CHINESE JOURNAL OF MATHEMATICS
Vol. 23, No. 3, pp. 257-274, September 1995

THE ALMOST PERIODIC SOLUTIONS OF NONAUTONOMOUS ABSTRACT DIFFERENTIAL EQUATIONS

Yu-Hsien Chang (張幼賢) and Jein-Shan Chen (陳界山)

Abstract. In this paper we present some results concerned with the problem of existence of almost-periodic and asymptotically almost-periodic solutions of the following nonautonomous abstract differential equation in Banach space:

$$\frac{dx}{dt} = A(t)x + f(t) \qquad t \in J\,,$$

where $J = [0, \infty)$ or $J = (-\infty, \infty)$, $f(t)$ is an almost periodic function or a S^p almost periodic function, $A(t)$ is a continuous operator on J, and $\{A(t) | t \in J\}$ generates a (totally) evolution system $\{U(t,s) | t, s \in J\}$.

1. Introduction and Preliminaries

The problem about the existence of almost-periodic solutions of abstract autonomous differential equations has been the subject of much activity over the past years (e.g. Krein [5], Henriquez [6], Hengartner [9], Zaidman [11]). However, the motivation of this paper is a resent paper of Henriquez [6], in which he proved the existence of asymptotically almost-periodic solutions for abstract autonomous differential equations.

In this paper we will present a generalization of some results contained in Krein's [5], Zaidman's [11] and Henriquez's [6] works. With some suitable assumptions on the evolution system and the forcing term, using the characterizations of

Received December 27, 1993.
1991 Mathematics Subject Classification: 34C25, 42A84, 34C30, 34D20.
Key words and phrases: Almost periodic function, evolution system.
*This work is partially supported by N.S.C. of R.O.C. (NSC 82-0208-M003-010).

(asymptotically) almost periodicity(L. Amerio & G. Prouse [1], Bescicovitch [2], Zaidman [10]) and results on Bohl exponents and e-dichotomy (Krein [5]), we obtain new results on the existence of almost-periodic or asymptotically almost-periodic solutions for abstract non-autonomous differential equations in Banach space.

Throughout this paper we will denote by X a real or complex Banach space endowed with norm $\|\cdot\|$. Some of the following preliminaries were proved in the references. However, for the completion we still list them here.

Definition 1. A continuous function $f(t) : J \to X$ is said to be almost-periodic (in short, *a.p.*) if for every $\varepsilon > 0$, there exists a set P_ε relatively dense in J such that

$$\|f(t+\tau) - f(t)\| \le \varepsilon$$

for every $t \in J$ and every $\tau \in P_\varepsilon$.

Definition 2. A continuous function $f(t) : R^+ \to X$ is called asymptotically almost periodic (in short, *a.a.p*) if there are functions $g(t) \in a.p.(R : X)$ and $q(t) \in C_0(R^+ : X)$ such that

$$f(t) = g(t) + q(t) \qquad \text{for every } \ t \ge 0\,,$$

where $C_0(R^+ : X)$ is the space of continuous functions from R^+ into X which vanish at infinity.

Definition 3. A function $f(t) \in L^p(J : X)$ $p \ge 1$ is called almost periodic in the sense of Stepanov (in short, $S^p - a.p.$) if the function $\tilde{f}(t) : J \to L^p([0,1]; X)$ defined by

$$\tilde{f}(t)(\eta) = f(t+\eta)\,, \quad t \in J\,, \quad \eta \in [0,1]$$

is almost periodic.

From these definitions one can easily have the following result:

Lemma 1. *In order that $f(t) \in a.a.p.(R^+ : X)$ it is necessary and sufficient that for every $\varepsilon > 0$ there is $T(\varepsilon) > 0$ such that the set of real numbers*

$$\left\{ \tau \,\middle|\, \tau \ge 0\,,\ \sup_{t \ge T(\varepsilon)} \|f(t+\tau) - f(t)\| \le \varepsilon \right\}$$

is relatively dense on R^+ (see e.g. Zaidman [11]).

We consider the non–autonomous differential equation

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in J = [0,\infty), \tag{1.1}$$

where $f(t) : J \to X$ is a continuous function and $A(\cdot) : J \to [X]$ is an integrally bounded (i.e.: $\sup_{t\in J} \int_t^{t+1} \|A(s)\| ds \le M \ (0 \le t < \infty)$) and continuous operator. We can define a Cauchy operator $U(t)$ of the homogeneous differential equation

$$\frac{dx}{dt} = A(t)x, \qquad t \in J, \tag{1.2}$$

by

$$U(t) = I + \int_0^t A(t_1)dt_1 + \sum_{n=2}^{\infty} \int_0^t \int_0^{t_n} \cdots \int_0^{t_2} A(t_n)\cdots A(t_1)dt_1 \cdots dt_n \, .$$

The solution of equation (1.2) is represented by $x(t) = U(t)x_0$, where x_0 is the initial value (Krein [5]). From this, we can define the evolution system $U(t,\tau)$ of equation (1.1) or (1.2) by $U(t,\tau) = U(t)U^{-1}(\tau)$. It is well–known that the evolution system $U(t,\tau)$ satisfies the following properties:

(a) $U(t,t) = I$;

(b) $U(t,s)U(s,\tau) = U(t,\tau)$;

(c) $U(t,\tau) = [U(\tau,t)]^{-1}$;

(d) $\|U(t,\tau)\| \le e^{\int_\tau^t \|A(s)\| ds}$,

and the solution of equation

$$(1.1)-(1.3): \begin{cases} \dfrac{dx}{dt} = A(t)x + f(t), & t > 0, \\ x(0) = x_0 \end{cases}$$

is represented by $x(t) = U(t)x_0 + \int_0^t U(t)U^{-1}(\tau)f(\tau)d\tau$ (Krein [5]).

Definition 4. Let $x(t) = U(t)x_0$ be a solution of equation (1.2). By the (upper) Bohl exponent $K_B(x_0)$ of this solution is meant the greatest lower bound of all those numbers ρ for which there exist numbers N_ρ such that

$$\|x(t)\| \le N_\rho e^{\rho(t-\tau)} \|x(\tau)\| \tag{1.5}$$

for any $t, \tau \in [0, \infty)$, $t \geq \tau$. If such numbers ρ do not exist, we put $K_B(x_0) = \infty$. In exactly the same way the lower Bohl exponent $K'_B(x_0)$ of a solution $x(t)$ is the least upper bound of those numbers ρ' for which there exists a numbers $N_{\rho'} > 0$ such that

$$\|x(t)\| \leq N_{\rho'} \cdot e^{\rho'(t-\tau)} \|x(\tau)\| \qquad (0 \leq \tau \leq t < \infty). \tag{1.6}$$

From these definitions and properties one can easily get the following Lemma (see e.g. Krein [5]):

Lemma 2. *Let $x(t)$ be a solution of equation* (1.2). *Then the following formulas hold:*

$$\begin{cases} K_B(x_0) = \overline{\lim\limits_{\tau\to\infty, t-\tau\to\infty}} \dfrac{\ln\|x(t)\| - \ln\|x(\tau)\|}{t-\tau}, \\ K'_B(x_0) = \underline{\lim\limits_{\tau\to\infty, t-\tau\to\infty}} \dfrac{\ln\|x(t)\| - \ln\|x(\tau)\|}{t-\tau}. \end{cases} \tag{1.7}$$

Suppose now P is a projection in X and $X_P = PX$ is the corresponding subspace. We consider the totality of solutions $x(t) = U(t)x_0$ of equation (1.2) that are initially in $X_P : x_0 \in X_P$. By the upper (lower) Bohl exponent $K_B(P)$ ($K'_B(P)$) of this totality of solutions is meant the greatest lower (least upper) bound of the exponents $\rho(\rho')$ for which formula (1.5) ((1.6)) is valid for all of the solutions $x(t) = U(t)x_0$ with $x_0 \in X_P$ and a number $N_\rho > 0$ not depending on x_0. We will call the exponents $K_B(P)$ and $K'_B(P)$ the upper and lower Bohl exponents of equation (1.2), respectively, corresponding to the projection P. In particular, when $P = I$ we will simply call the Bohl exponents of equation (1.2) and use the notation

$$K_B = K_B(I), \qquad K'_B = K'_B(I).$$

Then it is obvious that

$$k'_B \leq k'_B(P) \leq k_B(P) \leq k_B.$$

Lemma 3. *In order for the upper (lower) Bohl exponent of equation* (1.2) *to be finite (i.e. $K_B < \infty$ ($K'_B > -\infty$) it is necessary and sufficient that*

$$\begin{cases} K = \sup\limits_{0\leq t-\tau\leq 1} \|U(t,\tau)\| < \infty, \\ K' = \sup\limits_{0\leq t-\tau\leq 1} \|U(\tau,t)\| < \infty. \end{cases}$$

Instead of proving this Lemma directly we prove more general result:

Remark 1. Suppose $m_P = \sup_{0\le t<\infty} \|U(t)PU^{-1}(t)\| < \infty$, then the necessary and sufficient condition for $K_B < \infty$ ($K'_B > -\infty$) is that

$$\begin{cases} k_P &= \sup_{0\le t-\tau\le 1} \|U(t)PU^{-1}(\tau)\| < \infty\,, \\ k'_P &= \sup_{0\le t-\tau\le 1} \|U(\tau)PU^{-1}(t)\| < \infty\,. \end{cases}$$

(*note that for* $P = I$, $m_P = 1 < \infty$).

Proof. Suppose $K_B(P) < \infty$. Since $x(t) = U(t)x_0$ for some $x_0 \in X_P$ and

$$\|U(t)Px\| \le N_\rho e^{\rho(t-\tau)}\|U(\tau)Px\| \quad (t \ge \tau) \quad \text{for every } x \in X\,,$$

letting $x = U^{-1}(\tau)y$, we have from the hypothesis that

$$\|U(t)PU^{-1}(\tau)y\| \le N_\rho e^{\rho(t-\tau)}\|U(\tau)PU^{-1}(\tau)\| \le N_\rho e^{\rho(t-\tau)} m_P\|y\|$$

and hence

$$\|U(t)PU^{-1}(\tau)\| \le N_\rho m_P e^{\rho(t-\tau)} \quad (t \ge \tau)\,.$$

Take the supremum, we have

$$k_P = \sup_{0\le t-\tau\le 1} \|U(t)PU^{-1}(\tau)\| < \infty\,.$$

Similarlly, we can prove that $k'_P < \infty$ whenever $k'_B(P) < \infty$.

On the other hand, if $k_P < \infty$, take n to be the largest nonnegative integer not greater than $t-\tau$; and set $\tau_k = \tau + k$ $(k = 1, 2, \cdots, n)$, $\tau_{n+1} = t$. Then

$$\begin{aligned} U(t)PU^{-1}(\tau) &= U(\tau_{n+1})PU^{-1}(\tau_n)U(\tau_n)PU^{-1}(\tau_{n-1})\cdots U(\tau_1)PU^{-1}(\tau) \\ &= \prod_{k=1}^{n+1} U(\tau_k)PU^{-1}(\tau_{k-1})\,, \end{aligned}$$

$$\begin{aligned} \|U(t)PU^{-1}(\tau)\| &\le \prod_{k=1}^{n+1}\|U(\tau_k)PU^{-1}(\tau_{k-1})\| \le K_P^{n+1} \\ &\le K_P \cdot e^{n\ln K_P} \le K_P \cdot e^{(t-\tau)\ln K_P}\,. \end{aligned}$$

So, for any $x_0 \in X_P$,

$$\|U(t)x_0\| = \|U(t)PU(\tau)U^{-1}(\tau)x_0\| \le K_P \cdot e^{(t-\tau)\ln K_P}\|U(\tau)x_0\|\,,$$

and hence $k_B(P) \le \ln K_P < \infty$.

Lemma 4. *If $A(t)$ is integrally bounded, then the Bohl exponents of equation* (1.2) *are finite.*

Proof. It is easy to see (e.g. Pazy [8]) that

$$e^{-\int_s^t \|A(\tau)\|d\tau} \le \|U^{\pm}(t,s)\| \le e^{\int_s^t \|A(\tau)\|d\tau} \qquad \text{for all} \quad t \ge s\,,$$

and hence

$$K = \sup_{0 \le t-\tau \le 1} \|U(t,\tau)\| < \infty\,.$$

Thus, $k_B < \infty$.

Lemma 5. *If $m_P = \sup_{0 \le t < \infty} \|U(t)PU^{-1}(t)\| < \infty$ and the Bohl exponents are finite, they are representable by the formulas*

$$\begin{cases} k_B &= \overline{\lim_{\tau,s\to\infty}} \dfrac{\ln\|U(\tau+s,\tau)\|}{s}\,, \\ -k'_B &= \overline{\lim_{\tau,s\to\infty}} \dfrac{\ln\|U(\tau,\tau+s)\|}{s}\,. \end{cases}$$

Instead of proving this Lemma directly we prove more general result:

Remark 2. If $m_P = \sup_{0 \le t < \infty} \|U(t)PU^{-1}(t)\| < \infty$ and the Bohl exponents are finite, then

$$\begin{cases} k_B(P) &= \overline{\lim_{\tau,s\to\infty}} \dfrac{\ln\|U(\tau+s)PU^{-1}(\tau)\|}{s}\,, \\ -k'_B(P) &= \overline{\lim_{\tau,s\to\infty}} \dfrac{\ln\|U(\tau)PU^{-1}(\tau+s)\|}{s}\,. \end{cases}$$

Proof. Set

$$u = \overline{\lim_{\tau,s\to\infty}} \frac{\ln\|U(\tau+s)PU^{-1}(\tau)\|}{s}\,.$$

At first, let $\rho > k_B(P)$ (as in the proof of Remark 1) which satisfies that

$$\|U(t)PU^{-1}(\tau)\| \le N_\rho m_P e^{\rho(t-\tau)} \qquad (t \ge \tau)\,.$$

So, $u \le \rho$ and hence $u \le k_B(P) < \infty$. Second, for every $\rho > u$, we can easily see that $\rho \ge k_B(P)$, which implies $u \ge k_B(P)$. Hence the conclution of the first assertion is proved. The proof of the second assertion is the same as the the first one, and hence the conclusion of this remark is true.

Lemma 6. *Suppose the Bohl exponent k_B of the equation (1.2) is finite. In order for it to be negative it is necessary and sufficient that there exist positive numbers T and $q < 1$ for which the following condition is satisfied:*

for every $x \in X$ and $t \geq 0$ there exists a number $\theta_{x,t} \in [0,T]$ with the property that

$$(*) \qquad \|U(t+\theta_{x,t},t)x\| \leq q\|x\|\,.$$

Proof. For convenience, we say the equation (1.2) has the property $B(v,N)$ provided there exist $v \in \Re$, $N > 0$ such that all solutions $x(t)$ of equation (1.2) satisfy that

$$\|x(t)\| \leq Ne^{-v(t-\tau)}\|x(\tau)\|\,, \qquad (t \geq \tau);$$

$$\text{i.e.} \quad \|U(t,\tau)\| \leq Ne^{-v(t-\tau)}\,, \qquad (t \geq \tau)\,.$$

If k_B is negative, then there exist positive numbers $v, N > 0$ such that the equation (1.2) has property $B(v,N)$. So, for every $T > 0$ such that $Ne^{-vT} < 1$, we have

$$\|U(t+T,t)\| \leq Ne^{-vT}\,.$$

Take $\theta_{x,t} = T$, $q = Ne^{-vT}$. Then $\|U(t+\theta_{x,t},t)x\| \leq q\|x\|$.
Let $0 \leq t_0 < t < \infty$, from the continuity of $U(\tau,\tau')$, there is a θ such that

$$\|U(\tau,\tau')\| \leq \frac{1}{q} \qquad \text{whenever } \tau,\tau' \text{ satisfy } t_0 \leq \tau,\tau' < 2t, |\tau-\tau'| \leq \theta\,.$$

For any $x \in X$,

$$\|x\| = \|U(\tau,\tau')U(\tau',\tau)x\| < \frac{1}{q}\|U(\tau',\tau)x\|\,.$$

This implies that $\|U(\tau',\tau)x\| > q\|x\|$.
So, $\theta_{x,t}$, whenever $t_0 \leq \tau < \tau + \theta_{x,t} \leq 2t$.
From the hypothesis, for any $x_0 \in X$, there exist

$$t_1 = t_0 + \theta_{x_0,t_0};\ \ x_1 = U(t_1,t_0)x_0\,,$$

$$t_2 = t_1 + \theta_{x_1,t_1};\ \ x_2 = U(t_2,t_1)x_1 = U(t_2,t_0)x_0\,,$$

....

$$t_{k+1} = t_k + \theta_{x_k,t_k};\ \ x_{k+1} = U(t_{k+1},t_k)x_k = U(t_{k+1},t_0)x_0\,.$$

After finite steps, say m steps ($m < (t-t_0)/\theta$), we can get $t_m < t < t_{m+1}$. Since $U(t,t_0)x_0 = U(t,t_m)x_m$, $\|x_m\| \leq q^m\|x_0\|$ and k_B is finite we have

$$\|U(t,t_0)x_0\| \leq kq^m\|x_0\|, \quad \text{where} \quad k = \sup_{0\leq t-\tau\leq T} \|U(t,\tau)\| < \infty .$$

The fact θx_k, $t_k \leq T$, $k = 1, 2, \cdots, m+1$, implies

$$t \leq t_0 + (m+1)T \qquad \text{i.e. } m+1 \geq \frac{t-t_0}{T},$$

and hence

$$\|U(t,t_0)x_0\| \leq \frac{k}{q} q^{(t-t_0)/T}\|x_0\| = Ne^{-v(t-t_0)}\|x_0\|,$$

$$\text{where} \quad N = \frac{k}{q}, \quad v = \frac{\ln q^{-1}}{T}.$$

Thus, the Bohl exponent of equation (1.2) is negative. The proof of this Lemma is complete now.

Lemma 7. *Suppose equation* (1.2) *has a finite Bohl exponent* k_B *and* p *is any positive number. The Bohl exponent of the equation is negative precisely when there exists a positive constant* C *for which*

$$(*)' \qquad \left\{\int_\tau^\infty \|U(t,\tau)x\|^p dt\right\}^{\frac{1}{p}} \leq C\|x\| \qquad (t_0 \leq \tau < \infty).$$

Proof. Suppose the Bohl exponent of equation (1.2) is negative, then $\|U(t,\tau)\| \leq Ne^{-v(t-\tau)}$, and hence

$$\left\{\int_\tau^\infty \|U(t,\tau)x\|^p dt\right\}^{\frac{1}{p}} \leq \frac{N}{vP}\|x\|.$$

We only need to show that $(*)'$ implies the $(*)$ in Lemma 6. Suppose, on the contrary, for any $0 < q < 1$ and any $T > 0$ there exist x_0, τ_0 such that

$$\|U(t,\tau_0)x_0\| > q\|x_0\|, \qquad t \in [\tau_0, \tau_0 + T].$$

This implies

$$\int_{\tau_0}^\infty \|U(t,\tau_0)x_0\|^p dt \geq \int_{\tau_0}^{\tau_0+T} \|U(t,\tau_0)x_0\|^p dt \geq q^p\|x_0\|^p T.$$

Take T large enough such that $q^pT > C^p$. This contradicts $(*)'$. Thus, $(*)'$ implies the $(*)$ in Lemma 6 and hence this Lemma is proved.

Definition 5. An operator function $A(t)$ is said to be precompactly valued if its range is precompact in $[X]$, (i.e. if every sequence $\{A(t_n)\}$ contains a subsequence converging to an operator A of $[X]$, where $[X]$ denotes all linear operatos on X).

Definition 6. An operator C is called an ω–limit operator of $A(t)$ if there exists a sequence $t_n \to \infty$ such that $A(t_n) \to C$.

Definition 7. we say an operator function $A(t)$ satisfies $S_{\varepsilon,L}$ condition for some $\varepsilon > 0$ and $L > 0$ if there exists a nember $T > 0$ such that the inequality $\|A(s) - A(t)\| \le \varepsilon$ is satisfied when $s, t \ge T$, $|s - t| \le L$. A function $A(t)$ is said to be stationary at infinity if it satisfies condition $S_{\varepsilon,L}$ for any arbitarily small $\varepsilon > 0$ and some positive L.

Lemma 8. *Suppose $A(t)$ is a precompactly valued operator function that is stationary at infinity. In order for the Bohl exponent k_B of equation* (1.2) *to be negative it is necessary and sufficient that the spectra of the ω–limit operators of $A(t)$ lie in some halfplane $Re\lambda \le -v_0\,(v_0 > 0)$.*

Proof. To see this Lemma is true, we prove it in the following several steps:

Step 1: we will show that if $A(t)$ is a precompactly valued operator function and all spectra of the ω–limit operators of $A(t)$ are lying in the same half plane $\mathrm{Re}\lambda \le -v_0\ (v_0 > 0)$, then there is a $T_0 > 0$ such that

$$\|e^{A(t)\tau}\| \le N_0 e^{-v_0 \tau} \quad \text{whenever} \quad t > T_0\,,$$

where N_0, v_0 is independent of t. Moreover, suppose for sufficiently small $\varepsilon > 0$ and sufficiently large $L > 0$, $A(t)$ satisfies condition $S_{\varepsilon,L}$ ($\varepsilon < \frac{v_0}{N_0}$, $L > \ln \frac{N_0}{v_0 - N_0\varepsilon}$), it will be proved that the equation (1.2) has negative Bohl exponent.

The first part of this assertion was proved by Krein (see [5]). He also proved the following statement: Suppose $U_k(t,s)$ $(k = 1, 2)$ are evolution operators of the equations

$$\frac{dx}{dt} = A_k(t)x\,, \qquad k = 1, 2\,.$$

If

$$\|U_1(t,s)\| \le Ne^{-v_1(t-s)} \quad \text{where} \quad N > 0, v_1 \in \Re\,,\ t \ge s\,,$$

then the following hold:

$$\|U_2(t,s) - U_1(t,s)\| \le Ne^{-v_1(t-s)}(e^{N\int_s^t \|A_2(\tau)-A_1(\tau)\|d\tau} - 1),$$

$$\|U_2(t,s)\| \le Ne^{-v_1(t-s)}e^{N\int_s^t \|A_2(\tau)-A_1(\tau)\|d\tau}, \quad t \ge s.$$

From the assumption we have that

$$\|A(t) - A(\tau)\| < \varepsilon \qquad (\tau \le t \le \tau + L) \text{ whenever } \tau \text{ is large enough}.$$

By taking $A_1(t) \equiv A(\tau)$, $A_2(t) = A(t)$ $(\tau \le t \le \tau + L)$ and using the facts

$$U_1(t,s) = e^{(t-s)A(\tau)} \quad \text{and} \quad \|e^{(t-s)A(\tau)}\| \le N_0 e^{-v_0(t-s)},$$

we have that

$$\|U(t,s)\| \le N_0 e^{-v(t-s)} \quad (\tau \le t \le \tau + L), \quad \text{where } v = \tau_0 - N_0\varepsilon;$$

and hence

$$\|U(\tau + L, \tau)\| \le N_0 e^{-vL}.$$

Since $A(t)$ is precompactly valued, it is bounded and integrally bounded. Thus, from Lemma 6 the Bohl exponent k_B of the equation (1.2) is negative, and the assertion of step 1 is proved completely now.

Step 2: We will prove that if $A(t)$ is precompactly valued, the Bohl exponent k_B is negative and $A(t)$ satisfies condition $S_{\varepsilon,L}$ for sufficiently small $\varepsilon > 0$ and sufficiently large $L > 0$, then all spectra of the ω-limit operators of $A(t)$ are lying in the same half plane $\mathrm{Re}\lambda \le -v_0$ $(v_0 > 0)$.

Suppose C is a ω-limit operator of $A(t)$ (i.e. ther exists a sequence $t_n \to \infty$ such that $A(t_n) \to C$), then for any small $\delta > 0$, there is a sufficiently large n such that $\|C - A(t_n)\| < \delta$. Since $A(t)$ satisfies condition $S_{\varepsilon,L}$, $\|A(t_n) - A(t)\| \le \varepsilon$ for large n, and hence $\|C - A(t)\| < \varepsilon + \delta$ $(t_n \le t \le t_n + L)$. So, from the result of Krein in Step 1 we have $\|e^{CL}\| \le Ne^{-v't}$, where $v' = v - N(\varepsilon + \delta)$. Hence, the spectrum $\sigma(C)$ of C lies in the half plane

$$\mathrm{Re}\,\lambda \le \frac{\ln N}{L} - v' = -v_0, \quad \text{where } v_0 > 0 \text{ is independent of } C,$$

and the assertion of Step 2 is proved. The conclusion of this Lemma follows immediately.

Definition 8. Let X_1, X_2 be a pair of nonzero disjoint subspaces of a Banach space X (i.e. $X_1 \cap X_2 = \{0\}$). We define the angular distance between X_1, X_2 as

$$Sn(X_1, X_2) = \inf_{x_i \in X_i, \|x_i\|=1} \|x_1 + x_2\|.$$

Lemma 9. *Suppose the space X decomposes into a direct sum $X = X_1 + X_2$ of closed subspaces and P_1, $P_2 = I - P_1$ are the corresponding supplementary projections. Then the following is valid:*

$$\frac{1}{\|P_k\|} \leq Sn(X_1, X_2) \leq \frac{2}{\|P_k\|} \qquad (k = 1, 2).$$

Hence, the boundedness from above of a set $\{P_1, P_2\}$ of projections in a Banach space X is equivalent to the bonundedness from below of the set $\{Sn(PX, (I-P)X)\}$ of angular distances between the subspace PX and its complement $(I-P)X$.

Proof. Take any $\delta > Sn(X_1, X_2)$, there exists $x_k \in X_k$, $\|x_k\| = 1$, $k = 1,2$ such that $\|x_1 + x_2\| < \delta$. Let $x = x_1 + x_2$. Then $P_k x = x_k$ and $1 = \|x_k\| \leq \|P_k\| \cdot \|x\| < \|P_k\| \cdot \delta$, and hence $1/\|P_k\| < \delta$. This implies $1/\|P_k\| < Sn(X_1, X_2)$. For any $x \in X$,

$$\begin{aligned} Sn(X_1, X_2) &\leq \left\| \frac{P_1 x}{\|P_1 x\|} + \frac{P_2 x}{\|P_2 x\|} \right\| = \frac{1}{\|P_1 x\|} \left\| P_1 x + \frac{\|P_1 x\|}{\|P_2 x\|} P_2 x \right\| \\ &= \frac{1}{\|P_1 x\|} \left\| x + \frac{\|P_1 x\| - \|P_2 x\|}{\|P_2 x\|} P_2 x \right\| \\ &\leq \frac{1}{\|P_1 x\|} \left\{ \|x\| + \left\| \frac{\|P_1 x\| + \|P_2 x\|}{\|P_2 x\|} P_2 x \right\| \right\} \\ &\leq 2 \frac{\|x\|}{\|P_1 x\|} \end{aligned}$$

and hence

$$Sn(X_1, X_2) \leq 2 \inf_{x \in X} \frac{\|x\|}{\|P_1 x\|} = \frac{2}{\|P_1\|}$$

Similarly, one can show that

$$Sn(X_1, X_2) \leq \frac{2}{\|P_2\|}.$$

Thus, the assertion of this Lemma is true.

Definition 9. we say that equation

$$\frac{dx}{dt} = A(t)x, \qquad t \in J = (-\infty, \infty) \tag{1.2'}$$

is e–dichotomic on J with exponents $v_1 > 0$, $v_2 > 0$ if for some $t_0 \in J$ the space X decomposes into a direct sum $X = X_1(t_0) + X_2(t_0)$ of closed subspaces such that the following conditions are satisfied:

(a) The solutions $x_1(t) = U(t,t_0)x_1^0$ of equation (1.2′) in the subspace $X_1(t_0)$ at $t = t_0$ ($x_1^0 \in X_1(t_0)$) are subject to the estimate

$$\|x_1(t)\| \leq N_1 e^{-v_1(t-s)}\|x_1(s)\|, \qquad t \geq s, \quad t,s \in J,$$

with some exponent $v_1 > 0$.

(b) The solutions $x_2(t) = U(t,t_0)x_2^0$ of equation (1.2′) in the subspace $X_2(t_0)$ at $t = t_0$ ($x_2^0 \in X_2(t_0)$) are subject to the estimate

$$\|x_2(t)\| \leq N_2 e^{-v_2(s-t)}\|x_2(s)\|, \qquad t \leq s, \quad t,s \in J,$$

with some exponent $v_2 > 0$.

(c) The angular distance between the subspaces $X_1(t) = U(t,t_0)X_1(t_0)$ and $X_2(t) = U(t,t_0)X_2(t_0)$ cannot become arbitrarily small under a variation of t; more precisely, there exists a constant $r > 0$ such that

$$Sn(X_1(t), X_2(t)) \geq r, \qquad t \in J.$$

Lemma 10. *In order for equation* (1.2′) *to be e-dichotomic on $J = R$ with exponents $v_1 > 0$, $v_2 > 0$, it is necessary and sufficient that the conditions*

$$(*)'') \qquad \begin{cases} \|U(t)P_1U^{-1}(s)\| \leq \tilde{N}_1 e^{-v_1(t-s)} & (t \geq s), \\ \|U(t)P_2U^{-1}(s)\| \leq \tilde{N}_2 e^{-v_2(s-t)} & (s \geq t) \end{cases}$$

with certain constants $\tilde{N}_k$ be satisfied on this interval.

Proof. Let $P_k(t)$ be the corresponding supplementary projections of the subspaces $X_k(t)$, $k = 1,2$. Taking $t = s$, we have

$$\|P_k(t)\| = \|U(t)P_kU^{-1}(t)\| \leq \tilde{N}_k.$$

From Lemma 9, the condition (c) of Definition 9 is satisfied. For $x(0) = P_1x(0)$ and $t \geq s$,

$$\|x(t)\| = \|U(t)P_1x(0)\| = \|U(t)P_1U^{-1}(s)x(s)\| \leq \tilde{N}_1 \cdot e^{-v_1(t-s)}\|x(s)\|.$$

Hence, the condition (a) of Definition 9 is satisfied. Similarly, the condition (b) of Definition 9 is also satisfied.

On the other hand, taking $x_1^0 = P_1U^{-1}(s)x$ ($x \in X$) in the condition (a) of Definition 9, we have from Lemma 9 and the condition (c) of Definition 9

$$\begin{aligned}\|U(t)P_1U^{-1}(s)x\| &= \|U(t)x_1^0\| \le N_1e^{-v_1(t-s)}\|U(s)x_1^0\| \\ &\le N_1e^{-v_1(t-s)}\|U(s)P_1U^{-1}(s)x\| \\ &\le N_1Me^{-v_1(t-s)}\|x\|.\end{aligned}$$

Similarly, for $t \le s$, the second assertion of $(*)''$ also holds and Lemma 10 is proved now.

We now consider the non-autonomous equation:

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in J = (-\infty, \infty). \tag{1.4}$$

Let P_1, P_2 be a pair of mutually complementary projections: $P_1 + P_2 = I$. If $U(t)$ is the Cauchy operator of equation (1.4), we put

$$G(t,s) = \begin{cases} U(t)P_1U^{-1}(s), & t > s, \\ -U(t)P_2U^{-1}(s), & t < s. \end{cases}$$

It immediately follows from the definition of the $G(t,s)$ that $G(t,s)$ satisfies the following properties:

(a) $\dfrac{\partial G(t,s)}{\partial t} = A(t)G(t,s), t \neq s;$

(b) $\dfrac{\partial G(t,s)}{\partial s} = -G(t,s)A(s), t \neq s;$

(c) $G(s^+,s) - G(s^-,s) = I;$

(d) $G(t,t^+) - G(t,t^-) = -I;$

(e) $x(t) = \int_{-\infty}^{\infty} G(t,s)f(s)ds$ is the solution of equation (1.4).

2. Main Results

Theorem 1. *If $A(\cdot) : J \to [X]$ is continuous on J and is integrally bounded, $f(t) : J \to X$ is $S^1 - a.p.$, and for any $p > 0$ there exists a positive number $C > 0$ such that*

$$\left\{\int_\tau^\infty \|U(t,\tau)x\|^p dt\right\}^{\frac{1}{p}} \le C\|x\| \qquad (t_0 \le \tau < \infty),$$

then the solution of equation (1.1) – (1.3):

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t) \\ x(0) = x_0 \end{cases}$$

is asymptotically almost periodic.

Proof. From Lemma 7 it follows that $k_B < \infty$ and $k_B < 0$, i.e.

$$\|U(t,s)\| \le Ne^{-\alpha(t-s)}, \qquad N > 0, \alpha > 0, \ t \ge s.$$

Since the solution of equation (1.1)–(1.3) is formulated by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s)ds,$$

and $\lim_{t\to\infty} \|U(t,0)x_0\| \le \lim_{t\to\infty} Ne^{-\alpha(t-s)}\|x_0\| = 0$, we have $U(t,0)x_0 \in C_0(R^+ : X)$. Let $u(t) = \int_0^t U(t,s)f(s)ds$, we will prove that $u(t) \in$ *a.a.p.*. Hence we will conclude that $x(t) = U(t,0)x_0 + u(t)$ belongs to *a.a.p.*.

Since

$$\begin{aligned} u(t+\tau) &= \int_0^{t+\tau} U(t+\tau,s)f(s)ds \\ &= \int_{-\tau}^0 U(t+\tau,s+\tau)f(s+\tau)ds \\ &\quad + \int_0^t U(t+\tau,s+\tau)f(s+\tau)ds \\ &= \int_0^\tau U(t+\tau,\tau-s)f(\tau-s)ds \\ &\quad + \int_0^t U(t+\tau,s+\tau)f(s+\tau)ds, \end{aligned}$$

we have that

$$\begin{aligned} u(t+\tau) - u(t) = &\int_0^\tau U(t+\tau,\tau-s)f(\tau-s)ds \\ &+ \int_0^t [U(t+\tau,s+\tau)f(s+\tau) - U(t,s)f(s)]ds. \end{aligned}$$

From the assumption $\|U(t,s)\| \le Ne^{-\alpha(t-s)}$, it follows that

$$\begin{aligned} \|u(t+\tau) - u(t)\| &\le \int_0^\tau Ne^{-\alpha(t+s)}\|f(\tau-s)\|ds \\ &\quad + \int_0^t Ne^{-\alpha(t-s)}\|f(s+\tau) - f(s)\|ds \\ &= I_1 + I_2, \end{aligned}$$

where I_1, I_2 are two constants. By choosing n such that $n \le t < n+1$, we have

$$\begin{aligned} I_1 &\le Ne^{-\alpha t} \cdot \sum_{k=0}^{\infty} e^{-\alpha k} \int_k^{k+1} \|f(s)\| ds \\ &\le Ne^{-\alpha t} \cdot \sum_{k=0}^{\infty} e^{-\alpha k} \cdot C = \frac{NC \cdot e^{-\alpha t}}{1 - e^{-\alpha}} \end{aligned}$$

and

$$\begin{aligned} I_2 &= Ne^{-\alpha t} \int_0^t e^{\alpha s} \|f(s+\tau) - f(s)\| ds \\ &\le Ne^{-\alpha t} \cdot \sum_{k=0}^{n} e^{\alpha(k+1)} \int_k^{k+1} \|f(s+\tau) - f(s)\| ds \\ &\le Ne^{-\alpha t} \sum_{k=0}^{n} e^{\alpha(k+1)} \cdot \varepsilon' \le \frac{Ne^{\alpha} \cdot \varepsilon'}{1 - e^{-\alpha}}, \end{aligned}$$

and hence we obtain that

$$\|u(t+\tau) - u(t)\| \le \frac{NCe^{-\alpha t}}{1 - e^{-\alpha}} + \frac{Ne^{\alpha} \cdot \varepsilon'}{1 - e^{-\alpha}}.$$

From the last inequality we deduce that for every $\varepsilon > 0$ there exist a relatively dense set P_ε ($P_\varepsilon = P_{\varepsilon'}$, for ε' appropriate) and a constant $T_\varepsilon > 0$ such that

$$\|u(t+\tau) - u(t)\| \le \varepsilon$$

for all $t \ge T_\varepsilon$, $\tau \in P_\varepsilon$. It follows from Lemma 1 that $u(t) \in a.a.p.$ and Theorem 1 is proved.

Remark 3. Obviously, any *a.p.* function is a $S^p - a.p.$ function and from the fact that $1 \le p_1 \le p_2$ implies $S^{p_1} \subseteq S^{p_2}$ (see e.g. Amerio and Prouse [1]), any $S^p - a.p.$ function is a $S^1 - a.p.$ function. Theorem 1 also can be similarily proved provided $A(t)$ is replaced by a periodic or bounded operator and $f(t)$ is replaced by a $S^p - a.p.$ function or an *a.p.* function in Theorem 1.

Theorem 2. *If $A(\cdot) : J \to [X]$ is continuous on J and is integrally bounded, $f(t) : J \to X$ is $S^1 - a.p.$, and if there exist positive numbers T and $q < 1$ for which the following condition is satisfied:*

> *for every $x \in X$ and $t \ge 0$ there exists a number $\theta_{x,t} \in [0, T]$ with the property that*

$$\|U(t+\theta_{x,t},t)x\| \leq q\|x\|,$$

then the solution of equation (1.1) – (1.3):

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t), \ t > 0, \\ x(0) = x_0 \end{cases}$$

is asymptotically almost periodic.

Theorem 3. *Suppose $A(t)$ is a precompactly valued operator function that is stationary at infinity, and all the spectra of the ω-limit oprtators of $A(t)$ lie in some halfplane $Re\lambda \leq -v_0$ $(v_0 > 0)$, and $f(t) : J \to X$ is S^1 – a.p.. Then the solution of equation* (1.1) – (1.3):

$$\begin{cases} \dfrac{dx}{dt} = A(t)x + f(t), \ t > 0, \\ x(0) = x_0 \end{cases}$$

is asymptotically almost periodic.

Applying Lemma 4, Lemma 6, Lemma 8 and the technique used in the proof of Theorem 1, one can easily get Theorem 2 and Theorem 3. The detail of the proof is omitted here.

Theorem 4. *Suppose equation* (1.2′) *is e–dichotomic on $J = R$ with exponents $v_1 > 0$, $v_2 > 0$, and $f(t)$ is a.p.. Then the solution of equation* (1.4):

$$\frac{dx}{dt} = A(t)x + f(t), \qquad t \in J = (-\infty, \infty)$$

is almost periodic.

Proof. Since the solution of equation (1.4) can be represented by

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} G(t,s)f(s)ds \\ &= \int_{-\infty}^{t} U(t)P_1U^{-1}(s)f(s)ds \\ &\quad + \int_{t}^{\infty} -U(t)P_2U^{-1}(s)f(s)ds, \end{aligned}$$

we can write

$$
\begin{aligned}
x(t+\tau) &= \int_{-\infty}^{t+\tau} U(t+\tau)P_1U^{-1}(s)f(s)ds \\
&\quad - \int_{t+\tau}^{\infty} U(t+\tau)P_2U^{-1}(s)f(s)ds \\
&= \int_{-\infty}^{t} U(t+\tau)P_1U^{-1}(s+\tau)f(s+\tau)ds \\
&\quad - \int_{t}^{\infty} U(t+\tau)P_2U^{-1}(s+\tau)f(s+\tau)ds \,.
\end{aligned}
$$

From Lemma 10, it follows that

$$
\begin{cases}
\|U(t)P_1U^{-1}(s)\| \le \tilde{N}_1 e^{-v_1(t-s)} & (t \ge s) \\
\|U(t)P_2U^{-1}(s)\| \le \tilde{N}_2 e^{-v_2(s-t)} & (s \ge t)
\end{cases} .
$$

Hence, we obtain that

$$
\begin{aligned}
\|x(t+\tau)-x(t)\| \le & \int_{-\infty}^{t} \tilde{N} e^{-v_1(t-s)} \|f(s+\tau)-f(s)\|ds \\
& + \int_{t}^{\infty} \tilde{N}_2 e^{-v_2(s-t)} \|f(s+\tau)-f(s)\|ds \,.
\end{aligned}
$$

From the fact that $f(t)$ is $a.p.$, we obtain the estimate:

$$
\|x(t+\tau)-x(t)\| \le \frac{\tilde{N}_1\varepsilon}{v_1} + \frac{\tilde{N}_2\varepsilon}{v_2} .
$$

It follows that the solution of equation (1.4) is almost periodic and Theorem 4 is proved.

References

1. M. Amerio and G. Prouse, Almost Periodic Function and Functional Equations, Van Nostrand–Reinhold, New York, 1971.
2. A. S. Besicovitch, Almost Periodic Functions, Dover Publications, New York, 1958.
3. P. L. Butzer and H. Berens, Semigroups of Operators and Approximation, Springer Verlag, 1967.
4. W. A. Coppel, Stability and Asymptotic Behavior of Differential Equations, D. C. Heath and Company, 1965.

5. J. L. Daleckii and M. G. Krein, Stability of solutions of differential equations in Banach spaces, in Transl. Math. Monographs, Vol. 43, Amer. Math. Soc., Providence, RI, 1974.

6. H. Henriquez, Asymptotically Almost–Periodic Solutions of Abstract Differential Equations, *J. Math. Anal. Appl.* **160** (1991), 157–175.

7. R. Nagel, One–parameter Semigroups of Positive Operators, Lecture Notes in Math., Vol. 1184, Springer–Verlag, Berlin, 1986.

8. A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer–Verlag, New York, 1983.

9. A. S. Rao and W. Hengartner, On the existence of a unique almost periodic solution of an abstract differential equation, *J. London Math. Soc.* (2) **8** (1974), 577–581.

10. S. D. Zaidman, Almost Periodic Function in Abstract Spaces, Pitman, London, 1985.

11. S. D. Zaidman, Abstract Differential Equations, Pitman, London, 1979.

12. V. V. Zhikov, Remarks On Compactness Conditions Related to the Work of M. I. Kadets on Integration of Abstract Almost–Periodic Functions, *Funct. Anal. and its Appl.* **5** (1971), 26–30.

Institute of Mathematics, National Taiwan Normal University,
Taipei, Taiwan