Motivations
Finding an analytic proof of Smale conjecture: the topological space consisted of unknotted loops is homotopic to the topological space of round circles. However,
- A topological proof of Smale conjecture was given by Hatcher in 1983 [3].
- It’s more natural and intuitive to have an analytic proof of Smale conjecture instead of an abstract topological proof (see [2]).
- This poster is based on the joint work in [6].

Introduction
Assume $f : \mathbb{R}/2 \pi \rightarrow \mathbb{R}^3$ is sufficiently smooth. Let $\gamma = \partial_t f$, $d\gamma = \gamma$ be the arclength element, and $d_s = \gamma/|\gamma|$, the arclength differentiation. Denote by $T = \partial_t f$ the unit tangent vector, by $l$ the set of arclength parameter of $f$, and by $a = \partial_s f$ the curvature vector of $f$. Define the bending energy of $f$, $\mathcal{E}_b[f]$, by
\[
\int |\gamma| \, ds,
\]
and the so-called Möbius energy (which is a kind of electrostatic energy) of $f$, $\mathcal{E}_M[f]$, by
\[
\int_{\mathbb{S}^1} \frac{|f(s)| - f(s)|^2}{1 - D(f(s), s)^2} \, ds,
\]
where $D(f(s), s)$ denotes the minimum length of subarcs of $f$ with end points $f(s)$ and $f(s)$, and $s$ both denote the arclength parameter of $f$. Let the total energy of $f$ be
\[
\mathcal{E}_{E,\text{type}}[f] := a \cdot E[f] + \gamma \cdot \mathcal{E}_M[f] + \lambda \cdot C[f].
\]
Main Theorem
Let $f_{0}$ be a given smooth initial loop in the Euclidean 3-space. Assume $f_{0}$ is the solution of
\[
\partial_t f = -\nabla \mathcal{E}_{E,\text{type}}[f],
\]
where $a, \gamma, \lambda$ are non-negative constants.
Then,
1. the solution of evolution equation $f_t$ remains smooth for all $t > 0$; and
2. the asymptotic solution, $f_{\infty}$, is an equilibrium configuration of $\mathcal{E}_{E,\text{type}}$, i.e., $f_{\infty}$ satisfies
\[
\delta \mathcal{E}_{E,\text{type}}[f_{\infty}] = 0.
\]

In the proof of the main theorem, the mathematical analysis follows [7], namely it is based on $L^2$ curvature estimates and Gagliardo-Nirenberg interpolation inequalities.

Numerical Simulations
The algorithm here is an extension of that in [7], where we exploited the divergence form of the main part in the evolution equation and the partition into a second-order parabolic-elliptic system for the position vector $f$ and the curvature vector $\gamma$. We choose $\alpha = \lambda = \gamma$.
\[
\partial_t f + \partial_s (\partial_s + |\gamma|^2 T) + \gamma H_f = \lambda \mathcal{H}_f,
\]
and discretize the problem using a semi-implicit scheme in time and piecewise-affine finite elements for the space dependence.

Computational Experiments
Below we show examples in exhibit interesting dynamical behaviour for the gradient flow equation in Eqs.(5) and (6).

A (1,12)-knot initial curve. Below shows an example of competition between the elastic energy and the Möbius energy during unattaching an unknotted loop of (1,12)-knot type into a round circle. Notice that the shape changes rapidly during $t = 0.011$ and $t = 0.013$. Here we choose $\alpha = \gamma = \lambda$.

Curve straightening flow for the same (2,3)-knot initial curve. The figures below show that the curve-straightening flow for the same initial curve of nearly flat trefoil doesn’t prevent from self-intersection. Here we choose $\alpha = \lambda = \gamma$.

A figure-eight type initial curve. The figures below show that the part of two nearby arcs of an almost flat figure-eight was initially pulled apart. Here we choose $\alpha = \gamma = \lambda$.

A (2,3)-knot initial curve. The figures below show that a nearly flat trefoil remains in the same knot type during the flow. Here we choose $\alpha = \gamma = \lambda$.

A Remained Problem
- Are round circles the only equilibrium configurations of the elastic knot energy, $\mathcal{E}_{E,\text{type}}$, in the class of unknotted loops?

Other Applications
There are two aspects in applications. We hope to explore them in the future.
1. Constructing approximate solutions of the N-body problem (see [1]).
2. Modeling over-damped dynamics of elastic rods (e.g., bio-polymers, see [7] and [5]).

References