# Exact formula for the second-order tangent set of the second-order cone complementarity set

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**Abstract.** The second-order tangent set is an important concept in describing the curvature of the set involved. Due to the existence of the complementarity condition, the second-order cone (SOC) complementarity set is a nonconvex set. Moreover, unlike the vector complementarity set, the SOC complementarity set is not even the union of finitely many polyhedral convex sets. Despite these difficulties, we succeed in showing that like the vector complementarity set, the SOC complementarity set is second-order directionally differentiable and an exact formula for the second-order tangent set of the SOC complementarity set can be given. We derive these results by establishing the relationship between the second-order tangent set of the SOC complementarity set and the second-order directional derivative of the projection operator over the second-order cone, and calculating the second-order directional derivative of the projection operator over the second-order cone. As an application, we derive second-order necessary optimality conditions for the mathematical program with second-order cone complementarity constraints.

**Keywords:** projection operator, second-order directional derivatives, second-order tangent sets, second-order cone complementarity sets, second-order necessary optimality conditions, mathematical program with second-order cone complementarity constraints.

**AMS subject classifications.** 90C26, 90C33, 49J52, 46G05.

## 1 Introduction

In optimization, an important issue is how to approximate the feasible region using derivatives of the function and the tangent cone of the set involved. Such needs arise in optimality conditions, constraint qualifications and stability analysis when the problem data are perturbed. In the same way that second-order derivatives provide quadratic approximations whereas first-order derivatives only provide linear approximation to a given function, second-order tangent sets provide better approximation than tangent cones to a set at a point, in particular when the given set is not a polyhedral set or the union of finitely many

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polyhedral sets. As a result, the second-order tangent sets have been used successfully in second-order optimality conditions, stability analysis, and metric subregularity (see e.g. [2, 3, 4, 7, 8, 10, 14, 16] and references therein). More recently, Gfrerer and Mordukhovich [11] use the second-order tangent set to give an estimate of the upper curvature of a set, which is used to study the Robinson regularity of parametric constraint systems.

In optimization, one often has to deal with a feasible region in the form $C := \{x \mid F(x) \in \Theta\}$, where $F : \mathbb{R}^n \to \mathbb{R}^m$ is a second-order continuously differentiable mapping and $\Theta$ is a closed set in $\mathbb{R}^m$. By [18, Proposition 13.13], under a constraint qualification, the second-order tangent set of the feasible region $C$ can be characterized as

$$\left.\begin{array}{r} d \in T_C(x) \\ w \in T_C^2(x, d) \end{array}\right\} \iff \left\{\begin{array}{l} \nabla F(x)d \in T_\Theta(F(x)) \\ \nabla F(x)w + d^T \nabla^2 F(x)d \in T_\Theta^2(F(x); \nabla F(x)d), \end{array}\right. \tag{1}$$

where $T_C, T_C^2$ denote the tangent cone and the second-order tangent set, respectively (see Definition 2.1). In the case when $\Theta = \mathbb{R}_-^{m_1} \times \{0\}^{m_2}$, $m_1 + m_2 = m$, the system is described by inequality and equality constraints. In this case, since the set $\Theta$ is polyhedral, the second-order tangent set of $\mathbb{R}_-^{m_1} \times \{0\}^{m_2}$ is a polyhedral set, and hence the second-order tangent set of the feasible region is a system of equalities and inequalities involving the second-order derivatives of the constraint mapping $F$ (see, e.g., Bonnans and Shapiro [4, Formula (3.81)]), provided a constraint qualification holds. In recent years, the second-order cone programming (SOCP) has attracted much attention due to a broad range of applications in fields from engineering, control and finance to robust optimization and combinatorial optimization (see e.g., [1] for introduction to the theories and its applications).

Consider the second-order cone defined as

$$\mathcal{K} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} \mid \|x_2\| \leq x_1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Bonnans and Ramírez gave the characterization for the second-order tangent set [3, Lemma 27], and using it to formulate second-order necessary and sufficient optimality conditions for nonlinear SOCPs. Since the second-order cone is not polyhedral, the second-order tangent set is not polyhedral [3].

In recent years, there are more and more researches on the second-order cone (SOC) complementarity system defined as

$$\mathcal{K} \ni G(z) \perp H(z) \in \mathcal{K},$$

where $u \perp v$ means the vectors $u$ and $v$ are perpendicular, $G(z), H(z) : \mathbb{R}^n \to \mathbb{R}^m$. One of the sources of the SOC complementarity system is the Karush-Kuhn-Tucker (KKT) optimality condition for the second-order cone programming (see e.g. [1, 5]), and the other is the equilibrium system for a Nash game where the constraints involving second-order cones (see e.g. [13]). We call the closed cone

$$\Omega := \{(x, y) \in \mathbb{R}^{2m} \mid \mathcal{K} \ni x \perp y \in \mathcal{K}\},$$

the SOC complementarity set (or the complementarity set associated with the second-order cone, c.f. [15]). Using the SOC complementarity set, the SOC complementarity system can be reformulated as $(G(z), H(z)) \in \Omega$. Due to the existence of the complementarity condition, the SOC complementarity set is a nonconvex set. Moreover, due to the nonpolyhedral structure of the second-order cone $\mathcal{K}$, the SOC complementarity set is also nonpolyhedral. Hence the SOC complementarity set is a difficult object to study in the variational analysis.

The main goal of this paper is to provide a precise formula for the second-order tangent set to the SOC complementarity set $\Omega$. The projection operator over the second-order cone $\Pi_{\mathcal{K}}(x) := \arg\min_{x'\in\mathcal{K}} \|x' - x\|$ is one of our main tools in the subsequent analysis. It is well-known that the metric projection operator $\Pi_{\mathcal{K}}(x)$ provides an alternative characterization of the SOC complementarity set:

$$(x, y) \in \Omega \Longleftrightarrow \Pi_{\mathcal{K}}(x - y) = x. \tag{2}$$

The projection operator $\Pi_{\mathcal{K}}(x)$ is known to be first-order directionally differentiable (see e.g. [17, Lemma 2]) and the connection between its tangent cone and its directional derivative has been given (see [15, 21]): for any $(x, y) \in \Omega$,

$$(d, w) \in T_\Omega(x, y) \Longleftrightarrow \Pi'_{\mathcal{K}}(x - y; d - w) = d. \tag{3}$$

Using this connection, it has been shown that the SOC complementarity set $\Omega$ is geometrically derivable and the exact formula for its tangent cone is given; see, e.g., [21, Theorem 5.1]. Moreover, the coderivative of the projection operator $\Pi_{\mathcal{K}}$ allows us to characterize the various normal cones as in [20, Proposition 2.1] and show that the SOC complementarity set is not only geometrically derivable but also directionally regular [21, Theorem 6.1]. So far by using the first-order variational analysis, it has been revealed that although the SOC complementarity set is neither a convex set nor the union of finitely many polyhedral convex sets, it enjoys certain nice properties that a convex set or the union of finitely many polyhedral convex sets has. In this paper, we continue to investigate the second-order variational properties of the SOC complementarity cone. Our main contributions are as follows:

- We derive the exact formula for the second-order directional derivative of the projection operator over second-order cone. We further establish the connection between the second-order tangent set and the second-order directional derivative of the projection operator: for any $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$,

$$(p, q) \in T^2_\Omega((x, y); (d, w)) \Longleftrightarrow \Pi''_{\mathcal{K}}(x - y; d - w, p - q) = p. \tag{4}$$

- We show that the SOC complementarity set is second-order directionally differentiable (see Definition 2.2). Note that this nice property is not even enjoyed by a convex set (see [4, Example 3.31]).

- Using the characterization (4) and the precise formula for the second-order directional derivative of the projection operation over the second-order cone, we derive the exact formula for the second-order tangent set of the SOC complementarity set. Compared with the usual vector complementarity set, our research shows that the task of establishing the formula of second-order tangent set to the second-order cone complementarity set, which has nonpolyhedral and nonconvex structure, is not trivial.

- Based on the exact formula of the second-order tangent set of $\Omega$, we develop the second-order optimality conditions for the mathematical program with second-order cone complementarity constraints (SOCMPCC).

We organize our paper as follows. Section 2 contains the preliminaries. In Section 3, we calculate the second-order directional derivative of the projection operator over the second-order cone. Section 4 is devoted to the exact formula of the second-order tangent set to the SOC complementarity set. The second-order optimality conditions of SOCMPCC are discussed in Section 5.

## 2 Preliminaries

In this section, we clarify the notation and recall some background materials. First, we denote by $\mathbb{R}_+$ and $\mathbb{R}_{++}$ the set of nonnegative scalars and positive scalars respectively, i.e., $\mathbb{R}_+ := \{\alpha |\ \alpha \geq 0\}$ and $\mathbb{R}_{++} := \{\alpha |\ \alpha > 0\}$. For a set $C$, denote by $\mathrm{int}C$, $\mathrm{cl}C$, $\mathrm{bd}C$, $\mathrm{co}C$, $C^c$ its interior, closure, boundary, convex hull, and its complement, respectively. For a closed set $C \subseteq \mathbb{R}^n$, let $C^\circ$ and $\sigma(\cdot|C)$ stand for the polar cone and the support function of $C$, respectively, i.e., $C^\circ = \{v |\, \langle v, w\rangle \leq 0,\ \forall w \in C\}$ and $\sigma(z|C) = \sup\{\langle z, x\rangle |\, x \in C\}$ for $z \in \mathbb{R}^n$. Denote by $\mathrm{lin}C$ the largest subspace $L$ such that $C + L \subseteq C$. For a vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we denote $x^\circ$ the polar set of the set $\{x\}$ and $\hat{x} := (x_1, -x_2)$, the reflection of vector $x$ on the $x_1$ axis. For a nonzero vector $x$, we denote by $\bar{x} := x/\|x\|$. Let $o(\lambda) : \mathbb{R}_+ \to \mathbb{R}^m$ stand for a mapping with the property that $o(\lambda)/\lambda \to 0$ when $\lambda \downarrow 0$. For a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ and vectors $x, d \in \mathbb{R}^n$, we denote by $\nabla F(x) \in \mathbb{R}^{m\times n}$ the Jacobian of $F$ at $x$, by $\nabla^2 F(x)$ the second-order derivative of $F$ at $x$, and by $\nabla^2 F(x)(d, d)$ the quadratic form corresponding to $\nabla^2 F(x)$. The directional derivative of $F$ at $x$ in direction $d$ is defined as

$$F'(x; d) := \lim_{t\downarrow 0} \frac{F(x + td) - F(x)}{t},$$

provided that the above limit exists. If $F$ is directionally differentiable at $x$ in direction $d$, its parabolic second-order directional derivative is defined as

$$F''(x; d, w) := \lim_{t\downarrow 0} \frac{F(x + td + \frac{1}{2}t^2 w) - F(x) - tF'(x; d)}{\frac{1}{2}t^2},$$

provided that the above limit exists. Moreover if the following limit exists

$$F''(x; d, w) = \lim_{\substack{t\downarrow 0 \\ w'\to w}} \frac{F(x + td + \frac{1}{2}t^2 w') - F(x) - tF'(x; d)}{\frac{1}{2}t^2},$$

then $F$ is said to be parabolical second-order directionally differentiable at $x$ in the direction $d$ in the sense of Hadamard. In general, the concept of parabolical second-order directional differentiability in the Hadamard sense is stronger than that of parabolical second-order directional differentiability. However, when $F$ is locally Lipschitz at $x$, these two concepts coincide. It is known that if $F$ is parabolical second-order directional differentiable in the Hadamard sense at $x$ along $d, w$, then

$$F(x + td + \frac{1}{2}t^2 w + o(t^2)) = F(x) + tF'(x; d) + \frac{1}{2}t^2 F''(x; d, w) + o(t^2).$$

**Definition 2.1 (Tangent Cones)** *Let $S \subseteq \mathbb{R}^m$ and $x \in S$. The regular/Clarke, inner and (Bouligand-Severi) tangent/contingent cone to $S$ at $x$ are defined respectively as*

$$\widehat{T}_S(x) := \liminf_{\substack{x' \xrightarrow{S} x \\ t\downarrow 0}} \frac{S - x'}{t} = \left\{ d \in \mathbb{R}^m \,\middle|\, \forall\, t_k \downarrow 0,\ x_k \xrightarrow{S} x,\ \exists d_k \to d \ \text{ with } \ x_k + t_k d_k \in S \right\},$$

$$T^i_S(x) := \liminf_{t\downarrow 0} \frac{S - x}{t} = \left\{ d \in \mathbb{R}^m \,\middle|\, \forall\, t_k \downarrow 0,\ \exists d_k \to d \ \text{ with } \ x + t_k d_k \in S \right\},$$

$$T_S(x) := \limsup_{t\downarrow 0} \frac{S - x}{t} = \left\{ d \in \mathbb{R}^m \,\middle|\, \exists\, t_k \downarrow 0,\ d_k \to d \ \text{ with } \ x + t_k d_k \in S \right\}.$$

*The inner and outer second-order tangent sets to $S$ at $x$ in direction $d$ are defined respectively as*

$$\begin{aligned} T_S^{i,2}(x;d) &:= \left\{ w \in \mathbb{R}^m \,\middle|\, \operatorname{dist}\left( x + td + \frac{1}{2}t^2 w, S \right) = o(t^2), \quad t \geq 0 \right\}, \\ T_S^2(x;d) &:= \left\{ w \in \mathbb{R}^m \,\middle|\, \exists\; t_n \downarrow 0 \text{ such that } \operatorname{dist}\left( x + t_n d + \frac{1}{2}t_n^2 w, S \right) = o(t_n^2) \right\}. \end{aligned}$$

While for a nonconvex set $S$, the contingent cone $T_S(x)$ may be nonconvex, it is known that the regular/Clarke tangent cone $\widehat{T}_S(x)$ is always closed and convex. By definition, since the distance function of a convex set is convex, it is easy to see that the inner second-order tangent set is always convex when the set $S$ is convex. On the other hand, the outer second-order tangent set may be nonconvex even when the set $S$ is convex (see [4, Example 3.35]). Note that $T_S^{i,2}(x;d) \subseteq T_S^2(x;d)$ and the outer second-order tangent set $T_S^2(x;d)$ needs not be a cone (it may be empty; see e.g. an example in [18, page 592]). If $T_S^{i,2}(x;d) = T_S^2(x;d)$, we simply call $T_S^2(x;d)$ the second-order tangent set to $S$ in direction $d$.

**Definition 2.2** *[4, Definition 3.32] A set $S$ is said to be second-order directionally differentiable at $x \in S$ in a direction $d \in T_S(x)$, if $T_S^i(x) = T_S(x)$ and $T_S^{i,2}(x;d) = T_S^2(x;d)$.*

**Definition 2.3 (Normal Cones)** *Let $S \subseteq \mathbb{R}^m$ and $x \in S$. The regular/Fréchet, limiting/Mordukhovich, and Clarke normal cone of $S$ at $x$ are defined respectively as*

$$\begin{aligned} \widehat{N}_S(x) &:= \left\{ v \in \mathbb{R}^m \mid \langle v, x' - x \rangle \leq o(\|x' - x\|) \ \forall x' \in S \right\}, \\ N_S(x) &:= \limsup_{x' \xrightarrow{S} x} \widehat{N}_S(x') = \left\{ \lim_{k\to\infty} v_k \mid v_k \in \widehat{N}_S(x_k), \quad x_k \xrightarrow{S} x \right\}, \\ N_S^c(x) &:= clco N_S(x). \end{aligned}$$

**Lemma 2.1 (Tangent-Normal Polarity)** *(see [18, Theorem 6.28], [6]) For a closed set $S \subseteq \mathbb{R}^m$ and $x \in S$, $\widehat{T}_S(x) = (N_S(x))^\circ = (N_S^c(x))^\circ, \widehat{N}_S(x) = (T_S(x))^\circ$, $(\widehat{T}_S(x))^\circ = N_S^c(x)$.*

We recall some known results concerning the second-order cone $\mathcal{K}$ in $\mathbb{R}^m$. The topological interior and the boundary of $\mathcal{K}$ are

$$\operatorname{int}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | x_1 > \|x_2\|\} \quad \text{and} \quad \operatorname{bd}\mathcal{K} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1} | x_1 = \|x_2\|\},$$

respectively. Similar to the eigenvalue decomposition of a matrix, for any given vector $x := (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$, $x$ can be decomposed as (see e.g [9])

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of $x$ respectively, given by

$$\lambda_i(x) := x_1 + (-1)^i \|x_2\| \quad \text{and} \quad u_x^{(i)} := \begin{cases} \frac{1}{2}(1, (-1)^i \bar{x}_2) & \text{if} \ \ x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if} \ \ x_2 = 0, \end{cases}$$

with $w$ being a fixed unit vector in $\mathbb{R}^{m-1}$.

**Lemma 2.2** *(see e.g. [19, Proposition 2.2]) For any $x, y \in bd\mathcal{K}\backslash\{0\}$, the following equivalence holds:*

$$x^T y = 0 \Longleftrightarrow y = k\hat{x} \text{ with } k = y_1/x_1 > 0 \Longleftrightarrow y = k\hat{x} \text{ with } k \in \mathbb{R}_{++}.$$

For a given real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define the SOC function $f^{\text{soc}} : \mathbb{R}^m \to \mathbb{R}^m$ as

$$f^{\text{soc}}(z) := f(\lambda_1(z))\, u_z^{(1)} + f(\lambda_2(z))\, u_z^{(2)}. \tag{5}$$

For $z \in \mathbb{R}^m$, let $\Pi_{\mathcal{K}}(z)$ be the metric projection of $z$ onto $\mathcal{K}$. Then by [9], it can be calculated as

$$\Pi_{\mathcal{K}}(z) = \lambda_1(z)_+ u_z^{(1)} + \lambda_2(z)_+ u_z^{(2)}, \tag{6}$$

where $\alpha_+ := \max\{\alpha, 0\}$ is the nonnegative part of the number $\alpha \in \mathbb{R}$. Hence the projection operator $\Pi_{\mathcal{K}}(\cdot)$ is an SOC function corresponds to the plus function $f(\alpha) := \alpha_+$.

# 3 Second-order directional derivative of the projection operator over the second-order cone

As commented in the introduction, there exists a close relationship between the second-order tangent set of the SOC complementarity set and the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$; see (4). Therefore, to obtain the exact formula of the second-order tangent set, we need to calculate the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$. This task is done in this section, which is of independent interest. For the convenience of notations, we sometime use $\Phi(x)$ instead of $\bar{x}$ to stand for $x/\|x\|$ as $x \neq 0$. It is easy to verify (see e.g. [23, Theorem 3.1]) that $\Phi$ is second-order continuously differentiable at $x \neq 0$ with

$$\begin{aligned}
\nabla\Phi(x) &= (I - \bar{x}\bar{x}^T)/\|x\|, \\
\nabla^2\Phi(x)(w, w) &= -2\frac{\bar{x}^T w}{\|x\|^2} w + w^T\left(\frac{3\bar{x}\bar{x}^T - I}{\|x\|^3}\right) wx \\
&= -2\frac{\bar{x}^T w}{\|x\|}\nabla\Phi(x)(w) - \frac{1}{\|x\|} w^T \nabla\Phi(x) w\bar{x},
\end{aligned}$$

where $I$ is the identity matrix in $\mathbb{R}^{m\times m}$.

Since the second-order cone $\mathcal{K}$ is a special circular cone $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \cos\theta\|x\| \le x_1\}$$

with $\theta = 45°$, the SOC function $f^{\text{soc}}$ is a special case of the circular cone function $f^{\mathcal{L}_\theta}$ studied in [23] with $\theta = 45°$. The following result follows from [23, Theorem 3.3] immediately.

**Lemma 3.1** *Suppose that $f : \mathbb{R} \to \mathbb{R}$. Then, the SOC function $f^{\text{soc}}$ is parabolic second-order directionally differentiable at $x$ in the Hadamard sense if and only if $f$ is parabolic second-order directionally differentiable at $\lambda_i(x)$ in the Hadamard sense for $i = 1, 2$. Moreover,*

**(i)** *if $x_2 = 0$ and $d_2 = 0$, then*

$$(f^{\text{soc}})''(x; d, w) = f''(x_1; d_1, w_1 - \|w_2\|)\, u_w^{(1)} + f''(x_1; d_1, w_1 + \|w_2\|)\, u_w^{(2)};$$

**(ii)** *if $x_2 = 0$ and $d_2 \neq 0$, then*

$$
\begin{aligned}
\left(f^{\mathrm{soc}}\right)''(x;d,w) &= f''\left(x_1; d_1 - \|d_2\|, w_1 - \bar{d}_2^T w_2\right) u_d^{(1)} \\
&\quad + f''\left(x_1; d_1 + \|d_2\|, w_1 + \bar{d}_2^T w_2\right) u_d^{(2)} \\
&\quad + \frac{1}{2}\Big(f'(x_1; d_1 + \|d_2\|) - f'(x_1; d_1 - \|d_2\|)\Big) \begin{pmatrix} 0 \\ \nabla\Phi(d_2) w_2 \end{pmatrix};
\end{aligned}
$$

**(iii)** *if $x_2 \neq 0$, then*

$$
\begin{aligned}
&\left(f^{\mathrm{soc}}\right)''(x;d,w) \\
&= f''\left(x_1 - \|x_2\|; d_1 - \bar{x}_2^T d_2, w_1 - \left[\bar{x}_2^T w_2 + d_2^T \nabla\Phi(x_2) d_2\right]\right) u_x^{(1)} \\
&\quad + f''\left(x_1 + \|x_2\|; d_1 + \bar{x}_2^T d_2, w_1 + \left[\bar{x}_2^T w_2 + d_2^T \nabla\Phi(x_2) d_2\right]\right) u_x^{(2)} \\
&\quad + \Big(f'(x_1 + \|x_2\|; d_1 + \bar{x}_2^T d_2) - f'(x_1 - \|x_2\|; d_1 - \bar{x}_2^T d_2)\Big) \begin{pmatrix} 0 \\ \nabla\Phi(x_2) d_2 \end{pmatrix} \\
&\quad + \frac{1}{2}\Big(f(x_1 + \|x_2\|) - f(x_1 - \|x_2\|)\Big) \begin{pmatrix} 0 \\ \nabla\Phi(x_2) w_2 + \nabla^2\Phi(x_2)(d_2, d_2) \end{pmatrix}.
\end{aligned}
$$

Since the projection operator $\Pi_{\mathcal{K}}(\cdot)$ is the SOC function corresponding to the plus function $f(\alpha) := \alpha_+$, we will need the second-order directional derivative of the plus function.

**Lemma 3.2** *(see e.g. [22]) Let $f(\alpha) := \alpha_+$ for $\alpha \in \mathbb{R}$. Then $f$ is parabolic second-order directionally differentiable at $x$ in the Hadamard sense and*

$$
f'(x;d) = \begin{cases} d & \text{if } x > 0, \\ d_+ & \text{if } x = 0, \\ 0 & \text{if } x < 0, \end{cases} \quad \text{and} \quad f''(x;d,w) = \begin{cases} w & \text{if } x > 0 \text{ or } x = 0, d > 0, \\ 0 & \text{if } x < 0 \text{ or } x = 0, d < 0, \\ w_+ & \text{if } x = d = 0. \end{cases}
$$

Since in the formula of the second-order directional derivative of the projection operator, we will need the tangent cone and the second-order tangent set for the set $\mathcal{K}$ and its polar $\mathcal{K}^\circ$, for convenience we summarize their formulas in the following two lemmas.

**Lemma 3.3** *[3, Lemma 25 and Lemma 27] For any $x \in \mathcal{K}$, one has*

$$
T_{\mathcal{K}}(x) = \begin{cases} \mathbb{R}^m & \text{if } x \in \mathrm{int}\mathcal{K}; \\ \mathcal{K} & \text{if } x = 0; \\ \{d \in \mathbb{R}^m \mid -d_1 + \bar{x}_2^T d_2 \leq 0\} & \text{if } x \in \mathrm{bd}\mathcal{K}\backslash\{0\}. \end{cases}
$$

*For any $x \in \mathcal{K}$ and $d \in T_{\mathcal{K}}(x)$,*

$$
T^2_{\mathcal{K}}(x;d) = \begin{cases} \mathbb{R}^m & \text{if } d \in \mathrm{int}T_{\mathcal{K}}(x); \\ T_{\mathcal{K}}(d) & \text{if } x = 0; \\ \{w \mid w_2^T x_2 - w_1 x_1 \leq d_1^2 - \|d_2\|^2\} & \text{if } x \in \mathrm{bd}\mathcal{K}\backslash\{0\} \textit{ and } d \in \mathrm{bd}T_{\mathcal{K}}(x). \end{cases}
$$

Applying [3, Lemma 25 and Lemma 27] to $\mathcal{K}^\circ = -\mathcal{K}$ yields the following result.

**Lemma 3.4** *For $x \in \mathcal{K}^\circ$, one has*

$$T_{\mathcal{K}^\circ}(x) = \begin{cases} \mathbb{R}^m & \text{if } x \in \mathrm{int}\mathcal{K}^\circ; \\ \mathcal{K}^\circ & \text{if } x = 0; \\ \{d \in \mathbb{R}^m | d_1 + \bar{x}_2^T d_2 \le 0\} & \text{if } x \in \mathrm{bd}\mathcal{K}^\circ \backslash \{0\}. \end{cases}$$

*For $x \in \mathcal{K}^\circ$ and $d \in T_{\mathcal{K}^\circ}(x)$, one has*

$$T^2_{\mathcal{K}^\circ}(x; d) = \begin{cases} \mathbb{R}^m & \text{if } d \in \mathrm{int}T_{\mathcal{K}^\circ}(x); \\ T_{\mathcal{K}^\circ}(d) & \text{if } x = 0; \\ \{w | w_2^T x_2 - w_1 x_1 \le d_1^2 - \|d_2\|^2\} & \text{if } x \in \mathrm{bd}\mathcal{K}^\circ \backslash \{0\} \textit{ and } d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x). \end{cases}$$

We are now ready to give the second-order directional derivative of the projection operator.

**Theorem 3.1** *The projection operator $\Pi_{\mathcal{K}}$ is parabolic second-order directionally differentiable in the Hadamard sense. Moreover, for any $x, d, w \in \mathbb{R}^m$, the second-order directional derivative can be calculated as in the following six cases.*

**Case (i)** $x \in \mathrm{int}\mathcal{K}$. $\Pi''_{\mathcal{K}}(x; d, w) = w$.

**Case (ii)** $x \in \mathrm{int}\mathcal{K}^\circ$. $\Pi''_{\mathcal{K}}(x; d, w) = 0$.

**Case (iii)** $x = 0$.

$$\Pi''_{\mathcal{K}}(x; d, w) = \begin{cases} w & \text{if } d \in \mathrm{int}\mathcal{K}, \\ 0 & \text{if } d \in \mathrm{int}\mathcal{K}^\circ, \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{d}_2^T w_2 \\ \left[w_1 - \frac{d_1}{\|d_2\|}\bar{d}_2^T w_2\right]\bar{d}_2 + \left[1 + \frac{d_1}{\|d_2\|}\right] w_2 \end{pmatrix} & \text{if } d \in (\mathcal{K} \cup \mathcal{K}^\circ)^c, \\ w & \text{if } d \in \mathrm{bd}\mathcal{K}\backslash\{0\}, w \in T_{\mathcal{K}}(d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{d}_2^T w_2 \\ 2w_2 + (w_1 - \bar{d}_2^T w_2)\bar{d}_2 \end{pmatrix} & \text{if } d \in \mathrm{bd}\mathcal{K}\backslash\{0\}, w \notin T_{\mathcal{K}}(d), \\ 0 & \text{if } d \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}, w \in T_{\mathcal{K}^\circ}(d), \\ \frac{1}{2}(w_1 + \bar{d}_2^T w_2)\begin{pmatrix} 1 \\ \bar{d}_2 \end{pmatrix} & \text{if } d \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}, w \notin T_{\mathcal{K}^\circ}(d), \\ \Pi_{\mathcal{K}}(w) & \text{if } d = 0. \end{cases}$$

**Case (iv)** $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

$$\Pi''_{\mathcal{K}}(x; d, w) =$$

$$\begin{cases} w & \text{if } d \in \mathrm{int}T_{\mathcal{K}}(x), \\ w & \text{if } d \in \mathrm{bd}T_{\mathcal{K}}(x), w \in T^2_{\mathcal{K}}(x; d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left[w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right]\bar{x}_2 + 2w_2 \end{pmatrix} & \text{if } d \in \mathrm{bd}T_{\mathcal{K}}(x), w \notin T^2_{\mathcal{K}}(x; d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2 + 2d_1\bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 + 2w_2 + 2\frac{d_1 - \bar{x}_2^T d_2}{\|x_2\|} d_2 \end{pmatrix} & \text{if } d \in T_{\mathcal{K}}(x)^c. \end{cases}$$

**Case (v)** $x \in \mathrm{bd}\mathcal{K}^{\circ}\backslash\{0\}$.

$$\Pi''_{\mathcal{K}}(x;d,w) = \begin{cases} 0 & \text{if } d \in \mathrm{int}T_{\mathcal{K}^\circ}(x), \\ 0 & \text{if } d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x), w \in T^2_{\mathcal{K}^\circ}(x;d), \\ \frac{1}{2}\left(w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right)\begin{pmatrix} 1 \\ \bar{x}_2 \end{pmatrix} & \text{if } d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x), w \notin T^2_{\mathcal{K}^\circ}(x;d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2 - 2d_1\bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 + 2\frac{d_1 + \bar{x}_2^T d_2}{\|x_2\|}d_2 \end{pmatrix} & \text{if } d \in T_{\mathcal{K}^\circ}(x)^c. \end{cases}$$

**Case (vi)** $x \in (\mathcal{K} \cup \mathcal{K}^\circ)^c$.

$$\Pi''_{\mathcal{K}}(x;d,w) = \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - (\bar{x}_2^T d_2)^2}{\|x_2\|} \\ \left[w_1 - \frac{x_1}{\|x_2\|}\bar{x}_2^T w_2 - \frac{x_1}{\|x_2\|^2}\left(\|d_2\|^2 - 3(\bar{x}_2^T d_2)^2\right) - 2d_1\frac{\bar{x}_2^T d_2}{\|x_2\|}\right]\bar{x}_2 + 2\frac{\|x_2\|d_1 - x_1\bar{x}_2^T d_2}{\|x_2\|^2}d_2 + \left[1 + \frac{x_1}{\|x_2\|}\right]w_2 \end{pmatrix}.$$

**Proof.** By (5)-(6), the projection operator $\Pi_{\mathcal{K}}$ is the SOC function $f^{soc}$ with $f(t) := t_+$. Applying Lemmas 3.1 and 3.2 will give the parabolic second-order directional differentiability of $\Pi_{\mathcal{K}}$ in the Hadamard sense and a formula for $\Pi''_{\mathcal{K}}$. However in some cases the formula obtained will still involve the plus operator $(\cdot)_+$. In this theorem we aim at obtaining the exact formula as proposed. For some cases, e.g., in the cases $x \in \mathrm{int}\mathcal{K}$; $x \in \mathrm{int}\mathcal{K}^\circ$; $x = 0, d \in \mathrm{int}\mathcal{K}$; $x = 0, d \in \mathrm{int}\mathcal{K}^\circ$; $x = 0, d = 0$, we can prove the results by directly using the definition of second-order directional derivative. In some other cases, e.g., in the cases $x = 0, d \in \mathrm{bd}\mathcal{K}\backslash\{0\}$; $x = 0, d \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}$; $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}, d \in \mathrm{bd}T_{\mathcal{K}}(x)$; $x \in \mathrm{bd}\mathcal{K}^\circ\backslash\{0\}, d \in \mathrm{bd}T_{\mathcal{K}^\circ}(x)$, we can further use the representation of tangent cones in Lemmas 3.3 and 3.4 to obtain the proposed exact formula. For simplicity, we only prove some of the cases. The others can be obtained by following similar arguments.

**Case** $x \in \mathrm{int}\mathcal{K}$. In this case $\Pi_{\mathcal{K}}(x) = x$, $\Pi'_{\mathcal{K}}(x;d) = d$ and $\Pi_{\mathcal{K}}(x+td+\frac{1}{2}t^2w) = x+td+\frac{1}{2}t^2w$ for $t > 0$ sufficiently small. Hence

$$\Pi''_{\mathcal{K}}(x;d,w) := \lim_{t\downarrow 0}\frac{\Pi(x+td+\frac{1}{2}t^2w) - \Pi_{\mathcal{K}}(x) - t\Pi'_{\mathcal{K}}(x;d)}{\frac{1}{2}t^2} = w.$$

**Case** $x = 0$ and $d \in \mathrm{int}\mathcal{K}$. In this case $\Pi_{\mathcal{K}}(x) = 0$ and $\Pi'_{\mathcal{K}}(x;d) = d$. Note that

$$\Pi_{\mathcal{K}}(x+td+\frac{1}{2}t^2w) = \Pi_{\mathcal{K}}(td+\frac{1}{2}t^2w) = td+\frac{1}{2}t^2w,$$

for $t > 0$ sufficiently small. Hence $\Pi''_{\mathcal{K}}(x;d,w) = w$.

**Case** $x = 0$ and $d = 0$. It is obvious that $\Pi_{\mathcal{K}}(0) = 0, \Pi'_{\mathcal{K}}(0;0) = 0$ and $\Pi_{\mathcal{K}}(x+td+\frac{1}{2}t^2w) = \Pi_{\mathcal{K}}(\frac{1}{2}t^2w) = \frac{1}{2}t^2\Pi_{\mathcal{K}}(w)$. Hence $\Pi''_{\mathcal{K}}(x;d,w) = \Pi_{\mathcal{K}}(w)$.

**Case** $x = 0$ and $d \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then $\Pi_{\mathcal{K}}(x) = 0$ and $d_1 = \|d_2\| \neq 0$. Directly applying Lemmas 3.1(ii) and 3.2 yield

$$\Pi''_{\mathcal{K}}(x;d,w) \;=\; \frac{1}{2}(w_1 - \bar{d}_2^T w_2)_+\begin{pmatrix} 1 \\ -\bar{d}_2 \end{pmatrix} + \frac{1}{2}(w_1 + \bar{d}_2^T w_2)\begin{pmatrix} 1 \\ \bar{d}_2 \end{pmatrix} + \begin{pmatrix} 0 \\ (I - \bar{d}_2\bar{d}_2^T)w_2 \end{pmatrix}. \quad (7)$$

Recall from Lemma 3.3 that $w \in T_{\mathcal{K}}(d)$ if and only if $w_1 \geq \bar{d}_2^T w_2$. It follows from (7) that

$$\Pi''_{\mathcal{K}}(x; d, w) = \begin{cases} w & \text{if } w \in T_{\mathcal{K}}(d), \\ \frac{1}{2}\begin{pmatrix} w_1 + \bar{d}_2^T w_2 \\ 2w_2 + (w_1 - \bar{d}_2^T w_2)\bar{d}_2 \end{pmatrix} & \text{if } w \notin T_{\mathcal{K}}(d). \end{cases}$$

**Case** $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $d \in \mathrm{bd}T_{\mathcal{K}}(x)$. Then $x_1 = \|x_2\| \neq 0.$ and $-d_1 + \bar{x}_2^T d_2 = 0$. Directly applying Lemmas 3.1(iii) and 3.2 yield

$$\begin{aligned} &\Pi''_{\mathcal{K}}(x; d, w) \\ = \ & \frac{1}{2}\left(w_1 - \left[\bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right]\right)_+ \begin{pmatrix} 1 \\ -\bar{x}_2 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right)\bar{x}_2 + 2w_2 \end{pmatrix}. \end{aligned} \tag{8}$$

Recall from Lemma 3.3 that $w \in T^2_{\mathcal{K}}(x; d)$ if and only if $w_2^T x_2 - w_1 x_1 \leq d_1^2 - \|d_2\|^2$. Hence it follows from (8) that $\Pi''_{\mathcal{K}}(x; d, w) = w$ if $w \in T^2_{\mathcal{K}}(x; d)$ and

$$\Pi''_{\mathcal{K}}(x; d, w) = \frac{1}{2}\begin{pmatrix} w_1 + \bar{x}_2^T w_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(w_1 - \bar{x}_2^T w_2 - \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right)\bar{x}_2 + 2w_2 \end{pmatrix}$$

if $w \notin T^2_{\mathcal{K}}(x; d)$. $\square$

## 4 Second-order tangent set for the SOC complementarity set

This section is devoted to deriving the exact formula for the second-order tangent set to the SOC complementarity set. To this end, we first build its connection with the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$, whose existence is guaranteed by virtue of Theorem 3.1.

**Proposition 4.1** *For any $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$, one has*

$$T^{i,2}_\Omega\big((x, y); (d, w)\big) = T^2_\Omega\big((x, y); (d, w)\big) = \big\{(p, q) \,|\, \Pi''_{\mathcal{K}}(x - y; d - w, p - q) = p\big\}.$$

**Proof.** Since $T^{i,2}_\Omega\big((x, y); (d, w)\big) \subseteq T^2_\Omega\big((x, y); (d, w)\big)$, it suffices to show

$$T^2_\Omega\big((x, y); (d, w)\big) \subseteq \Upsilon\big((x, y); (d, w)\big) \subseteq T^{i,2}_\Omega\big((x, y); (d, w)\big),$$

where $\Upsilon\big((x, y); (d, w)\big) := \big\{(p, q) \,|\, \Pi''_{\mathcal{K}}(x - y; d - w, p - q) = p\big\}$. Let $(p, q) \in T^2_\Omega\big((x, y); (d, w)\big)$. Then by definition, there exist $t_n \downarrow 0$, $(\alpha(t_n), \beta(t_n)) = o(t_n^2)$ such that $(x, y) + t_n(d, w) + \frac{1}{2}t_n^2(p, q) + (\alpha(t_n), \beta(t_n)) \in \Omega$. By the equivalence in (2), it follows that

$$\begin{aligned} &\Pi_{\mathcal{K}}\Big(x - y + t_n(d - w) + \frac{1}{2}t_n^2(p - q) + \alpha(t_n) - \beta(t_n)\Big) \\ &= \ x + t_n d + \frac{1}{2}t_n^2 p + \alpha(t_n) \\ &= \ \Pi_{\mathcal{K}}(x - y) + t_n \Pi'_{\mathcal{K}}(x - y; d - w) + \frac{1}{2}t_n^2 p + \alpha(t_n), \end{aligned}$$

where the last equality follows from the equivalence in (2) and (3). Hence, $\Pi''_{\mathcal{K}}(x-y;d-w,p-q)=p$, i.e., $(p,q)\in\Upsilon\big((x,y);(d,w)\big)$.

Now, take $(p,q)\in\Upsilon\big((x,y);(d,w)\big)$, i.e., $\Pi''_{\mathcal{K}}(x-y;d-w,p-q)=p$. For $t>0$, define

$$r(t):=\Pi_{\mathcal{K}}(x-y+t(d-w)+\frac{1}{2}t^2(p-q))-\Pi_{\mathcal{K}}(x-y)-t\Pi'_{\mathcal{K}}(x-y;d-w)-\frac{1}{2}t^2\Pi''_{\mathcal{K}}(x-y;d-w,p-q).$$

Then $r(t)=o(t^2)$ according to the second-order directional differentiability of $\Pi_{\mathcal{K}}$ by Theorem 3.1. Note that

$$\begin{aligned}
& \Pi_{\mathcal{K}}\big(x-y+t(d-w)+\frac{1}{2}t^2p+r(t)-\frac{1}{2}t^2q-r(t)\big)\\
= & \ \Pi_{\mathcal{K}}(x-y+t(d-w)+\frac{1}{2}t^2(p-q))\\
= & \ \Pi_{\mathcal{K}}(x-y)+t\Pi'_{\mathcal{K}}(x-y;d-w)+\frac{1}{2}t^2\Pi''_{\mathcal{K}}(x-y;d-w,p-q)+r(t)\\
= & \ x+td+\frac{1}{2}t^2p+r(t),
\end{aligned}$$

where the last equality follows from the equivalence in (2) and (3). This together with equivalence (2) yields that

$$\left(x+td+\frac{1}{2}t^2p+r(t),y+tw+\frac{1}{2}t^2q+r(t)\right)\in\Omega.$$

It means $(p,q)\in T^{i,2}_{\Omega}\big((x,y);(d,w)\big)$. The proof is complete. $\square$

**Remark 4.1** *The proof of equivalence (4) in Proposition 4.1 is very similar to that of equivalence (3) as in [21, Proposition 5.2]. Note that although the equivalence (3) was shown in [15, Proposition 3.1], the proof in [21, Proposition 5.2] is much more concise without going over each possible cases as in [15, Proposition 3.1]. Moreover from the proof of [21, Proposition 5.2], one can see that the equivalence (3) holds for any general convex cone $\mathcal{K}$ as long as the projection operator $\Pi_{\mathcal{K}}$ satisfies the Lipschitz continuity and directional differentiability. Similarly from the proof of Proposition 4.1, we can see that equivalence (4) in Proposition 4.1 holds for any general convex cone $\mathcal{K}$ whenever the projection operator $\Pi_{\mathcal{K}}$ satisfies the Lipschitz continuity and parabolic second-order directional differentiability in the Hadamard sense.*

The above result tells us that for characterizing the structure of the second-order tangent set to $\Omega$, we need to study the expression of the second-order directional derivative of the projection operator $\Pi_{\mathcal{K}}$, which has been obtained in Theorem 3.1. With these preparations, the explicit expression of the second-order tangent set to $\Omega$ is given below. For convenience, we recall the formula for the tangent cone first.

**Lemma 4.1** *[21, Theorem 5.1] For any $(x,y)\in\Omega$,*

$$\begin{aligned}
& T^i_{\Omega}(x,y)=T_{\Omega}(x,y)\\
= & \left\{(d,w)\ \middle|\ \begin{array}{ll}
d\in\mathbb{R}^m,\ w=0, & \text{if } x\in\operatorname{int}\mathcal{K},\ y=0;\\
d=0,\ w\in\mathbb{R}^m, & \text{if } x=0,\ y\in\operatorname{int}\mathcal{K};\\
x_1\hat{w}-y_1d\in\mathbb{R}x,\quad d\perp y,\ w\perp x, & \text{if } x,y\in\operatorname{bd}\mathcal{K}\backslash\{0\};\\
d\in T_{\mathcal{K}}(x),\ w=0 \text{ or } d\perp\hat{x}\ ,w\in\mathbb{R}_+\hat{x}, & \text{if } x\in\operatorname{bd}\mathcal{K}\backslash\{0\},\ y=0;\\
d=0,\ w\in T_{\mathcal{K}}(y) \text{ or } d\in\mathbb{R}_+\hat{y},\ w\perp\hat{y}, & \text{if } x=0,\ y\in\operatorname{bd}\mathcal{K}\backslash\{0\};\\
d\in\mathcal{K},\ w\in\mathcal{K},\ d\perp w, & \text{if } x=0,\ y=0.
\end{array}\right\}.
\end{aligned}$$

According to Proposition 4.1 and Lemma 4.1, we obtain the following result.

**Theorem 4.1** *The set $\Omega$ is second-order directionally differentiable at every $(x, y) \in \Omega$ in every direction $(d, w) \in T_\Omega(x, y)$.*

**Remark 4.2** *It is well-known that for a convex set, the tangent cone and inner tangent cone coincide, but the inner and outer second-order tangent sets can be different; see [4, Example 3.31]. Here we show that SOC complementarity set $\Omega$, although it is nonconvex, is second-order directionally differentiable, i.e., the tangent cone and inner tangent cone coincide, and the inner and outer second-order tangent sets coincide as well.*

The inner and outer second-order tangent set to product sets have been studied in [4, Page 168]. Particularly, for $C := C_1 \times \cdots \times C_m$ with $C_i \in \mathbb{R}^{n_i}$, at certain $x = (x_1, \ldots, x_m)$ with $x_i \in C_i$, according to [4],

$$T_C^{i,2}(x; d) = T_{C_1}^{i,2}(x_1; d_1) \times \ldots T_{C_m}^{i,2}(x_m; d_m)$$

and

$$T_C^2(x; d) \subset T_{C_1}^2(x_1; d_1) \times \ldots T_{C_m}^2(x_m; d_m). \tag{9}$$

If all except at most one of $C_i$ are second-order directional differentiable, then the equality holds in (9). Noting that second-order cone complementarity set is second-order directional differentiable, Theorem 4.1 can be then extended to the Cartesian product of finitely many second-order cone complementarity sets.

**Corollary 4.1** *Suppose that $\Omega_1, \cdots, \Omega_l$ are all SOC complementarity sets. Then the Cartesian product $\Omega := \Omega_1 \times \Omega_2 \times \cdots \times \Omega_l$ is second-order directionally differentiable at every $(x, y) \in \Omega$ in every direction $(d, w) \in T_\Omega(x, y)$ and*

$$T_\Omega^2((x, y); (d, w)) = T_{\Omega_1}^2((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_1}^2((x_l, y_l); (d_l, w_l)).$$

**Proof.** Since $(d, w) \in T_\Omega(x, y) = T_{\Omega_1}(x_1, y_1) \times \cdots \times T_{\Omega_l}(x_l, y_l)$, $(d_i, w_i) \in T_{\Omega_i}(x_i, y_i)$ for $i = 1, \ldots, l$. Take $(p, q) \in T_\Omega^2((x, y); (d, w))$. Hence

$$\begin{aligned} T_\Omega^2((x, y); (d, w)) &\subseteq T_{\Omega_1}^2((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_1}^2((x_l, y_l); (d_l, w_l)) \\ &= T_{\Omega_1}^{i,2}((x_1, y_1); (d_1, w_1)) \times \cdots \times T_{\Omega_l}^{i,2}((x_l, y_l); (d_l, w_l)) \\ &= T_\Omega^{i,2}((x, y); (d, w)), \end{aligned}$$

where the first inclusion and the second equation follows from [4, Page 168], and the first equation comes from Theorem 4.1. □

**Theorem 4.2** *For any $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$, the formula of the second-order tangent set for the SOC complementarity set can be described as in the following six cases.*

**Case (i)** $x \in \operatorname{int}\mathcal{K}$ and $y = 0$. $T_\Omega^2\big((x, y); (d, w)\big) = \mathbb{R}^m \times \{0\}$.

**Case (ii)** $x = 0$ and $y \in \operatorname{int}\mathcal{K}$. $T_\Omega^2\big((x, y); (d, w)\big) = \{0\} \times \mathbb{R}^m$.

**Case (iii)** $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

$$T^2_\Omega\big((x,y);(d,w)\big) = \left\{ (p,q) \;\middle|\; \begin{array}{l} p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d), \quad q \in \mathrm{bd}T^2_{\mathcal{K}}(y;w), \\ (x_1w_1 - y_1d_1)\left(\frac{w_2 - w_1\bar{y}_2}{y_1} - \frac{d_2 - d_1\bar{x}_2}{x_1}\right) - p_1y_2 - q_1x_2 = x_1q_2 + y_1p_2 \end{array} \right\}.$$

**Case (iv)** $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ *and* $y = 0$.

$$T^2_\Omega\big((x,y);(d,w)\big) = \left\{ (p,q) \;\middle|\; \begin{array}{ll} q = 0, & \text{if } d \in \mathrm{int}T_{\mathcal{K}}(x),\ w = 0; \\ p \in T^2_{\mathcal{K}}(x;d),\ q = 0, \text{ or } \ p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d),\ q \in \mathbb{R}_+\hat{x} & \text{if } d \in \mathrm{bd}T_{\mathcal{K}}(x),\ w = 0; \\ p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d),\ -q_1\bar{x}_2 - 2\frac{w_1d_2}{\|x_2\|} - 2\frac{d_1w_2}{\|x_2\|} = q_2, & \text{if } d \perp \hat{x},\ w \in \mathbb{R}_{++}\hat{x}. \end{array} \right\}.$$

**Case (v)** $x = 0$ *and* $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

$$T^2_\Omega\big((x,y);(d,w)\big) = \left\{ (p,q) \;\middle|\; \begin{array}{ll} p = 0, & \text{if } d = 0,\ w \in \mathrm{int}T_{\mathcal{K}}(y); \\ p = 0,\ q \in T^2_{\mathcal{K}}(y;w), \text{ or } p \in \mathbb{R}_+\hat{y}, \quad q \in \mathrm{bd}T^2_{\mathcal{K}}(y;w) & \text{if } d = 0,\ w \in \mathrm{bd}T_{\mathcal{K}}(y); \\ q \in \mathrm{bd}T^2_{\mathcal{K}}(y;w),\ -p_1\bar{y}_2 - 2\frac{w_1d_2}{\|y_2\|} - 2\frac{d_1w_2}{\|y_2\|} = p_2, & \text{if } d \in \mathbb{R}_{++}\hat{y},\ w \perp \hat{y}. \end{array} \right\}.$$

**Case (vi)** $x = y = 0$. $T^2_\Omega\big((x,y);(d,w)\big) = T_\Omega(d,w)$.

**Proof.** By Proposition 4.1, to describe an element $(p,q) \in T^2_\Omega((x,y);(d,w))$, it suffices to describe an element $(p,q)$ satisfying $\Pi''_{\mathcal{K}}(x-y;d-w,p-q) = p$. For simplicity, we denote by $z := x-y, \xi := d-w$ and $\eta := p-q$.

**Case (i)** $x \in \mathrm{int}\mathcal{K}$ and $y = 0$. Since $z = x-y \in \mathrm{int}\mathcal{K}$, by Theorem 3.1(i), we have $\Pi''_{\mathcal{K}}(x-y;d-w,p-q) = p-q$. It follows that

$$\Pi''_{\mathcal{K}}(x-y;d-w,p-q) = p \Longleftrightarrow q = 0.$$

Hence $T^2_\Omega\big((x,y);(d,w)\big) = \mathbb{R}^m \times \{0\}$.

**Case (ii)** $x = 0$ and $y \in \mathrm{int}\mathcal{K}$. Since $z = x-y \in -\mathrm{int}\mathcal{K}$, by Theorem 3.1(ii), we know $\Pi''_{\mathcal{K}}(z;d-w,p-q) = 0$. It follows that

$$\Pi''_{\mathcal{K}}(x-y;d-w,p-q) = p \Longleftrightarrow p = 0.$$

Hence $T^2_\Omega\big((x,y);(d,w)\big) = \{0\} \times \mathbb{R}^m$.

**Case (iii)** $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $x^Ty = 0$. In this case $x_1 = \|x_2\| \neq 0$ and by Lemma 2.2,

$$z = x-y = (x_1,x_2) - k(x_1,-x_2) = ((1-k)x_1,(1+k)x_2), \quad k = y_1/x_1. \tag{10}$$

This yields $z_1 + \|z_2\| = 2x_1 > 0$ and $z_1 - \|z_2\| = -2kx_1 < 0$, i.e., $z \in (\mathcal{K} \cup \mathcal{K}^\circ)^c$. Then by Theorem 3.1(vi), $\Pi''_{\mathcal{K}}(z;\xi,\eta) = p$ where $p = (p_1,p_2) \in \mathbb{R} \times \mathbb{R}^{m-1}$ if and only if

$$p_1 = \frac{1}{2}\left(\eta_1 + \bar{z}_2^T\eta_2 + \frac{\|\xi_2\|^2 - (\bar{z}_2^T\xi_2)^2}{\|z_2\|}\right), \tag{11}$$

$$
\begin{aligned}
p_2 \;&=\; \frac{1}{2}\left(\eta_1 - \frac{z_1}{\|z_2\|}\bar{z}_2^T\eta_2 - \frac{z_1}{\|z_2\|^2}\left[\|\xi_2\|^2 - 3(\bar{z}_2^T\xi_2)^2\right] - 2\xi_1\frac{\bar{z}_2^T\xi_2}{\|z_2\|}\right)\bar{z}_2 \\
&\quad + \frac{\|z_2\|\xi_1 - z_1\bar{z}_2^T\xi_2}{\|z_2\|^2}\xi_2 + \frac{1}{2}\left(1 + \frac{z_1}{\|z_2\|}\right)\eta_2. \qquad (12)
\end{aligned}
$$

We now try to derive an equivalent expression for (11) and (12). Since $(d, w) \in T_\Omega(x, y)$, according to Lemma 4.1, $x \perp w$, $y \perp d$ and there exists $\beta \in \mathbb{R}$ such that $x_1\hat{w} - y_1 d = \beta x$, from which and $x_1 = \|x_2\| \neq 0$ we have

$$
w_1 = kd_1 + \beta, \quad w_2 = -kd_2 - \beta\bar{x}_2, \qquad (13)
$$

and

$$
\bar{x}_2^T w_2 = -w_1, \quad \bar{x}_2^T d_2 = d_1. \qquad (14)
$$

Note that $\bar{z}_2 = \bar{x}_2$ by (10). Hence it follows from (13) and (14) that

$$
\begin{aligned}
&\bar{z}_2^T\xi_2 = \bar{x}_2^T(d_2 - w_2) = d_1 + w_1 = (1 + k)d_1 + \beta, \qquad (15)\\
&\|\xi_2\|^2 = \|d_2 - w_2\|^2 = \|(1 + k)d_2 + \beta\bar{x}_2\|^2 = (k + 1)^2\|d_2\|^2 + 2\beta(k + 1)d_1 + \beta^2,\\
&\xi_1 = d_1 - w_1 = (1 - k)d_1 - \beta. \qquad (16)
\end{aligned}
$$

Hence (11) can be rewritten as

$$
p_1 = -q_1 + \bar{x}_2^T(p_2 - q_2) + \frac{x_1 + y_1}{x_1^2}\Big(\|d_2\|^2 - d_1^2\Big). \qquad (17)
$$

The term in front of $\bar{z}_2$ in (12) becomes

$$
\begin{aligned}
& \frac{1}{2}\left(\eta_1 - \frac{z_1}{\|z_2\|}\bar{z}_2^T\eta_2 - \frac{z_1}{\|z_2\|^2}\left[\|\xi_2\|^2 - 3(\bar{z}_2^T\xi_2)^2\right] - 2\xi_1\frac{\bar{z}_2^T\xi_2}{\|z_2\|}\right) \\
=\;& \frac{1}{2}\left(\eta_1 + \bar{z}_2^T\eta_2 + \frac{\|\xi_2\|^2 - (\bar{z}_2^T\xi_2)^2}{\|z_2\|}\right) - \frac{z_1 + \|z_2\|}{2\|z_2\|}\bar{z}_2^T\eta_2 \\
& + \frac{z_1 + \|z_2\|}{2\|z_2\|^2}\left[(\bar{z}_2^T\xi_2)^2 - \|\xi_2\|^2\right] + \left[\frac{z_1}{\|z_2\|^2}\big(\bar{z}_2^T\xi_2\big)^2 - \frac{\xi_1}{\|z_2\|}\big(\bar{z}_2^T\xi_2\big)\right] \\
=\;& \frac{y_1p_1 - x_1q_1}{x_1 + y_1} + \left[\frac{x_1 - y_1}{(x_1 + y_1)^2}(d_1 + w_1)^2 - \frac{1}{x_1 + y_1}(d_1^2 - w_1^2)\right] \\
=\;& \frac{y_1p_1 - x_1q_1}{x_1 + y_1} + 2\frac{x_1w_1^2 + x_1d_1w_1 - y_1d_1^2 - y_1d_1w_1}{(x_1 + y_1)^2},
\end{aligned}
$$

where the second equality uses (10), (11), and (15)-(17). It follows from (15) and (16) that the term in front of $\xi_2$ in (12) is

$$
\frac{\xi_1}{\|z_2\|} - \frac{z_1}{\|z_2\|}\frac{\bar{z}_2^T\xi_2}{\|z_2\|} = \frac{1}{x_1 + y_1}\big(d_1 - w_1\big) - \frac{x_1 - y_1}{(x_1 + y_1)^2}\big(d_1 + w_1\big) = 2\frac{y_1d_1 - x_1w_1}{(x_1 + y_1)^2}.
$$

The term in front of $\eta_2$ in (12) is $1/2\big(1 + (z_1/\|z_2\|)\big) = x_1/(x_1 + y_1)$. Hence (12) can be rewritten as

$$
p_2 \;=\; \left(\frac{y_1p_1 - x_1q_1}{x_1 + y_1} + 2\frac{x_1w_1^2 + x_1d_1w_1 - y_1d_1^2 - y_1d_1w_1}{(x_1 + y_1)^2}\right)\bar{x}_2 + 2\frac{y_1d_1 - x_1w_1}{(x_1 + y_1)^2}(d_2 - w_2)
$$

$$+\frac{x_1}{x_1+y_1}(p_2-q_2). \tag{18}$$

Further notice that $(y_1p_1-x_1q_1)\bar{x}_2=-p_1y_2-q_1x_2$ and

$$\begin{aligned}
&2\frac{y_1d_1-x_1w_1}{(x_1+y_1)^2}(d_2-w_2)+2\frac{x_1w_1^2+x_1d_1w_1-y_1d_1^2-y_1d_1w_1}{(x_1+y_1)^2}\bar{x}_2\\
=\;& 2\frac{-x_1\beta}{(x_1+y_1)^2}[(1+k)d_2+\beta\bar{x}_2]+2\frac{x_1\beta^2+y_1d_1\beta+x_1d_1\beta}{(x_1+y_1)^2}\bar{x}_2\\
=\;& 2\frac{\beta}{x_1+y_1}\big(-d_2+d_1\bar{x}_2\big)=2\frac{x_1w_1-y_1d_1}{x_1(x_1+y_1)}\big(-d_2+d_1\bar{x}_2\big),
\end{aligned}$$

where the first and third equations use (13) and the fact $y_1=kx_1$. Hence (18) can be rewritten as

$$\begin{aligned}
y_1p_2+x_1q_2 &= 2\frac{x_1w_1-y_1d_1}{x_1}(-d_2+d_1\bar{x}_2)-p_1y_2-q_1x_2\\
&= (x_1w_1-y_1d_1)\left(\frac{w_2-w_1\bar{y}_2}{y_1}+\frac{-d_2+d_1\bar{x}_2}{x_1}\right)-p_1y_2-q_1x_2, \qquad (19)
\end{aligned}$$

where the second step comes from the fact $(-d_2+d_1\bar{x}_2)/x_1=(w_2-w_1\bar{y}_2)/y_1$ due to (13). Hence (11) and (12) is equivalent to (17) and (19).

Now, multiplying (19) by $\bar{x}_2^T$ and using (14) yields

$$x_1\bar{x}_2^Tq_2+y_1\bar{x}_2^Tp_2=y_1p_1-x_1q_1. \tag{20}$$

Hence it follows from (17) and (20) that

$$p_1=\left(1+\frac{y_1}{x_1}\right)\bar{x}_2^Tp_2-\frac{y_1}{x_1}p_1+\frac{x_1+y_1}{x_1^2}\big(\|d_2\|^2-d_1^2\big),$$

i.e.,

$$p_1=\bar{x}_2^Tp_2+\frac{1}{x_1}\big(\|d_2\|^2-d_1^2\big)\Longleftrightarrow p\in \mathrm{bd}T_{\mathcal{K}}^2(x;d). \tag{21}$$

Since $d_1=(w_1-\beta)/k$, $d_2=-(w_2+\beta\bar{x}_2)/k$ by (13), and $w_2^T\bar{x}_2=-w_1$ by (14), we see

$$\frac{\|d_2\|^2-d_1^2}{x_1^2}=\frac{\|w_2\|^2-w_1^2}{y_1^2}. \tag{22}$$

Similarly, it follows from (17), (20), (22), and $\bar{x}_2=-\bar{y}_2$ that

$$q_1=-\frac{x_1}{y_1}q_1+\left(1+\frac{x_1}{y_1}\right)\bar{y}_2^Tq_2+\frac{x_1+y_1}{y_1^2}\big(\|w_2\|^2-w_1^2\big),$$

i.e.,

$$\frac{1}{y_1}\big(\|w_2\|^2-w_1^2\big)=q_1-\bar{y}_2^Tq_2\Longleftrightarrow q\in \mathrm{bd}T_{\mathcal{K}}^2(y;w). \tag{23}$$

Hence along the line

$$\left\{\begin{array}{l}(11)\\(12)\end{array}\right.\Longleftrightarrow\left\{\begin{array}{l}(17)\\(19)\end{array}\right.\Longleftrightarrow\left\{\begin{array}{l}(21),(23)\\(19)\end{array}\right.$$

the desired result follows.

**Case (iv)** $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$. In this case $z = x - y = x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

(**iv**)-1. $d \in \mathrm{int}T_{\mathcal{K}}(x)$ and $w = 0$. Then $\xi = d - w = d \in \mathrm{int}T_{\mathcal{K}}(x)$. Hence $\Pi''_{\mathcal{K}}(z;\xi,\eta) = \eta$ by Theorem 3.1(iv). It follows that $\Pi''_{\mathcal{K}}(x-y;d-w,p-q) = p$ if and only if $q = 0$.

(**iv**)-2. $d \in \mathrm{bd}T_{\mathcal{K}}(x)$ and $w = 0$. Then $\xi = d \in \mathrm{bd}T_{\mathcal{K}}(x)$. Hence $\Pi''_{\mathcal{K}}(z;\xi,\eta) = \Pi''_{\mathcal{K}}(x;d,\eta)$ and by Proposition 3.1(iv)

$$\begin{aligned}&\Pi''_{\mathcal{K}}(x;d,\eta)\\ &= \begin{cases} \eta & \text{if } \eta \in T^2_{\mathcal{K}}(x;d),\\ \frac{1}{2}\begin{pmatrix} \eta_1 + \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} \\ \left(\eta_1 - \bar{x}_2^T\eta_2 - \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right)\bar{x}_2 + 2\eta_2\end{pmatrix} & \text{if } \eta \notin T^2_{\mathcal{K}}(x;d).\end{cases}\end{aligned}$$

Note that $\eta \in T^2_{\mathcal{K}}(x;d) \Longleftrightarrow \eta_2^T x_2 - \eta_1 x_1 \le d_1^2 - \|d_2\|^2$ by Lemma 3.3. Hence $\Pi''_{\mathcal{K}}(x;d,p-q) = p$ if and only if either $p - q \in T^2_{\mathcal{K}}(x;d)$ and $q = 0$ or the following system holds

$$\begin{cases} \eta_1 - \bar{x}_2^T\eta_2 - \frac{\|d_2\|^2-d_1^2}{\|x_2\|} < 0,\\ \frac{1}{2}\left(\eta_1 + \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right]\right) = p_1,\\ \frac{1}{2}\left(\eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right]\right)\bar{x}_2 + \eta_2 = p_2.\end{cases} \tag{24}$$

We now further simplify the system (24).

$$\begin{aligned}(24) \quad &\Longleftrightarrow \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right] < 0,\\ \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|} = p_1 + q_1,\\ \frac{1}{2}(\eta_1 - p_1 - q_1)\bar{x}_2 = q_2.\end{cases} \Longleftrightarrow \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right] < 0,\\ \bar{x}_2^T p_2 - \bar{x}_2^T q_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|} = p_1 + q_1,\\ -q_1\bar{x}_2 = q_2.\end{cases}\\ &\Longleftrightarrow \begin{cases} \eta_1 - \left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|}\right] < 0,\\ \bar{x}_2^T p_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|} = p_1,\\ -q_1\bar{x}_2 = q_2.\end{cases} \Longleftrightarrow \begin{cases} q_1 > 0,\\ \bar{x}_2^T p_2 + \frac{\|d_2\|^2-d_1^2}{\|x_2\|} = p_1,\\ -q_1\bar{x}_2 = q_2.\end{cases}\\ &\Longleftrightarrow \begin{cases} q \in \mathbb{R}_{++}\hat{x},\\ p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d).\end{cases}\end{aligned}$$

Hence either $p \in T^2_{\mathcal{K}}(x;d)$ and $q = 0$ or $q \in \mathbb{R}_{++}\hat{x}$ and $p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d)$.

(**iv**)-3. $d \perp \hat{x}$ and $w \in \mathbb{R}_{++}\hat{x}$. Then $\bar{x}_2^T d_2 = d_1$ and $\bar{x}_2^T w_2 = -w_1$. Hence

$$\xi_1 - \bar{x}_2^T\xi_2 = d_1 - w_1 - \bar{x}_2^T(d_2 - w_2) = -2w_1 < 0, \tag{25}$$

which implies $\xi \in T_{\mathcal{K}}(x)^c$ by Lemma 3.3. Thus by Theorem 3.1(iv), $\Pi''_{\mathcal{K}}(z;\xi,\eta) = p$ takes the form

$$\begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T\eta_2 + \frac{\|\xi_2\|^2 - (\bar{x}_2^T\xi_2)^2}{\|x_2\|}\right) = p_1,\\ \frac{1}{2}\left(\eta_1 - \bar{x}_2^T\eta_2 - \frac{\|\xi_2\|^2 - 3(\bar{x}_2^T\xi_2)^2 + 2\xi_1\bar{x}_2^T\xi_2}{\|x_2\|}\right)\bar{x}_2 + \eta_2 + \frac{\xi_1 - \bar{x}_2^T\xi_2}{\|x_2\|}\xi_2 = p_2.\end{cases} \tag{26}$$

Note that

$$\xi_1\bar{x}_2^T\xi_2 - (\bar{x}_2^T\xi_2)^2 \;=\; (\xi_1 - \bar{x}_2^T\xi_2)\bar{x}_2^T\xi_2 = -2w_1(d_1 + w_1),$$

$$\|\xi_2\|^2 - (\bar{x}_2^T\xi_2)^2 \quad = \quad \|d_2 - w_2\|^2 - (d_1 + w_1)^2 = \|d_2\|^2 - d_1^2,$$

where we have used the fact $d \perp w$ and $w_2 = -w_1\bar{x}_2$ due to $d \perp \hat{x}$ and $w \in \mathbb{R}_{++}\hat{x}$. Therefore

$$\begin{aligned}
\frac{\|\xi_2\|^2 - 3(\bar{x}_2^T\xi_2)^2 + 2\xi_1\bar{x}_2^T\xi_2}{\|x_2\|} &= \frac{\|\xi_2\|^2 - (\bar{x}_2^T\xi_2)^2}{\|x_2\|} + 2\frac{\xi_1\bar{x}_2^T\xi_2 - (\bar{x}_2^T\xi_2)^2}{\|x_2\|} \\
&= \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} - 4\frac{w_1(d_1 + w_1)}{\|x_2\|}. \qquad (27)
\end{aligned}$$

Putting (25)-(27) into (26) yields

$$\begin{aligned}
&\Pi''_{\mathcal{K}}(z;\xi,\eta) = p \\
\Longleftrightarrow\ &\begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ \left(\frac{1}{2}\eta_1 - \frac{1}{2}\left[\bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right] + 2\frac{w_1}{\|x_2\|}(d_1 + w_1)\right)\bar{x}_2 - 2\frac{w_1}{\|x_2\|}(d_2 - w_2) = q_2. \end{cases} \\
\Longleftrightarrow\ &\begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ \left(-q_1 + 2\frac{w_1}{\|x_2\|}(d_1 + w_1)\right)\bar{x}_2 - 2\frac{w_1}{\|x_2\|}(d_2 - w_2) = q_2. \end{cases} \\
\Longleftrightarrow\ &\begin{cases} \frac{1}{2}\left(\eta_1 + \bar{x}_2^T\eta_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|}\right) = p_1, \\ -q_1\bar{x}_2 - 2\frac{d_1 w_2}{\|x_2\|} - 2\frac{w_1 d_2}{\|x_2\|} = q_2. \end{cases} \\
\Longleftrightarrow\ &\begin{cases} \bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1, \\ -q_1\bar{x}_2 - 2\frac{w_1 d_2}{\|x_2\|} - 2\frac{d_1 w_2}{\|x_2\|} = q_2. \end{cases}
\end{aligned}$$

where the third equivalence uses the fact $w_2 = -w_1\bar{x}_2$ due to $w \in \mathbb{R}_{++}\hat{x}$ and the last step follows from substituting the expression for $q_2$ in the second equation into the first one to obtain

$$\bar{x}_2^T q_2 = \bar{x}_2^T\left(-q_1\bar{x}_2 - 2\frac{w_1 d_2}{\|x_2\|} - 2\frac{d_1 w_2}{\|x_2\|}\right) = -q_1.$$

The desired result follows from noting that $p \in \mathrm{bd}T^2_{\mathcal{K}}(x;d)$ if and only if $\bar{x}_2^T p_2 + \frac{\|d_2\|^2 - d_1^2}{\|x_2\|} = p_1$ by virtue of Lemma 3.3.

**Case (v)** $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. The proof is omitted, since this case is symmetric to Case (iv).

**Case (vi)** $x = 0$ and $y = 0$. Since $\Omega$ is cone, according to the definition of second-order tangent set, we have

$$T^2_\Omega\big((0,0);(d,w)\big) = T_\Omega(d,w).$$

From all the above, the proof is complete. $\square$

# 5 Second-order optimality conditions for SOCMPCC

In this section, as an application of the second-order tangent set for the SOC complementarity set, we consider second-order optimality conditions for the mathematical programming with second-order cone complementarity constraints (SOCMPCC):

$$\min \ f(x) \quad \text{s.t.} \ \ \mathcal{K} \ni G(x) \perp H(x) \in \mathcal{K}, \qquad (28)$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $G, H : \mathbb{R}^n \to \mathbb{R}^m$ are second-order continuously differentiable. For simplicity, we restrict our attention on the simpler case, i.e., $\mathcal{K}$ is a $m$-dimensional second-order cone. All analysis can be easily carried over to more general cases where $\mathcal{K}$ is a Cartesian product of some second-order cones. SOCMPCC is an important class of optimization problems that has many applications. We refer the reader to [19, 21] and the reference within for applications and the first-order necessary optimality conditions.

Denote by $F(x) := (G(x), H(x))$. Then SOCMPCC (28) can be rewritten as

$$\min \ f(x) \quad \text{s.t.} \ \ F(x) \in \Omega. \tag{29}$$

For a convex set-constrained optimization problem in the form of (29) where $\Omega$ is replaced by a convex closed set $K$ (see [4, (3.93)]), second-order optimality conditions that involve the second-order tangent set to $K$ have been developed in [2, 4]. In particular when the convex set $K$ is not polyhedral, the second-order tangent set to $K$ is needed in the second-order optimality conditions. However, if set $\Omega$ in problem (29) is nonconvex, these optimality conditions are not applicable in general. In what follows, we will establish the second-order optimality conditions for the SOCMPCC, which is not a convex set-constrained optimization problem. We would like to emphasize that, even if the second-order cone complementarity set is nonconvex, its tangent cone and second-order tangent set have nice properties so that some of the theories in the second-order optimality conditions for a convex set-constrained optimization problem still hold. This observation relies heavily on the exact formula of tangent cone and second-order tangent set established in the previous section.

First we present some results needed for further analysis. Recall that the regular tangent cone is always convex. The following result shows that the regular tangent cone to the SOC complementarity set $\Omega$ is not only convex but is a subspace.

**Proposition 5.1** *For any $(x, y) \in \Omega$,*

$$\begin{aligned}
&\widehat{T}_\Omega(x,y) = \text{lin}T_\Omega(x,y)\\
&= \left\{ (d,w) \;\middle|\; \begin{array}{lll} d \in \mathbb{R}^m,\ w = 0, & & \text{if } x \in \text{int}\mathcal{K} \text{ and } y = 0;\\ d = 0,\ w \in \mathbb{R}^m, & & \text{if } x = 0 \text{ and } y \in \text{int}\mathcal{K};\\ x_1\hat{w} - y_1 d \in \mathbb{R}x, & d \perp y,\ w \perp x, & \text{if } x, y \in \text{bd}\mathcal{K}\backslash\{0\};\\ d \perp \hat{x},\ w = 0, & & \text{if } x \in \text{bd}\mathcal{K}\backslash\{0\} \text{ and } y = 0;\\ d = 0,\ w \perp \hat{y}, & & \text{if } x = 0 \text{ and } y \in \text{bd}\mathcal{K}\backslash\{0\};\\ d = 0,\ w = 0, & & \text{if } x = 0 \text{ and } y = 0. \end{array} \right\}.
\end{aligned}$$

**Proof.** The formula of $\text{lin}T_\Omega(x, y)$ is clear from that of $T_\Omega(x, y)$ in Lemma 4.1. According to the tangent-normal polarity as in Lemma 2.1, we can obtain the formula of $\widehat{T}_\Omega(x, y)$ by taking the polar of the limiting normal cone to $\Omega$ given in [20, Theorem 5.1]. The obtained formula of $\text{lin}T_\Omega(x, y)$ and $\widehat{T}_\Omega(x, y)$ shows that they have the same expression. $\square$

The exact formula established in Theorem 4.2 and Proposition 5.1 immediately imply the following results.

**Corollary 5.1** *For all $(x, y) \in \Omega$, $T_\Omega(x, y) + \widehat{T}_\Omega(x, y) = T_\Omega(x, y)$.*

**Proposition 5.2** *For $(x, y) \in \Omega$ and $(d, w) \in T_\Omega(x, y)$,*

$$T^2_\Omega\big((x, y); (d, w)\big) + \widehat{T}_\Omega(x, y) = T^2_\Omega\big((x, y); (d, w)\big).$$

**Proof.** The inclusion "$\supseteq$" is clear, since $(0,0)\in \widehat{T}_\Omega(x,y)$. For all cases except where $x,y\in \mathrm{bd}\mathcal{K}\backslash\{0\}$, it is easy to see that "$\subseteq$" can be achieved by using the formula of $T^2_\Omega\big((x,y);(d,w)\big)$ given in Theorem 4.2 and the formula of $\widehat{T}_\Omega(x,y)$ given in Proposition 5.1. Now consider the case where $x,y\in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Let $(p,q)\in T^2_\Omega\big((x,y);(d,w)\big)$ and $(u,v)\in \widehat{T}_\Omega(x,y)$. Since $p\in \mathrm{bd}T^2_{\mathcal{K}}(x;d)$, then $\hat{x}^Tp=\|d_2\|^2-d_1^2$ by Lemma 3.3. Hence $\hat{x}^T(p+u)=\hat{x}^Tp+\hat{x}^Tu=\|d_2\|^2-d_1^2$ due to the fact $u\perp\hat{x}$ (since $u\perp y$ by Proposition 5.1 and $y\in\mathbb{R}_{++}\hat{x}$ ). This means $p+u\in \mathrm{bd}T^2_{\mathcal{K}}(x;d)$. Similarly, we can obtain $q+v\in \mathrm{bd}T^2_{\mathcal{K}}(y;w)$. Since $(u,v)\in \widehat{T}_\Omega(x,y)$, it follows from Proposition 5.1 that there exists $\tau\in\mathbb{R}$ such that $x_1\hat{v}-y_1u=\tau x$. Thus

$$x_1v_2+y_1u_2=-\tau x_2=-\frac{x_1v_1-y_1u_1}{x_1}x_2=-v_1x_2-u_1y_2, \tag{30}$$

where the last step comes from Lemma 2.2. Since $x,y\in \mathrm{bd}\mathcal{K}\backslash\{0\}$, $(p,q)\in T^2_\Omega\big((x,y);(d,w)\big)$, it follows from Theorem 4.2 that

$$(p,q)\in T^2_\Omega\big((x,y);(d,w)\big)\Longleftrightarrow\left\{\begin{array}{l} p\in \mathrm{bd}T^2_{\mathcal{K}}(x;d), q\in \mathrm{bd}T^2_{\mathcal{K}}(y;w)\\ \xi-p_1y_2-q_1x_2=x_1q_2+y_1p_2\end{array}\right., \tag{31}$$

where $\xi:=(x_1w_1-y_1d_1)\left(\frac{w_2-w_1\bar{y}_2}{y_1}-\frac{d_2-d_1\bar{x}_2}{x_1}\right)$. This, together with (30), implies

$$\begin{aligned} x_1(q_2+v_2)+y_1(p_2+u_2) &= x_1q_2+y_1p_2-v_1x_2-u_1y_2\\ &= \xi-p_1y_2-q_1x_2-v_1x_2-u_1y_2\\ &= \xi-(p_1+u_1)y_2-(q_1+v_1)x_2. \end{aligned}$$

Hence together with $p+u\in \mathrm{bd}T^2_{\mathcal{K}}(x;d)$ and $q+v\in \mathrm{bd}T^2_{\mathcal{K}}(y;w)$, we have that $(p+u,q+v)\in T^2_\Omega\big((x,y);(d,w)\big)$ by virtue of (31). $\square$

With these preparations, we are now ready to develop a second-order necessary optimality condition for SOCMPCCs. Define the Lagrange function as $L(x,\lambda):=f(x)+\langle F(x),\lambda\rangle$ and the following three multiplier sets

$$\begin{aligned} \Lambda^c(x^*) &:= \{\lambda\,|\,\nabla_xL(x^*,\lambda)=0,\ \lambda\in N^c_\Omega(F(x^*))\},\\ \Lambda(x^*) &:= \{\lambda\,|\,\nabla_xL(x^*,\lambda)=0,\ \lambda\in N_\Omega(F(x^*))\},\\ \Lambda^F(x^*) &:= \{\lambda\,|\,\nabla_xL(x^*,\lambda)=0,\ \lambda\in \widehat{N}_\Omega(F(x^*))\}. \end{aligned}$$

Denote by $C(x^*):=\big\{d|\,\nabla f(x^*)d\le 0, \nabla F(x^*)d\in T_\Omega(F(x^*))\big\}$ the critical cone. Note that if there exists $\lambda\in\Lambda^F(x^*)$, then $C(x^*)=\big\{d|\,\nabla f(x^*)d=0, \nabla F(x^*)d\in T_\Omega(F(x^*))\big\}$.

**Theorem 5.1** *Let $x^*$ be a locally optimal solution of SOCMPCC. Suppose that the nondegeneracy condition*

$$\nabla F(x^*)\mathbb{R}^n+\mathrm{lin}T_\Omega(F(x^*))=\mathbb{R}^{2m} \tag{32}$$

*holds. Then $\Lambda^c(x^*)=\Lambda(x^*)=\Lambda^F(x^*)=\{\lambda_0\}$ and*

$$\nabla^2_{xx}L(x^*,\lambda_0)(d,d)-\sigma\left(\lambda_0|T^2_\Omega(F(x^*);\nabla F(x^*)d)\right)\ge 0,\quad \forall d\in C(x^*).$$

**Proof.** Step 1. We prove $\Lambda^c(x^*)=\Lambda(x^*)=\Lambda^F(x^*)=\{\lambda_0\}$. Since $\nabla F(x^*)\mathbb{R}^n+\mathrm{lin}T_\Omega(F(x^*))=\nabla F(x^*)\mathbb{R}^n+\widehat{T}_\Omega(F(x^*))=\mathbb{R}^{2m}$, taking polars on the both sides of the above equation, by the rule for polar cones [18, Corollary 11.25] and the fact that $(\widehat{T}_\Omega)^\circ = N^c_\Omega$, we have

$$\nabla F(x^*)^T\lambda = 0,\ \lambda \in N^c_\Omega(F(x^*)) \quad \Longrightarrow \quad \lambda = 0. \tag{33}$$

Suppose that $\lambda^1, \lambda^2 \in \Lambda^c(x^*)$. Then $\lambda^1 - \lambda^2$ satisfies $\nabla F(x^*)^T(\lambda^1 - \lambda^2) = 0$ and $\lambda^1 - \lambda^2 \in N^c_\Omega(F(x^*))$ since $N^c_\Omega(F(x))$ is subspace (because $\widehat{T}_\Omega(x,y)$ is a subspace and $N^c_\Omega = (\widehat{T}_\Omega)^\circ$). Thus $\lambda^1 = \lambda^2$ by (33). This means that $\Lambda^c(x^*)$ is a singleton. Since $\Lambda^F(x^*) \subseteq \Lambda(x^*) \subseteq \Lambda^c(x^*)$, it remains to show that $\Lambda^F(x^*)$ is nonempty. Since $N_\Omega \subseteq N^c_\Omega$, the condition (33) ensures

$$\nabla F(x^*)^T\lambda = 0,\ \lambda \in N_\Omega(F(x^*)) \quad \Longrightarrow \quad \lambda = 0, \tag{34}$$

which in turn implies that the system $F(x) - \Omega$ is metrically regular at $(x^*, 0)$. Thus according to [12, Theorem 4], Proposition 5.1, and Corollary 5.1, we have $\widehat{N}_S(x) = \nabla F(x)^T\widehat{N}_\Omega(F(x))$, where $S := \{x \,|\, F(x) \in \Omega\}$. As $x^*$ is a local optimal solution of problem (29), we have $0 \in \nabla f(x^*)^T + \widehat{N}_S(x^*) = \nabla f(x^*)^T + \nabla F(x^*)^T\widehat{N}_\Omega(F(x^*))$, which indicates that $\Lambda^F(x^*)$ is nonempty. Hence $\Lambda^c(x^*), \Lambda(x^*), \Lambda^F(x^*)$ are all singleton and coincide with each other. Let us denote the unique element by $\lambda_0$.

Step 2. We show that for all $d \in C(x^*)$ and for any convex subset $\mathcal{T}(d)$ in $T^2_\Omega(F(x^*); \nabla F(x^*)d)$, $\nabla^2_{xx}L(x^*, \lambda_0)(d,d) - \sigma(\lambda_0|\mathcal{T}(d)) \geq 0$. The idea of the proof is inspired by the arguments in [2, Theorem 3.1] and using the properties of tangent cone and second-order tangent set discussed above. For the sake of completeness, we give the detailed proof here. Consider the set $\Gamma(d) := \mathrm{cl}\{\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\}$. Since the regular tangent cone is convex, the set $\Gamma(d)$ is closed and convex. Moreover, it follows from Proposition 5.2 and the fact that the second-order tangent set is closed that $\Gamma(d) \subseteq T^2_\Omega(F(x^*); \nabla F(x^*)d)$. Because $x^*$ is locally optimal of problem (29), by definition of the second-order tangent cone, we can show that

$$\nabla f(x^*)w + \nabla^2 f(x^*)(d,d) \geq 0, \quad \forall d \in C(x^*), w \in T^2_C\big(x^*; d\big),$$

where $C$ denotes the feasible region of problem (29). Since (34) holds, by [18, Proposition 13.13], the chain rule for tangent sets (1) holds with $\Theta$ taken as $\Omega$. It follows that for all $d \in C(x^*)$, the following optimization problem

$$\begin{cases} \min\limits_w & \nabla f(x^*)w + \nabla^2 f(x^*)(d,d) \\ \text{s.t.} & \nabla F(x^*)w + \nabla^2 F(x^*)(d,d) \in T^2_\Omega\big(F(x^*); \nabla F(x^*)d\big) \end{cases}$$

has nonnegative optimal value. Since $\Gamma(d) \subseteq T^2_\Omega(F(x^*); \nabla F(x^*)d)$, it is clear that the following convex set constrained problem

$$\begin{cases} \min\limits_w & \nabla f(x^*)w + \nabla^2 f(x^*)(d,d) \\ \text{s.t.} & \nabla F(x^*)w + \nabla^2 F(x^*)(d,d) \in \Gamma(d) \end{cases} \tag{35}$$

has nonnegative optimal value as well. Since the optimization problem (35) can be put into the form of problem [4, (2.291)] involving an indicator function of set $\Gamma(d)$ and the dual problem of [4, (2.291)] is in the form of [4, (2.298)] and the conjugate function of an indicator function is the support function, the dual problem of (35) is

$$\max_\lambda \left\{ \inf_w \mathcal{L}(w,\lambda) - \sigma(\lambda|\Gamma(d)) \right\},$$

where $\mathcal{L}(w,\lambda) := \nabla_x L(x^*, \lambda)w + \nabla^2_{xx}L(x^*, \lambda)(d,d)$ is the Lagrange function of (35). Note that

$$\sigma(\lambda|\Gamma(d)) = \sigma\big(\lambda|\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\big) = \sigma(\lambda|\mathcal{T}(d)) + \sigma\big(\lambda|\widehat{T}_\Omega(F(x^*))\big) = +\infty,$$

whenever $\lambda \notin [\widehat{T}_\Omega(F(x))]^\circ = N^c_\Omega(F(x))$. Therefore, the dual problem of (35) is

$$\max_{\lambda \in \Lambda^c(x^*)} \left\{ \nabla^2_{xx} L(x^*, \lambda)(d,d) - \sigma\left(\lambda | \Gamma(d)\right) \right\} = \nabla^2_{xx} L(x^*, \lambda_0)(d,d) - \sigma\left(\lambda_0 | \Gamma(d)\right), \tag{36}$$

where the equality holds since $\Lambda^c(x^*) = \{\lambda_0\}$ by Step 1.

Since $\mathrm{lin} T_\Omega(F(x^*)) = \widehat{T}_\Omega(F(x^*))$ by Proposition 5.1 and $\mathrm{lin} T_\Omega(F(x^*))$ is a subspace, we have $\mathrm{lin} T_\Omega(F(x^*)) = -\widehat{T}_\Omega(F(x^*))$. Hence condition (32) is $\nabla F(x^*)\mathbb{R}^n - \widehat{T}_\Omega(F(x^*)) = \mathbb{R}^{2m}$, which in turn implies $\nabla F(x^*)\mathbb{R}^n - \big(\mathcal{T}(d) + \widehat{T}_\Omega(F(x^*))\big) = \mathbb{R}^{2m}$. Hence $\nabla F(\bar{x})\mathbb{R}^n - \Gamma(d) = \mathbb{R}^{2m}$. So the Robinson's constraint qualification (see [4, (2.313)]) for problem (35) holds. It ensures that the zero dual gap property holds (see [4, Theorem 2.165]). Hence the optimal value of the dual problem (36) is equal to the optimal value of problem (35) and hence nonnegative. In addition, noting $\mathcal{T}(d) \subseteq \Gamma(d)$, $\sigma(\lambda_0|\mathcal{T}(d)) \le \sigma(\lambda_0|\Gamma(d))$, which further implies that

$$\nabla^2_{xx} L(x^*, \lambda_0)(d,d) - \sigma\left(\lambda_0 | \mathcal{T}(d)\right) \ge 0. \tag{37}$$

Step 3. Note that $T^2_\Omega(F(x^*); \nabla F(x^*)d) = \bigcup_{a \in T^2_\Omega(F(x^*);\nabla F(x^*)d)} \{a\}$ is the union of convex sets. For each $a \in T^2_\Omega(F(x^*); \nabla F(x^*)d)$, by (37) we have

$$\nabla^2_{xx} L(x^*, \lambda_0)(d,d) - \langle \lambda_0, a \rangle \ge 0.$$

It then yields the desired result

$$\nabla^2_{xx} L(x^*, \lambda_0)(d,d) - \sigma\left(\lambda_0 | T^2_\Omega(F(x^*); \nabla F(x^*)d)\right) \ge 0.$$

□

**Remark 5.1** *The nondegeneracy condition (32), together with the special geometric structure of second-order cone complementarity set, can ensure not only the uniqueness of Lagrangian multiplier in Step 1, but also the zero-dual gap property between (35) and (36) in Step 2. The nondegeneracy condition, stronger than the Robinson's constraint qualification, is a generalization of linear independence constraint qualification in the conic case. We refer to [4, Proposition 4.75] for the detailed discussion on the relationship between nondegeneracy condition and uniqueness of multiplier in the convex case.*

We next derive the exact formula for the support function of the second-order tangent set to the SOC complementarity set needed in applying Theorem 5.1. Under the assumption of Theorem 5.1 we have $C(x^*) = \big\{d |\, \nabla f(x^*)d = 0, \nabla F(x^*)d \in T_\Omega(F(x^*))\big\}$. Thus $d \in C(x^*)$ if and only if $\nabla F(x^*)d \in T_\Omega(F(x^*))$ and $\langle \lambda_0, \nabla F(x^*)d \rangle = 0$. Therefore the following results will be useful.

**Proposition 5.3** *For $(x,y) \in \Omega$ and $(d,w) \in T_\Omega(x,y)$, take $(u,v) \in \widehat{N}_\Omega(x,y)$ such that $\langle (u,v), (d,w) \rangle = 0$. Then*

$$\sigma\big((u,v) | T^2_\Omega((x,y);(d,w))\big) = \left\{ \begin{array}{ll} 0, & \text{if } x \in \mathrm{int}\mathcal{K} \text{ and } y = 0; \\ 0, & \text{if } x = 0 \text{ and } y \in \mathrm{int}\mathcal{K}; \\ 0, & \text{if } x = 0 \text{ and } y = 0. \end{array} \right\}.$$

*If $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$, then*

$$\sigma\big((u,v) | T^2_\Omega((x,y);(d,w))\big)$$

$$= \left\{ \begin{array}{ll} 0, & \text{if } d \in \mathrm{int}T_{\mathcal{K}}(x) \text{ and } w = 0; \\ -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) - 2\frac{w_1 d_2^T v_2}{\|x_2\|} - 2\frac{d_1 w_2^T v_2}{\|x_2\|}, & \text{if } d \in \mathrm{bd}T_{\mathcal{K}}(x) \text{ and } w \in \mathbb{R}_+\hat{x}. \end{array} \right\}.$$

*If $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, then*

$$\begin{aligned} &\sigma\big((u,v)|T^2_{\Omega}((x,y);(d,w))\big) \\ = &\left\{ \begin{array}{ll} 0, & \text{if } d = 0 \text{ and } w \in \mathrm{int}T_{\mathcal{K}}(y); \\ -\frac{v_1}{y_1}(w_1^2 - \|w_2\|^2) - 2\frac{d_1 w_2^T u_2}{\|y_2\|} - 2\frac{w_1 d_2^T u_2}{\|y_2\|}, & \text{if } d \in \mathbb{R}_+\hat{y} \text{ and } w \in \mathrm{bd}T_{\mathcal{K}}(y). \end{array} \right\}. \end{aligned}$$

*If $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, then*

$$\begin{aligned} &\sigma\big((u,v)|T^2_{\Omega}((x,y);(d,w))\big) \\ = &\frac{x_1u_1 + y_1v_1}{x_1^2}\Big(\|d_2\|^2 - d_1^2\Big) + \frac{x_1w_1 - y_1d_1}{x_1y_1}\Big(w^Tv - d^Tu\Big). \end{aligned}$$

**Proof.** For $(x,y) \in \Omega$, take $(d,w) \in T_{\Omega}(x,y)$, $(p,q) \in T^2_{\Omega}((x,y);(d,w))$ with the exact formula given in Theorem 4.2 and $(u,v) \in \widehat{N}_{\Omega}(x,y)$ whose exact formula can be found in [20, Theorem 3.1].

**Case (i)** $x \in \mathrm{int}\mathcal{K}$ and $y = 0$. In this case $u = 0$ and $q = 0$. Hence $\sigma\big((u,v)|T^2_{\Omega}((x,y);(d,w))\big) = \max\{\langle (u,v),(p,q)\rangle | (p,q) \in T^2_{\Omega}((x,y);(d,w))\} = 0$. The proof for the case of $x = 0$ and $y \in \mathrm{int}\mathcal{K}$ is similar and hence we omit it.

**Case (ii)** $x \in \mathrm{bd}\mathcal{K}\backslash\{0\}$ and $y = 0$. Then $u \in \mathbb{R}_-\hat{x}$ and $v \in \hat{x}^\circ$ by the formula of $\widehat{N}_{\Omega}(x,y)$.

(**ii**)-1. Suppose further that $d \in \mathrm{int}T_{\mathcal{K}}(x)$ and $w = 0$. Then by Theorem 4.2, $q = 0$. Since $0 = \langle u,d\rangle + \langle v,w\rangle = \langle u,d\rangle$, which together with the fact that $d \in \mathrm{int}T_{\mathcal{K}}(x)$ (i.e., $d^T\hat{x} > 0$) implies $u = 0$. Hence $\sigma\big((u,v)|T^2_{\Omega}((x,y);(d,w))\big) = \max\{\langle (u,v),(p,q)\rangle | (p,q) \in T^2_{\Omega}((x,y);(d,w))\} = 0$.

(**ii**)-2. Suppose further that $d \in \mathrm{bd}T_{\mathcal{K}}(x)$ and $w = 0$, then $q = 0$ or $q \in \mathbb{R}_+\hat{x}$ by Theorem 4.2. Hence

$$\begin{aligned} \sigma\big((u,v)|T^2_{\Omega}\big((x,y);(d,w)\big)\big) &= \max\Big\{\sigma(u|T^2_{\mathcal{K}}(x;d)), \sigma(u|\mathrm{bd}T^2_{\mathcal{K}}(x;d)) + \sigma(v|\mathbb{R}_+\hat{x})\Big\} \\ &= \sigma(u|T^2_{\mathcal{K}}(x;d)) = -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2), \end{aligned}$$

where the second equality holds because $\sigma(v|\mathbb{R}_+\hat{x}) = 0$ since $v \in \hat{x}^\circ$, and the last step comes from the fact that since $u \in \mathbb{R}_-\hat{x}$, $\langle u,p\rangle = \frac{u_1}{x_1}\langle \hat{x},p\rangle \le \frac{u_1}{x_1}(\|d_2\|^2 - d_1^2)$ for all $p \in T^2_{\mathcal{K}}(x;d)$ by Lemma 3.3, and the maximum can be attained by letting $p = \frac{\|d_2\|^2 - d_1^2}{2x_1^2}\hat{x}$.

Now consider the case where $d \in \mathrm{bd}T_{\mathcal{K}}(x)$ and and $w \in \mathbb{R}_{++}\hat{x}$. From the formula for $\mathrm{bd}T_{\mathcal{K}}(x)$ in this case, we get $d \perp \hat{x}$. Hence $\langle v,w\rangle = \langle (u,v),(d,w)\rangle = 0$ taking into the account that $u \in \mathbb{R}_-\hat{x}$. It further implies that $v \perp \hat{x}$ (i.e., $v_1 = \bar{x}_2^Tv_2$), because $w \in \mathbb{R}_{++}\hat{x}$. Hence

$$\begin{aligned} \sigma\big((u,v)|T^2_{\Omega}\big((x,y);(d,w)\big)\big) &= \sigma(u|\mathrm{bd}T^2_{\mathcal{K}}(x;d)) + \langle v,q\rangle \\ &= -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) + v_1q_1 - q_1v_2^T\bar{x}_2 - 2\frac{w_1d_2^Tv_2}{\|x_2\|} - 2\frac{d_1w_2^Tv_2}{\|x_2\|} \\ &= -\frac{u_1}{x_1}(d_1^2 - \|d_2\|^2) - 2\frac{w_1d_2^Tv_2}{\|x_2\|} - 2\frac{d_1w_2^Tv_2}{\|x_2\|}. \end{aligned}$$

**Case (iii)** $x = 0$ and $y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. The argument is similar to the above case.

**Case (iv)** $x, y \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Note that in this case since $x^T y = 0$, we have $y = k\hat{x}$ with $k := y_1/x_1$. Since $(u, v) \in \widehat{N}_\Omega(x, y)$ and $(d, w) \in T_\Omega(x, y)$, by the formula of $\widehat{N}_\Omega(x, y)$ and $T_\Omega(x, y)$ we have $v \perp y$, $d \perp y$, and there exist $\beta, \gamma \in \mathbb{R}$ such that $\hat{u} + kv = \beta x$ and $\hat{w} - kd = rx$. To simplify the notation, let

$$\xi := (x_1 w_1 - y_1 d_1) \left( \frac{w_2 + w_1 \bar{x}_2}{y_1} - \frac{d_2 - d_1 \bar{x}_2}{x_1} \right).$$

Since $v \perp y$, $y \in \mathbb{R}\hat{x}$ and $x_1 = \|x_2\| \neq 0$, we have $v_1 - \bar{x}_2^T v_2 = 0$. It follows that

$$v_2^T \xi = (x_1 w_1 - y_1 d_1) \left( \frac{w_2^T v_2 + w_1 v_1}{y_1} - \frac{d_2^T v_2 - d_1 v_1}{x_1} \right) = \frac{x_1 w_1 - y_1 d_1}{y_1} (w^T v - d^T u), \quad (38)$$

where in the last step we used the fact that $d^T \hat{v} = (1/k) d^T (\beta \hat{x} - u) = -(1/k) d^T u$ since $d \perp \hat{x}$. By the formula of $T^2_\Omega((x, y); (d, w))$ in Theorem 4.2 for this case we have

$$p \in \mathrm{bd}T^2_\mathcal{K}(x; d), \quad q \in \mathrm{bd}T^2_\mathcal{K}(y; w), \quad \xi - p_1 y_2 - q_1 x_2 = x_1 q_2 + y_1 p_2. \quad (39)$$

Therefore,

$$\begin{aligned}
&\langle u, p\rangle + \langle v, q\rangle \\
&= \langle \hat{u}, \hat{p}\rangle + \langle v, q\rangle = \langle \beta x - kv, \hat{p}\rangle + \langle v, q\rangle = \beta\langle \hat{x}, p\rangle + \langle v, q - k\hat{p}\rangle \\
&= \beta\langle \hat{x}, p\rangle + v_1(q_1 - kp_1) + v_2^T \left( \frac{\xi}{x_1} + (kp_1 - q_1)\bar{x}_2 \right) \\
&= \beta\langle \hat{x}, p\rangle + \frac{1}{x_1} v_2^T \xi = \frac{x_1 u_1 + y_1 v_1}{x_1^2} \big( \|d_2\|^2 - d_1^2 \big) + \frac{x_1 w_1 - y_1 d_1}{x_1 y_1} (w^T v - d^T u),
\end{aligned}$$

where the forth equality holds by virtue of (39), the fifty equality holds because $v_1 = v_2^T \bar{x}_2$ and the sixth equality holds due to (38) and (39). The desired formula follows.

**Case (v)** $x = 0$ and $y = 0$. In this case $d, w \in \mathcal{K}$, $d \perp w$, $(u, v) \in \widehat{N}_\Omega(x, y) = (-\mathcal{K}, -\mathcal{K})$ and $(p, q) \in T^2_\Omega\big((x, y); (d, w)\big) = T_\Omega(d, w)$.

(**v**)-1. $d = 0$ and $w \in \mathrm{int}\mathcal{K}$. Since $\langle v, w\rangle = \langle (u, v), (d, w)\rangle = 0$ and $v \in -\mathcal{K}$, we have $v = 0$. Since $d = 0$ and $w \in \mathrm{int}\mathcal{K}$, $(p, q) \in T^2_\Omega\big((x, y); (d, w)\big) = T_\Omega(d, w)$ implies that $p = 0$. Hence $\langle (u, v), (p, q)\rangle = 0$. It follows that $\sigma\big((u, v)|T^2_\Omega((x, y); (d, w))\big) = 0$.

(**v**)-2. $d \in \mathrm{int}\mathcal{K}$ and $w = 0$. It is similar to the above case.

(**v**)-3. $d, w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Then since $\langle (u, v), (d, w)\rangle = 0$ and $(u, v) \in (-\mathcal{K}, -\mathcal{K})$, we have $u \in \mathbb{R}_-\hat{d} = \mathbb{R}_- w$ and $v \in \mathbb{R}_-\hat{w} = \mathbb{R}_- d$. Since $(p, q) \in T_\Omega(d, w)$ and $d, w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, we have $p \perp w$ and $q \perp d$. Hence $p \perp u$ and $q \perp v$. So $\langle (u, v), (p, q)\rangle = 0$. It follows that $\sigma\big((u, v)|T^2_\Omega((x, y); (d, w))\big) = 0$.

(**v**)-4. $d = 0$ and $w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$. Since $\langle v, w\rangle = \langle (u, v), (d, w)\rangle = 0$ and $v \in -\mathcal{K}$, we have $v \in \mathbb{R}_-\hat{w}$. In this case since $(p, q) \in T_\Omega(d, w)$ with $d = 0$ and $w \in \mathrm{bd}\mathcal{K}\backslash\{0\}$, we have either $p = 0$ and $q \in T_\mathcal{K}(w)$ or $p \in \mathbb{R}_+\hat{w}$ and $q \perp \hat{w}$. If $p = 0$ and $q \in T_\mathcal{K}(w)$ (i.e., $\hat{w}^T q \geq 0$), then $\langle (u, v), (p, q)\rangle = \langle v, q\rangle \leq 0$ and the maximum is 0 which can be attained by letting $q = 0$. If $p \in \mathbb{R}_+\hat{w}$ and $q \perp \hat{w}$, then $\langle (u, v), (p, q)\rangle = \langle u, p\rangle \leq 0$, where the last step is due to $u \in -\mathcal{K}$ and $p \in \mathbb{R}_+\hat{w} \in \mathcal{K}$, and the maximum is 0 which can be attained by letting $p = 0$. It follows that $\sigma\big((u, v)|T^2_\Omega((x, y); (d, w))\big) = 0$.

(**v**)-5. $d \in \mathrm{bd}\mathcal{K}/\{0\}$ and $w = 0$. It is similar to the above case by symmetric argument.

($\mathbf{v}$)-6. $d=0$ and $w=0$. In this case $(p,q)\in T_\Omega(d,w)=\Omega$. Since $(u,v)\in(-\mathcal{K},-\mathcal{K})$, we have $\langle (u,v),(p,q)\rangle\le 0$ and the maximum is 0 which can be attained by letting $(p,q)=(0,0)$. It follows that $\sigma\big((u,v)|T^2_\Omega((x,y);(d,w))\big)=0$. $\square$

**Example 5.1** *Consider the following SOCMPCC.*

$$\begin{array}{ll} \min & f(x):=-x_2^2+x_1-x_4 \\ \text{s.t.} & \mathcal{K}\ni G(x):=(x_1,x_3-x_1,x_1-x_2)\perp(-x_2+1,x_1,x_1-x_4)=:H(x)\in\mathcal{K}. \end{array}$$

Since $x_1\ge x_1-x_2$ and $-x_2+1\ge 0$, we have $x_2\in[0,1]$. Hence $x_2^2\le x_2$. Since $-x_2+1\ge -x_1+x_4$, $-x_2+x_1-x_4\ge -1$. Thus $-x_2^2+x_1-x_4\ge -x_2+x_1-x_4\ge -1$. Hence $x^*=(0,0,0,1)$ is an optimal solution, and $G(x^*)=(0,0,0)$ and $H(x^*)=(1,0,-1)\in \mathrm{bd}\mathcal{K}\backslash\{0\}$.

Note that

$$\nabla G(x^*)=\begin{bmatrix} 1 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 1 & -1 & 0 & 0\end{bmatrix},\qquad \nabla H(x^*)=\begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 1 & 0 & 0 & -1\end{bmatrix}$$

and by the formula of the tangent cone in Lemma 4.1, we have

$$T_\Omega(G(x^*),H(x^*))=\left\{(d,w)\;\middle|\; \begin{array}{l} \text{either } d=0, -w_1-w_3\le 0\\ \text{or } d=t(1,0,1) \text{ for some } t\ge 0, w_1+w_3=0\end{array}\right\}. \tag{40}$$

It follows that

$$\mathrm{lin}T_\Omega(G(x^*),H(x^*))=\{((0,0,0),(\tau_1,\tau_2,-\tau_1))|\tau_1,\tau_2\in\mathbb{R}\}.$$

For any $v\in\mathbb{R}^6$, take $\xi=(v_1,v_1-v_3,v_1+v_2,v_3-v_4-v_6)\in\mathbb{R}^4$ and $\tau=(v_1-v_3+v_4,v_5-v_1,-(v_1-v_3+v_4))\in\mathbb{R}^3$, then

$$v=\begin{pmatrix}\nabla G(x^*)\\ \nabla H(x^*)\end{pmatrix}\xi+\begin{pmatrix}0\\ \tau\end{pmatrix}\in\nabla F(x^*)\mathbb{R}^4+\mathrm{lin}T_\Omega(F(x^*)).$$

Since $v$ is arbitrarily taken from $\mathbb{R}^6$, condition (32) holds.

The Lagrangian multiplier system is

$$\left\{\begin{array}{l} \begin{pmatrix}1\\0\\0\\-1\end{pmatrix}+\lambda_1^G\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\lambda_2^G\begin{pmatrix}-1\\0\\1\\0\end{pmatrix}+\lambda_3^G\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}+\lambda_1^H\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}+\lambda_2^H\begin{pmatrix}1\\0\\0\\0\end{pmatrix}+\lambda_3^H\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix},\\ \qquad\qquad (\lambda^G,\lambda^H)\in N_\Omega\big(G(x^*),H(x^*)\big).\end{array}\right.$$

Since $G(x^*)=(0,0,0)$ and $H(x^*)=(1,0,-1)\in\mathrm{bd}\mathcal{K}\backslash\{0\}$, we obtain the following expression of the limiting normal cone from [20, Theorem 5.1]

$$\begin{aligned} &N_\Omega\big(G(x^*),H(x^*)\big)\\ &=\{(u,v)|v=0 \ \text{ or } \ u_1+u_3=0, v=t(1,0,1), t\in\mathbb{R} \ \text{ or } \ u_1+u_3\le 0, v=t(1,0,1), t\le 0\}.\end{aligned}$$

Hence the only multipliers $(\lambda^G,\lambda^H)$ satisfying the Lagrangian multiplier system is $\lambda^G=(-1,0,1)$ and $\lambda^H=(-1,0,-1)$. Note that

$$C(x^*) \;=\; \{d\in\mathbb{R}^4|(d_1,d_3-d_1,d_1-d_2,-d_2,d_1,d_1-d_4)\in T_\Omega(G(x^*),H(x^*)),d_1\le d_4\}$$

$$= \quad \{d = (t, 0, t, t) | t \geq 0\},$$

where the second equality follows from (40).

Since $\nabla G(x^*)d = (t, 0, t), \nabla H(x^*)d = (0, t, 0)$ for any $d = (t, 0, t, t)$ with $t \geq 0$ in $C(x^*)$, by Proposition 5.3 we obtain

$$\sigma\big((\lambda^G, \lambda^H) | T^2_\Omega(G(x^*), H(x^*); \nabla G(x^*)d, \nabla H(x^*)d)\big) = -t^2 = -d_1^2.$$

Since $\nabla^2_{xx} L(x, \lambda) = \nabla^2 f(x) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we have

$$\nabla^2_{xx} L(x^*, \lambda)(d, d) = 0, \quad \forall d \in C(x^*), \tag{41}$$

and by Theorem 5.1

$$\begin{aligned} \Upsilon(x^*, \lambda)(d) \quad &:= \quad \nabla^2_{xx} L(x^*, \lambda)(d, d) - \sigma\big((\lambda^G, \lambda^H) | T^2_\Omega(G(x^*), H(x^*); \nabla G(x^*)d, \nabla H(x^*)d)\big) \\ &= \quad d_1^2 \geq 0, \quad \forall d \in C(x^*). \end{aligned} \tag{42}$$

(41) and (42) indicate that $\nabla^2_{xx} L(x^*, \lambda)$ is positive semidefinite over $C(x^*)$ while $\Upsilon(x^*, \lambda)$ is positive definite over $C(x^*)\backslash\{0\}$. In this example, the second-order necessary conditions involving the second-order tangent set (42) is stronger than the one not involving the second-order tangent set (41).


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