only if $\mathrm{Res}_D^B(\mu) \neq \mu^w$. Recall that every character $\mu$ of $B$ is given by

$$\mu \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \mu_1(\alpha)\mu_2(\delta),$$

for some $\mu_1$ and $\mu_2$ characters of $\mathbb{F}_q^\times$. Hence $\mathrm{Res}_D^B(\mu) = \mu^w$ if and only if $\mu_1 = \mu_2$.

**Lemma 2.5.2.** *If $\mu$ is a character of $B$ and $\mu = \mu^w$, then $\hat{\mu}$ has a 1-dimensional component.*

**Proof.** The assumption implies that $\mu(b) = \mu_1(\det(b))$ for every $b \in B$, where $\mu_1$ is a character of $\mathbb{F}_q^\times$. Consider the character $\tilde{\mu} = \mu_1 \circ \det$ of $G$. We have that $\langle \tilde{\mu}, \hat{\mu} \rangle = \langle \mathrm{Res}_B^G(\tilde{\mu}), \mu \rangle \geq 1$. □

We summarize our results as the following:

**Theorem 2.5.3.** *Let $\mu$ be a character of $B$ given by*

$$\mu \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \mu_1(\alpha)\mu_2(\delta),$$

*for some $\mu_1$ and $\mu_2$ characters of $\mathbb{F}_q^\times$ and let $\hat{\mu} = \mathrm{Ind}_B^G(\mu)$. Then*

1. *If $\mu_1 \neq \mu_2$, then $\hat{\mu}$ is an irreducible $(q+1)$-dimensional representation of $G$.*
2. *If $\mu_1 = \mu_2$, then $\hat{\mu}$ decomposes into a direct sum of a 1-dimensional representation and a q-dimensional representation.*

**Lemma 2.5.4.** *Let $\mu$ and $\mu'$ be two distinct characters of $B$. Then $\hat{\mu} = \hat{\mu'}$ if and only if $\mathrm{Res}_D^B(\mu') = \mu^w$.*

**Proof.** Applying Frobenius reciprocity twice, we obtain

$$(\hat{\mu'}, \hat{\mu}) = (\mu', \mathrm{Res}_B^G(\hat{\mu})) = (\mu', \mu) + (\mu', \mathrm{Ind}_D^B(\mu^w)) = (\mu', \mu) + (\mathrm{Res}_D^B(\mu'), \mu^w).$$

Our lemma follows easily. □

We shall now calculate the number of irreducible representation of $G$ which are components of induced representations of the form $\mathrm{Ind}_B^G(\mu)$. Suppose that $\mu$ corresponds to the pair of characters $(\mu_1, \mu_2)$ of $\mathbb{F}_q^\times$. By Theorem 2.5.3, we have two possibilities:

(1) $\mu_1 = \mu_2$. In this case we denote by $\rho'_{(\mu_1,\mu_1)}$ the corresponding one-dimensional irreducible component and $\rho_{(\mu_1,\mu_1)}$ the corresponding $q$-dimensional irreducible component of $\hat{\mu}$, respectively. $\mathbb{F}_q^\times$ has $q-1$ characters; hence, we obtain $q-1$ irreducible representations of $G$ of each type.

(2) $\mu_1 \neq \mu_2$. In this case $\hat{\mu}$ is an irreducible representation of dimension $q+1$ and we denote it by $\rho_{(\mu_1,\mu_2)}$. Then number of these $\mu$ is equal to the number of characters of $B$ minus the number of characters of type (1), *i.e.* $(q-1)^2-(q-1)$. Hence, by Lemma 2.5.4, we obtain in this way $(q-1)(q-2)/2$ irreducible representations of $G$ of dimension $q+1$.

We have therefore proved:

**Theorem 2.5.5.** *The irreducible representations of $G$, which are components of induced representations of the form $\mathrm{Ind}_B^G(\mu)$ where $\mu$ is a character of $B$, split up into the following cases:*

(1) $q-1$ *representations* $\rho'_{(\mu_1,\mu_1)}$ *of dimension one;*

(2) $q-1$ *representations* $\rho_{(\mu_1,\mu_1)}$ *of dimension* $q$*;*

(3) $\frac{1}{2}(q-1)(q-2)$ *representations* $\rho_{(\mu_1,\mu_2)}$ *with* $\mu_1 \neq \mu_2$ *of dimension* $q+1$.

**Corollary 2.5.6.** *The group* $\mathrm{GL}(2,\mathbb{F}_q)$ *has exactly* $q-1$ *characters.*

**Proof.** If $\chi$ is a character of $G$, then $\chi$ is a component of $\mathrm{Ind}_B^G(\mathrm{Res}_B^G(\chi))$. It follows by Theorem 2.5.5 (1) that $G$ has exactly $q-1$ characters. □

**Remark .** Given a character $\upsilon$ of $\mathbb{F}_q^\times$, the composite map $\upsilon\circ\det : G \to \mathbb{C}$, $s \mapsto \upsilon(\det(s))$ is a character of $G$. By Corollary 2.5.6, these are all the characters for $G$.

**Corollary 2.5.7.** *The subgroup* $\mathrm{SL}(2,\mathbb{F}_q) = \{g \in \mathrm{GL}(2,\mathbb{F}_q) \mid \det(g) = 1\}$ *is the commutator subgroup of* $\mathrm{GL}(2,\mathbb{F}_q)$.

**Proof.** $\mathrm{SL}(2,\mathbb{F}_q)$ is normal and $\mathrm{GL}(2,\mathbb{F}_q)/\mathrm{SL}(2,\mathbb{F}_q) \cong \mathbb{F}_q^\times$ which is abelian. Hence $\mathrm{SL}(2,\mathbb{F}_q)$ contains the commmutator subgroup of $\mathrm{GL}(2,\mathbb{F}_q)$. By Corollary 2.5.6, $[G:G'] = q-1$ and hence $\mathrm{SL}(2,\mathbb{F}_q) = G'$. □

## 2.6. The Jacquet Module of a Representation of $\mathrm{GL}(2,\mathbb{F}_q)$

In Section 1.12, we give a concrete construction of induced representation. We ought to use this construction to get some information about induced representation. In this section, we introduce the Jacquet module and provide another approach of last section.

We define the *Jacquet Module* of a representation $\rho$ of $G$ as

$$J(V_\rho) = \{v \in V_\rho \mid \rho_u(v) = v \ \text{ for every } u \in U\}.$$

The fact that $U$ is normal in $B$ implies that $B$ acts on $J(V_\rho)$. Indeed, if $v \in J(V_\rho)$, $b \in B$ and $u \in U$, then $b^{-1}ub \in U$; hence

$$\rho_u(\rho_b(v)) = \rho_b(\rho_{b^{-1}ub}(v)) = \rho_b(v).$$

It is also clear that if $\rho_1$ and $\rho_2$ are representations of $G$, then

$$J(V_{\rho_1} \oplus V_{\rho_2}) = J(V_{\rho_1}) \oplus J(V_{\rho_2}).$$

The importance of the Jacquet modules for our investigation lies in the following:

**Lemma 2.6.1.** *Let $\rho$ be a representation of $G$. Then $J(V_\rho) \neq 0$ if and only if there exists a character $\mu$ of $B$ such that $(\rho, \hat{\mu}) \neq 0$.*

**Proof.** Suppose that $J(V_\rho) \neq 0$. Then $J(V_\rho)$ can be considered as a non-trivial $\mathbb{C}[B/U]$-module via $\rho$. Since $B/U$ is abelian, it follows that there exists a character $\mu$ of $B$ and a non-zero element $v \in J(V_\rho)$ such that $\rho_b(v) = \mu(b)v$ for every $b \in B$. Hence $(\mathrm{Res}^G_B(\rho), \mu) \neq 0$. By the Frobenius reciprocity $(\rho, \hat{\mu}) \neq 0$ and half of the lemma is thus proved.

Now suppose that $(\rho, \hat{\mu}) \neq 0$. By Frobenius reciprocity there exists a non-zero element $v \in V_\rho$ such that $\rho_b(v) = \mu(b)v$ for every $b \in B$. Since $U$ is the commutator subgroup of $B$, $\mu$ is trivial on $U$. Hence $v \in J(V_\rho)$. □

We now investigate $J(V_{\hat{\mu}})$.

**Lemma 2.6.2.** *If $\mu$ is a character of $B$, then* $\dim(J(V_{\hat{\mu}})) = 2$.

**Proof.** By definition, $J(V_{\hat{\mu}})$ consists of all the functions $f : G \to \mathbb{C}$ that satisfy

$$f(bs) = \mu(b)f(s) \quad \text{and} \quad f(bu) = \hat{\mu}_u(f)(b) = f(b) \quad \text{for all } b \in B,\ s \in G \text{ and } u \in U.$$

In particular

$$f(b) = \mu(b)f(1) \quad \text{and} \quad f(bwu) = \mu(b)f(w) \quad \text{for all } b \in B \text{ and } u \in U.$$

Using the Bruhat decomposition, this implies that $f$ is determined by its values in 1 and $w$, where it can be arbitrary. It follows that $\dim(J(V_{\hat{\mu}})) = 2$. □

**Corollary 2.6.3** (*cf.* Proposition 2.5.1)**.** *If $\mu$ is a character of $B$, then $\hat{\mu}$ has at most two irreducible components.*

**Proof.** Let $\hat{\mu} = \rho_1 \oplus \cdots \oplus \rho_r$ be a decomposition of $\hat{\mu}$ into irreducible components. then $J(V_{\hat{\mu}}) = J(V_{\rho_1}) \oplus \cdots \oplus J(V_{\rho_r})$. By Lemma 2.6.1, $\dim(J(V_{\rho_i})) \geq 1$ for $i = 1 \ldots r$. On other hand $\dim(J(V_{\hat{\mu}})) = 2$. Hence $r \leq 2$. □

Now we can easily get the restriction of $\hat{\mu}$ to the subgroup $P$, which in some cense is more complicated by using the method in Section 1.14.

**Proposition 2.6.4.** *If $\mu$ is a character of $B$, then*

$$\mathrm{Res}^G_P(V_{\hat{\mu}}) = \mathrm{Res}^B_P(J(V_{\hat{\mu}})) \oplus V_\pi.$$

**Proof.** $J(V_{\hat{\mu}})$ is a $\mathbb{C}[B]$-module and hence is a $\mathbb{C}[P]$-module. Let $V$ be a $\mathbb{C}[P]$-complement to $J(V_{\hat{\mu}})$ in $V_{\hat{\mu}}$. Then $\dim(V) = q-1$, since $\dim(J(V_{\hat{\mu}})) = 2$. Further, $V$ has no one-dimensional $\mathbb{C}[P]$-submodule; indeed otherwise, there would exist a non-zero element $v \in V$ and a character $\chi$ of $P$ such that $\hat{\mu}_p v = \chi(p)v$, for every $p \in P$. In particular, we would have that $\hat{\mu}_u(v) = v$ for every $u \in U$, since $U$ is the commutator subgroup of $P$. Thus $v \in J(V_{\hat{\mu}})$, which is a contradiction. Therefore, by Theorem 2.2.1, $V$ is isomorphic to the unique irreducible $\mathbb{C}[P]$-module $V_\pi$ of dimension $q-1$. □

A canonical basis for $J(V_{\hat{\mu}})$ is the two functions $f_1, f_2 \in V_{\hat{\mu}}$ satisfying

$$\begin{array}{ll} f_1(1) = 1 & f_1(w) = 0 \\ f_2(1) = 0 & f_2(w) = 1 \end{array}.$$

For $f_1$, we have $\hat{\mu}_b(f_1)(1) = f_1(b) = \mu(b)f_1(1)$. Also by teh Bruhat decomposition there exists for every $b \in B$ elements $b_1 \in B$ and $u \in U$ such that $wb = b_1wu$. Hence $\hat{\mu}_b(f_1)(w) = f_1(bw) = f_1(b_1wu) = \mu(b_1)f_1(w) = 0$. Therefore, $\hat{\mu}_b(f_1) = \mu(b)f_1$ for every $b \in B$.

For $f_2$, we have $\hat{\mu}_b(f_2)(1) = f_2(b) = \mu(b)f_2(1) = 0$. Since $f_1$ and $f_2$ generate $J(V_{\hat{\mu}})$, for every $b \in B$ there exist $\alpha_1(b)$, $\alpha_2(b) \in \mathbb{C}$ such that

$$\hat{\mu}_b(f_2) = \alpha_1(b)f_1 + \alpha_2(b)f_2.$$

Evaluate at 1 we get $\alpha_1(b) = 0$ for all $b \in B$; hence $\hat{\mu}_b(f_2) = \alpha_2(b)f_2$ for all $b \in B$. It follows that $\alpha_2$ is a character of $B$. In particular, if $d \in D$, then $\hat{\mu}_d(f_2)(w) = f_2(wdww) = \mu(wdw)f_2(w)$. This implies that $\alpha_2(d) = \mu(wdw) = \mu^w(d)$ for all $d \in D$. Recall that every character of $D$ can be extend uniquely to a character of $B$, *i.e.* if

$$b = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix},$$

then $\alpha_2(b) = \alpha_2(d)$. Hence $\hat{\mu}_b(f_2)(w) = \alpha_2(b)f_2(w) = \alpha_2(d)f_2(w) = \mu^w(d)f_2(w) = \mu^w(b)f_2(w)$ (here, we extend the character $\mu^w$ of $D$ to the corresponding character of $B$). We summary our results as the following:

**Lemma 2.6.5.** *If $\mu$ is a character of $B$, then*

$$\hat{\mu}_b(f_1) = \mu(b)f_1 \quad \textit{and} \quad \hat{\mu}_b(f_2) = \mu^w(b)f_2.$$

The following lemmas give the exact information about the components of $\hat{\mu}$.

**Lemma 2.6.6** (Lemma 2.5.2)**.** *If $\mu$ is a character of $B$ and $\mu = \mu^w$, then $\hat{\mu}$ has a 1-dimensional component.*

**Proof.** The assumption implies that $\mu(b) = \mu_1(\det(b))$ for every $b \in B$, where $\mu_1$ is a character of $\mathbb{F}_q^\times$. Now define a function $f : G \to \mathbb{C}$ by $f(s) = \mu_1(\det(s))$, for $s \in G$. It is easy to see that $f \in V_{\hat{\mu}}$ and $f$ is an eigenvector of $G$ that belongs to the eigenvalue $\mu_1 \circ \det$. □

**Lemma 2.6.7.** *If $\mu$ is a character of $B$, then $\hat{\mu}$ has at most one 1-dimensional component.*

**Proof.** Assume that $\hat{\mu}$ has two 1-dimensional components. Then by Corollary 2.6.3 they are all the components of $\hat{\mu}$. It follows that $q + 1 = \dim(\hat{\mu}) = 2$, which is a contradiction. □

**Lemma 2.6.8.** *If $\mu$ is a character of $B$ and $\hat{\mu}$ is reducible, then $\hat{\mu}$ has a 1-dimensional component. Furthermore, $\mu = \mu^w$.*

**Proof.** Since $\hat{\mu}$ is reducible, by Corollary 2.6.3 we have that $V_{\hat{\mu}} = V \oplus V'$ where $V$ and $V'$ are non-trivial irreducible $\mathbb{C}[G]$-module. These are also $\mathbb{C}[P]$-modules and hence by Proposition 2.6.4 we can assume, without loss of generality, that $V_\pi \in V$. On the other hand, $0 \neq J(V) \subseteq J(V_{\hat{\mu}}) \cap V$. Hence $V_\pi \subsetneq V$. It follows that $\dim(V) \geq q$ and hence $\dim(V') = 1$.

We have proved that there exists a character $\chi$ of $G$ and a non-zero function $f : G \to \mathbb{C}$ in $V_{\hat{\mu}}$ such that $\hat{\mu}_s(f) = \chi(s)f$ for all $s \in G$. In particular, we have $f(1) \neq 0$; indeed otherwise, because there exists a positive integer $n$ such that $s^n = 1$ for all $s$, we would have $0 = f(1) = f(s \cdot s^{n-1}) = \hat{\mu}_{s^{n-1}}(f)(s) = \chi(s^{n-1})f(s)$. It follows that $f(s) = 0$ since $\chi(s^{n-1}) \neq 0$. This is a contradiction. Let $d \in D$. Then $\mu(d)f(1) = f(d) = \chi(d)f(1)$. Hence $\mu(d) = \chi(d)$ for every $d \in D$. It follows that $\mu^w(d) = \mu(wdw) = \chi(wdw) = \chi(d) = \mu(d)$, for every $d \in D$. Hence $\mu = \mu^w$. □

Summing up the Lemmas 2.5.2–2.6.8, we can obtain Theorem 2.5.3.

## 2.7. The conjugacy Classes of $\mathrm{GL}(2, \mathbb{F}_q)$

Before we start to investigate the irreducible representations of $G$, we would like to compute their number. By Theorem 1.8.2, this number is equal to the number of the conjugacy classes of $G$. Now, we give explicitly a representative for each of the conjugacy classes.

An element $g$ of $G$ has two eigenvalues. All the elements in the conjugacy class of $g$ have the same eigenvalues. There are therefore four cases:

- The eigenvalues of $g$ belong to $\mathbb{F}_q$.
  In this case $g$ is conjugate over $\mathbb{F}_q$ to a unique matrix in a *canonical Jordan form*. If both eigenvalues are equal to the same element $\alpha$ of $\mathbb{F}_q$, then we have:

(1) $g$ is diagonalizable. Hence the Jordan form is

$$c_1(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} .$$

There are $q-1$ classes of the form $c_1(\alpha)$.

(2) $g$ is not diagonalizable. In this case for any $v \in \mathbb{F}_q^2$ which is not an eigenvector of $g$, we have that $g(v) - \alpha v$ is an eigenvector of $g$ (by the Cayley-Hamilton theorem). Hence $v, g(v) - \alpha v$ form a basis of $\mathbb{F}_q^2$ over $\mathbb{F}_q$ and so the Jordan form is

$$c_2(\alpha) = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} .$$

There are $q-1$ classes of the form $c_2(\alpha)$.

If the eigenvalues are $\alpha, \beta \in \mathbb{F}_q$ and $\alpha \neq \beta$, then

(3) the Jordan form is

$$c_3(\alpha, \beta) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} .$$

Since $c_3(\alpha, \beta)$ and $c_3(\beta, \alpha)$ are in the same conjugacy class, there are $\frac{1}{2}(q-1)(q-2)$ classes of the form $c_3(\alpha, \beta)$.

- The eigenvalues of $g$ do not belong to $\mathbb{F}_q$

  In this case the two eigenvalues $\lambda$ and $\overline{\lambda}$ belong to the unique quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ and $\lambda, \overline{\lambda}$ are conjugate over $\mathbb{F}_q$. Let $v$ be a nonzero vector in $\mathbb{F}_q^2$. then $v$, $g(v)$ form a basis of $\mathbb{F}_q^2$ over $\mathbb{F}_q$. By Cayley-Hamilton theorem, when consider $g$ as a linear operator on $\mathbb{F}_q^2$ with respect to the basis $v$, $g(v)$,

  (4) $g$ is conjugate in $G$ to

$$c_4(\lambda) = \begin{pmatrix} 0 & -\lambda\overline{\lambda} \\ 1 & \lambda + \overline{\lambda} \end{pmatrix} .$$

Since $c_4(\lambda)$ is conjugate to $c_4(\gamma)$ if and only if $\lambda = \gamma$ or $\lambda = \overline{\gamma}$, there are $\frac{1}{2}(q^2 - q)$ classes of the form $c_4(\lambda)$.

# The Representations of $\mathrm{GL}(2, \mathbb{F}_q)$

## 3.1. Cuspidal Representations

Irreducible representations of $G$ that are not components of $\hat{\mu}$, with $\mu$ a character of $B$, are said to be *cuspidal*. By Lemma 2.6.1, an irreducible representation $\rho$ of $G$ is cuspidal if and only if $J(V_\rho) = 0$. Comparing Theorem 2.5.5 with the results in Section 2.7, we find that $G$ has $\frac{1}{2}(q^2 - q)$ cuspidal representations, exactly as the number of conjugacy classes of the form $c_4(\lambda)$.

**Lemma 3.1.1.** *Let $\rho$ be a cuspidal representation of $G$. Then $\mathrm{Res}^G_P(\rho) = r\pi$ for some positive integer $r$.*

**Proof.** If $v \in \mathrm{Res}^G_P(V_\rho)$ is an eigenvector, then using the similar argument as in the proof of Proposition 2.6.4, we have that $v \in J(V_\rho)$; thus $J(V_\rho) \neq 0$, contrary to the assumption that $\rho$ is cuspidal. Hence $\mathrm{Res}^G_P(V_\rho)$ cannot have 1-dimensional component and hence $\mathrm{Res}^G_P(\rho)$ must be a multiple of $\pi$. □

**Lemma 3.1.2.** *Let $\psi$ be a non-unit character of $U$. Then $\hat{\pi} = \mathrm{Ind}^G_P(\pi) = \mathrm{Ind}^G_U(\psi)$ has no multiple component.*

**Proof.** Recall that by Proposition 1.16.1, $\mathrm{Hom}_{\mathbb{C}[G]}(V_{\hat{\pi}}, V_{\hat{\pi}})$ is isomorphic to the algebra $\mathcal{A}$ of all functions $F : G \to \mathbb{C}$ satisfying

$$F(u_1 s u_2) = \psi(u_1 u_2) F(s) \quad \text{for } u_1, u_2 \in U \text{ and } s \in G,$$

and where multiplication between $F_1, F_2 \in \mathcal{A}$ is given by the formula

$$(F_1 \cdot F_2)(s) = \frac{1}{[G : U]} \sum_{t \in G} F_1(st^{-1}) F_2(t).$$

We shall show that $\mathcal{A}$ is abelian. This implies that $\hat{\pi}$ has no multiple components (*cf.* Section 1.16).

We start by defining an *involution* $s \mapsto s' = (wsw)^T$ on $G$, *i.e.*

$$s = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \longmapsto s' = \begin{pmatrix} \delta & \beta \\ \gamma & \alpha \end{pmatrix}$$

It is obvious that $(s_1 s_2)' = s_2' s_1'$ and $u' = u$ for every $s_1, s_2 \in G$ and $u \in U$. We continue by defining for an element $F \in \mathcal{A}$ a function $F' : G \to \mathbb{C}$ by $F'(s) = F(s')$ for every $s \in G$. It is easy to check that $F' \in \mathcal{A}$ and $(F_1 \cdot F_2)' = F_2' \cdot F_1'$. we shall show that $F = F'$. Hence $F_1 \cdot F_2 = F_2 \cdot F_1$. □

In order to prove that $F = F'$, it suffices to prove that $F$ and $F'$ coincide on representatives of the double cosets $U\backslash G/U$. Indeed, by Bruhat's decomposition and $B = UD$, we have that the above representatives are of the following two forms

$$(a)\quad \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}, \quad (b)\quad \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}$$

We only have to check for those of the form $(b)$ with $\alpha \neq \delta$. Indeed, acting with $F$ on both sides of

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\begin{pmatrix} 1 & \alpha^{-1}\delta\beta \\ 0 & 1 \end{pmatrix},$$

we have

$$\psi(\beta)F\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} = F\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\psi(\alpha^{-1}\delta\beta).$$

Since $\alpha \neq \delta$ and $\psi$ is a non-unit character, we have that $\psi(\beta) \neq \psi(\alpha^{-1}\delta\beta)$ for some $\beta \in \mathbb{F}_q$. This implies that $F$ vanishes on matrices of the form $(b)$ with $\alpha \neq \delta$ and so is $F'$.

**Proposition 3.1.3.** *Let $\rho$ be a representation of $G$. Then $\rho$ is cuspidal if and only if $\mathrm{Res}_P^G(\rho) = \pi$.*

**Proof.** If $\rho$ is cuspidal, then Lemma 3.1.2, $(\mathrm{Res}_P^G(\rho), \pi) = (\rho, \mathrm{Ind}_P^G(\pi)) = 1$. Hence $\mathrm{Res}_P^G(\rho) = \pi$, by Lemma 3.1.1.

On the other hand, if $\mathrm{Res}_P^G(\rho) = \pi$, then since $\pi$ is irreducible, $\rho$ is also irreducible. By Theorem 2.5.3, the components of $\hat{\mu}$ have the only possible dimensions 1, $q$, or $q+1$. However, $\dim(\rho) = q - 1$; hence, $\rho$ is cuspidal. ☐

**Remark .** By Theorem 2.5.3 and Lemma 3.1.1, we can use the fact that the number of cuspidal representations is $\frac{1}{2}(q^2 - q)$ to calculate directly that every cuspidal form has dimension $q - 1$.

## 3.2. Characters of $\mathbb{F}_{q^2}^\times$

$\mathbb{F}_{q^2}$ is the only quadratic extension of $\mathbb{F}_q$. If $\lambda$ is an element of $\mathbb{F}_{q^2}$, then $\overline{\lambda}$ denotes its unique conjugate over $\mathbb{F}_q$. Since the Galois group $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ is generated by the *Frobenius automorphism* $\lambda \mapsto \lambda^q$, we have in fact $\overline{\lambda} = \lambda^q$.

The function $\mathrm{N}(\lambda) = \lambda\overline{\lambda}$ is the *norm map* from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$. It is multiplicative, *i.e.* $\mathrm{N}(\lambda\lambda') = \mathrm{N}(\lambda)\mathrm{N}(\lambda')$.

**Lemma 3.2.1.** *The kernel of the norm map from $\mathbb{F}_{q^2}^\times$ to $\mathbb{F}_q^\times$ consists of $q+1$ elements. Furthermore, the norm map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ is surjective.*

**Proof.** The restriction of $N$ to $\mathbb{F}_{q^2}^\times$ is a homomorphism into $\mathbb{F}_q^\times$. Since $\mathrm{N}(\lambda) = \lambda^{q+1}$, the kernel of this homomorphism consists of elements satisfying $x^{q+1} - 1 = 0$, which has at most $q+1$ elements. Hence, the image of $N$ consists of at least $(q^2-1)/(q+1) = q-1$ elements. Therefore, it must be $\mathbb{F}_q^\times$. ☐

**Corollary 3.2.2** (Hilbert's Satz 90)**.** *If $\zeta$ is an element of $\mathbb{F}_{q^2}^\times$ such that $\mathrm{N}(\zeta) = 1$, then there exists a $\lambda \in \mathbb{F}_{q^2}^\times$ such that $\lambda\overline{\lambda}^{-1} = \zeta$.*

**Proof.** The set $E = \{\lambda \in \mathbb{F}_{q^2}^\times \mid \mathrm{N}(\lambda) = 1\}$, has exactly $q+1$ element. Consider the map $h : \mathbb{F}_{q^2}^\times \to E$ defined by $h(\lambda) = \lambda\overline{\lambda}^{-1}$. Its kernel is $\mathbb{F}_q^\times$. Hence the image of $h$ has $(q^2-1)/(q-1) = q+1$ elements, exactly as many as $E$ has. ☐

Let $\chi$ be a character of $\mathbb{F}_q^\times$. Composing $\chi$ with the norm mp $N$ from $\mathbb{F}_{q^2}^\times$ to $\mathbb{F}_q^\times$, we obtain a character $\tilde{\chi}$ of $\mathbb{F}_{q^2}^\times$:

$$\tilde{\chi}(\lambda) = \chi(\mathrm{N}(\lambda)),\ \lambda \in \mathbb{F}_{q^2}^\times.$$

If $\nu$ is a character of $\mathbb{F}_{q^2}^\times$, then $\overline{\nu}$ denotes its *conjugate* over $\mathbb{F}_q$, *i.e.*

$$\overline{\nu}(\lambda) = \nu(\overline{\lambda}),\ \ \lambda \in \mathbb{F}_{q^2}^\times.$$

A character $\nu$ of $\mathbb{F}_{q^2}^\times$ is said to be *decomposable* if $\nu = \tilde{\chi}$ for some character $\chi$ of $\mathbb{F}_q^\times$.

**Lemma 3.2.3.** *A character $\nu$ of $\mathbb{F}_{q^2}^\times$ is decomposable if and only if $\nu = \overline{\nu}$.*

**Proof.** If $\nu$ is decomposable, then since $\mathrm{N}(\lambda) = \mathrm{N}(\overline{\lambda})$, certainly we have $\nu(\lambda) = \nu(\overline{\lambda}) = \overline{\nu}(\lambda)$. Conversely, if $\nu = \overline{\nu}$, then we define a map $\chi : \mathrm{N}(\mathbb{F}_{q^2}^\times) \to \mathbb{C}$ by $\chi(\mathrm{N}(\lambda)) = \nu(\lambda)$. By Corollary 3.2.2, we can check that $\chi$ is well defined. The fact that $N$ is surjective now extends the domain of $\chi$ to $\mathbb{F}_q^\times$. Hence, $\chi$ is a character of $\mathbb{F}_q^\times$ and $\nu$ is therefore decomposable. $\square$

**Lemma 3.2.4.** *If $\nu$ is a non-decomposable character of $\mathbb{F}_{q^2}^\times$, then*

$$\sum_{\mathrm{N}(x)=\alpha} \nu(x) = 0 \ \ \textit{for every } \alpha \in \mathbb{F}_q^\times.$$

**Proof.** By Lemma 3.2.3, there exists $\zeta \in \mathbb{F}_{q^2}^\times$ such that $\nu(\zeta) \neq \nu(\overline{\zeta})$. Let $\lambda = \zeta/\overline{\zeta}$. Then

$$\sum_{\mathrm{N}(x)=\alpha} \nu(x) = \sum_{\mathrm{N}(x)=\alpha} \nu(\lambda x) = \nu(\lambda) \sum_{\mathrm{N}(x)=\alpha} \nu(x),$$

and our claim follows. $\square$

We shall need the analogue to Lemma 3.2.3 for the characters of additive group $\mathbb{F}_{q^2}^+$. The *trace function* $\mathrm{Tr} : \mathbb{F}_{q^2}^+ \to \mathbb{F}_q^+$ is defined by $\mathrm{Tr}(\lambda) = \lambda + \overline{\lambda}$.

**Lemma 3.2.5.** *The kernel of the trace map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ consists of $q$ elements. Furthermore, the trace map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$ is surjective.*

**Proof.** The trace function is additive and its kernel consists of elements in $\mathbb{F}_{q^2}$ satisfying $x^q + x = 0$. $\square$

**Corollary 3.2.6.** *If $\lambda \in \mathbb{F}_{q^2}^\times$, then for every $\alpha \in \mathbb{F}_q$ there exists an $x \in \mathbb{F}_{q^2}$ such that $\lambda x + \overline{\lambda}\overline{x} = \alpha$.*

**Proof.** There exists a $\zeta \in \mathbb{F}_{q^2}$ such that $\mathrm{Tr}(\zeta) = \alpha$. Choose $x = \zeta/\lambda$ will do. $\square$

## 3.3. The Small Weil Group

Let $F/E$ be a finite Galois extension. Its Galois group $G(F/E)$ acts on the multiple group $F^\times$ of $F$. Denote by $W(F/E) = G(F/E) \cdot F^\times$, the semi-direct product of $G(E/F)$ by $F^\times$. It consists of all pairs $(x, \sigma)$ where $x \in F^\times$ and $\sigma \in G(F/E)$. Multiplication is given by the formula

$$(x, \sigma) \cdot (y, \tau) = (x \cdot \sigma(y), \sigma\tau).$$

It is easy to check that the identity is $(1, 1)$ and the inverse is given by $(x, \sigma)^{-1} = (\sigma^{-1}(x), \sigma^{-1})$. The map $x \mapsto (x, 1)$ is an embedding of $F^\times$ in $W(F/E)$. We identify $F^\times$ with its image. $F^\times$ is normal in $W(F/E)$.

The group $W(F/E)$ is in general not abelian. A typical commutator is

$$(x, \sigma)(y, \tau)(x, \sigma)^{-1}(y, \tau)^{-1} = (x \cdot \sigma(y) \cdot \sigma\tau\sigma^{-1}(x^{-1}) \cdot \sigma\tau\sigma^{-1}\tau^{-1}(y^{-1}), \sigma\tau\sigma^{-1}\tau^{-1}).$$

In particular, if $G(F/E)$ is abelian, then we simply have

$$(x, \sigma)(y, \tau)(x, \sigma)^{-1}(y, \tau)^{-1} = (x \cdot \sigma(y) \cdot \tau(x^{-1}) \cdot y^{-1}, 1).$$

We now restrict our attention to the case where $E = \mathbb{F}_q$ and $F = \mathbb{F}_{q^2}$. In this case, $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is called the *small Weil group* of the extension $\mathbb{F}_{q^2}/\mathbb{F}_q$. It is a finite group having $2(q^2-1)$ elements, and it can be described as the free group generated by $\mathbb{F}_{q^2}^\times$ and $\varphi$ (the conjugation) with the relations

$$\varphi^2 = 1, \qquad \text{and} \qquad x\cdot\varphi = \varphi\cdot\varphi(x) = \varphi\cdot\overline{x}, \quad \text{for } x \in \mathbb{F}_{q^2}^\times.$$

(Here, we identify $\varphi$ by $(1,\varphi)$.)

We would like to establish a correspondence between the representations of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and the characters of $\mathbb{F}_{q^2}^\times$. We remark that since $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ contains the abelian normal subgroup $\mathbb{F}_{q^2}^\times$ of index 2, by Proposition 1.15.2 its irreducible representations are of dimension $\leq 2$.

The commutator subgroup $W(\mathbb{F}_{q^2}/\mathbb{F}_q)'$ of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ is the set $\{z/\overline{z} \mid z \in \mathbb{F}_{q^2}^\times\}$, which by Corollary 3.2.2 is equal to $\{x \in \mathbb{F}_{q^2}^\times \mid \mathrm{N}(x) = 1\}$. Hence by Lemma 3.2.1, we have proved:

**Lemma 3.3.1.** *$W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ has $2(q-1)$ characters.*

**Remark .** If $\tau$ is a character of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$, then $\tau(x) = \tau(\overline{x})$ for every $x \in \mathbb{F}_{q^2}^\times$ because $x/\overline{x} \in W(\mathbb{F}_{q^2}/\mathbb{F}_q)'$. On the other hand, starting from a character $\mu$ of $\mathbb{F}_q^\times$, we define characters $\tau_1$, $\tau_2$ of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ by $\tau_i(x) = \mu(\mathrm{N}(x))$ and $\tau_i(\varphi) = (-1)^i$. These are all the characters of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$.

Consider now a 2-dimensional representation $\tau$ of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Its restriction to $\mathbb{F}_{q^2}^\times$ decomposes into a direct sum of two characters. Let $\nu$ be one of them. By construction, there exists a vector $0 \neq v \in V_\tau$ such that $\tau_x(v) = \nu(x)v$, for every $x \in \mathbb{F}_{q^2}^\times$. Let $v' = \tau_\varphi(v)$. Then the relation $x\cdot\varphi = \varphi\cdot\overline{x}$ implies $\tau_x(v') = \nu(\overline{x})v'$, for every $x \in \mathbb{F}_{q^2}^\times$. Hence, $\overline{\nu}$ is also a component of the restriction of $\tau$ to $\mathbb{F}_{q^2}^\times$. There are two possibilities.

(1) $\nu \neq \overline{\nu}$. In this case, $v$ and $v'$ are linearly independent and we have

$$\mathrm{Res}^W_{\mathbb{F}_{q^2}^\times}(\tau) = \nu \oplus \overline{\nu}.$$

In this case we also have that $\tau$ is irreducible; indeed, otherwise $\nu$ must be equal to the restriction of one of the 1-dimensional component of $\tau$ to $\mathbb{F}_{q^2}^\times$, which we already knew is decomposable.

(2) $\nu = \overline{\nu}$. In this case, either $v'$ is a multiple of $v$ or $v'$ and $v$ are linearly independent. In the first case $v$ is an eigenvector of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$. This implies that $\tau$ is reducible. In the second case, $v$ and $v'$ generate $V_\tau$. Hence $\tau_x(w) = \nu(x)w$ for every $w \in V_\tau$. Let $w$ be an eigenvector of $\tau_\varphi$. Then $w$ is an eigenvector of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$, This implies again that $\tau$ is reducible.

We have thus proved:

**Lemma 3.3.2.** *Let $\tau$ be a two dimensional representation of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and let $\nu$ be a component of its restriction to $\mathbb{F}_{q^2}^\times$. Then $\nu$ is non-decomposable if and only if $\tau$ is irreducible.*

**Remark .** Let $\nu$ be a non-decomposable character of $\mathbb{F}_{q^2}^\times$. Define a 2-dimensional representation $\tau_\nu$ of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$ by

$$\tau_\nu(x) = \begin{pmatrix} \nu(x) & 0 \\ 0 & \nu(\overline{x}) \end{pmatrix} \qquad \text{and} \qquad \tau_\nu(\varphi) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$\tau_\nu$ is irreducible and $\tau_\nu = \tau_{\nu'}$ if and only if $\nu' = \nu$ or $\nu' = \overline{\nu}$. Theses are all the $\frac{1}{2}(q^2-q)$ two-dimensional representations of $W(\mathbb{F}_{q^2}/\mathbb{F}_q)$.

## 3.4. Constructing Cuspidal Representations from Non-decomposable Characters

Let $\nu$ be a non-decomposable character. We are going to define a representation $\rho$ that will turn out being a cuspidal representation. In order to define $\rho$ on $G$, it suffices to define $\rho$ as a map from $B \cup \{w'\}$

into the automorphism group of an appropriate vector space $V$ such that the restriction of $\rho$ to $B$ is a homomorphism and such that $\rho$ preserves the relations (1), (2) and (3) of Proposition 2.4.1.

The dimension of $\rho$ should be $q-1$. Hence it is convenient to take $V$ as the vector space of all functions $f : \mathbb{F}_q^\times \to \mathbb{C}$. On the other hand, $\mathrm{Res}_P^G(\rho) = \pi = \mathrm{Ind}_U^P(\psi)$, by the construction of induced representation, we only have to give values on a system of representatives of $P/U \cong A$. Identifying $A$ with $\mathbb{F}_q^\times$, and using the identity

$$\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha x & \beta x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha x & 0 \\ 0 & 1 \end{pmatrix}$$

we are led to the following definition:

$$\left[\rho\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}(f)\right](x) = \psi(\beta x)f(\alpha x).$$

Further, we would like to have $\rho$ coincides with $\nu$ on $Z$:

$$\left[\rho\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}(f)\right](x) = \nu(\delta)f(x).$$

It follows that we must define $\rho$ on $B$ by

$$\left[\rho\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}(f)\right](x) = \left[\rho\begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}\left(\rho\begin{pmatrix} \alpha\delta^{-1} & \beta\delta^{-1} \\ 0 & 1 \end{pmatrix}(f)\right)\right](x) = \nu(\delta)\psi(\beta\delta^{-1}x)f(\alpha\delta^{-1}x).$$

A straightforward calculation shows that $\rho$ is indeed a homomorphism of $B$ into $\mathrm{Aut}(V)$.

In order to define $\rho(w')$, we define a function $j : \mathbb{F}_q^\times \to \mathbb{C}$ by

$$j(x) = \frac{-1}{q}\sum_{\mathrm{N}(\lambda)=x,\ \lambda\in\mathbb{F}_{q^2}^\times} \psi(\mathrm{Tr}(\lambda))\nu(\lambda),$$

and for an $f \in V$ define $\rho(w')(f)$ by

$$[\rho(w')(f)](x) = \sum_{y\in\mathbb{F}_q^\times} \nu(y^{-1})j(xy)f(y).$$

Our task now is to prove that $\rho(w')$ together with the definition of $\rho$ on $B$ is compatible with the relations (1), (2) and (3) of Proposition 2.4.1. We remark the once we prove the identity (2) is true, we have $\rho(w') \in \mathrm{Aut}(V)$ automatically.

In order to prove that $\rho$ preserve identity (1) of Proposition 2.4.1, we compute for every $f \in V$:

$$\left[\rho\left(w'\begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\right)(f)\right](x) = \sum_{y\in\mathbb{F}_q^\times} \nu(y^{-1})j(xy)(\nu(\delta)f(\alpha\delta^{-1}y)),$$

and

$$\left[\rho\left(\begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}w'\right)(f)\right](x) = \nu(\alpha)(\sum_{z\in\mathbb{F}_q^\times} \nu(z^{-1})j(\delta\alpha^{-1}xz)f(z)).$$

Changing variables by $z = \alpha\delta^{-1}y$, we will see that they are equal.

In order to prove that $\rho$ preserve identity (2) of Proposition 2.4.1, we compute for every $f \in V$:

$$\begin{aligned}
[\rho(w')(\rho(w')(f))](x) &= \sum_{y\in\mathbb{F}_q^\times} \nu(y^{-1})j(xy)(\sum_{z\in\mathbb{F}_q^\times} \nu(z^{-1})j(yz)f(z)) \\
&= \sum_{z\in\mathbb{F}_q^\times} \nu(xz^{-1})f(z)(\sum_{y\in\mathbb{F}_q^\times} j(yz)j(xy)\nu(x^{-1}y^{-1})).
\end{aligned}$$

We can get

$$[\rho(w')(\rho(w')(f))](x) = \nu(-1)f(x) = \left[\rho\begin{pmatrix}\delta & 0\\ 0 & \alpha\end{pmatrix}(f)\right](x),$$

by using the following:

**Lemma 3.4.1.** *Given any $x \in \mathbb{F}_q^\times$, then*

$$\sum_{y\in\mathbb{F}_q^\times} j(yz)j(xy)\nu(x^{-1}y^{-1}) = \begin{cases}\nu(-1) & \text{if } z = x,\\ 0 & \text{if } z \neq x.\end{cases}$$

**Proof.** We start from the left-hand side:

$$\begin{aligned}\sum_{y\in\mathbb{F}_q^\times} j(yz)j(xy)\nu(x^{-1}y^{-1}) &= \frac{1}{q^2}\sum_{y\in\mathbb{F}_q^\times}\sum_{\substack{\mathrm{N}(t)=\frac{z}{x}(xy)\\ \mathrm{N}(s)=xy}}\psi(\mathrm{Tr}(t+s))\nu(tsx^{-1}y^{-1})\\ &= \frac{1}{q^2}\sum_{s\in\mathbb{F}_{q^2}^\times}\sum_{\mathrm{N}(t)=\frac{z}{x}\mathrm{N}(s)}\psi(\mathrm{Tr}(t+s))\nu(t/\overline{s}) \qquad \text{(using N is onto)}\\ &= \frac{1}{q^2}\sum_{\mathrm{N}(\lambda)=z/x}\nu(\lambda)\sum_{s\in\mathbb{F}_{q^2}^\times}\psi(\mathrm{Tr}(s(1+\overline{\lambda}))) \qquad \text{(by letting } \lambda = t/\overline{s})\end{aligned}$$

For a fixed $\lambda$, the map $s \mapsto \psi(\mathrm{Tr}(s(1+\overline{\lambda})))$ is a character of $\mathbb{F}_{q^2}^+$. Since the map $s \mapsto \mathrm{Tr}(s(1+\overline{\lambda}))$ maps $\mathbb{F}_{q^2}$ onto $\mathbb{F}_q$ if $\lambda \neq -1$ and $\psi$ is not the unit character, it follows that

$$\sum_{s\in\mathbb{F}_{q^2}^\times}\psi(\mathrm{Tr}(s(1+\overline{\lambda}))) = \begin{cases}-1 & \text{if } \lambda \neq -1,\\ q^2-1 & \text{if } \lambda = -1.\end{cases}$$

We now distinguish between two cases and suppose first that $z = x$. Then

$$\begin{aligned}\sum_{y\in\mathbb{F}_q^\times} j(yz)j(xy)\nu(x^{-1}y^{-1}) &= \frac{1}{q^2}\sum_{\substack{\mathrm{N}(\lambda)=1\\ \lambda\neq-1}}\nu(\lambda)\sum_{s\in\mathbb{F}_{q^2}^\times}\psi(\mathrm{Tr}(s(1+\overline{\lambda}))) + \frac{1}{q^2}\nu(-1)(q^2-1)\\ &= \frac{-1}{q^2}\sum_{\substack{\mathrm{N}(\lambda)=1\\ \lambda\neq-1}}\nu(\lambda) + \frac{1}{q^2}\nu(-1)(q^2-1)\\ &= \frac{1}{q^2}\nu(-1) + \frac{1}{q^2}\nu(-1)(q^2-1) \qquad \text{(by Lemma 3.2.4)}\\ &= \nu(-1)\end{aligned}$$

Now suppose that $z \neq x$. Then $\mathrm{N}(\lambda) = z/x$ implies that $\lambda \neq -1$. Hence, in this case by Lemma 3.2.4,

$$\sum_{y\in\mathbb{F}_q^\times} j(yz)j(xy)\nu(x^{-1}y^{-1}) = \frac{-1}{q^2}\sum_{\mathrm{N}(\lambda)=z/x}\nu(\lambda) = 0.$$

□

Finally we have to prove that $\rho$ preserves relation (3) of Proposition 2.4.1. We compute for every $f \in V$:

$$\begin{aligned}\left[\rho\left(w'\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}w'\right)(f)\right](x) &= \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)\psi(y)[\rho(w')(f)](y)\\ &= \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)\psi(y)(\sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})j(yz)f(z))\end{aligned}$$

and

$$\begin{aligned}\left[\rho\left(\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix} w'\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right)(f)\right](x) &= \nu(-1)\psi(-x)\left[\rho\left(w'\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\right)(f)\right](x)\\ &= \nu(-1)\psi(-x)\sum_{z\in\mathbb{F}_q^\times}\nu(z^{-1})j(xz)(\psi(-z)f(z))\end{aligned}$$

In order to prove that these two equalities are equal, we need the following:

**Lemma 3.4.2.** *Given any $x, z \in \mathbb{F}_q^\times$, we have that*

$$\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)j(yz)\psi(y) = \nu(-1)j(xz)\psi(-x)\psi(-z).$$

**Proof.** We start from the left-hand side.

$$\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)j(yz)\psi(y) = \frac{1}{q^2}\sum_{y\in\mathbb{F}_q^\times}\sum_{\substack{\mathrm{N}(t)=xy\\ \mathrm{N}(s)=yz}}\psi(\mathrm{Tr}(s+t)+y)\nu(sty^{-1})$$

Let $\lambda = sty^{-1}$. Then $\mathrm{N}(t) = xy$ and $\mathrm{N}(s) = yz$ imply that $\mathrm{N}(\lambda) = xz$. Using

$$\mathrm{Tr}(s+t)+y = z^{-1}\mathrm{N}(s+z+\lambda) - z\mathrm{N}(1+z^{-1}\lambda),$$

we have that

$$\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)j(yz)\psi(y) = \frac{1}{q^2}\sum_{\mathrm{N}(\lambda)=xz}\psi(-z\mathrm{N}(1+z^{-1}\lambda))\nu(\lambda)\sum_{s\in\mathbb{F}_{q^2}^\times}\psi(z^{-1}\mathrm{N}(s+z+\lambda)).$$

However,

$$\begin{aligned}\sum_{s\in\mathbb{F}_{q^2}^\times}\psi(z^{-1}\mathrm{N}(s+z+\lambda)) &= \sum_{r\in\mathbb{F}_{q^2};\ r\neq z+\lambda}\psi(z^{-1}\mathrm{N}(r))\\ &= \sum_{r\in\mathbb{F}_{q^2}^\times}\psi(z^{-1}\mathrm{N}(r)) + 1 - \psi(z^{-1}\mathrm{N}(z+\lambda))\\ &= (q+1)\sum_{\alpha\in\mathbb{F}_q^\times}\psi(z^{-1}\alpha) + 1 - \psi(z^{-1}\mathrm{N}(z+\lambda)) \qquad \text{(by Lemma 3.2.1)}\\ &= -q - \psi(z^{-1}\mathrm{N}(z+\lambda)).\end{aligned}$$

Therefore, we have that

$$\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)j(yz)\psi(y) = \frac{1}{q^2}\sum_{\mathrm{N}(\lambda)=xz}\psi(-z\mathrm{N}(1+z^{-1}\lambda))\nu(\lambda)(-q-\psi(z^{-1}\mathrm{N}(z+\lambda)))$$

$$= \frac{-1}{q}\sum_{\mathrm{N}(\lambda)=xz}\psi(-z\mathrm{N}(1+z^{-1}\lambda))\nu(\lambda) - \frac{1}{q^2}\sum_{\mathrm{N}(\lambda)=xz}\psi(-z\mathrm{N}(1+z^{-1}\lambda)+z^{-1}\mathrm{N}(z+\lambda))\nu(\lambda)$$

Note that under the assumption $\mathrm{N}(\lambda) = xz$, we have that

$$-z\mathrm{N}(1+z^{-1}\lambda) = -x - z - \mathrm{Tr}(\lambda) \quad\text{and}\quad -z\mathrm{N}(1+z^{-1}\lambda) + z^{-1}\mathrm{N}(z+\lambda) = 0$$

Hence, we may continue the chain of equalities by

$$\begin{aligned}
\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)j(yz)\psi(y) &= \frac{-1}{q}\psi(-x-z)\sum_{\mathrm{N}(\lambda)=xz}\psi(-\mathrm{Tr}(\lambda))\nu(\lambda)-\frac{1}{q^2}\sum_{\mathrm{N}(\lambda)=xz}\nu(\lambda)\\
&= \frac{-1}{q}\psi(-x-z)\sum_{\mathrm{N}(\lambda)=xz}\psi(-\mathrm{Tr}(\lambda))\nu(\lambda) \qquad \text{(by Lemma 3.2.4)}\\
&= \frac{-1}{q}\psi(-x-z)\nu(-1)\sum_{\mathrm{N}(\lambda')=xz}\psi(\mathrm{Tr}(\lambda'))\nu(\lambda') \quad \text{(by letting } \lambda=-\lambda'\text{)}\\
&= \nu(-1)j(xz)\psi(-x)\psi(-z).
\end{aligned}$$

$\square$

For later references let us also describe the action of $\rho$ on the element

$$s=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix},\quad \gamma\neq 0.$$

We use the identity

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\beta-\alpha\gamma^{-1}\delta & -\alpha\\ 0 & -\gamma\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \gamma^{-1}\delta\\ 0 & 1\end{pmatrix}.$$

For $f:\mathbb{F}_q^\times\to\mathbb{C}$,

$$\begin{aligned}
[\rho(s)(f)](x) &= \nu(-\gamma)\psi(\alpha\gamma^{-1}x)\left[\rho(w')\rho\begin{pmatrix}1 & \gamma^{-1}\delta\\ 0 & 1\end{pmatrix}(f)\right]((\beta-\alpha\gamma^{-1}\delta)(-\gamma)^{-1}x)\\
&= \nu(-\gamma)\psi(\alpha\gamma^{-1}x)\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j((\alpha\delta-\beta\gamma)\gamma^{-2}xy)\left[\rho\begin{pmatrix}1 & \gamma^{-1}\delta\\ 0 & 1\end{pmatrix}(f)\right](y)\\
&= \frac{-1}{q}\nu(-\gamma)\psi(\alpha\gamma^{-1}x)\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})\psi(\gamma^{-1}\delta y)f(y)\sum_{\mathrm{N}(\lambda)=\gamma^{-2}xy\det(s)}\psi(\mathrm{Tr}(\lambda))\nu(\lambda)\\
&= \frac{-1}{q}\sum_{y\in\mathbb{F}_q^\times}\left(\psi(\frac{\alpha x+\delta y}{\gamma})\sum_{\mathrm{N}(\lambda')=xy^{-1}\det(s)}\psi(-\frac{y}{\gamma}\mathrm{Tr}(\lambda'))\nu(\lambda')\right)f(y)\quad (\text{letting } \lambda'=-\frac{\gamma\lambda}{y})
\end{aligned}$$

We have therefore proved:

**Proposition 3.4.3.** *Let* $s=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{GL}(2,\mathbb{F}_q)$ *with* $\gamma\neq 0$*. Then we have*

$$[\rho(s)(f)](x)=\sum_{y\in\mathbb{F}_q^\times}k(x,y;s)f(y),$$

*where*

$$k(x,y;s)=\frac{-1}{q}\psi(\frac{\alpha x+\delta y}{\gamma})\sum_{\mathrm{N}(\lambda)=xy^{-1}\det(s)}\psi(-\frac{y}{\gamma}\mathrm{Tr}(\lambda))\nu(\lambda).$$

## 3.5. The Correspondence between Cuspidal Representations and Non-decomposable Characters

In last section, given a non-decomposable character $\nu$ of $\mathbb{F}_{q^2}^\times$ we associate a cuspidal representation. To distinguish its dependence with $\nu$, we denote such a representation by $\rho_\nu$.

**Proposition 3.5.1.** *If $\nu$ and $nu'$ are non-decomposable characters of $\mathbb{F}_{q^2}^\times$, then $\rho_\nu$ is isomorphic to $\rho_{\nu'}$ if and only if $\nu$ is conjugate to $\nu'$ over $\mathbb{F}_q$.*

**Proof.** Let $\rho = \rho_n u$, $\rho' = \rho_{\nu'}$ and let $j = j_\nu$, $j' = j_{\nu'}$ be the corresponding $j$ function, respectively. If $\nu' = \overline{\nu}$, then we have $\nu'(\alpha) = \nu(\alpha)$ for $\alpha \in \mathbb{F}_q^\times$. Hence, $\rho = \rho'$ on $B$. Further,

$$j'(x) = \frac{-1}{q} \sum_{\mathrm{N}(\lambda)=x} \psi(\mathrm{Tr}(\lambda))\nu'(\lambda) = \frac{-1}{q} \sum_{\mathrm{N}(\overline{\lambda})=x} \psi(\mathrm{Tr}(\overline{\lambda}))\nu(\overline{\lambda}) = \frac{-1}{q} \sum_{\mathrm{N}(\lambda)=x} \psi(\mathrm{Tr}(\lambda))\nu(\lambda) = j(x).$$

Hence $\rho'(w') = \rho(w')$. We conclude that $\rho = \rho'$.

Conversely, suppose that $\rho'$ is isomorphic to $\rho$. Then there exists an $\theta \in \mathrm{Aut}(V)$ such that $\rho'(s) \circ \theta = \theta \circ \rho(s)$, for all $s \in G$. However $\mathrm{Res}_P^G(\rho) = \pi = \mathrm{Res}_P^G(\rho')$ and $\pi$ is irreducible. By Schur's lemma (Proposition 1.3.1), $\theta$ is a homothety. Hence, $\rho'(s) = \rho(s)$ for every $s \in G$. In particular, $\rho$ and $\rho'$ are equal on $B$. Hence, $\nu(\alpha) = \nu'(\alpha)$ for every $\alpha \in \mathbb{F}_q^\times$. Further, $\rho(w') = \rho'(w')$; hence

$$\sum_{y \in \mathbb{F}_q^\times} \nu(y^{-1}) j(xy) f(y) = \sum_{y \in \mathbb{F}_q^\times} \nu'(y^{-1}) j'(xy) f(y),$$

for every $x \in \mathbb{F}_q^\times$ and every $f \in V$ (recall that $V$ is the space of functions $f : \mathbb{F}_q^\times \to \mathbb{C}$). This implies that $j(\alpha) = j'(\alpha)$ for every $\alpha \in \mathbb{F}_q$, *i.e.*

$$\sum_{\mathrm{N}(\lambda)=\alpha} \psi(\mathrm{Tr}(\lambda))\nu(\lambda) = \sum_{\mathrm{N}(\lambda)=\alpha} \psi(\mathrm{Tr}(\lambda))\nu'(\lambda).$$

For any $x \in \mathbb{F}_q^\times$, we also have that $j(x^2\alpha) = j'(x^2\alpha)$. Cancelling $\nu(x)$, we obtain

$$\sum_{\mathrm{N}(\lambda)=\alpha} \psi(x\mathrm{Tr}(\lambda))\nu(\lambda) = \sum_{\mathrm{N}(\lambda)=\alpha} \psi(x\mathrm{Tr}(\lambda))\nu'(\lambda), \qquad \text{for beingevery } \alpha, x \in \mathbb{F}_q^\times.$$

Now choose a generator $\lambda_0$ of the cyclic group $\mathbb{F}_{q^2}^\times$. For any $y \in \mathbb{F}_q$, there exists $\lambda \in \mathbb{F}_{q^2}$ such that $\mathrm{Tr}(\lambda) = y$ and $\mathrm{N}(\lambda) = \mathrm{N}(\lambda_0)$ (we use the fact that $\mathbb{F}_{q^2}$ is the unique quadratic extension of $\mathbb{F}_q$). The other solution is obvious $\overline{\lambda}$. Let $a_y = \nu(\lambda) + \nu(\overline{\lambda}) - \nu'(\lambda) - \nu'(\overline{\lambda})$. and let $\psi_y$ be the character of $\mathbb{F}_q^+$ defined by $\psi_y(x) = \psi(xy)$. Then we have that

$$\sum_{y \in \mathbb{F}_q} a_y \psi_y = 0.$$

Notice that if $y \neq y'$, then $\psi_y \neq \psi_{y'}$. Hence by Artin's lemma (Lemma 1.9.2), we have that $a_y = 0$ for every $y \in \mathbb{F}_q$. In particular, we have

$$\nu(\lambda_0) + \nu(\overline{\lambda_0}) = \nu'(\lambda_0) + \nu'(\overline{\lambda_0}).$$

Combining with

$$\nu(\lambda_0)\nu(\overline{\lambda_0}) = \nu(\mathrm{N}(\lambda_0)) = \nu'(\mathrm{N}(\lambda_0)) = \nu'(\lambda_0)\nu'(\overline{\lambda_0}),$$

we have either $\nu'(\lambda_0) = \nu(\lambda_0)$ or $\nu'(\lambda_0) = \nu(\overline{\lambda_0})$. This implies that $\nu' = \nu$ or $\nu' = \overline{\nu}$, since $\lambda_0$ is the generator of $\mathbb{F}_{q^2}^\times$.

□

**Remark .** There are totally $q^2 - q$ non-decomposable characters of $\mathbb{F}_{q^2}^\times$ and there are $\frac{1}{2}(q^2 - q)$ cuspidal representations of $\mathrm{GL}(2, \mathbb{F}_q)$. From Proposition 3.5.1, we know that every cuspidal representation of the form $\rho_\nu$, for some non-decomposable character $\nu$.

At this point, we would like to indicate an interesting duality between conjugacy classes of $\mathrm{GL}(2, \mathbb{F}_q)$ and characters of $\mathbb{F}_{q^2}^\times$. For example, the elements $\lambda \in \mathbb{F}_{q^2} - \mathbb{F}_q$ correspond to the conjugacy classes $c_4(\lambda)$ (*cf.* Section 2.7), whereas the characters $\nu$ of $\mathbb{F}_{q^2}\times$ that do not come from characters of $\mathbb{F}_q^\times$ (*i.e.* non-decomposable characters) correspond to the cuspidal representations $\rho_\nu$ of $G$. In both sets there are $\frac{1}{2}(q^2 - q)$.

We summarize these data in the following table.

| Elmt. of $\mathbb{F}_{q^2}^\times$ | Conj. Cl. | No. of Elmt. | Dim. | Irr. Rep. | Char. of $\widehat{\mathbb{F}_{q^2}^\times}$ |
|---|---|---|---|---|---|
| $\alpha \in \mathbb{F}_q^\times$ | $c_1(\alpha)$ | $q-1$ | $1$ | $\rho'_{(\mu_1,\mu_1)}$ | $\mu_1 \in \widehat{\mathbb{F}_q^\times}$ |
| | $c_2(\alpha)$ | $q-1$ | $q$ | $\rho_{(\mu_1,\mu_1)}$ | |
| $\alpha \neq \beta \in \mathbb{F}_q^\times$ | $c_3(\alpha)$ | $\frac{1}{2}(q-1)(q-2)$ | $q+1$ | $\rho_{(\mu_1,\mu_2)}$ | $\mu_1 \neq \mu_2 \in \widehat{\mathbb{F}_q^\times}$ |
| $\lambda \in \mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$ | $c_4(\lambda)$ | $\frac{1}{2}(q^2-q)$ | $q-1$ | $\rho_\nu$ | $\nu \in \widehat{\mathbb{F}_{q^2}^\times} - \widehat{\mathbb{F}_q^\times}$ |

## 3.6. Whittakers Models

Recall that we have fixed a non-unit character $\psi$ of $\mathbb{F}_q^+$, identified it with a character of $U$ and found that $\pi = \mathrm{Ind}_U^P(\psi)$ is a $(q-1)$-dimensional irreducible representation of $P$.

If $\chi$ is a character of $G$, then by Frobenius reciprocity

$$(\chi, \mathrm{Ind}_U^G(\psi)) = (\chi, \mathrm{Ind}_P^G(\pi)) = (\mathrm{Res}_P^G(\chi), \pi) = 0.$$

Hence all irreducible components of $\hat{\pi} = \mathrm{Ind}_U^G(\psi)$ are of dimension $> 1$ and each of them appears in multiplicity 1 by Lemma 3.1.2.

**Lemma 3.6.1.** *Let $\rho$ be an irreducible representation of $\mathrm{GL}(2, \mathbb{F}_q)$ of dimension $> 1$. Then*

$$\mathrm{Res}_P^G(V_\rho) = \mathrm{Res}_P^G(J(V_\rho)) \oplus V_\pi.$$

**Proof.** This has been prove when $\dim(\rho) = q-1$ (Lemma 3.1.1) and when $\dim(\rho) = q+1$ (Proposition 2.6.4). If $\dim(\rho) = q$, then there exists a character $\mu$ of $B$ and a character $\rho'$ of $G$ such that $\hat{\mu} = \mathrm{Ind}_B^G(\mu) = \rho' \oplus \rho$. Further, by Proposition 2.6.4 we have $\mathrm{Res}_P^G(V_{\hat{\mu}}) = \mathrm{Res}_P^G(J(V_{\hat{\mu}})) \oplus V_\pi$. Hence

$$\mathrm{Res}_P^G(V_{\rho'}) \oplus \mathrm{Res}_P^G(V_\rho) = \mathrm{Res}_P^G(J(V_{\rho'})) \oplus \mathrm{Res}_P^G(J(V_\rho)) \oplus V_\pi.$$

However, since $J(V_{\rho'}) \neq 0$ by Lemma 2.6.1, we have $\mathrm{Res}_P^G(V_{\rho'}) = \mathrm{Res}_P^G(J(V_{\rho'}))$, and hence the lemma. □

**Theorem 3.6.2.** $\mathrm{Ind}_U^G(\psi)$ *is the direct sum of all higher dimensional* $(> 1)$ *irreducible representations of $G$, each of multiplicity* 1.

**Proof.** For any higher dimensional irreducible representation $\rho$, since $\mathrm{Res}_P^G(J(V_\rho))$ is either 0 or decomposes into a direct sum of 1-dimensional $\mathbb{C}[P]$-submodules, by Lemma 3.6.1 we have

$$(\rho, \mathrm{Ind}_U^G(\psi)) = (\rho, \mathrm{Ind}_P^G(\pi)) = (\mathrm{Res}_P^G(\rho), \pi) = 1.$$

□

If now $\rho$ is an irreducible higher dimensional representation of $G$, the $V_\rho$ can be embedded in $\mathrm{Ind}_U^G(V_\psi)$. Thus, for every $v \in V_\rho$, there exists a unique function $W_v : G \to \mathbb{C}$ in $\mathrm{Ind}_U^G(V_\psi)$ called a *Whittaker function* of $\rho$ such that the following rules hold:

(1) $W_v = 0$ if and only if $v = 0$; $W_{cv+c'v'} = cW_v + c'W_{v'}$, for $c, c' \in \mathbb{C}$.

(2) $W_v(us) = \psi(u)W_v(s)$, for $u \in U$ and $s \in G$.

(3) $W_{\rho_s(v)}(r) = W_v(rs)$, for $r, s \in G$.

The set of all function $W_v$ form a $\mathbb{C}[G]$-submodule $W(\rho)$ of $\mathrm{Ind}_U^G(V_\psi)$ called the *Whittaker model* of $\rho$. By Theorem 3.6.2, this submodule is uniquely determined within $\mathrm{Ind}_U^G(V_\psi)$. Moreover, if $\rho'$ is another higher dimensional irreducible representation of $G$, then $W(\rho) \cap W(\rho') = \{0\}$.

## 3.7. The Γ-function of a representation of $G$

Let $\rho$ be a higher dimensional irreducible representation of $G$. The $P$-decomposition $V_\rho = J(V_\rho) \oplus V_\pi$ is also an $A$-decomposition, because $A$ is a subgroup of $P$. We study the action of $A$ through $\rho$ on $J(V_\rho)$ and $V_\pi$.

If $\rho = \rho_{(\mu_1,\mu_2)}$, where $\mu_1$ and $\mu_2$ are characters of $\mathbb{F}_q^\times$, then by Lemma 2.6.5 $\mu_1$ and $\mu_2$ are eigenvalues of $A$ on $J(V_\rho)$ (here, we identify $A$ with $\mathbb{F}_q^\times$). In this case, we call $\mu_1^{-1}$ and $\mu_2^{-1}$ the *exceptional characters* for $\rho$. We remark that if $\rho$ is cuspidal, then there is no exceptional character. In any case, if the inverse of a character $\upsilon$ of $\mathbb{F}_q^\times$ is not exceptional, then $\upsilon$ is not an eigenvalue of $A$ operating on $J(V_\rho)$ through $\rho$.

By the definition of $V_\pi$, every $f \in V_\pi$ is uniquely determined by its value on the representative $P/U \cong A$. Hence $\mathrm{Res}_A^P(V_\pi)$ is isomorphic to the space of all functions $f : \mathbb{F}_q^\times \to \mathbb{C}$, and $A$ acts on this space by the formula

$$\left[\rho\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}(f)\right](x) = f(x\alpha), \qquad \text{for } \alpha, x \in \mathbb{F}_q^\times.$$

**Lemma 3.7.1.** *Let $\rho$ be a higher dimensional irreducible representation of $G$. If a character $\upsilon$ of $\mathbb{F}_q^\times$ is not an exceptional character of $\rho$, then any two linear mappings $l_1$, $l_2 : V_\rho \to \mathbb{C}$ satisfying*

$$l_i(\rho\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}(v)) = \upsilon^{-1}(x)l_i(v) \qquad \textit{for every } x \in \mathbb{F}_q^\times \textit{ and } v \in V_\rho,$$

*are linearly dependent.*

**Proof.** Let $\zeta$ be a generator of the cyclic group $\mathbb{F}_q^\times$ and define the linear map $T : V_\rho \to V_\rho$ by

$$T(v) = \rho\begin{pmatrix}\zeta & 0\\ 0 & 1\end{pmatrix}(v) - \upsilon^{-1}(\zeta)v.$$

Then

$$\mathrm{Ker}(T) = \{v \in V_\rho \mid \rho\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}(v) = \upsilon^{-1}(\alpha)v \quad \text{for all } x \in \mathbb{F}_q^\times\},$$

*i.e.* the space of eigenvectors of $A$ belonging to the eigenvalue $\upsilon^{-1}$. Since $\upsilon$ is non-exceptional, $\upsilon^{-1}$ is not the eigenvalue of $A$ operating on $J(V_\rho)$ through $\rho$. By Lemma 3.6.1, we conclude that $\mathrm{Ker}(T) \subseteq \mathrm{Res}_A^P(V_\pi)$. However, any function $f : \mathbb{F}_q^\times \to \mathbb{C}$ satisfying

$$f(x\alpha) = \left[\rho\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}(f)\right](x) = \upsilon^{-1}(\alpha)f(x),$$

is uniquely defined by the value of $f(1)$; indeed, we have $f(\alpha) = \upsilon^{-1}(\alpha)f(1)$, for all $\alpha \in \mathbb{F}_q^\times$. This implies that $\dim \mathrm{Ker}(T) = 1$, and hence $\dim T(V_\rho) = \dim(V_\rho) - 1$.

Now, suppose $l_1$ and $l_2$ are not zero mapping. Clearly, $T(V_\rho) \subseteq \mathrm{Ker}(l_i)$ and $\dim \mathrm{Ker}(l_i) = \dim(V_\rho) - 1$. Therefore $\mathrm{Ker}(l_1) = T(V_\rho) = \mathrm{Ker}(l_2)$. This follows that $l_1$ and $l_2$ are linearly dependent. □

**Theorem 3.7.2.** *Let $\rho$ be a higher dimensional irreducible representation of $G$ and let $\upsilon$ be a character of $\mathbb{F}_q^\times$ which is not exceptional for $\rho$. Then there exists a complex number $\Gamma_\rho(\upsilon)$ such that for every Whittaker function $W_v$ of $\rho$, we have*

$$\Gamma_\rho(\upsilon) \sum_{x\in\mathbb{F}_q^\times} W_v\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\upsilon(x) = \sum_{x\in\mathbb{F}_q^\times} W_v\begin{pmatrix}0 & 1\\ x & 0\end{pmatrix}\upsilon(x).$$

**Proof.** Define the linear mappings $l_1$, $l_2 : V_\rho \to \mathbb{C}$ by

$$l_i(v) = \sum_{x\in\mathbb{F}_q^\times} W_v\left(w^i\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\right)\upsilon(x) \qquad \text{for } i = 1, 2 \text{ and } v \in V_\rho.$$

Then

$$\begin{aligned} l_i(\rho\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}(v)) &= \sum_{x\in\mathbb{F}_q^\times} W_v\left(w^i\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\upsilon(x) = \sum_{x\in\mathbb{F}_q^\times} W_v\left(w^i\begin{pmatrix}\alpha x & 0\\ 0 & 1\end{pmatrix}\right)\upsilon(x)\\ &= \upsilon(\alpha^{-1})\sum_{x\in\mathbb{F}_q^\times} W_v\left(w^i\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\right)\upsilon(x) = \upsilon^{-1}(\alpha)l_i(v) \qquad \forall\,\alpha\in\mathbb{F}_q^\times, \forall\, v\in V_\rho. \end{aligned}$$

$l_2$ is not a zero mapping; indeed by the proof of Lemma 3.7.1, there exists a nonzero $v \in \mathrm{Res}_A^P(V_\pi)$ such that $v$ is a eigenvector of $A$ belonging to $\upsilon^{-1}$. Hence

$$W_v\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} = W_{\upsilon^{-1}(x)v}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \upsilon^{-1}(x)W_v\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \text{for every } x\in\mathbb{F}_q^\times.$$

This implies that for this $v$, $l_2(v) = (q-1)W_v(1) \neq 0$. It follows from Lemma 3.7.1 that $l_1$ is a multiple of $l_2$ by a constant denoted by $\Gamma_\rho(\upsilon)$. □

The complex valued function $\Gamma_\rho(\upsilon)$ defined for every non-exceptional character $\upsilon$ of $\mathbb{F}_q^\times$ for $\rho$ will play an important role in the computation of the character table of $\mathrm{GL}(2,\mathbb{F}_q)$.

## 3.8. Determination of $\rho$ by $\Gamma_\rho$

Let $\rho$ be a higher dimensional irreducible representation of $G$. For every $v \in V_\rho$, let $W_v$ be the corresponding Whittaker function of $\rho$ and let $F(\mathbb{F}_q^\times,\mathbb{C})$ be the space of all functions $f:\mathbb{F}_q^\times \to \mathbb{C}$. Consider the homomorphism $\mathrm{R}: V_\rho \to F(\mathbb{F}_q^\times,\mathbb{C})$ defined by $\mathrm{R}(v) = W_v|_A$ (here, we identify $A$ with $\mathbb{F}_q^\times$). If we define an operation of $\mathbb{F}_q^\times$ on $F(\mathbb{F}_q^\times,\mathbb{C})$ by $(\alpha * f)(x) = f(x\alpha)$ and identify $A$ with $\mathbb{F}_q^\times$, then R is an $A$-homomorphism.

**Lemma 3.8.1.** *The homomorphism* R *is surjective and* $\mathrm{Ker}(\mathrm{R}) = J(V_\rho)$.

**Proof.** We start by determining Ker(R). Let $v \in J(V_\rho)$. Then for every $\alpha\in\mathbb{F}_q^\times$, we choose a $\beta\in\mathbb{F}_q$ such that $\psi(\alpha\beta)\neq 1$ (because $\psi$ is non-unit). Since $u = \begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \in U$, we have $\rho_u(v) = v$. Hence

$$W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = W_{\rho_u(v)}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = W_v\begin{pmatrix}\alpha & \alpha\beta\\ 0 & 1\end{pmatrix} = w_v\left(\begin{pmatrix}1 & \alpha\beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) = \psi(\alpha\beta)W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}.$$

This shows that $W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = 0$, for every $\alpha\in\mathbb{F}_q^\times$. Hence, $v\in\mathrm{Ker}(\mathrm{R})$.

To prove that $\mathrm{Ker}(\mathrm{R}) \subseteq J(V_\rho)$, we remark first that Ker(R) is a $\mathbb{C}[P]$-module; indeed, since $P = AU$ and $U$ is normal in $P$, for any $a\in A$, $p\in P$, there exist $a'\in A$ and $u\in U$ such that $ap = ua'$. Hence, if $v\in\mathrm{Ker}(\mathrm{R})$, then $W_{\rho_p(v)}(a) = W_v(ap) = \psi(u)W_v(a') = 0$. This implies that the $\mathbb{C}[P]$-submodule $V_\pi\cap\mathrm{Ker}(\mathrm{R})$ is either $\{0\}$ or $V_\pi$, since $V_\pi$ is $P$-irreducible. Assume $V_\pi\cap\mathrm{Ker}(\mathrm{R}) = V_\pi$. Then $J(V_\rho)\subseteq\mathrm{Ker}(\mathrm{R})$ and $V_\rho = J(V_\rho)\oplus V_\pi$ implies that R is a zero mapping. In particular, we have $W_v(1) = 0$ for all $v\in V_\rho$. Hence for every $v\in V_\rho$, $W_v(s) = W_{\rho_s(v)}(1) = 0$, for every $s\in G$. Thus $v = 0$, which is absurd. We conclude that $V_\pi\cap\mathrm{Ker}(\mathrm{R}) = \{0\}$. Hence $\mathrm{Ker}(\mathrm{R}) = J(V_\rho)$. This fact implies that $\dim(\mathrm{R}(V_\rho)) = \dim(V_\pi) = q-1 = \dim(F(\mathbb{F}_q^\times,\mathbb{C}))$. □

The center $Z$ of $G$ consists of the scalar matrices and is therefore canonical isomorphic to $\mathbb{F}_q^\times$. The restriction of $\rho$ to $Z$ can therefore be identified with a character $\upsilon_\rho$ of $\mathbb{F}_q^\times$, called the *central character* of $\rho$.

**Theorem 3.8.2.** *A cuspidal representation $\rho$ of $G$ is uniquely determined by its $\Gamma$-function and its central character.*

**Proof.** Let $\rho$ and $\rho'$ be two cuspidal representation of $G$. Suppose that $\rho$ and $\rho'$ coincide on $Z$. Then since $\mathrm{Res}_P^G(\rho) = \pi = \mathrm{Res}_P^G(\rho')$ and $B = ZP$, we conclude that $\rho$ and $\rho'$ coincide on $B$. Suppose in addition that $\Gamma_\rho = \Gamma_{\rho'}$. Because of Bruhat decomposition $G = B \cup BwU$, in order to prove that $\rho$ and $\rho'$ are isomorphic, it suffices to show that $\rho_w$ and $\rho'_w$ coincide.

What we have now is that there exists an isomorphism $\theta : V_\rho \to V_{\rho'}$ such that $\theta(\rho_s(v)) = \rho'_s(\theta(v))$ for every $v \in V_\rho$ and $s \in B$. Since $J(V_\rho) = J(V_{\rho'}) = \{0\}$, by Lemma 3.8.1 the maps $\mathrm{R} : V_\rho \to F(\mathbb{F}_q^\times, \mathbb{C})$ and $\mathrm{R}' : V_{\rho'} \to F(\mathbb{F}_q^\times, \mathbb{C})$ are $\mathbb{C}[A]$-isomorphisms. Given any character $\upsilon$ of $A$, we know that there exist $v \in V_\rho$ such that $\rho_a(v) = \upsilon(a)v$ for every $a \in A$. Therefore, $W_v(a) = W_{\rho_a(v)}(1) = \upsilon(a)W_v(1)$ and similarly because $\rho'_a(\theta(v)) = \upsilon(a)\theta(v)$, $W_{\theta(v)}(a) = \upsilon(a)W_{\theta(v)}(1)$. Multiplying a suitable constant, we can assume that $W_v(a) = W_{\theta(v)}(a)$ for every $a \in A$. Since $V_\rho$ is a direct sum of eigenvectors of $A$ belonging to characters of $A$, without loss of generality, we can assume that $W_v|_A = W_{\theta(v)}|_A$ for every $v \in V_\rho$.

Therefore, by Theorem 3.7.2 the assumption $\Gamma_\rho = \Gamma_{\rho'}$ implies that for every character $\upsilon$ of $\mathbb{F}_q^\times$ (there is no exceptional character for cuspidal representation),

$$\sum_{x \in \mathbb{F}_q^\times} W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \upsilon(x) = \sum_{x \in \mathbb{F}_q^\times} W_{\theta(v)} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \upsilon(x).$$

This implies by Artin's lemma that

$$W_v \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = W_{\theta(v)} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}, \qquad \text{for every } x \in \mathbb{F}_q^\times,\ v \in V_\rho.$$

Hence for every $x \in \mathbb{F}_q^\times$ let $z = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in Z$, and we have

$$\begin{aligned} W_{\rho_w(v)} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} &= W_v \left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = W_v \left( \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \right) = W_{\rho_z(v)} \begin{pmatrix} 0 & 1 \\ x^{-1} & 0 \end{pmatrix} \\ &= W_{\theta(\rho_z(v))} \begin{pmatrix} 0 & 1 \\ x^{-1} & 0 \end{pmatrix} = W_{\rho'_z(\theta(v))} \begin{pmatrix} 0 & 1 \\ x^{-1} & 0 \end{pmatrix} = W_{\rho'_w(\theta(v))} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

This shows that

$$\mathrm{R}'(\theta(\rho_w(v))) = W_{\theta(\rho_w(v))}|_A = W_{\rho_w(v)}|_A = W_{\rho'_w(\theta(v))}|_A = \mathrm{R}'(\rho'_w(\theta(v))).$$

Since $\mathrm{R}'$ is an isomorphism, we conclude $\theta(\rho_w(v)) = \rho'_w(\theta(v))$.

□

## 3.9. The Bessel Function of a representation

Let $\rho$ be a higher dimensional irreducible representation of $G$. Then for $q > 3$, we have $\dim(V_\rho) \geq q - 1 > 2 \geq \dim(J(V_\rho))$, and for $q = 3$ and $dim(V_\rho) = 2$, $\rho$ is cuspidal and hence $J(V_\rho) = \{0\}$. Therefore, $V_\rho \neq J(V_\rho)$ in all cases.

As $U$ is abelian, $\mathrm{Res}_U^G(\rho)$ decomposes into a direct sum of characters. Since $V_\rho \neq J(V_\rho)$, one of the characters must be non-unit. Fix a non-unit character $\psi$ of $\mathbb{F}_q^+$. Recall that for another non-unit character $\psi'$ of $\mathbb{F}_q^+$, there exists an $\alpha \in \mathbb{F}_q^\times$ such that $\psi'(x) = \psi(\alpha x)$ for all $x \in \mathbb{F}_q$. Suppose that $\psi'$ is an eigenvalue of $U$ through $\rho$ with an eigenvector $v' \in V_\rho$, *i.e.*

$$\rho \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} (v') = \psi'(\beta)v', \quad \forall\, \beta \in \mathbb{F}_q.$$

Replacing $v'$ by $v = \rho \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} (v')$, we get that for every $\beta \in \mathbb{F}_q$,

$$\rho \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} (v) = \rho \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \rho \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix} (v') = \psi'(\alpha^{-1}\beta) \rho \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} (v') = \psi(\beta) v.$$

Hence, $\psi$ is an eigenvalue of $U$ through $\rho$ with eigenvector $v$. Using similar method and using $dim(V_\rho)-\dim(J(V_\rho)) = q-1$, we conclude that every non-unit character of $\mathbb{F}_q^+$ appears exactly once as an eigenvalue of $U$ through $\rho$. On the other hand, the Whittaker model $W(\rho)$ of $\rho$ is considered as a $\mathbb{C}[G]$-submodule of $\mathrm{Ind}_U^G V_\psi$ for the fixed $\psi$ (*cf.* Section 3.6 condition (2)). It follows that if $\alpha \in \mathbb{F}_q^\times$, then for all $\beta \in \mathbb{F}_q$ since $\rho\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}(v) = \psi(\beta)v$, we have

$$\begin{aligned}\psi(\beta)W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} &= W_{\psi(\beta)v}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = W_v\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix}\right)\\ &= W_v\left(\begin{pmatrix}1 & \alpha\beta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) = \psi(\alpha\beta)W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

If $\alpha \neq 1$, then we may conclude that $W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} = 0$. Further, since $v \neq J(V_\rho)$, $W_v|_A \neq 0$ by Lemma 3.8.1 and therefore $W_v(1) \neq 0$. The vector is said to be a *Bessel vector* for $\rho$.

Let us sum up the results in the following:

**Lemma 3.9.1.** *Let $\rho$ be a higher dimensional irreducible representation of $G$. Then for a given non-unit character $\psi$ of $U$, we have $(\psi, \mathrm{Res}_U^G(\rho)) = 1$. A Bessel vector for $\rho$ is an eigenvector $v$ of $A$ belonging to $\psi$ which is unique up to a scalar multiple and satisfying*

$$W_v\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{cases}= 0 & \text{if } \alpha \neq 1 \text{ and } \alpha \in \mathbb{F}_q^\times,\\ \neq 0 & \text{if } \alpha = 1.\end{cases}$$

We use Bessel vector $v$ to define the *Bessel function* $J_\rho : G \to \mathbb{C}$ of $\rho$ by

$$J_\rho(s) = \frac{W_v(s)}{W_v(1)} \qquad \text{for every } s \in G.$$

Clearly $J_\rho(s)$ does not depend on which Bessel vector $v$ is used. Note that $J_\rho$ is also a Whittaker function for $\rho$. We have

$$J_\rho(su) = \frac{W_v(su)}{W_v(1)} = \frac{W_{\rho_u(v)}(s)}{W_v(1)} = \frac{\psi(u)W_v(s)}{W_v(1)} = \psi(u)J_\rho(s) = J_\rho(us), \quad \text{for } u \in U \text{ and } s \in G.$$

Also

$$J_\rho(a) = \begin{cases}0 & \text{if } a \neq 1 \text{ and } a \in A,\\ 1 & \text{if } a = 1.\end{cases}$$

Therefore, if a character $\upsilon$ of $\mathbb{F}_q^\times$ is not exceptional for $\rho$, we have by Theorem 3.7.2 that

$$\Gamma_\rho(\upsilon) = \sum_{x\in\mathbb{F}_q^\times} J_\rho\begin{pmatrix}0 & 1\\ x & 0\end{pmatrix}\upsilon(x).$$

One can use this formula to define $\Gamma_\rho(\upsilon)$ also for the exceptional character $\upsilon$.

## 3.10. The Computation of $\Gamma_\rho(\upsilon)$ for a Non-cuspidal $\rho$

Let $\rho$ be a higher dimensional irreducible representation of $G$ which is not cuspidal. Then $\rho$ is a component of $\hat{\mu} = \mathrm{Ind}_B^G(\mu)$, where $\mu$ is a character of $B$ which corresponds to the pair of characters $(\mu_1, \mu_2)$ of $\mathbb{F}_q^\times$. We may consider therefore $V_\rho$ as an irreducible $\mathbb{C}[G]$-submodule of $\mathrm{Ind}_B^G(V_\mu)$. Every element of $V\rho$ appears then as a function $f : G \to \mathbb{C}$ such that

$$f(bs) = \mu(b)f(s), \qquad \forall\, b \in B,\ s \in G.$$

The action of $G$ on $V_\rho$ is given by $\rho_r(f)(s) = f(sr)$ for $s, r \in G$.

We shall use this description of $V_\rho$ in order to give a concrete Whittaker model for $\rho$ in the space $\mathrm{Ind}_U^G(V_\psi)$. Thus, we shall define an injective $\mathbb{C}[G]$-linear map from $V_\rho$ into $\mathrm{Ind}_U^G(V_\psi)$. For every $f \in V_\rho$, let $W_f : G \to \mathbb{C}$ be the function defined by

$$W_f(s) = \sum_{u \in U} f\,(wus)\,\psi(u)^{-1}.$$

It is easy to check that

$$W_f\,(us) = \sum_{u' \in U} f\left(wu'us\right)\psi(u')^{-1} = \sum_{u'' \in U} f\left(wu''s\right)\psi(u''u^{-1})^{-1} = \psi(u)W_f(s), \quad \text{for } u \in U,\, s \in G$$

and

$$W_{\rho_r(f)(s)} = \sum_{u \in U} \rho_r(f)\,(wus)\,\psi(u)^{-1} = \sum_{u \in U} f\,(wusr)\,\psi(u)^{-1} = W_f(sr), \quad \text{for } r, s \in G.$$

Therefore, the map $f \mapsto W_f$ defines a $\mathbb{C}[G]$-linear map from $V_\rho$ into $\mathrm{Ind}_U^G(V_\psi)$. Since $\rho$ is irreducible, $V_\rho$ is the $\mathbb{C}[G]$-span of any non-zero $f \in V_\rho$. We show the map is injective by constructing a specific non-zero function $f \in V_\rho$ such that $W_f \neq 0$.

Using Bruhat's decomposition $G = B \cup BwU$, we define $f$ by

$$f(b) = 0 \quad \text{and} \quad f(bwu) = \mu(b)\psi(u), \quad \text{for } b \in B,\, u \in U.$$

Then $f$ is a non-zero element of $\mathrm{Ind}_B^G(V_\mu)$ and it satisfies

$$(*) \qquad f(su) = \psi(u)f(s) \qquad \text{for } u \in U \text{ and } s \in G.$$

Computing $W_f(u)$ for $u \in U$, we find

$$W_f(u) = \sum_{u' \in U} f\left(wu'u\right)\psi(u')^{-1} = \sum_{u' \in U} \psi\left(u'u\right)\psi(u')^{-1} = q\psi(u).$$

Hence $W_f \neq 0$. Now we only need to prove that $f \in V_\rho$.

If $\dim(\rho) = q+1$, then $V_\rho = \mathrm{Ind}_B^G(V_\mu)$, and there is nothing to prove. We can therefore assume that $\dim(\rho) = q-1$. In this case $\mathrm{Ind}_B^G(V_\mu) = V_{\rho'} \oplus V_\rho$, where $\rho'$ is a 1-dimensional representation of $G$. We can therefore write $f = f_1 + f_2$, where $f_1 \in V_{\rho'}$ and $f_2 \in V_\rho$. Since $\rho'$ is a 1-dimensional representation of $G$, for every $s \in G$, we have $f_1(s) = \rho'_s(f_1)(1) = \mu_1(\det(s))f_1(1)$ (*cf.* Lemma 2.5.2). In particular, we have $f_1(u) = f_1(1)$ for every $u \in U$. Now for every $u \in U$, by definition $\hat{\mu}_u(f_1)$ and $\psi(u)f_1$ both belong to $V_{\rho'}$ and similarly $\hat{\mu}_u(f_2)$ and $\psi(u)f_2$ both belong to $V_\rho$. Since by the equality $(*)$, $\hat{\mu}_u(f) = \psi(u)f$, we have that $\hat{\mu}_u(f_1) + \hat{\mu}_u(f_2) = \psi(u)f_1 + \psi(u)f_2$. By the direct sum decomposition, we conclude that $\hat{\mu}_u(f_1) = \psi(u)f_1$. In particular, we have $f_1(u) = \hat{\mu}_u(f_1)(1) = \psi(u)f_1(1)$, for every $u \in U$. It follows from $f_1(u) = f_1(1)$ that $f_1(1) = 0$. Since $f_1(s) = \mu_1(\det(s))f_1(1)$, we have $f_1 = 0$ and our contention is proved.

Note that $(*)$ implies that $f$ is an eigenvector belonging to $\psi$. Hence, $f$ is a Bessel vector for $\rho$. In order to compute $\Gamma_\rho(\upsilon)$, we now have to compute

$$W_f\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = \sum_{y \in \mathbb{F}_q} f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix}\right)\psi(y)^{-1}.$$

Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} & \text{if } y = 0, \\ \begin{pmatrix} -y^{-1} & x \\ 0 & yx \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & (yx)^{-1} \\ 0 & 1 \end{pmatrix} & \text{otherwise,} \end{cases}$$

we have that

$$W_f\begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} = \sum_{y \in \mathbb{F}_q^\times} \mu_1(-y^{-1})\mu_2(yx)\psi((yx)^{-1} - y) = \sum_{\alpha\beta = -1/x} \mu_1(\alpha)^{-1}\mu_2(\beta)^{-1}\psi(\alpha+\beta).$$

Also

$$W_f(1)=\sum_{u\in U} f(wu)\psi(u)^{-1}=\sum_{u\in U}\psi(u)\psi(u)^{-1}=q.$$

Hence

$$\begin{aligned}\Gamma_\rho(\upsilon) &= \frac{1}{q}\sum_{x\in\mathbb{F}_q^\times}\sum_{\alpha\beta=-1/x}\mu_1(\alpha)^{-1}\mu_2(\beta)^{-1}\psi(\alpha+\beta)\upsilon(x)\\ &= \frac{1}{q}\sum_{x\in\mathbb{F}_q^\times}\sum_{\alpha\beta=-1/x}\mu_1(\alpha)^{-1}\mu_2(\beta)^{-1}\psi(\alpha)\psi(\beta)\upsilon(-\alpha^{-1}\beta^{-1})\\ &= \frac{\upsilon(-1)}{q}\sum_{\alpha\in\mathbb{F}_q^\times}\mu_1(\alpha)^{-1}\upsilon(\alpha)^{-1}\psi(\alpha)\sum_{\beta\in\mathbb{F}_q^\times}\mu_2(\beta)^{-1}\upsilon(\beta)^{-1}\psi(\beta).\end{aligned}$$

Now recall that for a character $\chi$ of $\mathbb{F}_q^\times$ and a non-unit character $\phi$ of $\mathbb{F}_q^+$, one define the *Gauss sum*

$$G(\chi,\phi)=\sum_{x\in\mathbb{F}_q^\times}\chi(x)\phi(x).$$

We have therefore proved the following:

**Theorem 3.10.1.** *Let $\mu_1$ and $\mu_2$ be characters of $\mathbb{F}_q^\times$ and let $\rho=\rho_{(\mu_1,\mu_2)}$ be the corresponding irreducible representation of $G$. If $\upsilon$ is a character of $\mathbb{F}_q^\times$, then*

$$\Gamma_\rho(\upsilon)=\frac{\upsilon(-1)}{q}G(\mu_1^{-1}\upsilon^{-1},\psi)G(\mu_2^{-1}\upsilon^{-1},\psi).$$

**Remark .** It is well known that $\mid G(\chi,\phi)\mid=\sqrt{q}$. Hence $\mid\Gamma_\rho(\upsilon)\mid=1$.

If $\psi'$ is another non-unit character of $\mathbb{F}_q^+$, then there exists an $\alpha\in\mathbb{F}_q^\times$ such that $\psi'(x)=\psi(\alpha x)$. It follows that $G(\chi,\psi')=\chi(\alpha)^{-1}G(\chi,\psi)$. Hence, if we denote by $\Gamma'_\rho$ the $\Gamma$-function of $\rho$ obtained by using $\psi'$, then

$$\Gamma'_\rho(\upsilon)=\upsilon(\alpha)^2\mu_1(\alpha)\mu_2(\alpha)\Gamma_\rho(\upsilon).$$

## 3.11. The Computation of $\Gamma_\rho(\upsilon)$ for a Cuspidal $\rho$

Let $\nu$ be a non-decomposable character of $\mathbb{F}_{q^2}^\times$ and let $\rho=\rho_\nu$ be the corresponding cuspidal representation of $G$. Recall that we consider $V_\rho$ as the space of all functions $f:\mathbb{F}_q^\times\to\mathbb{C}$. We define the action of $\rho$ as the following.

$$\left[\rho\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}(f)\right](x)=\nu(\delta)\psi(\beta\delta^{-1}x)f(\alpha\delta^{-1}x)$$

and

$$[\rho(w')(f)](x)=\left[\rho\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}(f)\right](x)=\sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)f(y),$$

where $j=j_\nu$ is the function

$$j(x)=\frac{-1}{q}\sum_{\mathrm{N}(\lambda)=x,\ \lambda\in\mathbb{F}_{q^2}^\times}\psi(\mathrm{Tr}(\lambda))\nu(\lambda).$$

We shall use this description of $V_\rho$ in order to give a concrete Whittaker Model $W(\rho)$ in $\mathrm{Ind}_U^G(V_\psi)$. We define a function $\eta:G\to\mathbb{C}$ in $W(\rho)$ as the following.

$$\eta\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{cases}0 & \text{if } \gamma=0 \text{ and } \alpha\neq\delta,\\ \nu(\alpha)\psi(\alpha^{-1}\beta) & \text{if } \gamma=0 \text{ and } \alpha=\delta,\\ \frac{-1}{q}\psi(\frac{\alpha+\delta}{\gamma})\sum_{\mathrm{N}(\lambda)=\alpha\delta-\gamma\beta}\psi(-\frac{\mathrm{Tr}(\lambda)}{\gamma})\nu(\lambda) & \text{if } \gamma\neq 0.\end{cases}$$

We need to check that $\eta$ is in $\mathrm{Ind}_U^G(V_\psi)$. Indeed, by $\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \begin{pmatrix}\alpha+\zeta\gamma & \beta+\zeta\delta\\ \gamma & \delta\end{pmatrix}$ we have

$$\eta\left(\begin{pmatrix}1 & \zeta\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right) = \begin{cases}0 & \text{if } \gamma = 0 \text{ and } \alpha \neq \delta,\\ \nu(\alpha)\psi(\alpha^{-1}\beta+\zeta) & \text{if } \gamma = 0 \text{ and } \alpha = \delta,\\ \frac{-1}{q}\psi(\frac{\alpha+\delta}{\gamma}+\zeta)\sum_{\mathrm{N}(\lambda)=\alpha\delta-\gamma\beta}\psi(-\frac{\mathrm{Tr}(\lambda)}{\gamma})\nu(\lambda) & \text{if } \gamma \neq 0.\end{cases}$$

Thus, $\eta(us) = \psi(u)\eta(s)$, for all $u \in U$ and $s \in G$. Let $W(\eta)$ be the $\mathbb{C}[G]$-submodule of $\mathrm{Ind}_U^G(V_\psi)$ spanned by $\eta$. Define a map $\phi : W(\eta) \to V_\rho$ by

$$\phi(h)(x) = h\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}, \quad \text{for } h \in W(\eta) \text{ and } x \in \mathbb{F}_q^\times.$$

We are going to show that $\phi$ is a $\mathbb{C}[G]$-linear map. Since both $W(\eta)$ and $V_\rho$ are irreducible, this implies that $\phi$ is a $\mathbb{C}[G]$-isomorphism.

We only have to check $\phi(\hat{\pi}_s(\eta))(x) = \rho_s(\phi(\eta))(x)$ for the cases $s \in B$ and $s = w'$. For $s = \begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}$, we have

$$\phi(\hat{\pi}_s(\eta))(x) = \hat{\pi}_s(\eta)\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix} = \eta\left(\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}\right) = \eta\begin{pmatrix}x\alpha & x\delta\\ 0 & \delta\end{pmatrix} = \begin{cases}0 & \text{if } x \neq \alpha^{-1}\delta,\\ \nu(\delta)\psi(\alpha^{-1}\beta) & \text{if } x = \alpha^{-1}\delta.\end{cases}$$

and

$$\begin{aligned}\rho_s(\phi(\eta))(x) &= \left[\rho\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}(\phi(\eta))\right](x) = \nu(\delta)\psi(\beta\delta^{-1}x)(\phi(\eta))(\alpha\delta^{-1}x)\\ &= \nu(\delta)\psi(\beta\delta^{-1}x)\eta\begin{pmatrix}\alpha\delta^{-1}x & 0\\ 0 & 1\end{pmatrix} = \begin{cases}0 & \text{if } x \neq \alpha^{-1}\delta,\\ \nu(\delta)\psi(\alpha^{-1}\beta) & \text{if } x = \alpha^{-1}\delta.\end{cases}\end{aligned}$$

For $w' = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, we have

$$\phi(\hat{\pi}_{w'}(\eta))(x) = \eta\left(\begin{pmatrix}x & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right) = \eta\begin{pmatrix}0 & x\\ -1 & 0\end{pmatrix} = \frac{-1}{q}\sum_{\mathrm{N}(\lambda)=x}\psi(\mathrm{Tr}(\lambda))\nu(\lambda) = j(x)$$

and

$$\rho_{w'}(\phi(\eta))(x) = \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)\phi(\eta)(y) = \sum_{y\in\mathbb{F}_q^\times}\nu(y^{-1})j(xy)\eta\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix} = j(x)$$

We have thus proved that $W(\eta)$ is the Whittaker model of $\rho$. In fact, the homomorphism $\phi$ is the same as the homomorphism R defined in Section refsec:R. If we define an operation of $G$ on $F(\mathbb{F}_q^\times, \mathbb{C})$ by $\rho$ then R is an $G$-isomorphism.

**Lemma 3.11.1.** *The Wittaker function $\eta$ for $\rho$ is the Bessel function.*

**Proof.** This is true because $J_\rho(1) = \eta(1) = 1$ and $J_\rho(a) = \eta(a) = 0$ for $a \in A$ and $a \neq 0$. □

Now for every $f \in V_\rho = F(\mathbb{F}_q^\times, \mathbb{C})$, let $W_f : G \to \mathbb{C}$ be the function defined by

$$W_f(s) = \sum_{y\in\mathbb{F}_q^\times} f(y)\eta\left(s\begin{pmatrix}y^{-1} & 0\\ 0 & 1\end{pmatrix}\right), \qquad \text{for all } s \in G.$$

Then $W_f$ is the Witakker function for $\rho$ corresponding to $f$; indeed, we have $\mathrm{R}(W_f) = f$.

In order to compute $\Gamma_\rho(\upsilon)$, we now have to compute

$$\eta\begin{pmatrix}0 & 1\\ x & 0\end{pmatrix} = \frac{-1}{q}\sum_{\mathrm{N}(\lambda)=-x}\psi(-\frac{\mathrm{Tr}(\lambda)}{x})\nu(\lambda).$$

Hence, if $\upsilon$ is a character of $\mathbb{F}_q^\times$, then

$$\begin{aligned}\Gamma_\rho(\upsilon) &= \sum_{x\in\mathbb{F}_q^\times} \frac{-1}{q} \sum_{\mathrm{N}(\lambda)=-x} \psi(-\frac{\mathrm{Tr}(\lambda)}{x})\nu(\lambda)\upsilon(x) = \frac{-1}{q}\sum_{x\in\mathbb{F}_q^\times}\upsilon(x)\sum_{\mathrm{N}(\lambda)=-x}\psi(\lambda^{-1}+\overline{\lambda}^{-1})\nu(\lambda)\\ &= \frac{-1}{q}\sum_{\lambda\in\mathbb{F}_{q^2}^\times}\upsilon(-\lambda^{-1}\overline{\lambda}^{-1})\psi(\lambda+\overline{\lambda})\nu(\lambda) = \frac{-1}{q}\upsilon(-1)\sum_{\lambda\in\mathbb{F}_{q^2}^\times}\upsilon(\mathrm{N}(\lambda))^{-1}\psi(\mathrm{Tr}(\lambda))\nu(\lambda)\end{aligned}$$

**Theorem 3.11.2.** *Let $\nu$ be a non-decomposable character of $\mathbb{F}_{q^2}^\times$ and let $\rho=\rho_\nu$ be the corresponding cuspidal representation of $G$. Then*

$$\Gamma_\rho(\upsilon) = \frac{-1}{q}\upsilon(-1)\sum_{\lambda\in\mathbb{F}_{q^2}^\times}\upsilon(\mathrm{N}(\lambda))^{-1}\psi(\mathrm{Tr}(\lambda))\nu(\lambda) = \frac{-1}{q}\upsilon(-1)G_{\mathbb{F}_q^2}(\nu\cdot(\upsilon\circ\mathrm{N})^{-1},\psi\circ\mathrm{Tr})$$

*for every character $\upsilon$ of $\mathbb{F}_q^\times$.*

**Remark .** $G_{\mathbb{F}_{q^2}}$ is the Gauss sum for $\mathbb{F}_{q^2}$. As in the non cuspidal case, $\mid \Gamma_\rho(\upsilon)\mid = 1$, since $\mid G_{\mathbb{F}_{q^2}}\mid = q$. Also if $\psi'(x)=\psi(\alpha x)$ is another character of $\mathbb{F}_q^+$, then $\Gamma'_\rho(\upsilon)=\nu(\alpha)^{-1}\upsilon(\alpha)^2\Gamma_\rho(\upsilon)$.

## 3.12. The Characters Table of $\mathrm{GL}(2,\mathbb{F}_q)$

We conclude our exposition on the representations of $\mathrm{GL}(2,\mathbb{F}_q)$ with a computation of its characters table.

(A). Though $\hat{\pi}=\mathrm{Ind}_P^G(\pi)=\mathrm{Ind}_U^G(\psi)$ is not irreducible, it is the direct sum of all higher dimensional irreducible representations. We compute it first. By Theorem 1.13.1, we have

$$\chi_{\hat{\pi}}(s) = \frac{1}{\mid U\mid}\sum_{\substack{r\in G\\ r^{-1}sr\in U}}\psi(r^{-1}sr).$$

The only eigenvalue of elements of $U$ is 1. It follows that the only conjugacy class on which $\chi_{\hat{\pi}}$ may not vanish are $c_1(1)$ and $c_2(1)$. Clearly, $\chi_{\hat{\pi}}(1)=[G:U]=(q-1)^2(q+1)$. In order to compute $\chi_{\hat{\pi}}$ at $c_2(1)=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, we need:

**Lemma 3.12.1.**

$$s\begin{pmatrix}x&1\\0&x\end{pmatrix}s^{-1}\in B \Longleftrightarrow s\in B$$

**Proof.** Let $s=\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$ be an element of $G$ and let $s^{-1}=\begin{pmatrix}\alpha'&\beta'\\\gamma'&\delta'\end{pmatrix}$. Then $s\in B$ if and only if $\gamma=0$. The lemma follows therefore from:

$$\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\begin{pmatrix}x&1\\0&x\end{pmatrix}\begin{pmatrix}\alpha'&\beta'\\\gamma'&\delta'\end{pmatrix} = \begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\left[\begin{pmatrix}x&0\\0&x\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right]\begin{pmatrix}\alpha'&\beta'\\\gamma'&\delta'\end{pmatrix} = \begin{pmatrix}x+\alpha\gamma'&\alpha\delta'\\\gamma\gamma'&1+\gamma\delta'\end{pmatrix}$$

□

Lemma 3.12.1 implies that

$$\chi_{\hat{\pi}}\begin{pmatrix}1&1\\0&1\end{pmatrix} = \frac{1}{\mid U\mid}\sum_{s\in B}\psi\left(s^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}s\right) = \sum_{\alpha,\delta\in\mathbb{F}_q^\times}\psi(\alpha\delta^{-1}) = 1-q.$$

(B). Let $\mu$ be a character of $B$ which is defined by the pair of characters $(\mu_1,\mu_2)$ of $\mathbb{F}_q^\times$. Let $\hat{\mu}=\mathrm{Ind}_B^G(\mu)$ and compute

$$\chi_{\hat{\mu}}(s) = \frac{1}{\mid B\mid}\sum_{\substack{r\in G\\ r^{-1}sr\in B}}\mu(r^{-1}sr).$$

First,

$$\chi_{\hat\mu}\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix} = \frac{1}{\mid B\mid}\sum_{r\in G}\mu\left(r^{-1}\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix}r\right) = [G:B]\sum_{r\in G}\mu\begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix} = (q+1)\mu_1(x)\mu_2(x).$$

For $c_2(x)$, we use Lemma 3.12.1 again;

$$\chi_{\hat\mu}\begin{pmatrix} x & 1 \\ 0 & x\end{pmatrix} = \frac{1}{\mid B\mid}\sum_{r\in B}\mu\left(r^{-1}\begin{pmatrix} x & 1 \\ 0 & x\end{pmatrix}r\right) = \frac{q}{\mid B\mid}\sum_{\alpha,\delta\in\mathbb{F}_q^\times}\mu\begin{pmatrix} x & \alpha\delta^{-1} \\ 0 & x\end{pmatrix} = \mu_1(x)\mu_2(x).$$

In order to compute the value of $\chi_{\hat\mu}$ at $c_3(x,y)$, we need the following.

**Lemma 3.12.2.** *For* $x\neq y$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}^{-1} \in B \Longleftrightarrow \gamma = 0 \quad \text{or} \quad \delta = 0$$

**Proof.** Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}^{-1} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix}$ it is easy to check that if $\gamma\neq 0$ then $\alpha'=0$ if and only if $\delta=0$. Our lemma follows therefore from:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta'\end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & y\end{pmatrix} + \begin{pmatrix} \alpha\alpha'(x-y) & \alpha\beta'(x-y) \\ \gamma\alpha'(x-y) & \gamma\beta'(x-y)\end{pmatrix}.$$

□

Since $\begin{pmatrix} \alpha & \beta \\ \gamma & 0\end{pmatrix} = \begin{pmatrix} \beta & \alpha \\ 0 & \gamma\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$, we have

$$\chi_{\hat\mu}\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix} = \frac{1}{\mid B\mid}\sum_{r\in B}\mu\left(r^{-1}\begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix}r\right) + \frac{1}{\mid B\mid}\sum_{r\in B}\mu\left(r^{-1}\begin{pmatrix} y & 0 \\ 0 & x\end{pmatrix}r\right) = \mu_1(x)\mu_2(y)+\mu_1(y)\mu_2(x).$$

Finally, because the eigenvalues of elements $c_4(\lambda)$ do not belong to $\mathbb{F}_q$, $\chi_{\hat\mu}(c_4(\lambda))=0$.

(B1). If $\mu_1\neq\mu_2$, then $\hat\mu$ is irreducible. Its character has therefore been computed.

(B2). If $\mu_1=\mu_2$, then $\hat\chi=\rho\oplus\rho'$, where $\rho'$ is a 1-dimensional character given by $\rho'(s)=\mu_1(\det(s))$ (*cf.* Lemma 2.5.2). Hence $\chi_\rho(s)=\chi_{\hat\mu}(s)-\mu_1(\det(s))$.

(C). Let $\nu$ be a non-decomposable character of $\mathbb{F}_{q^2}^\times$ and let $\rho_\nu$ be the corresponding cuspidal representation. Let $W(\rho)$ be the Whittaker model for $\rho$. We know that elements in $W(\rho)$ are uniquely determined by their values on $A$. For $\alpha\in\mathbb{F}_q^\times$, define $\delta_\alpha : G\to\mathbb{C}$ be the unique element in $W(\rho)$ such that

$$\delta_\alpha\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} = \begin{cases} 1 & \text{if } x=\alpha, \\ 0 & \text{otherwise.}\end{cases}$$

The Bessel function $\eta=\delta_1$ and $\delta_\alpha=\rho\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}(\eta)$; indeed, we have

$$\left[\rho\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}(\eta)\right]\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix} = \eta\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}\right) = \begin{cases} 1 & \text{if } x=\alpha, \\ 0 & \text{otherwise.}\end{cases}$$

$\{\delta_\alpha \mid \alpha\in\mathbb{F}_q^\times\}$ is a basis of $W(\rho)$. We have

$$\begin{aligned}\rho(s)(\delta_\alpha) &= \sum_{x\in\mathbb{F}_q^\times}[\rho(s)(\delta_\alpha)]\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\delta_x = \sum_{x\in\mathbb{F}_q^\times}\left[\rho(s)\rho\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}(\eta)\right]\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}\delta_x \\ &= \sum_{x\in\mathbb{F}_q^\times}\eta\left(\begin{pmatrix} x & 0 \\ 0 & 1\end{pmatrix}s\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}\right)\delta_x.\end{aligned}$$

Hence,

$$\chi_\rho(s) = \sum_{\alpha\in\mathbb{F}_q^\times} \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} s \begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$$

The computation of $c_1(x)$ is now

$$\chi_\rho\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix} = \sum_{\alpha\in\mathbb{F}_q^\times} \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\right) = \sum_{\alpha\in\mathbb{F}_q^\times}\eta\begin{pmatrix}x & 0\\ 0 & x\end{pmatrix} = (q-1)\nu(x).$$

For $c_2(x)$, we have:

$$\chi_\rho\begin{pmatrix}x & 1\\ 0 & x\end{pmatrix} = \sum_{\alpha\in\mathbb{F}_q^\times} \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x & 1\\ 0 & x\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\right) = \sum_{\alpha\in\mathbb{F}_q^\times}\eta\begin{pmatrix}x & \alpha\\ 0 & x\end{pmatrix} = \nu(x)\sum_{\alpha\in\mathbb{F}_q^\times}\psi(x^{-1}\alpha) = -\nu(x).$$

For $c_3(x,y)$, we have:

$$\chi_\rho\begin{pmatrix}x & 1\\ 0 & x\end{pmatrix} = \sum_{\alpha\in\mathbb{F}_q^\times} \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & x\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\right) = \sum_{\alpha\in\mathbb{F}_q^\times}\eta\begin{pmatrix}y & 0\\ 0 & x\end{pmatrix} = 0.$$

For $c_4(z)$, we have:

$$\begin{aligned}
\chi_\rho\begin{pmatrix}0 & -\mathrm{N}(z)\\ 1 & \mathrm{Tr}(z)\end{pmatrix} &= \sum_{\alpha\in\mathbb{F}_q^\times} \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -\mathrm{N}(z)\\ 1 & \mathrm{Tr}(z)\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\right) = \sum_{\alpha\in\mathbb{F}_q^\times}\eta\begin{pmatrix}0 & -\alpha\mathrm{N}(z)\\ \alpha^{-1} & \mathrm{Tr}(z)\end{pmatrix}\\
&= \sum_{\alpha\in\mathbb{F}_q^\times}\frac{-1}{q}\psi(\alpha\mathrm{Tr}(z))\sum_{\mathrm{N}(\lambda)=\mathrm{N}(z)}\psi(-\alpha\mathrm{Tr}(\lambda))\nu(\lambda)\\
&= \frac{-1}{q}\sum_{\mathrm{N}(\lambda)=\mathrm{N}(z)}\nu(\lambda)\sum_{\alpha\in\mathbb{F}_q^\times}\psi(\alpha\mathrm{Tr}(z)-\alpha\mathrm{Tr}(\lambda))
\end{aligned}$$

Since $\mathrm{N}(\lambda)=\mathrm{N}(z)$, if $\lambda = z$ or $\lambda=\overline{z}$, then $\mathrm{Tr}(\lambda)=\mathrm{Tr}(z)$; hence $\sum_{\alpha\in\mathbb{F}_q^\times}\psi(\alpha\mathrm{Tr}(z)-\alpha\mathrm{Tr}(\lambda)) = q-1$. If $\lambda\neq z$ and $\lambda\neq\overline{z}$, then $\mathrm{Tr}(\lambda)\neq\mathrm{Tr}(z)$; hence $\sum_{\alpha\in\mathbb{F}_q^\times}\psi(\alpha\mathrm{Tr}(z)-\alpha\mathrm{Tr}(\lambda)) = -1$. It follows that

$$\begin{aligned}
\chi_\rho\begin{pmatrix}0 & -\mathrm{N}(z)\\ 1 & \mathrm{Tr}(z)\end{pmatrix} &= \frac{-1}{q}\Big((q-1)(\nu(z)+\nu(\overline{z})) - \sum_{\substack{\lambda\neq z,\overline{z}\\ \mathrm{N}(\lambda)=\mathrm{N}(z)}}\nu(\lambda)\Big)\\
&= \frac{-1}{q}\big(q(\nu(z)+\nu(\overline{z})) - \sum_{\mathrm{N}(\lambda)=\mathrm{N}(z)}\nu(\lambda)\big) = -\nu(z)-\nu(\overline{z}).
\end{aligned}$$

We sum up the character values in the following table.

| Rep. | $c_1(x)$ | $c_2(x)$ | $c_3(x,y)$ | $c_4(z)$ |
|---|---|---|---|---|
| $\rho_\nu$ | $(q-1)\nu(x)$ | $-\nu(x)$ | $0$ | $-\nu(z)-\nu(\overline{z})$ |
| $\rho_{(\mu_1,\mu_1)}$ | $q\mu_1(x)^2$ | $0$ | $\mu_1(xy)$ | $-\mu_1(z\overline{z})$ |
| $\rho_{(\mu_1,\mu_2)}$ | $(q+1)\mu_1(x)^2$ | $\mu_1(x)\mu_2(x)$ | $\mu_1(x)\mu_2(y)+\mu_1(y)\mu_2(x)$ | $0$ |
| $\mathrm{Ind}_U^G(\psi)$ | $(q-1)^2(q+1)\delta_{1,x}$ | $(1-q)\delta_{1,x}$ | $0$ | $0$ |

# Index

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