# ERROR BOUNDS FOR SYMMETRIC CONE COMPLEMENTARITY PROBLEMS

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ABSTRACT. In this paper, we investigate the issue of error bounds for symmetric cone complementarity problems (SCCPs). In particular, we show that the distance between an arbitrary point in Euclidean Jordan algebra and the solution set of the symmetric cone complementarity problem can be bounded above by some merit functions such as Fischer-Burmeister merit function, the natural residual function and the implicit Lagrangian function. The so-called $R_0$-type conditions, which are new and weaker than existing ones in the literature, are assumed to guarantee that such merit functions can provide local and global error bounds for SCCPs. Moreover, when SCCPs reduce to linear cases, we demonstrate such merit functions cannot serve as global error bounds under general monotone condition, which implicitly indicates that the proposed $R_0$-type conditions cannot be replaced by $P$-type conditions which include monotone condition as special cases.

1. **Introduction.** The symmetric cone complementarity problem (henceforth SCCP) is to find a vector $x \in \mathbb{V}$ such that

$$x \in \mathcal{K}, \quad F(x) \in \mathcal{K} \quad \text{and} \quad \langle x, F(x) \rangle = 0, \qquad (1)$$

where $\mathbb{V}$ is a Euclidean Jordan algebra, $\mathcal{K} \subset \mathbb{V}$ is a symmetric cone (see Section 2 for details), $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean inner product and $F$ is a continuous mapping from $\mathbb{V}$ into itself. When $F$ reduces to a linear transformation $L$, i.e., $F(x) = L(x) + q$ with $q \in \mathbb{V}$, the above symmetric cone complementarity problem becomes

$$x \in \mathcal{K}, \quad L(x) + q \in \mathcal{K} \quad \text{and} \quad \langle x, L(x) + q \rangle = 0,$$

which is called a symmetric cone linear complementarity problem and denoted by SCLCP.

In this paper, we focus on the issue of error bounds for symmetric cone complementarity problems. More specifically, we want to know, under what conditions,

the distance between an arbitrary point $x \in \mathbb{V}$ and the solution set of SCCPs can be bounded above by a merit function. Recall that a function $\psi : \mathbb{V} \to \mathbb{R}$ is called a *merit function* for SCCPs if $\psi(x) \geq 0$ for all $x$ and $\psi(x) = 0 \Leftrightarrow x$ solves SCCPs. Error bounds for complementarity problems have received increasing attention in the recent literature because they play important roles in sensitivity analysis where the problem data is subject to perturbation, and convergence analysis of some well-known iterative algorithms [7, 10, 11, 13] for solving the complementarity problems, see [16, 20]. Usually, finding global error bound through the following merit function

$$\psi_{\mathrm{NR}}(x) := \|\phi_{\mathrm{NR}}(x, F(x))\|^2 \quad \text{where} \quad \phi_{\mathrm{NR}}(x, y) = x - (x - y)_+ \quad \forall x, y \in \mathbb{V} \quad (2)$$

is a popular way because it is easier to compute, where $\phi_{\mathrm{NR}}$ is the natural residual complementarity function and $z_+$ is the the metric projection of $z \in \mathbb{V}$ onto the symmetric cone $\mathcal{K}$. There are other merit functions which can provide global error bounds such as Fischer-Burmeister merit function and the implicit Lagrangian function. In particular, for symmetric cone complementarity problems, such merit functions are defined as follows.

$$\psi_{\mathrm{FB}}(x) := \frac{1}{2} \|\phi_{\mathrm{FB}}(x, F(x))\|^2 \quad \text{where} \quad \phi_{\mathrm{FB}}(x, y) = (x + y) - (x^2 + y^2)^{\frac{1}{2}} \quad \forall x, y \in \mathbb{V} \tag{3}$$

and

$$\begin{aligned} \psi_{\mathrm{MS}}(x) \quad := \quad & 2\alpha \langle x, F(x) \rangle + \{ \|(-\alpha F(x) + x)_+\|^2 \\ & -\|x\|^2 + \|(-\alpha x + F(x))_+\|^2 - \|F(x)\|^2 \}. \end{aligned} \tag{4}$$

Here $x^2 = x \circ x$ denotes the Jordan product of $x$ and $x$, $x^{\frac{1}{2}}$ is the unique element that satisfies $(x^{\frac{1}{2}})^2 = x$, $\| \cdot \|$ denotes the standard Euclidean norm, and $\alpha > 0$ is a penalty parameter.

Issues regarding error bounds have been studied for classical linear or nonlinear complementarity problems. For linear complementarity problems (LCPs), $\psi_{\mathrm{NR}}(x)$, $\psi_{\mathrm{FB}}(x)$ and $\psi_{\mathrm{MS}}(x)$ are shown to be local error bounds for any LCPs [15, 16, 25], whereas $\psi_{\mathrm{NR}}(x)$ and $\psi_{\mathrm{MS}}(x)$ are shown to be global error bounds for LCPs under the condition of $R_0$-matrix [15, 19] or $P$-matrix [18]. In addition, Chen and Xiang [5] give a computable error bound for the $P$-matrix LCPs. In general, in order to obtain global error bounds for nonlinear complementarity problems (NCPs), $\psi_{\mathrm{NR}}(x)$ needs to satisfy some stronger conditions such as $F$ being a uniform $P$-function and Lipschitz continuity, or $F$ being a strongly monotone [4, 12]. Furthermore, the so-called $R_0$-type conditions for NCPs are investigated by Chen in [2].

It is known that symmetric cone complementarity problems provide a unified framework for nonlinear complementarity problems (NCPs), semidefinite complementarity problems (SDCPs) and second-order cone complementarity problems (SOCCPs). Along this line, there is some research work on error bounds for SCCPs. For instance, Chen [3] gives some conditions towards error bounds and bounded level sets for SOCCPs; Pan and Chen [22] consider error bound and bounded level sets of a one-parametric class of merit functions for SCCPs; Kong, Tuncel and Xiu [14] study error bounds of the implicit Lagrangian $\psi_{\mathrm{MS}}(x)$ for SCCPs. In general, one needs conditions such as $F$ has the uniform Cartesian $P$-property and is Lipschitz continuous. Besides, Liu, Zhang and Wang [17] study error bounds of a class of merit functions for SCCPs, where the transformation $F$ needs to be uniform $P^*$-property which is a more stringent condition. In this paper, motivated by [2],

we consider other conditions, which are indeed different from the aforementioned ones and called $R_0$-type conditions, to find error bounds for SCCPs.

The paper is organized as follows. In section 2, some preliminaries on Euclidean Jordan algebra associated with symmetric cone are introduced. Moreover, we define a class of $R_0$-type functions in Euclidean Jordan algebra $\mathbb{V}$. In section 3, we show the same growth of Fischer-Burmeister merit function $\psi_{\mathrm{FB}}(x)$, the natural residual function $\psi_{\mathrm{NR}}(x)$ and the implicit Lagrangian function $\psi_{\mathrm{MS}}(x)$. In sections 4 and 5, we provide local and global error bounds for SCCPs or SCLCPs with $R_0$-type conditions, respectively. Concluding remarks are given in section 6.

Throughout this paper, let $\mathbb{R}$ denote the space of real numbers. For an $x \in \mathbb{V}$, $(\cdot)_-$ be defined by $x_- := x_+ - x$. In fact, $x_-$ is the metric projection of $-x$ onto the symmetric cone $\mathcal{K}$ (see [26]). In this paper, we need the concept of $BD$-regular function [21]. For a locally Lipsctizian function $H : \mathbb{V} \to \mathbb{V}$, the set

$$\partial_B H(x) = \{\lim \nabla H(x_k): \ x_k \to x, x_k \in D_H\}$$

is called the $B$-subdifferential of $H$ at $x$, where $D_H$ denotes the set of points where $H$ is $F$-differentiable. The function $H$ is said to be $BD$-regular at $x$ if all the elements in $\partial_B H(x)$ are nonsingular. In addition, $S$ denotes the solution set of SCCPs and we assume that $S \neq \emptyset$.

2. **Preliminaries.** In this section, we briefly review some basic concepts and background materials on Euclidean Jordan algebra, which is a basic tool extensively used in the subsequent analysis. More details can be found in [6, 26].

A triple $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ ($\mathbb{V}$ for short) is called a *Euclidean Jordan algebra* where $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over $\mathbb{R}$ and $(x, y) \mapsto x \circ y :$ $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying

  **(i):** $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$
  **(ii):** $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$
  **(iii):** $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathbb{V}$

where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of $x$ and $y$. If a Jordan product only satisfies the conditions (i) and (ii) in the definition of Euclidean Jordan algebra $\mathbb{V}$, the algebra $\mathbb{V}$ is said to be a *Jordan algebra*. Throughout the paper we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element $e$ and with the property $x \circ e = x$ for all $x \in \mathbb{V}$. In a given Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* [6, Theorem III.2.1]. This means that $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \mathrm{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. Consider the Euclidean Jordan algebra $\mathbb{V}$ and the convex cone $\mathcal{K} \subset \mathbb{V}$. This $\mathcal{K}$ induces a partial order on $\mathbb{V}$, i.e., for any $x \in \mathbb{V}$,

$$x \in \mathcal{K} \ (x \in \mathrm{int}(\mathcal{K})) \iff x \succeq 0 \ (x \succ 0).$$

An element $c \in \mathbb{V}$ such that $c^2 = c$ is called an *idempotent* in $\mathbb{V}$; it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. We say that a finite set $\{e_1, e_2, \cdots, e_r\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e_i \circ e_j = 0 \ \text{ for } \ i \neq j, \ \text{ and } \ \sum_{i=1}^{r} e_i = e,$$

where $r$ is called the rank of $\mathbb{V}$. Now, we recall the spectral decomposition and Peirce decomposition of an element $x$ in $\mathbb{V}$.

**Theorem 2.1.** (The Spectral Decomposition Theorem) [6, Theorem III.1.2] *Let* $\mathbb{V}$ *be a Euclidean Jordan algebra. Then there is a number $r$ such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e_1, e_2, \cdots, e_r\}$ and real numbers $\lambda_1, \lambda_2, \cdots, \lambda_r$ with*

$$x = \lambda_1 e_1 + \cdots + \lambda_r e_r.$$

*Here, the numbers $\lambda_i$ ($i = 1, \cdots, r$) are the eigenvalues of $x$ and the expression $\lambda_1 e_1 + \cdots + \lambda_r e_r$ is the spectral decomposition (or the spectral expansion) of $x$.*

In a Euclidean Jordan algebra $\mathbb{V}$, let $\| \cdot \|$ be the norm induced by inner product $\|x\| := \sqrt{\langle x, x \rangle}$ for any $x \in \mathbb{V}$. Corresponding to the closed convex cone $\mathcal{K}$, let $\Pi_{\mathcal{K}}$ denote the *metric projection* onto $\mathcal{K}$, that is, for an $x \in \mathbb{V}$, $x^* = \Pi_{\mathcal{K}}(x)$ if and only if $x^* \in \mathcal{K}$ and $\|x - x^*\| \leq \|x - y\|$ for all $y \in \mathcal{K}$. It is well known that $x^*$ is unique. For any $x \in \mathbb{V}$, let $x_+$ denote the metric projection $\Pi_{\mathcal{K}}(x)$ of $x$ onto $\mathcal{K}$ in this paper. Combining the spectral decomposition of $x$ with the metric projection of $x$ onto $\mathcal{K}$, we have the expression of metric projection $x_+$ as follows [9]:

$$x_+ = \Pi_{\mathcal{K}}(x) = \max\{0, \lambda_1\}e_1 + \cdots + \max\{0, \lambda_r\}e_r,$$

and

$$x_- = \Pi_{\mathcal{K}}(x) = \max\{0, -\lambda_1\}e_1 + \cdots + \max\{0, -\lambda_r\}e_r.$$

Further, we have $x = x_+ - x_-$, $\langle x_+, x_- \rangle = 0$ and $x_+ \circ x_- = 0$. Corresponding to each $x \in \mathbb{V}$, let $\lambda_i(x)$ ($i = 1, 2, \cdots, r$) denote the eigenvalues of $x$. In the sequel, we write

$$\omega(x) := \max_{1 \leq i \leq r} \lambda_i(x) \quad \text{and} \quad \nu(x) := \min_{1 \leq i \leq r} \lambda_i(x).$$

With these notations, we note that

$$-x \in \mathcal{K} \iff \omega(x) \leq 0 \quad \text{and} \quad x \in \mathcal{K} \iff \nu(x) \geq 0.$$

We want to point out that different elements $x, y$ have their own Jordan frames in spectral decomposition, which are not easy to handle when we need to do operations for $x$ and $y$. Thus, we need another so-called Peirce decomposition to conquer such difficulty. In other words, in Peirce decomposition, different elements $x, y$ share the same Jordan frame. We elaborate them more as below.

**The Peirce decomposition:** Fix a Jordan frame $\{e_1, e_2, \cdots, e_r\}$ in a Euclidean Jordan algebra $\mathbb{V}$. For $i, j \in \{1, 2, \cdots, r\}$, we define the following eigenspaces

$$\mathbb{V}_{ii} := \{x \in \mathbb{V} \mid x \circ e_i = x\} = \mathbb{R}e_i$$

and

$$\mathbb{V}_{ij} := \left\{x \in \mathbb{V} \mid x \circ e_i = \frac{1}{2}x = x \circ e_j\right\} \quad \text{for} \ i \neq j.$$

**Theorem 2.2.** [6, Theorem IV.2.1] *The space $\mathbb{V}$ is the orthogonal direct sum of spaces $\mathbb{V}_{ij}(i \leq j)$. Furthermore,*

$$\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subset \mathbb{V}_{ii} + \mathbb{V}_{jj},$$
$$\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subset \mathbb{V}_{ik}, \quad \text{if} \ i \neq k,$$
$$\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}, \quad \text{if} \ \{i, j\} \cap \{k, l\} = \emptyset.$$

*Hence, given any Jordan frame $\{e_1, e_2, \cdots, e_r\}$, we can write any element $x \in \mathbb{V}$ as*

$$x = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij},$$

where $x_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$. The expression $\sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$ is called the Peirce decomposition of $x$.

Given a Euclidean Jordan algebra $\mathbb{V}$ with $dim(\mathbb{V}) = n > 1$, from Proposition III 4.4-4.5 and Theorem V.3.7 in [6], we know that any Euclidean Jordan algebra $\mathbb{V}$ and its corresponding symmetric cone $\mathcal{K}$ are, in a unique way, a direct sum of simple Euclidean Jordan algebras and the constituent symmetric cones therein, respectively, i.e.,

$$\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_m \quad \text{and} \quad \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_m,$$

where every $\mathbb{V}_i$ is a simple Euclidean Jordan algebra (that cannot be a direct sum of two Euclidean Jordan algebras) with the corresponding symmetric cone $\mathcal{K}_i$ for $i = 1, \cdots, m$, and $n = \sum_{i=1}^{m} n_i$ ($n_i$ is the dimension of $\mathbb{V}_i$). Therefore, for any $x = (x_1, \cdots, x_m)^T$ and $y = (y_1, \cdots, y_m)^T \in \mathbb{V}$ with $x_i, y_i \in \mathbb{V}_i$, we have

$$x \circ y = (x_1 \circ y_1, \cdots, x_m \circ y_m)^T \in \mathbb{V} \quad \text{and} \quad \langle x, y \rangle = \langle x_1, y_1 \rangle + \cdots + \langle x_m, y_m \rangle.$$

We end this section with some concepts on $R_0$-type functions, which are crucial to establishing global error bounds. First, for any $x \in \mathbb{V}$, let $\lambda_i(x)(i = 1, \cdots, r)$ denote the eigenvalues of $x$ and

$$\omega(x) := \max_{1 \le i \le r} \lambda_i(x).$$

**Definition 2.3.** A function $F : \mathbb{V} \to \mathbb{V}$ is called

**(a):** an $R_0^s$-function if for any sequence $\{x_k\}$ that satisfies

$$\|x_k\| \to \infty, \quad \frac{(-x_k)_+}{\|x_k\|} \to 0, \quad \frac{(-F(x_k))_+}{\|x_k\|} \to 0,$$

we have

$$\liminf_{k \to \infty} \frac{\omega(\phi_{\mathrm{NR}}(x_k, F(x_k)))}{\|x_k\|} > 0;$$

**(b):** an $R_{01}^s$-function if for any sequence $\{x_k\}$ that satisfies

$$\|x_k\| \to \infty, \quad \frac{(-x_k)_+}{\|x_k\|} \to 0, \quad \frac{(-F(x_k))_+}{\|x_k\|} \to 0,$$

we have

$$\liminf_{k \to \infty} \frac{\langle x_k, F(x_k) \rangle}{\|x_k\|} > 0;$$

**(c):** an $R_{02}^s$-function if for any sequence $\{x_k\}$ that satisfies

$$\|x_k\| \to \infty, \quad \frac{(-x_k)_+}{\|x_k\|} \to 0, \quad \frac{(-F(x_k))_+}{\|x_k\|} \to 0,$$

we have

$$\liminf_{k \to \infty} \frac{\omega(x_k \circ F(x_k))}{\|x_k\|} > 0.$$

From the property $\langle x, y \rangle \le \omega(x \circ y)\|e\|^2$ (see [26, Proposition 2.1(ii)]) and the above concepts, it is not hard to see that $R_{01}^s \Rightarrow R_{02}^s$. In addition, by applying the Peirce Decomposition Theorem, the following lemma shows another implication $R_0^s \Rightarrow R_{02}^s$.

**Lemma 2.4.** *If the function $F : \mathbb{V} \to \mathbb{V}$ is an $R_0^s$-function, then $F$ is an $R_{02}^s$-function.*

*Proof.* For the sake of simplicity, for any $x, y \in \mathbb{V}$, we let

$$x \sqcap y := x - (x - y)_+, \qquad x \sqcup y := y + (x - y)_+.$$

It is easy to verify that $x \sqcup y := y + (x - y)_+ = x + (y - x)_+$. Moreover, these are commutative operations with

$$(x \sqcap y) \circ (x \sqcup y) = x \circ y, \qquad x \sqcap y + x \sqcup y = x + y$$

and

$$x \sqcup y - x \sqcap y = |y - x| \in \mathcal{K}.$$

If we consider the element $x \sqcap y = x - (x - y)_+ \in \mathbb{V}$ and apply Spectral decomposition (Theorem 2.1), there exist a Jordan frame $\{e_1, e_2, \cdots, e_r\}$ and real numbers $\lambda_1, \lambda_2, \cdots, \lambda_r$ such that

$$x \sqcap y = \lambda_1 e_1 + \cdots + \lambda_r e_r.$$

On the other hand, considering the element $x \sqcup y = x + (y - x)_+ \in \mathbb{V}$ and applying Peirce decomposition (Theorem 2.2), we know

$$x \sqcup y = \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij}$$

with $x_i \in \mathbb{R}$ and $x_{ij} \in \mathbb{V}_{ij}$. Without loss of generality, let $\lambda_1 = \omega(x \sqcap y)$. To proceed the arguments, we first establish an inequality:

$$x_1 \geq \lambda_1.$$

Note that

$$(x \sqcup y - x \sqcap y) = \sum_{i=1}^{r} (x_i - \lambda_i) e_i + \sum_{i<j} x_{ij} \in \mathcal{K}.$$

Thus, it follows that

$$\langle x \sqcup y - x \sqcap y, e_1 \rangle = (x_1 - \lambda_1) \|e_1\|^2 \geq 0,$$

which yields $x_1 \geq \lambda_1$. Now suppose $R_0^s$ condition holds. Take a sequence $\{x_k\}$ satisfying the required condition in Definition 2.3 (c), i.e.,

$$\|x_k\| \to \infty, \quad \frac{(-x_k)_+}{\|x_k\|} \to 0, \quad \frac{(-y_k)_+}{\|x_k\|} \to 0,$$

where $y_k := F(x_k)$. From $R_0^s$ condition, we have

$$\liminf_{k \to \infty} \frac{\omega(x_k \sqcap y_k)}{\|x_k\|} = \liminf_{k \to \infty} \frac{\lambda_1}{\|x_k\|} > 0 \quad \text{and} \quad \lambda_1 > 0. \tag{5}$$

For the element $x_k \circ y_k \in \mathbb{V}$, applying Spectral decomposition (Theorem 2.1) again, there exist a Jordan frame $\{f_1, f_2, \cdots, f_r\}$ and real numbers $\mu_1, \mu_2, \cdots, \mu_r$ with $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r$ such that

$$x_k \circ y_k = \mu_1 f_1 + \cdots + \mu_r f_r.$$

Then, we have $\omega(x_k \circ y_k) = \mu_1$. On the other hand,

$$\begin{aligned}
x_k \circ y_k &= (x_k \sqcap y_k) \circ (x_k \sqcup y_k) \\
&= (\lambda_1 e_1 + \cdots + \lambda_r e_r) \circ \left( \sum_{i=1}^{r} x_i e_i + \sum_{i<j} x_{ij} \right) \\
&= \sum_{i=1}^{r} \lambda_i x_i e_i + \sum_{i=1}^{r} \lambda_i e_i \circ \left( \sum_{i<j} x_{ij} \right)
\end{aligned}$$

$$= \sum_{i=1}^{r} \lambda_i x_i e_i + \sum_{i=1}^{r} \frac{\lambda_i}{2} \sum_{i<j} x_{ij}.$$

Hence,

$$\begin{aligned}
\lambda_1 x_1 \langle e_1, e_1 \rangle &= \langle x_k \circ y_k, e_1 \rangle \\
&= \mu_1 \langle f_1, e_1 \rangle + \mu_2 \langle f_2, e_1 \rangle + \cdots + \mu_r \langle f_r, e_1 \rangle \\
&\leq \mu_1 \langle f_1, e_1 \rangle + \mu_1 \langle f_2, e_1 \rangle + \cdots + \mu_1 \langle f_r, e_1 \rangle \\
&\leq r\mu_1 \theta,
\end{aligned}$$

where $\theta = \max\{\langle f_1, e_1 \rangle, \cdots, \langle f_r, e_1 \rangle\}$. This leads to

$$\frac{\mu_1}{\|x_k\|} \geq \frac{\lambda_1 x_1 \langle e_1, e_1 \rangle}{r\theta \|x_k\|},$$

which combining with the formula (5) implies that

$$\liminf_{k \to \infty} \frac{\omega(x_k \circ y_k)}{\|x_k\|} = \liminf_{k \to \infty} \frac{\mu_1}{\|x_k\|} \geq \liminf_{k \to \infty} \frac{\lambda_1 x_1 \langle e_1, e_1 \rangle}{r\theta \|x_k\|} > 0,$$

where the second inequality holds due to $x_1 \geq \lambda_1 > 0$ and $\frac{\langle e_1, e_1 \rangle}{r\theta} > 0$. Therefore, the implication $R_0^s \Rightarrow R_{02}^s$ holds. □

Next, we introduce the so-called weak $R_0$-type functions, which will be used to establish bounded level sets for SCCPs.

**Definition 2.5.** A function $F : \mathbb{V} \to \mathbb{V}$ is called an $R_0^w$-function if for any sequence $\{x_k\}$ that satisfies

$$\|x_k\| \to \infty, \quad \limsup_{k \to \infty} \omega((-x_k)_+) < \infty, \quad \limsup_{k \to \infty} \omega((-F(x_k))_+) < \infty,$$

we have

$$\omega(x_k \sqcap F(x_k)) \to \infty.$$

When the mapping $F$ is a linear mapping, that is, $F(x) = L(x) + q$ for $q \in \mathbb{V}$, $R_0^s$-function and $R_0^w$-function are equivalent to $R_0$-property (or $R_0$-matrix) of $L$ (i.e., the SCLCP with $q = 0$ has a unique zero solution). Those proofs are similar to proofs for [2, Proposition 2.2]. Hence, we omit them. Moreover, by the definition 2.3 and 2.5, we have the following relation between $R_0^s$ and $R_0^w$.

**Theorem 2.6.** *For the function $F : \mathbb{V} \to \mathbb{V}$, we have*

$$R_0^s \Longrightarrow R_0^w.$$

*Proof.* Suppose $R_0^s$ condition holds. Take a sequence $\{x_k\}$ satisfying the required condition in Definition 2.5, i.e.,

$$\|x_k\| \to \infty, \quad \limsup_{k \to \infty} \omega((-x_k)_+) < \infty, \quad \limsup_{k \to \infty} \omega((-F(x_k))_+) < \infty.$$

It follows that

$$\|x_k\| \to \infty, \quad \frac{(-x_k)_+}{\|x_k\|} \to 0, \quad \frac{(-y_k)_+}{\|x_k\|} \to 0.$$

By the definition of $R_0^s$, we have

$$\liminf_{k \to \infty} \frac{\omega(x_k \sqcap y_k)}{\|x_k\|} > 0.$$

Combining with $\|x_k\| \to \infty$ implies that

$$\omega(x_k \sqcap y_k) \to \infty.$$

Therefore, the implication $R_0^s \Rightarrow R_0^w$ holds. $\qquad\qquad\qquad\qquad$ $\square$

3. **Growth behavior of $\psi_{\text{FB}}$, $\psi_{\text{NR}}$ and $\psi_{\text{MS}}$.** In this section, we show results that indicate the same growth of merit functions $\psi_{\text{FB}}$, $\psi_{\text{NR}}$ and $\psi_{\text{MS}}$. The following theorem tells us that $\psi_{\text{NR}}$ and $\psi_{\text{MS}}$ have the same growth behavior.

**Theorem 3.1.** *Let $\psi_{\text{NR}}$ and $\psi_{\text{MS}}$ be defined as in (2) and (4), respectively. For each $\alpha > 1$, the following holds*

$$2(\alpha - 1)\psi_{\text{NR}}(x) \le \psi_{\text{MS}}(x) \le 2\alpha(\alpha - 1)\psi_{\text{NR}}(x) \quad \forall x \in \mathbb{V}. \tag{6}$$

*Proof.* To begin the proof, we denote

$$f(x, \alpha) := -\langle \alpha F(x), (x - \alpha F(x))_+ - x \rangle - \frac{1}{2} \left\| (x - \alpha F(x))_+ - x \right\|^2.$$

We want to point out that there is another expression for $f(x, \alpha)$ as given below, see [8, Thm 3.1].

$$
\begin{aligned}
f(x, \alpha) &= \max_{y \in \mathcal{K}} - \left\langle \alpha F(x) + \frac{1}{2}(y - x), y - x \right\rangle \\
&= - \left\langle \alpha F(x) + \frac{1}{2}((x - \alpha F(x))_+ - x), (x - \alpha F(x))_+ - x \right\rangle \\
&\ge - \left\langle \alpha F(x) + \frac{1}{2}((x - F(x))_+ - x), (x - F(x))_+ - x \right\rangle.
\end{aligned} \tag{7}
$$

Now, we compute

$$
\begin{aligned}
\frac{1}{\alpha} f(x, \alpha) &= -\langle F(x), (x - \alpha F(x))_+ - x \rangle - \frac{1}{2\alpha} \left\| (x - \alpha F(x))_+ - x \right\|^2 \\
&= \langle x, F(x) \rangle + \frac{1}{\alpha} \langle x - \alpha F(x), (x - \alpha F(x))_+ \rangle \\
&\quad - \frac{1}{2\alpha} \left\| (x - \alpha F(x))_+ \right\|^2 - \frac{1}{2\alpha} \|x\|^2 \\
&= \langle x, F(x) \rangle + \frac{1}{2\alpha} \left( \left\| (x - \alpha F(x))_+ \right\|^2 - \|x\|^2 \right).
\end{aligned}
$$

Likewise,

$$f(x, 1) = - \left\langle F(x) + \frac{1}{2}((x - F(x))_+ - x), (x - F(x))_+ - x \right\rangle$$

and

$$\alpha f(x, \frac{1}{\alpha}) = -\frac{1}{2\alpha} \left( \left\| (-\alpha x + F(x))_+ \right\|^2 - \|F(x)\|^2 \right).$$

Combining the above two equations, we obtain an identity for $\psi_{\text{MS}}(x)$

$$\psi_{\text{MS}}(x) = 2\alpha \left( \frac{1}{\alpha} f(x, \alpha) - \alpha f(x, \frac{1}{\alpha}) \right). \tag{8}$$

To show the desired two inequalities, we proceed by two steps. The first step is to verify the left-hand side of (6). To see this,

$$
\begin{aligned}
\psi_{\text{MS}}(x) &= 2\alpha \left( \frac{1}{\alpha} f(x, \alpha) - \alpha f(x, \frac{1}{\alpha}) \right) \\
&= 2\alpha \left( \frac{1}{\alpha} f(x, \alpha) - f(x, 1) \right) + 2\alpha \left( f(x, 1) - \alpha f(x, \frac{1}{\alpha}) \right)
\end{aligned}
$$

$$\geq\ 2\alpha\bigg[-\langle F(x),(x-F(x))_+ -x\rangle - \frac{1}{2\alpha}\|(x-F(x))_+ -x\|^2$$

$$+\langle F(x),(x-F(x))_+ -x\rangle + \frac{1}{2}\|(x-F(x))_+ -x\|^2\bigg]$$

$$+2\alpha\left(f(x,1)-\alpha f(x,\frac{1}{\alpha})\right)$$

$$=\ 2\alpha\frac{\alpha-1}{2\alpha}\psi_{\mathrm{NR}}(x)+2\alpha\left(f(x,1)-\alpha f(x,\frac{1}{\alpha})\right)$$

$$=\ (\alpha-1)\psi_{\mathrm{NR}}(x)+2\alpha\bigg[-\langle F(x),(x-F(x))_+ -x\rangle$$

$$-\frac{1}{2}\|(x-F(x))_+ -x\|^2 + \left\langle F(x),(x-\frac{1}{\alpha}F(x))_+ -x\right\rangle$$

$$+\frac{\alpha}{2}\|(x-\frac{1}{\alpha}F(x))_+ -x\|^2\bigg]$$

$$\geq\ (\alpha-1)\psi_{\mathrm{NR}}(x)+2\alpha\frac{\alpha-1}{2\alpha}\psi_{\mathrm{NR}}(x)$$

$$=\ 2(\alpha-1)\psi_{\mathrm{NR}}(x),$$

where the first inequality follows from (7). Next, we verify the right-hand side of (6). To this end, we observe two things:

$$\frac{1}{\alpha}f(x,\alpha)-f(x,1)$$

$$=\ -\langle F(x),(x-\alpha F(x))_+ -x\rangle - \frac{1}{2\alpha}\|(x-\alpha F(x))_+ -x\|^2$$

$$+\langle F(x),(x-F(x))_+ -x\rangle + \frac{1}{2}\|(x-F(x))_+ -x\|^2$$

$$=\ \frac{\alpha-1}{2\alpha}\psi_{\mathrm{NR}}(x)+\frac{1}{2\alpha}\psi_{\mathrm{NR}}(x)-\frac{1}{2\alpha}\|(x-\alpha F(x))_+ -x\|^2$$

$$+\langle F(x),(x-F(x))_+ -(x-\alpha F(x))_+\rangle$$

$$=\ \frac{\alpha-1}{2}\psi_{\mathrm{NR}}(x)-\frac{(\alpha-1)^2}{2\alpha}\psi_{\mathrm{NR}}(x)$$

$$-\frac{1}{2\alpha}\|(x-\alpha F(x))_+ -(x-F(x))_+\|^2 + \frac{1}{\alpha}\|(x-F(x))_+ -x\|^2$$

$$-\frac{1}{\alpha}\langle(x-\alpha F(x))_+ -x,(x-F(x))_+ -x\rangle$$

$$+\langle F(x),(x-F(x))_+ -(x-\alpha F(x))_+\rangle$$

$$=\ \frac{\alpha-1}{2}\psi_{\mathrm{NR}}(x)$$

$$-\frac{1}{2\alpha}\|(\alpha-1)(x-(x-F(x))_+) +(x-\alpha F(x))_+ -(x-F(x))_+\|^2$$

$$-\langle(x-F(x))_+ -x+F(x),(x-\alpha F(x))_+ -(x-F(x))_+\rangle$$

$$\leq\ \frac{\alpha-1}{2}\psi_{\mathrm{NR}}(x)$$

and

$$f(x,1)-\alpha f(x,\frac{1}{\alpha})$$

$$
\begin{aligned}
&= -\langle F(x), (x - F(x))_+ - x \rangle - \frac{1}{2}\|(x - F(x))_+ - x\|^2 \\
&\quad + \langle F(x), (x - \frac{1}{\alpha}F(x))_+ - x \rangle + \frac{\alpha}{2}\left\|(x - \frac{1}{\alpha}F(x))_+ - x\right\|^2 \\
&= \max_{y \in \mathcal{K}} -\langle F(x) + \frac{1}{2}(y - x), y - x \rangle + \alpha \min_{y \in \mathcal{K}}\langle \frac{1}{\alpha}F(x) + \frac{1}{2}(y - x), y - x \rangle \\
&\leq -\left\langle F(x) + \frac{1}{2}((x - F(x))_+ - x), (x - F(x))_+ - x \right\rangle \\
&\quad + \left\langle F(x) + \frac{\alpha}{2}((x - F(x))_+ - x), (x - F(x))_+ - x \right\rangle \\
&= \frac{\alpha - 1}{2}\|(x - F(x))_+ - x\|^2 \\
&= \frac{\alpha - 1}{2}\psi_{\text{NR}}(x).
\end{aligned}
$$

The above two expressions together with the identity (8) yield

$$
\begin{aligned}
\psi_{\text{MS}}(x) &\leq 2\alpha\left(\frac{\alpha - 1}{2}\psi_{\text{NR}}(x) + \frac{\alpha - 1}{2}\psi_{\text{NR}}(x)\right) \\
&= 2\alpha(\alpha - 1)\psi_{\text{NR}}(x).
\end{aligned}
$$

Thus, the proof is complete.                                           □

The same growth of the $\psi_{\text{NR}}$ and $\psi_{\text{FB}}$ is already proved in [1, Proposition 3.1] that we present it as below theorem.

**Theorem 3.2.** *Let* $\psi_{\text{NR}}$ *and* $\psi_{\text{FB}}$ *be defined as in (2) and (3), respectively. Then,*

$$
\left(2 - \sqrt{2}\right)\|\phi_{\text{NR}}(x, y)\| \leq \|\phi_{\text{FB}}(x, y)\| \leq \left(2 + \sqrt{2}\right)\|\phi_{\text{NR}}(x, y)\|
$$

*for any* $x, y \in \mathbb{V}$.

Combining Theorem 3.1 and Theorem 3.2, we can reach the conclusion that $\psi_{\text{FB}}$, $\psi_{\text{NR}}$, and $\psi_{\text{MS}}$ have the same growth.

4. **Local Error Bounds.** This section contains the proofs of boundedness of level set and local error bounds for SCCPs. To obtain such properties, we first present the definition of the local error bound and two lemmas that play important roles in the following analysis.

**Definition 4.1.** For the residual function $r(x) = \|\phi_{\text{NR}}(x, F(x))\|$, the function $r(x)$ is a local error bound if there exist constants $c > 0$ and $\delta > 0$ such that for each $x \in \{x \in \mathbb{V} \mid d(x, S) \leq \delta\}$, there holds

$$
d(x, S) \leq cr(x),
$$

where $S$ denote the solution set of the problem (1) and $d(x, S) = \inf_{y \in S}\|x - y\|$.

**Lemma 4.2.** *Let* $\phi_{\text{FB}}$ *be defined as in (3). Then, for any* $x, y \in \mathbb{V}$,

$$
\|(\phi_{\text{FB}}(x, y))_+\|^2 \geq \frac{1}{2}\left(\|(-x)_+\|^2 + \|(-y)_+\|^2\right).
$$

*Proof.* This is the result of [22, Lemma 5.2].                        □

**Lemma 4.3.** *Let $\phi_{\mathrm{NR}}$ be defined as in (2). Then, for any $x, y \in \mathbb{V}$, there is a constant $\beta > 0$ such that*

$$\|\phi_{\mathrm{NR}}(x,y)\|^2 \geq \frac{\beta}{2} \left( \|(-x)_+\|^2 + \|(-y)_+\|^2 \right).$$

*Proof.* By applying Theorem 3.2 and Lemma 4.2, the desired result is obtained immediately. □

For simplicity, we denote $r(x) := \|\phi_{\mathrm{NR}}(x, F(x))\|$ in the remaining part of this paper and call it a residual function. It is trivial that $r(x) = (\psi_{\mathrm{NR}}(x))^{\frac{1}{2}}$ for any $x \in \mathbb{V}$.

**Theorem 4.4.** *Consider the residual function $r(x) = \|\phi_{\mathrm{NR}}(x, F(x))\|$. If $F$ is an $R_0^w$-function, then the level set $\mathcal{L}(\gamma) := \{x \in \mathbb{V}|\ r(x) \leq \gamma\}$ is bounded for all $\gamma \geq 0$.*

*Proof.* Suppose there is an unbounded sequence $\{x_k\} \subseteq \mathcal{L}(\gamma)$ for some $\gamma > 0$. If $\limsup \omega((-x_k)_+) = \infty$, then (through a subsequence) $\|(-x_k)_+\| \to \infty$, by Lemma 4.3, which implies that $r(x_k) \to \infty$. This contradicts the boundness of $\mathcal{L}(\gamma)$. A similar contradiction ensues if $\limsup \omega((F(-x_k))_+) = \infty$. Thus, for the specified unbounded sequence $\{x_k\}$ satisfing the condition in Definition 2.5, by Definition 2.5, we also obtain that $\omega(\phi_{\mathrm{NR}}(x_k, F(x_k))) \to \infty$. With $r(x_k) = \|\phi_{\mathrm{NR}}(x_k, F(x_k))\|$, it is easy to see that $r(x_k) \to \infty$. This leads to a contradiction. Consequently, the level set $\mathcal{L}(\gamma) := \{x \in \mathbb{V}|\ r(x) \leq \gamma\}$ is bounded for all $\gamma \geq 0$. □

Theorem 4.4 says that $r(x)$ has property of bounded level set under $R_0$-type condition. However, $r(x)$ cannot serve as local error bound under $R_0$-type condition only, even for NCP case which is a special case of SCCPs. An example is given in [2] that illustrates $r(x)$ cannot be a local error bound for an $R_0$-type NCP ($F$ is $R_0$-type function). More specifically, consider $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = x^3$, it is easy to verify that $F$ is an $R_0^s$-function, and the corresponding NCP has a bounded solution set $S = \{0\}$. However, $r(x)$ cannot be a local error bound. A question arises here: Under what additional condition, can $r(x)$ be a local error bound for SCCPs? The following theorem answers this question by providing a sufficient condition for SCCPs.

**Theorem 4.5.** *Consider the residual function $r(x) = \|\phi_{\mathrm{NR}}(x, F(x))\|$. Suppose that the solution set $S$ of SCCPs is nonempty and that $\phi_{\mathrm{NR}}$ is BD-regular at all solutions of SCCPs. Then, $r(x)$ is a local error bound if it has a local bounded level set.*

*Proof.* Since $r(x)$ has a local bounded level set, there exists $\varepsilon > 0$ such that the level set $\mathcal{L}(\varepsilon) = \{x|\ r(x) \leq \varepsilon\}$ is bounded. Thus the set $\mathcal{L}(\varepsilon) = \{x|\ r(x) \leq \varepsilon\}$ is compact. Suppose that the conclusion is wrong. Then, there exists a sequence $\{x_k\} \subset \mathcal{L}(\varepsilon)$ such that

$$\frac{r(x_k)}{\mathrm{dist}(x_k, S)} \to 0 \quad \text{as} \quad k \to \infty.$$

Here $\mathrm{dist}(x_k, S)$ denotes the distance between $x_k$ and $S$. Thus, $r(x_k) \to 0$ and it follows from compactness of $\mathcal{L}(\varepsilon)$ that there is a convergent subsequence. Without loss of generality, let $\{x_k\}$ be a convergent sequence, and $\bar{x}$ be its limit, that is, $x_k \to \bar{x} \in \mathcal{L}(\varepsilon)$. Then, $r(\bar{x}) = 0$, which implies $\bar{x} \in S$. It turns out that

$$\frac{r(x_k)}{\|x_k - \bar{x}\|} \to 0 \quad \text{as} \quad k \to \infty. \tag{9}$$

From [24], we know that $\phi_{\mathrm{NR}}(x, F(x))$ is semismooth. By applying [21, Proposition 3] and $BD$-regular property of $\phi_{\mathrm{NR}}(x, F(x))$, there exist constants $c > 0$ and $\delta > 0$ such that $r(x) \geq c\|x - \bar{x}\|$ for any $x$ with $\|x - \bar{x}\| < \delta$. This contradicts (9). Consequently, the residual function $r(x)$ is a local error bound for SCCPs.          □

Results analogous to Theorem 4.5 can be stated for the other two merit functions, with Theorem 3.1 and 3.2, we may conclude that $\psi_{\mathrm{FB}}$ and $\psi_{\mathrm{MS}}$ are local error bounds for SCCPs.

5. **Global Error Bound.** In this section, we find a global error bound for SCCPs by using an $R_0$-type condition and a $BD$-regular condition. To achieve these results, we present the following definition and a technical lemma.

**Definition 5.1.** For the residual function $r(x) = \|\phi_{\mathrm{NR}}(x, F(x))\|$, the function $r(x)$ is a global error bound if there exist constant $c > 0$ such that for each $x \in \mathbb{V}$,

$$d(x, S) \leq cr(x),$$

where $S$ denote the solution set of the problem (1) and $d(x, S) = \inf_{y \in S} \|x - y\|$.

**Lemma 5.2.** *Let $\{x_k\}$ be any sequence such that $\|x_k\| \to \infty$. If $F$ is an $R_0^s$-function, then*

$$\liminf_{k \to \infty} \frac{r(x_k)}{\|x_k\|} > 0.$$

*Proof.* Suppose that the result is false. There exists a subsequence $x_{n_k}$ with $\|x_{n_k}\| \to \infty$ such that

$$\frac{r(x_{n_k})}{\|x_{n_k}\|} \to 0. \tag{10}$$

From Lemma 4.3, it follows that

$$\frac{(-x_{n_k})_+}{\|x_{n_k}\|} \to 0 \quad \text{and} \quad \frac{(-F(x_{n_k}))_+}{\|x_{n_k}\|} \to 0.$$

This together with the definition of $R_0^s$-function implies

$$\liminf_{k \to \infty} \frac{\omega(\phi_{\mathrm{NR}}(x_{n_k}, F(x_{n_k})))}{\|x_{n_k}\|} > 0,$$

which contradicts the formula (10). Consequently, we have the desired result.          □

**Theorem 5.3.** *Suppose that $F$ is an $R_0^s$-function and that $\phi_{\mathrm{NR}}$ is $BD$-regular at all solutions of SCCPs. Then, there exists a $\kappa > 0$ such that for any $x \in \mathbb{V}$*

$$\mathrm{dist}(x, S) \leq \kappa r(x),$$

*where $S$ is the solution set of SCCPs, $\mathrm{dist}(x, S)$ denotes the distance between $x$ and $S$.*

*Proof.* By the definition of $R_0^s$-function, Theorem 4.4 and Theorem 4.5, we claim that $r(x)$ is a local error bound so there exist $c > 0$ and $\delta > 0$ such that

$$r(x) < \delta \Rightarrow d(x, S) \leq cr(x).$$

Suppose $r(x)$ does not have the global error bound property. Then, there exists $x_k$ such that for any fixed $\bar{x} \in S$,

$$\|x_k - \bar{x}\| \geq \mathrm{dist}(x_k, S) > kr(x_k) \tag{11}$$

for all $k$. Clearly, the inequality $r(x_k) < \delta$ cannot hold for infinitely many $k's$, else $kr(x_k) < d(x_k, S) \leq cr(x_k)$ implies that $k \leq c$ for infinitely many $k's$. Therefore, $r(x_k) \geq \delta$ for all large $k$. Now,

$$\|x_k - \bar{x}\| \geq d(x_k, S) \geq kr(x_k) \geq k\delta$$

for infinitely many $k's$. This implies that $\|x_k\| \to \infty$. Now divide the inequality and take the limit $k \to \infty$, we have

$$1 = \lim_{k \to \infty} \frac{\|x_k - \bar{x}\|}{\|x_k\|} \geq \lim_{k \to \infty} k \frac{r(x_k)}{\|x_k\|} \to \infty,$$

where the last implication holds because $F$ is an $R_0^s$-function and Lemma 5.2. This clearly is a contradiction. $\square$

Adopting Theorem 3.1, Theorem 3.2 and Theorem 5.3, we have the following corollary for SCCPs.

**Corollary 1.** *Under the same conditions as in Theorem 5.3, both the merit function $\psi_{\mathrm{FB}}(x)$ and the implicit Lagrangian function $\psi_{\mathrm{MS}}(x)$ are global error bounds for SCCPs.*

When $F : \mathbb{V} \to \mathbb{V}$ is a linear mapping, that is, $F(x) = L(x) + q$ with $q \in \mathbb{V}$, if $L$ has $R_0$-property, then $r(x)$ being a local error bound can be improved as being a global error bound for SCLCPs, which is shown in the following theorem.

**Theorem 5.4.** *Suppose that $r(x)$ is a local error bound for SCLCPs and the linear transformation $L$ has $R_0$-property. Then, there exists $k > 0$ such that $\mathrm{dist}(x, S) \leq kr(x)$ for every $x \in \mathbb{V}$.*

*Proof.* Suppose that the conclusion is false. Then, for any integer $k > 0$, there exists an $x_k \in \mathbb{R}^n$ such that $\mathrm{dist}(x_k, S) > kr(x_k)$. Let $z(x_k)$ denote the closest solution of SCLCPs to $x_k$. Choosing a fixed solution $x_0 \in S$, we have

$$\|x_k - x_0\| \geq \|x_k - z(x_k)\| \geq \mathrm{dist}(x_k, S) > kr(x_k). \tag{12}$$

Since $r(x)$ is a local error bound, it implies that there exist some integer $K > 0$ and $\delta > 0$ such that for all $k > K$, $r(x_k) > \delta$. If not, then for every integer $K > 0$ and any $\delta > 0$, there exist some $k > K$ such that $r(x_k) \leq \delta$. By property of local error bound of $r(x)$, we have

$$\frac{\delta}{k}\|x_k - z(x_k)\| > \delta r(x_k) \geq \|x_k - z(x_k)\|.$$

Thus, we obtain $\frac{\delta}{k} > 1$. As $k$ goes to infinity, this leads to a contradiction. Consequently, $r(x_k) > \delta$. This together with (12) implies that $\|x_k - x_0\| \geq \|x_k - z(x_k)\| > k\delta$ which says that $\|x_k\| \to \infty$ as $k \to \infty$. Now, we consider the sequence $\{\frac{x_k}{\|x_k\|}\}$. There exist a subsequence $\{x_{k_i}\}$ such that

$$\lim_{i \to \infty} \frac{x_{k_i}}{\|x_{k_i}\|} = x.$$

Hence, it follows from (12) that

$$
\begin{aligned}
1 &= \lim_{i \to \infty} \frac{\|x_{k_i} - x_0\|}{\|x_{k_i}\|} \\
&\geq \lim_{i \to \infty} k_i \frac{r(x_{k_i})}{\|x_{k_i}\|}
\end{aligned}
$$

$$\begin{aligned}
&= \lim_{i \to \infty} k_i \left\| \frac{x_{k_i}}{\|x_{k_i}\|} - \left( \frac{x_{k_i}}{\|x_{k_i}\|} - L \frac{x_{k_i}}{\|x_{k_i}\|} - \frac{q}{\|x_{k_i}\|} \right)_+ \right\| \\
&= \lim_{i \to \infty} k_i \| x - (x - L(x))_+ \|.
\end{aligned}$$

This implies that $\|x - (x - L(x))_+\| = 0$, which shows that $x$ is a nonzero solution of SCLCPs with $q \in \mathbb{V}$. It contradicts the $R_0$-property of $L$. Then, the proof is complete. $\square$

There is one thing worthy of pointing it out. If we replace the condition of $R_0$-property into the monotonicity for the linear transformation $L$, the conclusion of Theorem 5.4 may not hold. This can be illustrated by the following example by using the implicit Lagrangian function $\psi_{\text{MS}}$.

**Example 5.1.** *Let* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined as*

$$L := \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad q := \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

It is easy to prove that the symmetric cone is $\mathbb{R}^2_+$ and the corresponding SCLCP has a unique solution $x^* = (0,0)^T$. Choosing $x_k = \left( \frac{k}{\sqrt{2}}, \frac{k}{\sqrt{2}} \right)^T$, $k \geq 0$ gives $F(x_k) = L(x_k) + q = (2,0)^T$. Then, for any $k > 2\sqrt{2}\alpha$ with $\alpha > 1$, we have

$$\begin{aligned}
\psi_{\text{MS}}(x_k) &= 4\alpha \left( \frac{k}{\sqrt{2}} \right) + \left( -2\alpha + \frac{k}{\sqrt{2}} \right)^2 + \left( \frac{k}{\sqrt{2}} \right)^2 - 2 \left( \frac{k}{\sqrt{2}} \right)^2 - 4 \\
&= 4 \left( \alpha^2 - 1 \right).
\end{aligned}$$

However, $\text{dist}(x_k, S) = \|x_k\| = k$. This implies $\text{dist}(x_k, S) > \psi_{\text{MS}}(x_k)$ as $k \to \infty$, which explains that $\psi_{\text{MS}}(x)$ cannot serve as global error bound for SCLCPs.

6. **Concluding Remarks.** In this paper, we have established some local and global error bounds for symmetric cone complementarity problems under the so-called $R_0$-type conditions. These new results on error bounds are based on the Fischer-Burmeister merit function, the natural residual function, and the implicit Lagrangian function. For symmetric cone linear complementarity problems, we have pointed out that global error bound do not exist under the condition of linear transformation $L$ being monotone, which implicitly indicates that the proposed $R_0$-type conditions cannot be replaced by monotone condition as special cases.

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