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Growth Behavior of Two Classes of Merit Functions for Symmetric Cone Complementarity Problems

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Abstract In the solution methods of the symmetric cone complementarity problem (SCCP), the squared norm of a complementarity function serves naturally as a merit function for the problem itself or the equivalent system of equations reformulation. In this paper, we study the growth behavior of two classes of such merit functions, which are induced by the smooth EP complementarity functions and the smooth implicit Lagrangian complementarity function, respectively. We show that, for the linear symmetric cone complementarity problem (SCLCP), both the EP merit functions and the implicit Lagrangian merit function are coercive if the underlying linear transformation has the P-property; for the general SCCP, the EP merit functions are coercive only if the underlying mapping has the uniform Jordan P-property, whereas the coerciveness of the implicit Lagrangian merit function requires an additional condition for the mapping, for example, the Lipschitz continuity or the assumption as in (45).

Keywords Symmetric cone complementarity problem · Jordan algebra · EP merit functions · Implicit Lagrangian function · Coerciveness

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1 Introduction

Given a Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$, where '$\circ$' denotes the Jordan product and $\mathbb{V}$ is a finite-dimensional vector space over the real field $\mathbb{R}$ equipped with the inner product $\langle\cdot,\cdot\rangle$, let $\mathcal{K}$ be a symmetric cone in $\mathbb{V}$ and let $F:\mathbb{V}\to\mathbb{V}$ be a continuous mapping. The symmetric cone complementarity problem (SCCP) is to find $\zeta\in\mathbb{V}$ such that

$$\zeta\in\mathcal{K},\quad F(\zeta)\in\mathcal{K},\quad \langle\zeta, F(\zeta)\rangle = 0. \tag{1}$$

The model provides a simple unified framework for various existing complementarity problems such as the nonlinear complementarity problem over nonnegative orthant cone (NCP), the second-order cone complementarity problem (SOCCP) and the semidefinite complementarity problem (SDCP), and hence has extensive applications in engineering, economics, management science, and other fields; see [1–4] and references therein. When $F(\zeta) = L(\zeta) + b$, $L:\mathbb{V}\to\mathbb{V}$ being a linear transformation and $b\in\mathbb{V}$, the SCCP becomes the linear complementarity problem over symmetric cones (SCLCP),

$$\zeta\in\mathcal{K},\quad L(\zeta)+b\in\mathcal{K},\quad \langle\zeta, L(\zeta)+b\rangle = 0. \tag{2}$$

Recently, there is much interest in the study of merit functions or complementarity functions associated with symmetric cones and the development of the merit function approach or the smoothing method for solving the SCCP. For example, Liu, Zhang and Wang [5] extended a class of merit functions proposed in [6] to the SCCP, Kong, Tuncel and Xiu [7] studied the extension of the implicit Lagrangian function proposed by Mangasarian and Solodov [8] to symmetric cones; Kong, Sun and Xiu [9] proposed a regularized smoothing method by the natural residual complementarity function associated with symmetric cones; and Huang and Ni [10] developed a smoothing-type algorithm with the regularized CHKS smoothing function over the symmetric cone.

A mapping $\phi:\mathbb{V}\times\mathbb{V}\to\mathbb{V}$ is called a complementarity function associated with the symmetric cone $\mathcal{K}$ if the following equivalence holds:

$$\phi(x,y)=0 \iff x\in\mathcal{K},\ y\in\mathcal{K},\quad \langle x,y\rangle = 0. \tag{3}$$

By Propositions III.4.4–4.5 and Theorem V.3.7 of [11], the Euclidean Jordan algebra $\mathbb{V}$ and the corresponding symmetric cone $\mathcal{K}$ can be written as

$$\mathbb{V} = \mathbb{V}_1\times\mathbb{V}_2\times\cdots\times\mathbb{V}_m \quad\text{and}\quad \mathcal{K} = \mathcal{K}^1\times\mathcal{K}^2\times\cdots\times\mathcal{K}^m, \tag{4}$$

where each $(\mathbb{V}_i, \circ, \langle\cdot,\cdot\rangle)$ is a simple Euclidean Jordan algebra and $\mathcal{K}^i$ is the symmetric cone in $\mathbb{V}_i$. Moreover, for any $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)\in\mathbb{V}$ with $x_i, y_i\in\mathbb{V}_i$,

$$x\circ y = (x_1\circ y_1,\dots,x_m\circ y_m) \quad\text{and}\quad \langle x,y\rangle = \langle x_1,y_1\rangle+\cdots+\langle x_m,y_m\rangle.$$

Therefore, the characterization (3) of complementarity function is equivalent to

$$\phi(x,y)=0 \iff x_i\in\mathcal{K}^i,\ y_i\in\mathcal{K}^i,\quad \langle x_i,y_i\rangle = 0 \quad\text{for all } i=1,2,\dots,m. \tag{5}$$

This means that, if ϕ is a complementarity function associated with the cone $\mathcal{K}$, then $\phi(x,y)$ for any $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_m)\in\mathbb{V}$ with $x_i,y_i\in\mathbb{V}_i$ can be written as

$$\phi(x,y):=\left(\phi^{(1)}(x_1,y_1),\phi^{(2)}(x_2,y_2),\dots,\phi^{(m)}(x_m,y_m)\right),$$

where $\phi^{(i)}:\mathbb{V}_i\times\mathbb{V}_i\to\mathbb{V}_i$ is a complementarity function associated with $\mathcal{K}^i$, i.e.,

$$\phi^{(i)}(x_i,y_i)=0 \quad\Longleftrightarrow\quad x_i\in\mathcal{K}^i,\ y_i\in\mathcal{K}^i,\quad \langle x_i,y_i\rangle=0. \tag{6}$$

Consequently, the SCCP can be reformulated as the following system of equations:

$$\Phi(\zeta):=\phi(\zeta,F(\zeta))=\begin{pmatrix}\phi^{(1)}(\zeta_1,F_1(\zeta))\\ \vdots\\ \phi^{(m)}(\zeta_m,F_m(\zeta))\end{pmatrix}=0,$$

which naturally induces a merit function $f:\mathbb{V}\to\mathbb{R}_+$ for the SCCP, defined as

$$f(\zeta):=(1/2)\|\Phi(\zeta)\|^2=(1/2)\sum_{i=1}^{m}\|\phi^{(i)}(\zeta_i,F_i(\zeta))\|^2.$$

In the rest of this paper, corresponding to the Cartesian structure of $\mathbb{V}$, we always write $F=(F_1,\dots,F_m)$ with $F_i:\mathbb{V}\to\mathbb{V}_i$ and $\zeta=(\zeta_1,\dots,\zeta_m)$ with $\zeta_i\in\mathbb{V}_i$.

The merit function f is often involved in the design of the merit function methods or the equation reformulation methods for the SCCP. For these methods, the coerciveness of f plays a crucial role in establishing the global convergence results. In this paper, we will study the growth behavior of two classes of such merit functions, which respectively correspond to the EP-functions introduced by Evtushenko and Purtov [12] and the implicit Lagrangian function by Mangasarian and Solodov [8]. The EP-functions over the symmetric cone $\mathcal{K}$ were first introduced by Kong and Xiu [13], defined by

$$\phi_\alpha(x,y):=-x\circ y+(1/2\alpha)\left[(x+y)_-\right]^2,\qquad 0<\alpha\le 1, \tag{7}$$

$$\phi_\beta(x,y):=-x\circ y+(1/2\beta)\left[(x_-)^2+(y_-)^2\right],\qquad 0<\beta<1, \tag{8}$$

where $(\cdot)_-$ denotes the minimum metric projection onto $-\mathcal{K}$. They showed that ϕ_α and ϕ_β are continuously differentiable and strongly semismooth complementarity functions associated with $\mathcal{K}$. In addition, Kong, Tuncel and Xiu [7] extended the implicit Lagrangian function to the symmetric cone $\mathcal{K}$ and studied its continuous differentiability and strongly semismoothness. The function is defined as follows:

$$\phi_{\mathrm{MS}}(x,y):=x\circ y+(1/2\alpha)\left\{[(x-\alpha y)_+]^2-x^2+[(y-\alpha x)_+]^2-y^2\right\}, \tag{9}$$

where $\alpha>0$ $(\neq 1)$ is a fixed constant, and $(\cdot)_+$ denotes the minimum metric projection on $\mathcal{K}$. Particularly, for the implicit Lagrangian merit function of the SCCP, they presented a mild stationary point condition and proved that it can provide a global error bound under the uniform Cartesian P-property and Lipschitz continuity of F.

This paper is mainly concerned with the growth behavior of the merit functions induced by the above three types of smooth complementarity functions, that is,

$$f_\alpha(\zeta) := (1/2)\|\phi_\alpha(\zeta, F(\zeta))\|^2 = (1/2)\sum_{i=1}^{m} \|\phi_\alpha^{(i)}(\zeta_i, F_i(\zeta))\|^2, \tag{10}$$

$$f_\beta(\zeta) := (1/2)\|\phi_\beta(\zeta, F(\zeta))\|^2 = (1/2)\sum_{i=1}^{m} \|\phi_\beta^{(i)}(\zeta_i, F_i(\zeta))\|^2, \tag{11}$$

$$f_{\rm MS}(\zeta) := (1/2)\|\phi_{\rm MS}(\zeta, F(\zeta))\|^2 = (1/2)\sum_{i=1}^{m} \|\phi_{\rm MS}^{(i)}(\zeta_i, F_i(\zeta))\|^2, \tag{12}$$

where $\phi_\alpha^{(i)}, \phi_\beta^{(i)}, \phi_{\rm MS}^{(i)}$ defined as in (7), (8), (9), respectively, are a complementarity function associated with $\mathcal{K}^i$. Specifically, we show that for the SCLCP (2), the EP merit functions f_α and f_β and the implicit Lagrangian function $f_{\rm MS}$ are coercive only if the linear transformation L has the P-property; for the general SCCP, f_α and f_β are coercive if the mapping F has the uniform Jordan P-property, but the coerciveness of $f_{\rm MS}$ needs an additional condition of F, for example, the Lipschitz continuity or the assumption as in (45). When $\mathbb{V} = \mathbb{R}^n$ and "$\circ$" denotes the componentwise product of the vectors, the obtaining results precisely reduce to those of Theorems 2.1 and 2.3 in [14] and Theorem 4.1 in [15]. However, for the general Euclidean Jordan algebra even the Lorentz algebra, to the best of our knowledge, similar results have not been established for these merit functions.

Throughout this paper, $\|\cdot\|$ represents the norm induced by the inner product $\langle\cdot,\cdot\rangle$, $\mathrm{int}(\mathcal{K})$ denotes the interior of the symmetric cone $\mathcal{K}$, and $(x_1, \ldots, x_m) \in \mathbb{V}_1 \times \cdots \times \mathbb{V}_m$ is viewed as a column vector in $\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_m$. For any $x \in \mathbb{V}$, $(x)_+$ and $(x)_-$ denotes the metric projection of x onto $\mathcal{K}$ and $-\mathcal{K}$, respectively, i.e., $(x)_+ := \mathrm{argmin}_{y\in\mathcal{K}}\{\|x-y\|\}$.

2 Preliminaries

This section recalls some concepts and materials of Euclidean Jordan algebras that will be used in the subsequent analysis. More detailed expositions of Euclidean Jordan algebras can be found in Koecher's lecture notes [16] and the monograph by Faraut and Korányi [11]. Besides, one can find excellent summaries in [17–19].

A Euclidean Jordan algebra is a triple $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$, where $(\mathbb{V}, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a finite-dimensional inner product space over the real field $\mathbb{R}$ and $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is a bilinear mapping satisfying the following three conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$, where $x^2 := x \circ x$;
(iii) $\langle x \circ y, z\rangle_{\mathbb{V}} = \langle y, x \circ z\rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

We assume that there is an element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$ and call e the unit element. Let $\zeta(x)$ be the degree of the minimal polynomial of $x \in \mathbb{V}$, which can be equivalently defined as $\zeta(x) := \min\{k : \{e, x, x^2, \ldots, x^k\}$ are linearly dependent$\}$. Since $\zeta(x) \le \dim(\mathbb{V})$ where $\dim(\mathbb{V})$ denotes the dimension of $\mathbb{V}$, the rank of $(\mathbb{V}, \circ)$ is well defined by $q := \max\{\zeta(x) : x \in \mathbb{V}\}$. In a Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$, we denote $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ by the set of squares. From Theorem III.2.1 of [11], $\mathcal{K}$ is a symmetric cone. This means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior $\operatorname{int}(\mathcal{K})$, and for any $x, y \in \operatorname{int}(\mathcal{K})$, there exists an invertible linear transformation $\mathcal{T} : \mathbb{V} \to \mathbb{V}$ such that $\mathcal{T}(\mathcal{K}) = \mathcal{K}$.

A Euclidean Jordan algebra is said to be simple if it is not the direct sum of two Euclidean Jordan algebras. By Propositions III.4.4–III.4.5 and Theorem V.3.7 of [11], any Euclidean Jordan algebra is, in a unique way, a direct sum of simple Euclidean Jordan algebras. Moreover, the symmetric cone in a given Euclidean Jordan algebra is, in a unique way, a direct sum of symmetric cones in the constituent simple Euclidean Jordan algebras. Here are two popular examples of simple Euclidean Jordan algebras. One is the algebra $\mathbb{S}^n$ of $n \times n$ real symmetric matrices with the inner product $\langle X, Y\rangle_{\mathbb{S}^n} := \operatorname{Tr}(XY)$ and the Jordan product $X \circ Y := (XY + YX)/2$, where $\operatorname{Tr}(X)$ is the trace of X and XY is the usual matrix multiplication of X and Y. In this case, the unit element is the identity matrix in $\mathbb{S}^n$ and the cone $\mathcal{K}$ is the set of all positive semidefinite matrices. The other is the Lorentz algebra $\mathfrak{L}^n$, also called the quadratic forms algebra, with $\mathbb{V} = \mathbb{R}^n$, $\langle\cdot,\cdot\rangle_{\mathbb{V}}$ being the usual inner product in $\mathbb{R}^n$ and the Jordan product defined by

$$x \circ y := (\langle x, y\rangle_{\mathbb{R}^n}, x_1 y_2 + y_1 x_2)\,, \tag{13}$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Under this case, the unit element $e = (1, 0, \ldots, 0) \in \mathbb{R}^n$, and the associate cone, called the Lorentz cone (or the second-order cone), is given by $\mathcal{K} := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : \|x_2\| \le x_1\}$.

Recall that an element $c \in \mathbb{V}$ is said to be idempotent if $c^2 = c$. Two idempotents c and d are said to be orthogonal if $c \circ d = 0$. We say that $\{c_1, c_2, \ldots, c_k\}$ is a complete system of orthogonal idempotents if

$$c_j^2 = c_j, \qquad c_j \circ c_i = 0 \quad \text{if } j \ne i,\ j, i = 1, 2, \ldots, k, \quad \text{and} \quad \sum_{j=1}^{k} c_j = e.$$

A nonzero idempotent is said to be primitive if it cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a Jordan frame. Then, we have the following spectral decomposition theorem (see Theorem III.1.2 in [11]).

Theorem 2.1 *Suppose that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a Euclidean Jordan algebra with rank q. Then, for each $x \in \mathbb{V}$, there exist a Jordan frame $\{c_1, c_2, \ldots, c_q\}$ and real numbers $\lambda_1(x), \lambda_2(x), \ldots, \lambda_q(x)$ such that $x = \sum_{j=1}^{q} \lambda_j(x) c_j$. The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues of x.*

In the sequel, we denote by $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$ the maximum eigenvalue and the minimum eigenvalue of x respectively and by $\operatorname{tr}(x) := \sum_{j=1}^{q} \lambda_j(x)$ the trace of x.

By Proposition III.1.5 of [11], a Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ)$ over $\mathbb{R}$ with a unit element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\mathrm{tr}(x \circ y)$ is positive definite. Therefore, we may define an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{V}$ by

$$\langle x, y \rangle := \mathrm{tr}(x \circ y), \quad \forall x, y \in \mathbb{V}.$$

Let $\| \cdot \|$ be the norm on $\mathbb{V}$ induced by the inner product $\langle \cdot, \cdot \rangle$, namely,

$$\|x\| := \sqrt{\langle x, x \rangle} = \left(\sum_{j=1}^{q} \lambda_j^2(x) \right)^{1/2}, \quad \forall x \in \mathbb{V}.$$

Then, by the Schwartz inequality, it is easy to verify that

$$\|x \circ y\| \le \|x\| \cdot \|y\|, \quad \forall x, y \in \mathbb{V}. \tag{14}$$

For a given $x \in \mathbb{V}$, we define the linear operator $\mathcal{L} : \mathbb{V} \to \mathbb{V}$ by

$$\mathcal{L}(x)y := x \circ y, \quad \text{for every } y \in \mathbb{V}.$$

Since the inner product $\langle \cdot, \cdot \rangle$ is associative by the associativity of $\mathrm{tr}(\cdot)$ (see Proposition II.4.3 of [11]), i.e., for all $x, y, z \in \mathbb{V}$, it holds that $\langle x, y \circ z \rangle = \langle y, x \circ z \rangle$, the linear operator $\mathcal{L}(x)$ for each $x \in \mathbb{V}$ is symmetric with respect to $\langle \cdot, \cdot \rangle$ in the sense that

$$\langle \mathcal{L}(x)y, z \rangle = \langle y, \mathcal{L}(x)z \rangle, \quad \forall y, z \in \mathbb{V}.$$

We say that elements x and y operator commute if $\mathcal{L}(x)$ and $\mathcal{L}(y)$ commute, i.e.,

$$\mathcal{L}(x)\mathcal{L}(y) = \mathcal{L}(y)\mathcal{L}(x).$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a real-valued function. Then, it is natural to define a vector-valued function associated with the Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$ by

$$\varphi_{\mathbb{V}}(x) := \varphi(\lambda_1(x))c_1 + \varphi(\lambda_2(x))c_2 + \cdots + \varphi(\lambda_q(x))c_q, \tag{15}$$

where $x \in \mathbb{V}$ has the spectral decomposition $x = \sum_{j=1}^{q} \lambda_j(x)c_j$. The function $\varphi_{\mathbb{V}}$ is also called the Löwner operator in [19] and shown to inherit many properties from φ. Especially, when $\varphi(t)$ is chosen as $\max\{0, t\}$ and $\min\{0, t\}$ for $t \in \mathbb{R}$, respectively, $\varphi_{\mathbb{V}}$ becomes the metric projection operator onto $\mathcal{K}$ and $-\mathcal{K}$,

$$(x)_+ := \sum_{j=1}^{q} \max\left\{0, \lambda_j(x)\right\}c_j, \qquad (x)_- := \sum_{j=1}^{q} \min\left\{0, \lambda_j(x)\right\}c_j. \tag{16}$$

It is easy to verify that $x = (x)_+ + (x)_-$, $|x| = (x)_+ - (x)_-$ and $\|x\|^2 = \|(x)_+\|^2 + \|(x)_-\|^2$.

An important part in the theory of Euclidean Jordan algebras is the Peirce decomposition theorem which is stated as follows (see Theorem IV.2.1 of [11]).

Theorem 2.2 *Let $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ be a Euclidean Jordan algebra with rank q and let $\{c_1, c_2, \ldots, c_q\}$ be a Jordan frame in $\mathbb{V}$. For $i, j \in \{1, 2, \ldots, q\}$, define*

$$\mathbb{V}_{ii} := \big\{x \in \mathbb{V} :\ x \circ c_i = x\big\}, \qquad \mathbb{V}_{ij} := \big\{x \in \mathbb{V} :\ x \circ c_i = (1/2)x = x \circ c_j\big\}, \quad i \neq j.$$

Then, the space $\mathbb{V}$ is the orthogonal direct sum of subspaces $\mathbb{V}_{ij}$ $(i \leq j)$. Furthermore,

(a) $\mathbb{V}_{ij} \circ \mathbb{V}_{ij} \subseteq \mathbb{V}_{ii} + \mathbb{V}_{jj}$;
(b) $\mathbb{V}_{ij} \circ \mathbb{V}_{jk} \subseteq \mathbb{V}_{ik}$ *if* $i \neq k$;
(c) $\mathbb{V}_{ij} \circ \mathbb{V}_{kl} = \{0\}$ *if* $\{i, j\} \cap \{k, l\} = \emptyset$.

To close this section, we recall the concepts of the P-property and the uniform Jordan P-property for a linear transformation and a nonlinear mapping.

Definition 2.1 A linear transformation $L : \mathbb{V} \to \mathbb{V}$ is said to have the P-property if

$$\left.\begin{array}{l} \zeta \text{ and } L(\zeta) \text{ operator commute} \\ \zeta \circ L(\zeta) \in -\mathcal{K} \end{array}\right\} \quad \Longrightarrow \quad x = 0.$$

Definition 2.2 A mapping $F = (F_1, \ldots, F_m)$ with $F_i : \mathbb{V} \to \mathbb{V}_i$ is said to have:

(i) the uniform Cartesian P-property if there is a positive scalar ρ such that, for any $\zeta, \xi \in \mathbb{V}$, there is an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$\langle \zeta_\nu - \xi_\nu,\ F_\nu(\zeta) - F_\nu(\xi)\rangle \geq \rho\|\zeta - \xi\|^2,$$

(ii) the uniform Jordan P-property if there is a positive scalar ρ such that, for any $\zeta, \xi \in \mathbb{V}$, there is an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$\lambda_{\max}\left[(\zeta_\nu - \xi_\nu) \circ (F_\nu(\zeta) - F_\nu(\xi))\right] \geq \rho\|\zeta - \xi\|^2.$$

Unless otherwise stated, in the subsequent analysis, we assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a simple Euclidean Jordan algebra of rank q and $\dim(\mathbb{V}) = n$.

3 Coerciveness of f_α and f_β

In this section, we study the conditions under which the EP merit functions f_α and f_β are coercive. For this purpose, we need Lemma 3.1 of [13], which is stated as follows.

Lemma 3.1 *For a given Jordan frame $\{c_1, c_2, \ldots, c_q\}$, if $z \in \mathbb{V}$ can be written as*

$$z = \sum_{i=1}^{q} z_i c_i + \sum_{1 \leq i < j \leq q} z_{ij},$$

with $z_i \in \mathbb{R}$ *for* $i = 1, 2, \dots, q$ *and* $z_{ij} \in \mathbb{V}_{ij}$ *for* $1 \le i < j \le q$, *then*

$$z_+ = \sum_{i=1}^{q} s_i c_i + \sum_{1 \le i < j \le q} s_{ij}, \qquad z_- = \sum_{i=1}^{q} w_i c_i + \sum_{1 \le i < j \le q} w_{ij},$$

where $s_i \ge (z_i)_+ \ge 0, 0 \ge (z_i)_- \ge w_i$ *with* $s_i + w_i = z_i$ *for* $i = 1, \dots, q$, *and* $s_{ij}, w_{ij} \in \mathbb{V}_{ij}$ *with* $s_{ij} + w_{ij} = z_{ij}$ *for* $1 \le i < j \le q$.

The following lemma summarizes some important inequalities involved in the maximum eigenvalue and the minimum eigenvalue for any $x \in \mathbb{V}$. Since their proofs can be found in Lemma 14 of [17] and Proposition 2.1 of [20], we here omit them.

Lemma 3.2 *For any* $x, y \in \mathbb{V}$, *the following inequalities always hold*:

(a) $\lambda_{\min}(x)\|c\|^2 \le \langle x, c\rangle \le \lambda_{\max}(x)\|c\|^2$ *for any nonzero idempotent* c;
(b) $|\lambda_{\max}(x+y) - \lambda_{\max}(x)| \le \|y\|$ *and* $|\lambda_{\min}(x+y) - \lambda_{\min}(x)| \le \|y\|$;
(c) $\lambda_{\max}(x+y) \le \lambda_{\max}(x) + \lambda_{\max}(y)$ *and* $\lambda_{\min}(x+y) \ge \lambda_{\min}(x) + \lambda_{\min}(y)$.

Using Lemmas 3.1–3.2, we may establish a lower bound for $\|\phi_\alpha(x, y)\|$ and $\|\phi_\beta(x, y)\|$.

Lemma 3.3 *Let* ϕ_α *and* ϕ_β *be given by* (7) *and* (8), *respectively. Then, for any* $x, y \in \mathbb{V}$,

$$\|\phi_\alpha(x, y)\| \ge \left[(2\alpha - \alpha^2)/(2\alpha)\right] \max\left\{[(\lambda_{\min}(x))_-]^2,\ [(\lambda_{\min}(y))_-]^2\right\}, \tag{17}$$

$$\|\phi_\beta(x, y)\| \ge \left[(1 - \beta^2)/(2\beta)\right] \max\left\{[(\lambda_{\min}(x))_-]^2,\ [(\lambda_{\min}(y))_-]^2\right\}. \tag{18}$$

Proof Suppose that x has the spectral decomposition $x = \sum_{i=1}^{q} x_i c_i$ with $x_i \in \mathbb{R}$ and $\{c_1, c_2, \dots, c_q\}$ being a Jordan frame. From Theorem 2.2, $y \in \mathbb{V}$ can be expressed by

$$y = \sum_{i=1}^{q} y_i c_i + \sum_{1 \le i < j \le q} y_{ij}, \tag{19}$$

where $y_i \in \mathbb{R}$ for $i = 1, 2, \dots, q$ and $y_{ij} \in \mathbb{V}_{ij}$. Therefore, for any $l \in \{1, 2, \dots, q\}$,

$$\begin{aligned} \langle c_l, x \circ y\rangle = \langle c_l \circ x, y\rangle &= \left\langle x_l c_l, \sum_{i=1}^{q} y_i c_i + \sum_{1 \le i < j \le q} y_{ij}\right\rangle \\ &= x_l \left\langle c_l, \sum_{i=1}^{q} y_i c_i\right\rangle + x_l \left\langle c_l, \sum_{1 \le i < j \le q} y_{ij}\right\rangle \\ &= x_l y_l, \end{aligned} \tag{20}$$

where the last equality is due to the fact that $\langle c_l, \sum_{1 \le i < j \le q} y_{ij}\rangle = 0$ by the orthogonality of V_{ij} $(i \le j)$.

We next prove the inequality (17). From (19) and the spectral decomposition of x,

$$x + y = \sum_{i=1}^{q}(x_i + y_i)c_i + \sum_{1\le i<j\le q} y_{ij},$$

which together with Lemma 3.1 implies that

$$(x + y)_- = \sum_{i=1}^{q} u_i c_i + \sum_{1\le i<j\le q} u_{ij},$$

where $u_i \le (x_i + y_i)_- \le 0$ for $i = 1, 2, \ldots, q$ and $u_{ij} \in \mathbb{V}_{ij}$. By this, we can compute

$$\begin{aligned}
\left\langle c_l, [(x+y)_-]^2 \right\rangle &= \left\langle c_l \circ \left(\sum_{i=1}^{q} u_i c_i + \sum_{1\le i<j\le q} u_{ij} \right),\ (x+y)_- \right\rangle \\
&= \left\langle u_l c_l + \left(c_l \circ \sum_{1\le i<j\le q} u_{ij} \right),\ \sum_{i=1}^{q} u_i c_i + \sum_{1\le i<j\le q} u_{ij} \right\rangle \\
&= u_l^2 + u_l \left\langle c_l, \sum_{1\le i<j\le q} u_{ij} \right\rangle + \left\langle \sum_{1\le i<j\le q} u_{ij},\ c_l \circ \sum_{i=1}^{q} u_i c_i \right\rangle \\
&\quad + \left\langle c_l \circ \sum_{1\le i<j\le q} u_{ij},\ \sum_{1\le i<j\le q} u_{ij} \right\rangle \\
&= u_l^2 + \left\langle c_l, \Big(\sum_{1\le i<j\le q} u_{ij} \Big)^2 \right\rangle, \quad \forall l = 1, 2, \ldots, q,
\end{aligned} \tag{21}$$

where the last equality is due to the fact that $\langle c_l, \sum_{1\le i<j\le q} u_{ij} \rangle = 0$ by the orthogonality of V_{ij} $(i \le j)$. Now, using (20)–(21), we obtain that

$$\begin{aligned}
\langle c_l, -\phi_\alpha(x, y) \rangle &= \left\langle c_l, x \circ y - (1/2\alpha) \left[(x+y)_- \right]^2 \right\rangle \\
&= x_l y_l - (1/2\alpha) \left[u_l^2 + \left\langle c_l, \Big(\sum_{1\le i<j\le q} u_{ij} \Big)^2 \right\rangle \right] \\
&\le x_l y_l - (1/2\alpha) \left[(x_l + y_l)_- \right]^2, \quad \forall l = 1, 2, \ldots, q,
\end{aligned} \tag{22}$$

where the inequality is due to the following facts

$$u_l \le (x_l + y_l)_- \le 0 \quad \text{and} \quad \left\langle c_l, \Big(\sum_{1\le i<j\le q} u_{ij} \Big)^2 \right\rangle \ge 0.$$

On the other hand, from Lemma 3.2(a) we have that

$$\langle c_l, -\phi_\alpha(x,y)\rangle \geq \lambda_{\min}(-\phi_\alpha(x,y))\|c_l\|^2 = \lambda_{\min}(-\phi_\alpha(x,y)), \quad \forall l = 1,2,\ldots,q. \tag{23}$$

Thus, combining (22) with (23), it follows that

$$2\alpha\lambda_{\min}(-\phi_\alpha(x,y)) \leq 2\alpha x_l y_l - [(x_l + y_l)_-]^2, \quad \forall l = 1,2,\ldots,q.$$

Let $\lambda_{\min}(x) = x_\nu$ with $\nu \in \{1,2,\ldots,q\}$. Then, we have particularly that

$$2\alpha\lambda_{\min}(-\phi_\alpha(x,y)) \leq 2\alpha\lambda_{\min}(x)y_\nu - [(\lambda_{\min}(x) + y_\nu)_-]^2. \tag{24}$$

We next proceed to the proof for two cases: $\lambda_{\min}(x) \leq 0$ and $\lambda_{\min}(x) > 0$.

Case (i): $\lambda_{\min}(x) \leq 0$. Under this case, we prove the following inequality:

$$2\alpha\lambda_{\min}(x)y_\nu - [(\lambda_{\min}(x) + y_\nu)_-]^2 \leq -(2\alpha - \alpha^2)[(\lambda_{\min}(x))_-]^2, \tag{25}$$

which, together with (24), implies immediately

$$\|\phi_\alpha(x,y)\| \geq \left|\lambda_{\min}(-\phi_\alpha(x,y))\right| \geq \left[(2\alpha - \alpha^2)/(2\alpha)\right][(\lambda_{\min}(x))_-]^2. \tag{26}$$

In fact, if $\lambda_{\min}(x) + y_\nu \geq 0$, then we can deduce that

$$\begin{aligned} 2\alpha\lambda_{\min}(x)y_\nu - [(\lambda_{\min}(x) + y_\nu)_-]^2 &= 2\alpha(\lambda_{\min}(x))_-(y_\nu)_+ \\ &\leq -(2\alpha - \alpha^2)[(\lambda_{\min}(x))_-]^2; \end{aligned}$$

otherwise, we have that

$$\begin{aligned} 2\alpha\lambda_{\min}(x)y_\nu - [(\lambda_{\min}(x) + y_\nu)_-]^2 &= 2\alpha\lambda_{\min}(x)y_\nu - [(\lambda_{\min}(x) + y_\nu)]^2 \\ &\leq -(2\alpha - \alpha^2)[\lambda_{\min}(x)]^2 \\ &= -(2\alpha - \alpha^2)[(\lambda_{\min}(x))_-]^2. \end{aligned}$$

Case (ii): $\lambda_{\min}(x) > 0$. Under this case, the inequality (26) clearly holds.
Summing up the above discussions, the inequality (26) holds for any $x, y \in \mathbb{V}$. In view of the symmetry of x and y in $\phi_\alpha(x,y)$, we also have that

$$\|\phi_\alpha(x,y)\| \geq \left[(2\alpha - \alpha^2)/(2\alpha)\right][(\lambda_{\min}(y))_-]^2,$$

for any $x, y \in \mathbb{V}$. Thus, the proof of the inequality (17) is completed.

We next prove the inequality (18). By the spectral decomposition of x, we have that $(x_-)^2 = \sum_{i=1}^q [(x_i)_-]^2 c_i$, which in turn implies

$$\langle c_l, (x_-)^2\rangle = [(x_l)_-]^2, \quad \forall l = 1,2,\ldots,q. \tag{27}$$

In addition, from Lemma 3.1 and the expression of y given by (19), it follows that

$$y_- = \sum_{i=1}^q v_i c_i + \sum_{1\leq i<j\leq q} v_{ij},$$

where $v_i \le (y_i)_- \le 0$ for $i = 1, 2, \ldots, q$ and $v_{ij} \in \mathbb{V}_{ij}$. By the same arguments as (21),

$$\langle c_l, (y_-)^2 \rangle = v_l^2 + \left\langle c_l, \Big(\sum_{1 \le i < j \le q} v_{ij} \Big)^2 \right\rangle, \quad \forall l = 1, 2, \ldots, q. \tag{28}$$

Now, from (20), (27) and (28), it follows that

$$\begin{aligned}
\langle c_l, -\phi_\beta(x, y) \rangle &= \left\langle c_l, x \circ y - (1/2\beta) \left[(x_-)^2 + (y_-)^2 \right] \right\rangle \\
&= x_l y_l - (1/2\beta) \left[((x_l)_-)^2 + v_l^2 + \left\langle c_l, \Big(\sum_{1 \le i < j \le q} v_{ij} \Big)^2 \right\rangle \right] \\
&\le x_l y_l - (1/2\beta) \left[((x_l)_-)^2 + (v_l)^2 \right] \\
&\le x_l y_l - (1/2\beta) \left[((x_l)_-)^2 + ((y_l)_-)^2 \right], \quad \forall l = 1, 2, \ldots, q,
\end{aligned}$$

where the first inequality is due to the nonnegativity of $\langle c_l, (\sum_{1 \le i < j \le q} v_{ij})^2 \rangle$ and the second one is due to the fact that $v_l \le (y_l)_- \le 0$. On the other hand, by Lemma 3.2(a),

$$\langle c_l, -\phi_\beta(x, y) \rangle \ge \lambda_{\min}(-\phi_\beta(x, y)) \|c_l\|^2 = \lambda_{\min}(-\phi_\beta(x, y)), \quad \forall l = 1, 2, \ldots, q.$$

Combining the last two inequalities leads immediately to

$$\lambda_{\min}(-\phi_\beta(x, y)) \le x_l y_l - (1/2\beta) \left[((x_l)_-)^2 + ((y_l)_-)^2 \right], \quad \forall l = 1, 2, \ldots, q.$$

Let $\lambda_{\min}(x) = x_\nu$ with $\nu \in \{1, 2, \ldots, q\}$, and suppose that $\lambda_{\min}(x) \le 0$. Then,

$$\begin{aligned}
\lambda_{\min}(-\phi_\beta(x, y)) &\le \lambda_{\min}(x) y_\nu - (1/2\beta) \left[((\lambda_{\min}(x))_-)^2 + ((y_\nu)_-)^2 \right] \\
&\le [(\lambda_{\min}(x))_-][(y_\nu)_-] - (1/2\beta) \left[((\lambda_{\min}(x))_-)^2 + ((y_\nu)_-)^2 \right] \\
&= -(1/2\beta) \left\{ \left[\beta (\lambda_{\min}(x))_- - (y_\nu)_- \right]^2 + (1 - \beta^2)[(\lambda_{\min}(x))_-]^2 \right\} \\
&\le - \left[(1 - \beta^2)/(2\beta) \right] [(\lambda_{\min}(x))_-]^2,
\end{aligned}$$

which in turn implies

$$\|\phi_\beta(x, y)\| \ge \left| \lambda_{\min}(-\phi_\beta(x, y)) \right| \ge \left[(1 - \beta^2)/(2\beta) \right] [(\lambda_{\min}(x))_-]^2. \tag{29}$$

If $\lambda_{\min}(x) = x_\nu > 0$, then the inequality (29) is obvious. Thus, (29) holds for any $x, y \in \mathbb{V}$. In view of the symmetry of x and y in $\phi_\beta(x, y)$, we also have

$$\|\phi_\beta(x, y)\| \ge |\lambda_{\min}(-\phi_\beta(x, y))| \ge \left[(1 - \beta^2)/(2\beta) \right] [(\lambda_{\min}(y))_-]^2$$

for any $x, y \in \mathbb{V}$. Consequently, the desired result follows. □

The following proposition characterizes an important property for the smooth EP complementarity functions ϕ_α and ϕ_β under a unified framework.

Proposition 3.1 *Let ϕ_α and ϕ_β be given as in (7) and (8), respectively. Let $\{x^k\} \subset \mathbb{V}$ and $\{y^k\} \subset \mathbb{V}$ be sequences satisfying one of the following conditions*:

(i) *either* $\lambda_{\min}(x^k) \to -\infty$ *or* $\lambda_{\min}(y^k) \to -\infty$;
(ii) $\lambda_{\min}(x^k), \lambda_{\min}(y^k) > -\infty$, $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$ *and* $\|x^k \circ y^k\| \to +\infty$.

Then, $\|\phi_\alpha(x^k, y^k)\| \to +\infty$ *and* $\|\phi_\beta(x^k, y^k)\| \to +\infty$.

Proof Under Case (i) the assertion is direct by Lemma 3.3. In what follows, we will prove the assertion under Case (ii). Notice that, in this case, the sequences $\{x^k\}$, $\{y^k\}$ and $\{x^k + y^k\}$ are all bounded below since $\lambda_{\min}(x^k), \lambda_{\min}(y^k) > -\infty$ and $\lambda_{\min}(x^k + y^k) \ge \lambda_{\min}(x^k) + \lambda_{\min}(y^k) > -\infty$. Therefore, the sequences $\{[(x^k + y^k)_-]^2\}$, $\{((x^k)_-)^2\}$ and $\{((y^k)_-)^2\}$ are bounded. In addition, we also have $\lambda_{\min}(x^k \circ y^k) \to -\infty$ or $\lambda_{\max}(x^k \circ y^k) \to +\infty$, since $\|x^k \circ y^k\| \to +\infty$.

If $\lambda_{\min}(x^k \circ y^k) \to -\infty$ as $k \to \infty$, then by Lemma 3.2(c) there holds that

$$\begin{aligned}\lambda_{\min}(-\phi_\alpha(x, y)) &= \lambda_{\min}\left[(x^k \circ y^k) - (1/2\alpha)((x^k + y^k)_-)^2\right] \\ &\le \lambda_{\min}(x^k \circ y^k) + (1/2\alpha)\left\|((x^k + y^k)_-)^2\right\|, \\ \lambda_{\min}(-\phi_\beta(x, y)) &= \lambda_{\min}\left[(x^k \circ y^k) - (1/2\beta)\left(((x^k)_-)^2 + ((y^k)_-)^2\right)\right] \\ &\le \lambda_{\min}(x^k \circ y^k) + (1/2\beta)\left\|((x^k)_-)^2 + ((y^k)_-)^2\right\|,\end{aligned}$$

which, together with the boundedness of $\|((x^k + y^k)_-)^2\|$ and $\|((x^k)_-)^2 + ((y^k)_-)^2\|$, implies that $\lambda_{\min}(-\phi_\alpha(x^k, y^k)) \to -\infty$ and $\lambda_{\min}(-\phi_\beta(x^k, y^k)) \to -\infty$. Since

$$\|\phi_\alpha(x^k, y^k)\| \ge |\lambda_{\min}(-\phi_\alpha(x, y))| \quad \text{and} \quad \|\phi_\beta(x^k, y^k)\| \ge |\lambda_{\min}(-\phi_\beta(x, y))|,$$

we obtain immediately that $\|\phi_\alpha(x^k, y^k)\| \to +\infty$ and $\|\phi_\beta(x^k, y^k)\| \to +\infty$.

If $\lambda_{\max}(x^k \circ y^k) \to +\infty$ as $k \to \infty$, from Lemma 3.2(c) it then follows that

$$\begin{aligned}\lambda_{\max}(-\phi_\alpha(x, y)) &= \lambda_{\max}\left[(x^k \circ y^k) - (1/2\alpha)((x^k + y^k)_-)^2\right] \\ &\ge \lambda_{\max}(x^k \circ y^k) - (1/2\alpha)\left\|((x^k + y^k)_-)^2\right\|, \\ \lambda_{\max}(-\phi_\beta(x, y)) &= \lambda_{\max}\left[(x^k \circ y^k) - (1/2\beta)\left(((x^k)_-)^2 + ((y^k)_-)^2\right)\right] \\ &\ge \lambda_{\max}(x^k \circ y^k) - (1/2\beta)\left\|((x^k)_-)^2 + ((y^k)_-)^2\right\|,\end{aligned}$$

which, by the boundedness of $\|((x^k + y^k)_-)^2\|$ and $\|((x^k)_-)^2 + ((y^k)_-)^2\|$, implies that $\lambda_{\max}(-\phi_\alpha(x^k, y^k)) \to +\infty$ and $\lambda_{\max}(-\phi_\beta(x^k, y^k)) \to +\infty$. Noting that

$$\|\phi_\alpha(x^k, y^k)\| \ge |\lambda_{\max}(-\phi_\alpha(x^k, y^k))| \quad \text{and} \quad \|\phi_\beta(x^k, y^k)\| \ge |\lambda_{\max}(-\phi_\beta(x^k, y^k))|,$$

we obtain readily that $\|\phi_\alpha(x^k, y^k)\| \to +\infty$ and $\|\phi_\beta(x^k, y^k)\| \to +\infty$. □

When $\mathbb{V} = \mathbb{R}^n$, '$\circ$' being the componentwise product of the vectors, $\|x^k \circ y^k\| \to +\infty$ automatically holds if $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$, and Proposition 3.1 reduces to the result of Lemma 2.5 in [21] for the NCPs. However, for the general Euclidean Jordan algebra, this condition is necessary as illustrated by the following example.

Example 3.1 Consider the Lorentz algebra $\mathcal{L}^n = (\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ introduced in Sect. 2. Assume that $n = 3$ and take the sequences $\{x^k\}$ and $\{y^k\}$ as follows:

$$x^k = \begin{pmatrix} k \\ k \\ 0 \end{pmatrix}, \qquad y^k = \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix}, \qquad \text{for each } k.$$

It is easy to verify that $\lambda_{\min}(x^k) = 0, \lambda_{\min}(y^k) = 0,\ \lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$, but $\|x^k \circ y^k\| \nrightarrow +\infty$. For such $\{x^k\}$ and $\{y^k\}$, by computation we have that $\|\phi_\alpha(x^k, y^k)\| = 0$ and $\|\phi_\beta(x^k, y^k)\| = 0$, i.e. the conclusion of Proposition 3.1 does not hold.

In the subsequent analysis, we use often the continuity of the Jordan product stated by the following lemma. Since the proof can be found in [10], we omit it.

Lemma 3.4 *Let $\{x^k\}$ and $\{y^k\}$ be the sequences such that $x^k \to \bar{x}$ and $y^k \to \bar{y}$ when $k \to \infty$. Then, we have that $x^k \circ y^k \to \bar{x} \circ \bar{y}$.*

Now, we are in a position to establish the coerciveness of f_α and f_β. Assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a general Euclidean Jordan algebra. We consider first the SCLCP case.

Theorem 3.1 *Let f_α and f_β be given by (10) and (12), respectively. If $F(\zeta) = L(\zeta) + b$, with the linear transformation L having the P-property, then f_α and f_β are coercive.*

Proof Let $\{\zeta^k\}$ be a sequence such that $\|\zeta^k\| \to +\infty$. We need only to prove that

$$f_\alpha(\zeta^k) \to +\infty, \qquad f_\beta(\zeta^k) \to +\infty. \tag{30}$$

By passing to a subsequence if necessary, we assume that $\zeta^k/\|\zeta^k\| \to \bar{\zeta}$, and consequently $(L(\zeta^k) + b)/\|\zeta^k\| \to L(\bar{\zeta})$. If $\lambda_{\min}(\zeta^k) \to -\infty$, then from Proposition 3.1 it follows that $\|\phi_\alpha(\zeta^k, L(\zeta^k) + b)\|, \|\phi_\beta(\zeta^k, L(\zeta^k) + b)\| \to +\infty$, which in turn implies (30).

Now, assume that $\{\zeta^k\}$ is bounded below. We argue that the sequence $\{L(\zeta^k) + b\}$ is unbounded by contradiction. Suppose that $\{L(\zeta^k) + b\}$ is bounded. Then,

$$L(\bar{\zeta}) = \lim_{k\to\infty} \left[(L(\zeta^k) + b)/\|\zeta^k\| \right] = 0 \in \mathcal{K}.$$

Since $\{\zeta^k\}$ is bounded below and $\lambda_{\max}(\zeta^k) \to +\infty$ by $\|\zeta^k\| \to +\infty$, there is an element $\bar{d} \in \mathbb{V}$ such that $(\zeta^k - \bar{d})/\|\zeta^k - \bar{d}\| \in \mathcal{K}$ for each k. Noting that $\mathcal{K}$ is closed, we have

$$\lim_{k\to\infty} (\zeta^k - \bar{d})/\|\zeta^k - \bar{d}\| = \bar{\zeta}/\|\bar{\zeta}\| = \bar{\zeta} \in \mathcal{K}.$$

Thus, $\bar{\zeta} \in \mathcal{K}$, $L(\bar{\zeta}) \in \mathcal{K}$ and $\bar{\zeta} \circ L(\bar{\zeta}) = 0$. From Proposition 6 of [18], it follows that $\bar{\zeta}$ and $L(\bar{\zeta})$ operator commute. This, together with $\bar{\zeta} \circ L(\bar{\zeta}) = 0 \in -\mathcal{K}$ and the P-property of L, implies that $\bar{\zeta} = 0$, yielding a contradiction to $\|\bar{\zeta}\| = 1$. Hence, the sequence $\{L(\zeta^k) + b\}$ is unbounded. Without loss of generality, assume that $\|L(\zeta^k) + b\| \to +\infty$.

If $\lambda_{\min}(L(\zeta^k) + b) \to -\infty$, then using Proposition 3.1 yields the desired result (30). We next assume that the sequence $\{L(\zeta^k) + b\}$ is bounded below. We prove that

$$(\zeta^k/\|\zeta^k\|) \circ \left[(L(\zeta^k) + b)/\|\zeta^k\|\right] \nrightarrow 0. \tag{31}$$

Suppose that (31) does not hold; then, from Lemma 3.4, it follows that

$$\bar{\zeta} \circ L(\bar{\zeta}) = \lim_{k\to+\infty} \left[(\zeta^k - d)/\|\zeta^k\|\right] \circ \left[(L(\zeta^k) + b - d)/\|\zeta^k\|\right] = 0 \quad \forall d \in \mathbb{V}. \tag{32}$$

Since $\{\zeta^k\}$ and $\{L(\zeta^k) + b\}$ are bounded below and $\lambda_{\max}(\zeta^k), \lambda_{\max}(L(\zeta^k) + b) \to +\infty$, there is an element $\tilde{d}$ such that $\zeta^k - \tilde{d} \in \mathcal{K}$ and $L(\zeta^k) + b - \tilde{d} \in \mathcal{K}$ for each k. Therefore,

$$\left[(\zeta^k - \tilde{d})/\|\zeta^k\|\right] \in \mathcal{K}, \qquad \left[(L(\zeta^k) + b - \tilde{d})/\|\zeta^k\|\right] \in \mathcal{K}, \quad \forall k.$$

Noting that $\mathcal{K}$ is closed, $\bar{\zeta} = \lim_{k\to\infty}(\zeta^k - \tilde{d})/\|\zeta^k\|$ and $L(\bar{\zeta}) = \lim_{k\to\infty}[(L(\zeta^k) + b - \tilde{d})/\|\zeta^k\|]$, we have

$$\bar{\zeta} \in \mathcal{K}, \qquad L(\bar{\zeta}) \in \mathcal{K}. \tag{33}$$

From (32) and (33) and Proposition 6 of [18], it follows that $\bar{\zeta}$ and $L(\bar{\zeta})$ operator commute. Using the P-property of L and noting that $\bar{\zeta} \circ L(\bar{\zeta}) = 0 \in -\mathcal{K}$, we then obtain $\bar{\zeta} = 0$, which clearly contradicts $\|\bar{\zeta}\| = 1$. Therefore, (31) holds. Since $\|\zeta^k\| \to +\infty$, we have $\|\zeta^k \circ (L(\zeta^k) + b)\| \to +\infty$. Combining with $\lambda_{\min}(\zeta^k), \lambda_{\min}(L(\zeta^k) + b) > -\infty$ and $\|\zeta^k\|, \|L(\zeta^k) + b\| \to +\infty$, it follows that the sequences $\{\zeta^k\}$ and $\{L(\zeta^k) + b\}$ satisfy condition (ii) of Proposition 3.1. This means that the result (30) holds. □

Theorem 3.2 *Let f_α and f_β be defined as in* (10) *and* (12), *respectively. If the mapping F has the uniform Jordan P-property, then f_α and f_β are coercive.*

Proof The proof technique is similar to that of Theorem 4.1 in [15]. For completeness, we include it. Let $\{\zeta^k\}$ be a sequence such that $\|\zeta^k\| \to +\infty$. Corresponding to the Cartesian structure of $\mathbb{V}$, let $\zeta^k = (\zeta_1^k, \ldots, \zeta_m^k)$ with $\zeta_i^k \in \mathbb{V}_i$ for each k. Define

$$J := \left\{i \in \{1, 2, \ldots, m\} \mid \{\zeta_i^k\} \text{ is unbounded}\right\}.$$

Clearly, the set $J \neq \emptyset$, since $\{\zeta^k\}$ is unbounded. Let $\{\xi^k\}$ be a bounded sequence with $\xi^k = (\xi_1^k, \ldots, \xi_m^k)$ and $\xi_i^k \in \mathbb{V}_i$ for $i = 1, 2, \ldots, m$, where ξ_i^k for each k is defined as follows:

$$\xi_i^k = \begin{cases} 0, & \text{if } i \in J, \\ \zeta_i^k, & \text{otherwise,} \end{cases}$$

with $i = 1, 2, \ldots, m$. Since F has the uniform Jordan P-property, there is a constant $\rho > 0$ such that

$$\begin{aligned} \rho\|\zeta^k - \xi^k\|^2 &\le \max_{i=1,\ldots,m} \lambda_{\max}\left[(\zeta_i^k - \xi_i^k) \circ (F_i(\zeta^k) - F_i(\xi^k))\right] \\ &= \lambda_{\max}\left[\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\right] \\ &\le \|\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\| \\ &\le \|\zeta_\nu^k\|\|F_\nu(\zeta^k) - F_\nu(\xi^k)\|, \end{aligned} \tag{34}$$

where ν is an index from $\{1, 2, \ldots, m\}$ for which the maximum is attained and the last inequality is due to (14). Clearly, $\nu \in J$ by the definition of $\{\xi^k\}$; consequently, $\{\zeta_\nu^k\}$ is unbounded. Without loss of generality, we assume that

$$\|\zeta_\nu^k\| \to +\infty. \tag{35}$$

Since

$$\|\zeta^k - \xi^k\|^2 \ge \|\zeta_\nu^k - \xi_\nu^k\|^2 = \|\zeta_\nu^k\|^2, \quad \text{for each } k, \tag{36}$$

dividing both sides of (34) by $\|\zeta_\nu^k\|$ then yields that

$$\rho\|\zeta_\nu^k\| \le \|F_\nu(\zeta^k) - F_\nu(\xi^k)\| \le \|F_\nu(\zeta^k)\| + \|F_\nu(\xi^k)\|.$$

Notice that $\{F(\xi^k)\}$ is bounded, since the mapping F is continuous and $\{\xi^k\}$ is bounded. Hence, the last inequality implies immediately

$$\|F_\nu(\zeta^k)\| \to +\infty. \tag{37}$$

In addition, we can verify by contradiction that

$$\|\zeta_\nu^k \circ F_\nu(\zeta^k)\| \to +\infty. \tag{38}$$

In fact, if $\{\|\zeta_\nu^k \circ F_\nu(\zeta^k)\|\}$ is bounded, then on the one hand we have

$$\lim_{k\to\infty} \|\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\| / \|\zeta_\nu^k\|^2 = 0,$$

but on the other hand, the inequality (36) implies

$$\lim_{k\to+\infty} \rho\|\zeta^k - \xi^k\|^2 / \|\zeta_\nu^k\|^2 \ge \rho > 0,$$

which clearly contradicts the third inequality in (34). Thus, from (35), (37), (38), the sequences $\{\zeta_\nu^k\}$ and $\{F_\nu(\zeta^k)\}$ satisfy the conditions of Proposition 3.1. Therefore, there necessarily holds that $\|\phi_\alpha^{(\nu)}(\zeta_\nu^k, F_\nu(\zeta^k))\| \to +\infty$ and $\|\phi_\beta^{(\nu)}(\zeta_\nu^k, F_\nu(\zeta^k))\| \to +\infty$, which in turn implies $f_\alpha(\zeta^k) \to +\infty$ and $f_\beta(\zeta^k) \to +\infty$ as $k \to \infty$. □

From Definition 2.2 and Lemma 3.2(a), clearly, the uniform Cartesian P-property implies the uniform Jordan P-property. Hence, the functions f_α and f_β are also coercive if F has the uniform Cartesian P-property. In addition, when $\mathbb{V} = \mathbb{R}^n$, '$\circ$' being the componentwise product of the vectors, the uniform Cartesian P-property and the uniform Jordan P-property of F are equivalent to saying that F is a uniform P-function; (see p. 299 of [1]), and now Theorem 3.2 recovers the known result of [14].

4 Coerciveness of f_{MS}

In this section, we study the coerciveness of the implicit Lagrangian merit function f_{MS} with the help of the natural residual complementarity function over symmetric cones,

$$r_\alpha(x, y) := x - (x - (1/\alpha)y)_+, \quad \forall x, y \in \mathbb{V} \text{ and } \alpha > 0. \tag{39}$$

To this end, we characterize first the growth behavior of the residual function r_α.

Lemma 4.1 *Let r_α be defined as in (39). Then, for any $x, y \in \mathbb{V}$, we have*

$$\lambda_{\min}(r_\alpha(x, y)) \le \min\left\{\lambda_{\min}(x),\ (1/\alpha)\lambda_{\min}(y)\right\}.$$

Proof For any $x, y \in \mathbb{V}$, from the definition of r_α and Lemma 3.2(c), we have

$$\begin{aligned}\lambda_{\min}(x) &= \lambda_{\min}\left[r_\alpha(x, y) + (x - (1/\alpha)y)_+\right]\\ &\ge \lambda_{\min}(r_\alpha(x, y)) + \lambda_{\min}\left[(x - (1/\alpha)y)_+\right],\end{aligned}$$

which implies that

$$\lambda_{\min}(r_\alpha(x, y)) \le \lambda_{\min}(x) - \lambda_{\min}\left[(x - (1/\alpha)y)_+\right] \le \lambda_{\min}(x). \tag{40}$$

On the other hand, we notice that the function r_α can be rewritten as

$$r_\alpha(x, y) = (x - (1/\alpha)y)_- + (1/\alpha)y.$$

Consequently,

$$\begin{aligned}(1/\alpha)\lambda_{\min}(y) &= \lambda_{\min}\left[r_\alpha(x, y) - (x - (1/\alpha)y)_-\right]\\ &\ge \lambda_{\min}(r_\alpha(x, y)) + \lambda_{\min}\left[-(x - (1/\alpha)y)_-\right]\\ &= \lambda_{\min}(r_\alpha(x, y)) + \lambda_{\min}\left[(-x + (1/\alpha)y)_+\right].\end{aligned}$$

This implies that

$$\lambda_{\min}(r_\alpha(x, y)) \le (1/\alpha)\lambda_{\min}(y) - \lambda_{\min}\left[(-x + (1/\alpha)y)_+\right] \le (1/\alpha)\lambda_{\min}(y). \tag{41}$$

From (40) and (41), we obtain immediately the first inequality. □

Proposition 4.1 *Let r_α be defined as in (39). Let $\{x^k\} \subset \mathbb{V}$ and $\{y^k\} \subset \mathbb{V}$ be the sequences satisfying one of the following conditions*

(i) *either $\lambda_{\min}(x^k) \to -\infty$ or $\lambda_{\min}(y^k) \to -\infty$;*
(ii) *$\lambda_{\min}(x^k), \lambda_{\min}(y^k) > -\infty$, $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$ and $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$.*

Then, $\|r_\alpha(x^k, y^k)\| \to +\infty$.

Proof If Case (i) holds, the result is direct by Lemma 4.1 and the fact that

$$\|r_\alpha(x^k, y^k)\| \geq |\lambda_{\min}[r_\alpha(x^k, y^k)]|.$$

It remains to prove the desired result under Case (ii). Suppose that the sequence $\{r_\alpha(x^k, y^k)\}$ is bounded. From the definition of r_α, we have that

$$\begin{aligned} r_\alpha(x^k, y^k) &= x^k - (1/2)\left(x^k - (1/\alpha)y^k\right) - (1/2)\left|x^k - (1/\alpha)y^k\right| \\ &= (1/2)\left(x^k + (1/\alpha)y^k\right) - (1/2)\left|x^k - (1/\alpha)y^k\right|. \end{aligned}$$

Therefore,

$$\left|x^k - (1/\alpha)y^k\right| = \left(x^k + (1/\alpha)y^k\right) - 2r_\alpha(x^k, y^k).$$

Squaring the two sides of the last equation then yields that

$$(1/\alpha)x^k \circ y^k = r_\alpha(x^k, y^k) \circ \left(x^k + (1/\alpha)y^k\right) - [r_\alpha(x^k, y^k)]^2.$$

Dividing the two sides by $\|x^k\|\|y^k\|$ and using the boundedness of $\{r_\alpha(x^k, y^k)\}$, we obtain

$$\lim_{k\to\infty} (x^k/\|x^k\|) \circ (y^k/\|y^k\|) = 0.$$

This contradicts the given assumption that $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$. □

When $\mathbb{V} = \mathbb{R}^n$, '$\circ$' being the componentwise product of the vectors, the condition $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$ implies $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$; consequently, Proposition 4.1 gives an important property of the natural residual NCP function or the minimum NCP function; see Lemma 2.5 of [21]. But, for the general Euclidean Jordan algebra, the following example illustrates that $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$ is necessary.

Example 4.1 Consider the Lorentz algebra $\mathcal{L}^n = (\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle_{\mathbb{R}^n})$ with $n = 3$. Take the sequences $\{x^k\}$ and $\{y^k\}$ as follows:

$$x^k = \begin{pmatrix} k \\ -(k+1) \\ (1/\alpha) \end{pmatrix}, \qquad y^k = \begin{pmatrix} k \\ k-1 \\ 1 \end{pmatrix}, \quad \text{for each } k.$$

It is easy to verify that $\lambda_{\min}(x^k) \to -1, \lambda_{\min}(y^k) = 1$ and $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$, but

$$x^k/\|x^k\| \to \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}, \qquad y^k/\|y^k\| \to \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix},$$
$$(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \to 0.$$

Therefore, the sequences $\{x^k\}$ and $\{y^k\}$ do not satisfy the assumption $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$. For such sequences, by computation, we have that

$$r_\alpha(x^k, y^k) = \begin{pmatrix} k \\ -(k+1) \\ (1/\alpha) \end{pmatrix} - \begin{pmatrix} k + (1/2) - (1/2\alpha) \\ -k - (1/2) + (1/2\alpha) \\ 0 \end{pmatrix} = \begin{pmatrix} (1/2\alpha) - (1/2) \\ -(1/2\alpha) - (1/2) \\ (1/\alpha) \end{pmatrix}.$$

Clearly, $\|r_\alpha(x^k, y^k)\| \nrightarrow +\infty$, i.e., the conclusion of Proposition 4.1 does not hold.

The next lemma states that $\|\phi_{\mathrm{MS}}(x, y)\|$ can be bounded by $\|r_\alpha(x, y)\|$ or $\|r_{1/\alpha}(x, y)\|$ from below.

Lemma 4.2 *Let ϕ_{MS} and r_α be defined as in (9) and (39), respectively. Then, for any $x, y \in \mathbb{V}$,*

$$\|\phi_{\mathrm{MS}}(x, y)\| \ge \max\Big\{[(\alpha^2 - 1)/(2\alpha\|e\|)]\|r_\alpha(x, y)\|^2,$$
$$[(1 - \alpha^2)/(2\alpha\|e\|)]\|r_{1/\alpha}(x, y)\|^2\Big\}.$$

Proof First, for any $x, y \in \mathbb{V}$, the following identity always holds:

$$\begin{aligned}\langle e, \phi_{\mathrm{MS}}(x, y)\rangle &= \langle x, y\rangle + (1/2\alpha)\Big\{\|(x - \alpha y)_+\|^2 - \|x\|^2 + \|(y - \alpha x)_+\|^2 - \|y\|^2\Big\} \\ &= \langle y, (x - (1/\alpha)y)_+\rangle + (\alpha/2)\|x - (x - (1/\alpha)y)_+\|^2 \\ &\quad - \langle y, (x - \alpha y)_+\rangle - (1/2\alpha)\,\|x - (x - \alpha y)_+\|^2. \end{aligned} \tag{42}$$

In fact, for any $x, y \in \mathbb{V}$, we can compute that

$$\begin{aligned}&\langle y, (x - (1/\alpha)y)_+\rangle + (\alpha/2)\|x - (x - (1/\alpha)y)_+\|^2 \\ &\quad = \langle y, (1/\alpha)(\alpha x - y)_+ - x\rangle + \langle y, x\rangle + (\alpha/2)\|(1/\alpha)(\alpha x - y)_+ - x\|^2 \\ &\quad = (\alpha/2)\|(1/\alpha)(\alpha x - y)_+ - x + (1/\alpha)y\|^2 + \langle y, x\rangle - (1/2\alpha)\|y\|^2 \\ &\quad = (1/2\alpha)\|-(y - \alpha x)_- + (y - \alpha x)\|^2 + \langle y, x\rangle - (1/2\alpha)\|y\|^2 \\ &\quad = (1/2\alpha)\|(y - \alpha x)_+\|^2 + \langle y, x\rangle - (1/2\alpha)\|y\|^2\end{aligned}$$

and

$$\begin{aligned}&\langle y,(x-\alpha y)_+\rangle+(1/2\alpha)\,\|x-(x-\alpha y)_+\|^2\\&\quad=(1/2\alpha)\|(x-\alpha y)_+\|^2-(1/\alpha)\,\langle x-\alpha y,(x-\alpha y)_+\rangle+(1/2\alpha)\|x\|^2\\&\quad=-(1/2\alpha)\,\|(x-\alpha y)_+\|^2+(1/2\alpha)\|x\|^2.\end{aligned}$$

The two equalities imply immediately (42). Now, consider the optimization problem

$$\min_{z\in\mathcal{K}}\langle y,z\rangle+(1/2\alpha)\langle z-x,z-x\rangle.$$

It is easy to verify that $z^*=(x-\alpha y)_+$ is the unique optimal solution, whereas $(x-(1/\alpha)y)_+$ is a feasible solution. Therefore, we have that

$$\begin{aligned}&\langle y,(x-\alpha y)_+\rangle+(1/2\alpha)\,\|x-(x-\alpha y)_+\|^2\\&\quad\le\langle y,(x-(1/\alpha)y)_+\rangle+(1/2\alpha)\|x-(x-(1/\alpha)y)_+\|^2.\end{aligned}$$

Combining this inequality with (42) yields

$$\langle e,\phi_{\mathrm{MS}}(x,y)\rangle\ge[(\alpha^2-1)/(2\alpha)]\|x-(x-(1/\alpha)y)_+\|^2,$$

which implies

$$\|\phi_{\mathrm{MS}}(x,y)\|\ge\langle e/\|e\|,\phi_{\mathrm{MS}}(x,y)\rangle\ge\Big[(\alpha^2-1)/(2\alpha\|e\|)\Big]\,\|r_\alpha(x,y)\|^2. \tag{43}$$

In addition, consider the following strictly convex optimization problem;

$$\min_{z\in\mathcal{K}}\langle y,z\rangle+(\alpha/2)\langle z-x,z-x\rangle.$$

We can verify that $z^*=(x-(1/\alpha)y)_+$ is the unique optimal solution, whereas $(x-\alpha y)_+$ is a feasible solution. Consequently, we have that

$$\begin{aligned}&\langle y,(x-(1/\alpha)y)_+\rangle+(\alpha/2)\|x-(x-(1/\alpha)y)_+\|^2\\&\quad\le\langle y,(x-\alpha y)_+\rangle+(\alpha/2)\|x-(x-\alpha y)_+\|^2.\end{aligned}$$

Combining this inequality with (42) then yields that

$$\langle e,\phi_{\mathrm{MS}}(x,y)\rangle\le\Big[(\alpha^2-1)/(2\alpha)\Big]\,\|x-(x-\alpha y)_+\|^2,$$

which in turn implies that

$$\|\phi_{\mathrm{MS}}(x,y)\|\ge-\langle e/\|e\|,\phi_{\mathrm{MS}}(x,y)\rangle\ge\Big[(1-\alpha^2)/(2\alpha\|e\|)\Big]\,\|r_{\frac{1}{\alpha}}(x,y)\|^2. \tag{44}$$

From (43) and (44), we obtain the desired result. The proof is thus complete. □

Note that, in Lemma 4.2, $\|e\|=\sqrt{q}$, since the rank of $\mathbb{V}$ is assumed to be q. Now, by Proposition 4.1 and Lemma 4.2, we readily have the following property of ϕ_{MS}.

Proposition 4.2 *Let ϕ_{MS} be defined as in (9). Let $\{x^k\} \subset \mathbb{V}$ and $\{y^k\} \subset \mathbb{V}$ be the sequences satisfying one of the following conditions*:

(i) *either* $\lambda_{\min}(x^k) \to -\infty$ *or* $\lambda_{\min}(y^k) \to -\infty$;
(ii) $\lambda_{\min}(x^k), \lambda_{\min}(y^k) > -\infty$, $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$ *and* $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$.

Then, $\|\phi_{\mathrm{MS}}(x^k, y^k)\| \to +\infty$.

Similar to Proposition 4.1, when $\mathbb{V} = \mathbb{R}^n$, $\circ$ being the componentwise product, the assumption $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$ is automatically satisfied, and from Proposition 4.2 we readily obtain the result of Lemma 6.2 of [22] for the NCPs. However, for the general Euclidean Jordan algebra, the following example shows that the assumption $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$ is also necessary.

Example 4.2 Consider the Lorentz algebra $\mathfrak{L}^n = (\mathbb{R}^n, \circ, \langle \cdot, \cdot \rangle_{\mathbb{R}^n})$ with $n = 3$ and take the sequences $\{x^k\}$ and $\{y^k\}$ as follows:

$$x^k = \begin{pmatrix} k \\ -k \\ 0 \end{pmatrix}, \qquad y^k = \begin{pmatrix} k^2 \\ k^2+1 \\ 0 \end{pmatrix}, \qquad \text{for each } k.$$

It is easy to verify that $\lambda_{\min}(x^k) = 0$, $\lambda_{\min}(y^k) = -1$ and $\lambda_{\max}(x^k), \lambda_{\max}(y^k) \to +\infty$, but

$$x^k/\|x^k\| \to (1/\sqrt{2}) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \qquad (y^k/\|y^k\|) \to (1/\sqrt{2}) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$
$$(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \to 0.$$

This shows that the sequences $\{x^k\}$ and $\{y^k\}$ do not satisfy the assumption $(x^k/\|x^k\|) \circ (y^k/\|y^k\|) \nrightarrow 0$. For such $\{x^k\}$ and $\{y^k\}$, we can show that

$$((x^k - \alpha y^k)_+)^2 - (x^k)^2 = \begin{pmatrix} 2k\alpha + (\alpha^2/2) \\ -2k\alpha - (\alpha^2/2) \\ 0 \end{pmatrix},$$

$$((y^k - \alpha x^k)_+)^2 - (y^k)^2 = \begin{pmatrix} -(1/2) \\ (1/2) \\ 0 \end{pmatrix},$$

$$\begin{aligned} \phi_{\mathrm{MS}}(x^k, y^k) &= \begin{pmatrix} -k \\ k \\ 0 \end{pmatrix} + (1/2\alpha) \left[\begin{pmatrix} 2k\alpha + (\alpha^2/2) \\ -2k\alpha - (\alpha^2/2) \\ 0 \end{pmatrix} + \begin{pmatrix} -(1/2) \\ (1/2) \\ 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} (\alpha/4) - (1/4\alpha) \\ -(\alpha/4) + (1/4\alpha) \\ 0 \end{pmatrix}. \end{aligned}$$

Clearly, $\|\phi_{\rm MS}(x^k, y^k)\| \nrightarrow \infty$, i.e., the result of Proposition 4.2 does not hold.

Now, assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ is a general Euclidean Jordan algebra. We establish the coercive properties of the merit function $f_{\rm MS}$ for the SCLCP and the SCCP.

Theorem 4.1 *Let $f_{\rm MS}$ be given by (12). If $F(\zeta) = L(\zeta) + b$, with the linear transformation L having the P-property, then the function $f_{\rm MS}$ is coercive.*

Proof Let $\{\zeta^k\}$ be a sequence such that $\|\zeta^k\| \to +\infty$. By passing to a subsequence if necessary, we can assume that $\zeta^k/\|\zeta^k\| \to \bar\zeta$, and hence $(L(\zeta^k)+b)/\|\zeta^k\| \to L(\bar\zeta)$. By the proof of Theorem 3.1, $L(\bar\zeta) \neq 0$ and $\{L(\zeta^k)+b\}$ is unbounded. Without loss of generality, assume that $\|L(\zeta^k)+b\| \to +\infty$.

If $\lambda_{\min}(\zeta^k) \to -\infty$ or $\lambda_{\min}(L(\zeta^k)+b) \to -\infty$, then using Proposition 4.2 yields

$$\|\phi_{\rm MS}(\zeta^k, L(\zeta^k)+b)\| \to +\infty, \qquad f_{\rm MS}(\zeta^k) \to +\infty.$$

We next assume that the sequences $\{\zeta^k\}$ and $\{L(\zeta^k)+b\}$ are bounded below. Since $\lambda_{\max}(\zeta^k), \lambda_{\max}(L(\zeta^k)+b) \to +\infty$ by $\|\zeta^k\|, \|L(\zeta^k)+b\| \to +\infty$, there is necessarily an element d such that $\zeta^k - d \in \mathcal{K}$ and $L(\zeta^k)+b-d \in \mathcal{K}$ for each k, implying that

$$(\zeta^k - d)/\|\zeta^k\| \in \mathcal{K} \quad \text{and} \quad (L(\zeta^k)+b-d)/\|L(\zeta^k)+b\| \in \mathcal{K}, \quad \text{for each } k.$$

Using the fact that $\mathcal{K}$ is a closed convex cone and noting that

$$\bar\zeta = \lim_{k\to\infty}(\zeta^k - d)/\|\zeta^k\|, \qquad L(\bar\zeta)/\|L(\bar\zeta)\| = \lim_{k\to\infty}(L(\zeta^k)+b-d)/\|L(\zeta^k)+b\|,$$

we have that $\bar\zeta \in \mathcal{K}$ and $L(\bar\zeta)/\|L(\bar\zeta)\| \in \mathcal{K}$. Suppose that $(\zeta^k/\|\zeta^k\|) \circ (L(\zeta^k)+b)/\|L(\zeta^k)+b\| \to 0$. Then, from Lemma 3.4, it follows that $\bar\zeta \circ (L(\bar\zeta)/\|L(\bar\zeta)\|) = 0$. Consequently,

$$\bar\zeta \in \mathcal{K}, L(\bar\zeta) \in \mathcal{K} \quad \text{and} \quad \bar\zeta \circ L(\bar\zeta) = 0.$$

By Proposition 6 of [18], $\bar\zeta$ and $L(\bar\zeta)$ operator commute. This, together with $\bar\zeta \circ L(\bar\zeta) = 0 \in -\mathcal{K}$ and the P-property of L, means that $\bar\zeta = 0$, which is impossible, since $\|\bar\zeta\| = 1$. Thus, $(\zeta^k/\|\zeta^k\|) \circ (L(\zeta^k)+b)/\|L(\zeta^k)+b\|) \nrightarrow 0$. Notice that $\lambda_{\min}(\zeta^k), \lambda_{\min}(L(\zeta^k)+b) > -\infty$ and $\|\zeta^k\|, \|L(\zeta^k)+b\| \to +\infty$; hence, the sequences $\{\zeta^k\}$ and $\{L(\zeta^k)+b\}$ satisfy the condition (ii) of Proposition 4.2, which implies that $f_{\rm MS}(\zeta^k) \to +\infty$. □

Theorem 4.2 *The function $f_{\rm MS}$ is coercive under one of the following conditions*:

(C1) *The mapping F has the uniform Jordan P-property and the Lipschitz continuity.*

(C2) *F has the uniform Jordan P-property and, for any $\{\zeta^k\}$, if there exists an index $i \in \{1, 2, \ldots, m\}$ such that $\lambda_{\max}(\zeta_i^k) \to +\infty$ and $\lambda_{\max}(F_i(\zeta^k)) \to +\infty$, then*

$$\limsup_{k\to\infty}\left\langle \zeta_i^k/\|\zeta_i^k\|, F_i(\zeta^k)/\|F_i(\zeta^k)\|\right\rangle > 0. \tag{45}$$

Proof The proof is similar to that of Theorem 4.1 in [15], and we include it for completeness. Let $\{\zeta^k\} \subset \mathbb{V}$ be any sequence such that $\|\zeta^k\| \to +\infty$. Corresponding to the structure of $\mathbb{V}$, we write $\zeta^k = (\zeta_1^k, \dots, \zeta_m^k)$ with $\zeta_i^k \in \mathbb{V}_i$ for each k. Define

$$J := \left\{ i \in \{1, 2, \dots, m\} \mid \{\zeta_i^k\} \text{ is unbounded} \right\}.$$

Clearly, the set $J \neq \emptyset$ since $\{\zeta^k\}$ is unbounded. Let $\{\xi^k\}$ be a bounded sequence with $\xi^k = (\xi_1^k, \dots, \xi_m^k)$ and $\xi_i^k \in \mathbb{V}_i$ for $i = 1, 2, \dots, m$, where ξ_i^k for each k is defined as

$$\xi_i^k = \begin{cases} 0, & \text{if } i \in J, \\ \zeta_i^k, & \text{otherwise,} \end{cases}$$

with $i = 1, 2, \dots, m$. If Condition C1 holds, then by the uniform Jordan P-property, there is a $\rho > 0$ such that

$$\begin{aligned} \rho\|\zeta^k - \xi^k\|^2 &\leq \max_{i=1,\dots,m} \lambda_{\max}\left[(\zeta_i^k - \xi_i^k) \circ (F_i(\zeta^k) - F_i(\xi^k))\right] \\ &= \lambda_{\max}\left[\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\right] \\ &\leq \|\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\| \\ &\leq \|\zeta_\nu^k\|\|F_\nu(\zeta^k) - F_\nu(\xi^k)\|, \end{aligned} \tag{46}$$

where ν is an index from $\{1, 2, \dots, m\}$ for which the maximum is attained; by the definition of $\{\xi^k\}$, clearly, $\nu \in J$, and the last inequality is due to (14). Since $\nu \in J$, $\{\zeta_\nu^k\}$ is unbounded. Without loss of generality, assume that

$$\|\zeta_\nu^k\| \to +\infty. \tag{47}$$

Notice that

$$\|\zeta^k - \xi^k\|^2 \geq \|\zeta_\nu^k - \xi_\nu^k\|^2 = \|\zeta_\nu^k\|^2, \quad \forall k.$$

Dividing the both sides of (46) by $\|\zeta_\nu^k\|$ then yields

$$\rho\|\zeta_\nu^k\| \leq \|F_\nu(\zeta^k) - F_\nu(\xi^k)\| \leq \|F_\nu(\zeta^k)\| + \|F_\nu(\xi^k)\|,$$

which, together with the boundedness of $\{F_\nu(\xi^k)\}$, implies that

$$\|F_\nu(\zeta^k)\| \to +\infty. \tag{48}$$

From (47) and (48), we thus obtain that

$$\|\zeta_\nu^k\| \to +\infty, \qquad \|F_\nu(\zeta^k)\| \to +\infty. \tag{49}$$

We next show that $(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|) \nrightarrow 0$. If this does not hold, by the continuity of $\lambda_{\max}(\cdot)$, we have that $\lambda_{\max}[(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|] \to 0$.

Consequently,

$$\begin{aligned}
&\lim_{k\to\infty} \lambda_{\max}\left[\zeta_\nu^k \circ (F_\nu(\zeta^k) - F_\nu(\xi^k))\right]/[\|\zeta_\nu^k\|\|F_\nu(\zeta^k)\|] \\
&\quad\le \lim_{k\to\infty} \lambda_{\max}\left[(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|)\right] \\
&\qquad + \lim_{k\to\infty} \lambda_{\max}\left[-\zeta_\nu^k \circ F_\nu(\xi^k)\right]/\left[\|\zeta_\nu^k\|\|F_\nu(\zeta^k)\|\right] \\
&\quad = 0,
\end{aligned} \tag{50}$$

where the inequality is due to Lemma 3.2(c). On the other hand, from the Lipschitz continuity of the mapping F, there exists a scalar $\gamma > 0$ such that

$$\|F(\zeta^k) - F(0)\| \le \gamma\|\zeta^k - 0\| = \gamma\|\zeta^k\|, \quad \text{for each } k,$$

which in turn implies that

$$\|F_\nu(\zeta^k)\| \le \|F_\nu(\zeta^k) - F_\nu(0)\| + \|F_\nu(0)\| \le \gamma\|\zeta^k\| + \|F_\nu(0)\|, \quad \forall k.$$

From the last inequality, we obtain that

$$\begin{aligned}
&\lim_{k\to\infty} \rho\|\zeta^k - \xi^k\|^2/[\|\zeta_\nu^k\|\|F_\nu(\zeta^k)\|] \\
&\quad\ge \lim_{k\to\infty} \rho\|\zeta^k - \xi^k\|^2/[\|\zeta^k\|(\gamma\|\zeta^k\| + \|F_\nu(0)\|)] = \frac{\rho}{\gamma} > 0.
\end{aligned}$$

This, together with (50), gives a contradiction to the first inequality in (46). Thus, the sequences $\{\zeta_\nu^k\}$ and $\{F_\nu(\zeta^k)\}$ satisfy the conditions of Proposition 4.2. Consequently,

$$\|\phi_{\mathrm{MS}}^{(\nu)}(\zeta_\nu^k, F_\nu(\zeta^k))\| \to +\infty \quad \text{and} \quad f_{\mathrm{MS}}(\zeta^k) \to +\infty.$$

If Condition C2 is satisfied, then from the above discussions we see that (46)–(49) still hold. If $\lambda_{\min}(\zeta_\nu^k) \to -\infty$ or $\lambda_{\min}(F_\nu(\zeta^k)) \to -\infty$, then using Lemma 4.1 and Lemma 4.2 readily yields that $\phi_{\mathrm{MS}}^{(\nu)}(\zeta_\nu^k, F_\nu(\zeta^k)) \to +\infty$, hence $f_{\mathrm{MS}}(\zeta^k) \to +\infty$. Otherwise, by (49), we have $\lambda_{\max}(\zeta_\nu^k) \to +\infty$ and $\lambda_{\max}(F_\nu(\zeta^k)) \to +\infty$. From the given assumption, it then follows that

$$\limsup_{k\to\infty} \left\langle \zeta_\nu^k/\|\zeta_\nu^k\|, F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\| \right\rangle > 0,$$

which, by Lemma 3.2(a), implies that

$$\limsup_{k\to\infty} \lambda_{\max}\left[(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|)\right] > 0.$$

This shows that $(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|) \nrightarrow 0$. Hence, the sequences $\{\zeta_\nu^k\}$ and $\{F_\nu(\zeta^k)\}$ satisfy the conditions of Proposition 4.2. Consequently, $\|\phi_{\mathrm{MS}}^{(\nu)}(\zeta_\nu^k, F_\nu(\zeta^k))\| \to +\infty$ and $f_{\mathrm{MS}}(\zeta^k) \to +\infty$. The proof is then completed. □

Notice that, when $\mathbb{V} = \mathbb{R}^n$, $\circ$ being the componentwise product of the vectors, the assumption (45) is automatically satisfied and the uniform Jordan P-property of F is equivalent to saying that F is a uniform P-function. Thus, Theorem 4.2 reduces to the known result of Theorem 4.1 of [15] for the NCPs. However, for the general Euclidean Jordan algebra, besides the uniform Jordan P-property of F, it require that F is Lipschitz continuous or satisfies the assumption (45) so that $(\zeta_\nu^k/\|\zeta_\nu^k\|) \circ (F_\nu(\zeta^k)/\|F_\nu(\zeta^k)\|) \nrightarrow 0$.

In addition, using Proposition 4.1 and the same arguments as in Theorems 4.1–4.2, we can obtain the coerciveness of the natural residual merit function for the SCCP,

$$R_\alpha(\zeta) := (1/2)\|r_\alpha(\zeta, F(\zeta))\|^2. \tag{51}$$

Theorem 4.3 *The function R_α defined by (51) is coercive under Condition* C1 *or* C2 *of Theorem 4.2. If $F(\zeta) = L(\zeta) + b$ with the linear transformation L having the P-property, then R_α is also coercive.*

Furthermore, from Lemma 4.2 we have that the growth rate of f_{MS} is higher than that of the natural residual merit function R_α. See the corollary below.

Corollary 4.1 *Let $\{\zeta^k\}$ be a sequence such that $\|\zeta^k\| \to +\infty$. If F satisfies Condition* C1 *or* C2 *of Theorem 4.2, then $R_\alpha(\zeta^k) \to +\infty$, $f_{\mathrm{MS}}(\zeta^k) \to +\infty$ and*

$$f_{\mathrm{MS}}(\zeta^k)/[R_\alpha(\zeta^k)]^{1+\sigma} \to +\infty, \quad 0 \le \sigma < 1.$$

5 Conclusions

In this paper, by using the P-properties of a mapping, we established the coerciveness of two classes of merit functions for the SCCP, i.e., the EP merit functions f_α and f_β and the implicit Lagrangian merit function f_{MS}. The obtained results characterize the growth behavior of the corresponding merit functions under a unified framework, and also provide a theoretical basis for the global convergence of the merit function approach and the equation reformulation method based on these functions. In addition, the results of this paper partially extend the work of [14] to the setting of symmetric cones.

References

1. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems. Springer, New York (2003)
2. Alizadeh, F., Goldfarb, D.: Second-order cone programming. Math. Program. **95**, 3–51 (2003)
3. Lobo, M.S., Vandenberghe, L., Boyd, S., Lebret, H.: Application of second-order cone programming. Linear Algebra Appl. **284**, 193–228 (1998)
4. Vandenberghe, L., Boyd, S.: A primal-dual potential reduction method for problems involving matrix inequalities. Math. Program. **69**, 205–236 (1995)

5. Liu, Y., Zhang, L., Wang, Y.: Some properties of a class of merit functions for symmetric cone complementarity problems. Asia-Pac. J. Oper. Res. **23**, 473–496 (2006)
6. Kanzow, C., Yamashita, Y., Fukushima, M.: New NCP functions and their properties. J. Optim. Theory Appl. **97**, 115–135 (1997)
7. Kong, L.C., Tuncel, L., Xiu, N.H.: Vector-valued implicit Lagrangian for symmetric cone complementarity problems. Asia-Pac. J. Oper. Res. (2008, in press)
8. Mangasarian, O.L., Solodov, M.V.: Nonlinear complementarity as unconstrained and constrained minimization. Math. Program. **62**, 277–297 (1993)
9. Kong, L.C., Sun, J., Xiu, N.H.: A regularized smoothing Newton method for symmetric cone complementarity problems. SIAM J. Optim. **19**, 1028–1047 (2008)
10. Huang, Z.H., Ni, T.: Coerciveness of a smoothing function for complementarity problems over symmetric cones with applications. Comput. Optim. Appl. (2008, in press)
11. Faraut, J., Korányi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs. Oxford University Press, London (1994)
12. Evtushenko, Y.G., Purtov, V.A.: Sufficient conditions for a minimum for nonlinear programming problems. Sov. Math. Dokl. **30**, 313–316 (1984)
13. Kong, L.C., Xiu, N.H.: New smooth C-functions for symmetric cone complementarity problems. Optim. Lett. **1**, 391–400 (2007)
14. Tseng, P.: Growth behavior of a class of merit functions for the nonlinear complementarity problem. J. Optim. Theory Appl. **89**, 17–37 (1996)
15. Kanzow, C., Fukushima, M.: Theoretical and numerical investigation of the D-gap function for box constrained variational inequalities. Math. Program. **83**, 55–87 (1998)
16. Koecher, M.: The Minnesota Notes on Jordan Algebras and Their Applications. Springer, Berlin (1999). Edited and annotated by A. Brieg and S. Walcher
17. Schmieta, S.H., Alizadeh, F.: Extension of primal-dual interior point algorithms for symmetric cones. Math. Program. **96**, 409–438 (2003)
18. Gowda, M.S., Sznajder, R., Tao, J.: Some P-properties for linear transformations on Euclidean Jordan algebras. Linear Algebra Appl. **393**, 203–232 (2004)
19. Sun, D., Sun, J.: Löwner's operator and spectral functions on Euclidean Jordan algebras. Math. Oper. Res. **33**, 421–445 (2008)
20. Tao, J., Gowda, M.S.: Some P-properties for nonlinear transformations on Euclidean Jordan algebras. Math. Oper. Res. **30**, 985–1004 (2005)
21. Kanzow, C.: Global convergence properties of some iterative methods for linear complementarity problems. SIAM J. Optim. **6**, 326–341 (1996)
22. Kanzow, C., Fukushima, M.: Equivalence of the generalized complementarity problem to differentiable unconstrained minimization. J. Optim. Theory Appl. **90**, 581–603 (1996)