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# The $H$-differentiability and calmness of circular cone functions

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**Abstract** Let $\mathcal{L}_\theta$ be the circular cone in $\mathbb{R}^n$ which includes second-order cone as a special case. For any function $f$ from $\mathbb{R}$ to $\mathbb{R}$, one can define a corresponding vector-valued function $f^{\mathcal{L}_\theta}$ on $\mathbb{R}^n$ by applying $f$ to the spectral values of the spectral decomposition of $x \in \mathbb{R}^n$ with respect to $\mathcal{L}_\theta$. The main results of this paper are regarding the $H$-differentiability and calmness of circular cone function $f^{\mathcal{L}_\theta}$. Specifically, we investigate the relations of $H$-differentiability and calmness between $f$ and $f^{\mathcal{L}_\theta}$. In addition, we propose a merit function approach for solving the circular cone complementarity problems under $H$-differentiability. These results are crucial to subsequent study regarding various analysis towards optimizations associated with circular cone.

**Keywords** Circular cone · $H$-differentiable · Calmness

**Mathematics Subject Classification** 26A27 · 26B05 · 26B35 · 49J52 · 90C33 · 65K05

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# 1 Introduction

Conic programming has drawn a lot of attention in the last decade. Generally speaking, the research works in this field are divided into two directions. One is on the general convex cone, see the excellent monograph written by Bonnans and Shapiro [3]; while the other focuses on some specific convex cones. In the latter case, much attention is paid to the so-called *symmetric cone*, which includes positive semi-definite matrices cone [26] and second-order cone [1] as special cases. The Jordan algebraic structure associated with symmetric cones allows us to deal with them in an unified way [13]. However, there exists a lot of cones which are convex but non-symmetric; for example, $p$-order cone [2], $L^p$-cone [15], and copositive cone [12], etc. For these non-symmetric cones, until now we don't know how to tackle with them in an unified framework. In fact, it needs to investigate them one-by-one because their structures are rather different. In this paper, we focus on a special non-symmetric convex cone, called *circular cone*. More precisely, the circular cone [5,27,28] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its half-aperture angle be $\theta$ with $\theta \in (0, \frac{\pi}{2})$. Then, the $n$-dimensional circular cone denoted by $\mathcal{L}_\theta$ can be expressed as

$$\mathcal{L}_\theta := \{x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \cos\theta \|x\| \le x_1\}.$$

Note that $\mathcal{L}_{45^\circ}$ corresponds the well-known second-order cone $\mathcal{K}^n$ (SOC, for short), which is given by

$$\mathcal{K}^n := \{x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\}.$$

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, there is a spectral decomposition for $x$ associated with circular cone case [27, Theorem 3.1], which is given as

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)} \tag{1}$$

where

$$\begin{cases} \lambda_1(x) = x_1 - \|x_2\| \cot\theta \\ \lambda_2(x) = x_1 + \|x_2\| \tan\theta \end{cases} \tag{2}$$

and

$$\begin{cases} u_x^{(1)} = \dfrac{1}{1+\cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta I \end{bmatrix} \begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\bar{x}_2 \end{bmatrix} \\ u_x^{(2)} = \dfrac{1}{1+\tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta I \end{bmatrix} \begin{bmatrix} 1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\bar{x}_2 \end{bmatrix} \end{cases} \tag{3}$$

with $\bar{x}_2 = x_2/\|x_2\|$ if $x_2 \neq 0$, and $\bar{x}_2$ being any vector $w$ in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $x_2 = 0$. With this spectral factorization (1), for any given $f : \mathbb{R} \to \mathbb{R}$, we can define the following vector-valued function associated with circular cone (which we call it "circular cone function" in general):

$$f^{\mathcal{L}_\theta}(x) := f(\lambda_1(x))\, u_x^{(1)} + f(\lambda_2(x))\, u_x^{(2)}. \tag{4}$$

Indeed, we can write out an explicit expression for (4) by plugging in $\lambda_i(x)$ and $u_x^{(i)}$:

$$f^{\mathcal{L}_\theta}(x) = \begin{bmatrix} \dfrac{f(x_1 - \|x_2\|\cot\theta)}{1+\cot^2\theta} + \dfrac{f(x_1 + \|x_2\|\tan\theta)}{1+\tan^2\theta} \\ \left( -\dfrac{f(x_1 - \|x_2\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \dfrac{f(x_1 + \|x_2\|\tan\theta)\tan\theta}{1+\tan^2\theta} \right) \bar{x}_2 \end{bmatrix}. \tag{5}$$

In particular, the formula (1) and (4) reduce to the well-known spectral decomposition and spectral function associated with second-order cone programming; see [6,9,10] for more details. A natural question is what properties of $f^{\mathcal{L}_\theta}$ are inhered from $f$ and vice versa. Once this is done, then we can analyze the property of the vector-valued function $f^{\mathcal{L}_\theta}$ by just study the single variable scalar function $f$. This reduces the difficulty of our analysis significantly, since $f$ is a real valued function on $\mathbb{R}$. In [5,28], the authors have answered this question in part, i.e., they show that the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, and ($\rho$-order) semismoothness are each inherited by $f^{\mathcal{L}_\theta}$ from $f$, and vice versa. However, it should be noted that the properties of Fréchet differentiability, continuous differentiability, and ($\rho$-order) semismoothness are requiring the condition of the locally Lipschitz continuity in advance. Hence, we hope to further study the properties between of $f$ and $f^{\mathcal{L}_\theta}$ without imposing Lipschitz continuity. Inspired by these points, we study the properties of *Calmness* and *$H$-differentiability*. Moreover, the exact formula of calmness modulus and $H$-differential are also established between $f^{\mathcal{L}_\theta}$ and $f$. In addition, we propose a merit function approach for solving the circular cone complementarity problems under $H$-differentiability.

It is well known that there exists a variety of definitions regarding nonsmoothness for extending the classical concept of differentiability. Why do we focus on the $H$-differentiability? We clarify our motivation as below. As indicated in [3, Chapter 4], the topic of studying nonsmoothness is a natural thing in optimization field; for example, for each fixed $x \in \mathbb{R}$ consider the following optimization problem:

$$\min_{t\in\Omega} tx \quad \text{subject to} \quad \Omega := [-1, 1].$$

Clearly, the optimal value function is $-|x|$, which is not differentiable at the origin. Note that the data involved in the above problem is rather simple and is smoothing. An important concept in the field of nonsmooth analysis is the generalized Jacobian for locally Lipschitz functions; see [11]. How to deal with the non-Lipschitz function? By this motivation, the concept of "$H$-differentiability" is introduced. It is well known that the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitz continuous function, the Bouligand subdifferential of a semismooth function, and the $C$-differential of a $C$-differentiable function are all examples of $H$-differentials; see [22] for more detailed discussion.

In [28], the following important relationship between circular cone and second-order cone is discovered. In particular, there holds that $\mathcal{L}_\theta = A^{-1}\mathcal{K}^n$ where

$$A = \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}.$$

This simple and basic relation helps us to study the normal cone, tangent cone, second-order tangent cone, second-order regularity of circular cone by using the corresponding results in second-order cone [27]. It however does not means that the extension of the results from second-order cone to circular cone is trivial. Indeed, the following two cases are possible: (i) one category of results is independent of the angle (i.e., still holds in the framework of circular cone); (ii) the second category is dependent of the angle, for example, for $x, y \in \mathcal{L}_\theta$, the inequality

$$\det(e + x + y) \leq \det(e + x)\det(e + y),$$

where $\det(x) := \lambda_1(x)\lambda_2(x)$ and $e := (1, 0, \ldots, 0) \in \mathbb{R}^n$, holds in the second-order cone setting. But, for circular cone setting, we show that this inequality fails for $\theta \in (0, 45°)$, but holds for $\theta \in [45°, 90°)$; see [29] for more information. In addition, it is surprised that a necessary condition for $f$ to be the $\mathcal{L}_\theta$-convexity is that $\theta \in [45°, 90°)$. Moreover, the exact formula of various derivative of projection over the circular cone $\mathcal{L}_\theta$ cannot be obtained by simple using the above basic relationship between the circular cone and second-order cone; see [30]. The aforementioned facts and observations give us new insight on circular cone and attract our attention to figuring out what role played by the angle $\theta$ in different settings.

To end this section, we say a few words about notations used in this paper. Define the ball of radius $\delta > 0$ centered at $x$ as $\mathbb{B}(x, \delta) := \{v \in \mathbb{R}^n \,|\, \|v - x\| \le \delta\}$. For convenience of notation, the unit ball at origin is written as $\mathbb{B}$, i.e., $\mathbb{B} := \mathbb{B}(0, 1) = \{v \in \mathbb{R}^n \,|\, \|v\| \le 1\}$. For a vector $x \in \mathbb{R}^n$ and a matrix $M \in \mathbb{R}^{n\times n}$, let $\|x\|$ stand for the Euclidean norm and $\|M\|$ for the norm induced by $\|\cdot\|$, i.e., $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ and $\|M\| = \max\limits_{\|x\|=1} \|Mx\|$.

## 2 Preliminaries

In this section, we review some basic concepts and materials about $H$-differentiability and calmness that will be used in subsequent analysis. We start with the concept of calmness.

**Definition 2.1** A mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is called to be calm at $\bar{x}$ if there exist $\delta > 0$ and $L > 0$ such that

$$\|F(x) - F(\bar{x})\| \le L\|x - \bar{x}\|, \quad \forall x \in \mathbb{B}(\bar{x}, \delta). \tag{6}$$

This is equivalent to saying

$$\mathrm{cam}(F)(\bar{x}) := \limsup_{\substack{x\to\bar{x}\\ x\neq\bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} < +\infty. \tag{7}$$

Here we call $\mathrm{cam}(F)(\bar{x})$ the calm modulus of $F$ at $\bar{x}$.

Note that (7) means that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|F(x) - F(\bar{x})\| \le (\mathrm{cam}(F)(\bar{x}) + \epsilon)\, \|x - \bar{x}\|, \qquad \forall x \in \mathbb{B}(\bar{x}, \delta). \tag{8}$$

Recall from [21, Chapter 9] that $F$ is locally Lipschitz at $\bar{x}$ if and only if $\mathrm{lip}(F)(\bar{x}) < +\infty$, where

$$\mathrm{lip}(F)(\bar{x}) := \limsup_{\substack{x,x'\to\bar{x}\\ x\neq x'}} \frac{\|F(x) - F(x')\|}{\|x - x'\|}.$$

It is clear that $\mathrm{cam}(F)(\bar{x}) \le \mathrm{lip}(F)(\bar{x})$. However, the inequality can be strict. To see this, we check the function

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

from which we see $\mathrm{cam}(f)(0) = 1 < +\infty = \mathrm{lip}(f)(0)$. As mentioned in [21, page 351], the inequality (6) only involves comparisons between $\bar{x}$ and nearpoint $x$, not between all possible pairs of points $x$ and $x'$ in some neighborhood of $\bar{x}$. Indeed, the locally Lipschitz continuity can be viewed as "locally uniform calmness", while Lipschitz continuity is viewed as "uniform calmness".

Next, we talk about the concepts of $H$-differentiability and $H$-differential of a function, which were first proposed by Gowda and Ravindran in [17]. Their motivation was to study a generalization (to nonsmooth case) of a result of Gale and Nikaido [14] which asserts that if the Jacobian matrix of a differentiable function $f$ from a closed rectangle $K \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$ is a $P$-matrix at each point of $K$, then $f$ is one-to-one on $K$.

**Definition 2.2** Given a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where $\Omega$ is an open set in $\mathbb{R}^n$ and $\bar{x} \in \Omega$, we say that a nonempty subset $T(\bar{x})$, also denoted by $T_F(\bar{x})$, of $\mathbb{R}^{m\times n}$ is an $H$-differential of $F$ at $\bar{x}$ if for every sequence $\{x^k\} \subseteq \Omega$ converging to $\bar{x}$, there exist a subsequence $\{x^{k_j}\}$ and a matrix $A \in T(\bar{x})$ such that

$$F(x^{k_j}) - F(\bar{x}) - A(x^{k_j} - \bar{x}) = o(\|x^{k_j} - \bar{x}\|). \tag{9}$$

We say that $F$ is $H$-differentiable at $\bar{x}$ if the $H$-differential of $F$ at $\bar{x}$ is nonempty.

As remarked in [16,17,22–25], the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitz continuous function, the Bouligand subdifferential of a semismooth function, and the $C$-differential of an $C$-differentiable function are all examples of $H$-differentials. In addition, any superset of an $H$-differential is an $H$-differential, $H$-differentiability implies continuity, and $H$-differentials satisfy simple sum, product, and chain rules. The class of $H$-differentiable functions is wider than the class of semismooth functions, since the former is not required to be locally Lipschitz continuous or directionally differentiable.

Now we point out that there is a useful equivalent expression for condition (9). For simplicity, let "$\twoheadrightarrow$" denote the convergence in the sense of taking some subsequence. With this notation, we see that condition (9) can be equivalently described as follows: For every sequence $\{x + t_k d^k\}$ with $t_k \downarrow 0$ and $\|d^k\| = 1$ for all $k$, there exist $t_{k_j} \downarrow 0$ and $d^{k_j} \to d$ and $A \in T_F(\bar{x})$ such that

$$\frac{F(\bar{x} + t_{k_j} d^{k_j}) - F(\bar{x})}{t_{k_j}} \to Ad, \tag{10}$$

i.e.,

$$\frac{F(\bar{x} + t_k d^k) - F(\bar{x})}{t_k} \twoheadrightarrow Ad.$$

Below are summaries of some well-known facts about $H$-differentiability, for more details, please refer to [16,17,22–25].

*Remark 2.1* (i) Any superset of an $H$-differential is an $H$-differential.
(ii) $H$-differentiability implies continuity.
(iii) If a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is $H$-differentiable at a point $\bar{x}$, then $F$ is calm at $\bar{x}$.

Note that the set $T_F(\bar{x})$ plays an important role in the definition of $H$-differentiability. For example, the converse statement of Remark 2.1(iii) holds by taking $T_F(\bar{x}) := \mathbb{R}^{m\times n}$ [25, page 281]. For completeness, we provide a simple proof for this claim as follows. Suppose that $F$ is $H$-differentiable at $\bar{x}$ with $T_F(\bar{x}) = \mathbb{R}^{m\times n}$. If (6) fails to hold, then we can find a sequence $\{x^k\}$ converging to $\bar{x}$ such that

$$\frac{\|F(x^k) - F(\bar{x})\|}{\|x^k - \bar{x}\|} \to +\infty. \tag{11}$$

For this sequence $\{x^k\}$, by the definition of $H$-differentiability of $F$ at $\bar{x}$, there exists a sequence $\{x^{k_j}\}$ such that

$$F(x^{k_j}) - F(\bar{x}) - A(x^{k_j} - \bar{x}) = o(\|x^{k_j} - \bar{x}\|).$$

This implies

$$\begin{aligned} \frac{\|F(x^{k_j}) - F(\bar{x})\|}{\|x^{k_j} - \bar{x}\|} &= \frac{\|A(x^{k_j} - \bar{x}) + o(\|x^{k_j} - \bar{x}\|)\|}{\|x^{k_j} - \bar{x}\|} \\ &\leq \|A\| + \frac{o(\|x^{k_j} - \bar{x}\|)}{\|x^{k_j} - \bar{x}\|} \\ &\to \|A\|, \end{aligned}$$

which contradicts (11). To see the converse, we take an arbitrary sequence $\{x^k\}$ satisfying $x^k \neq \bar{x}$ and $x^k \to \bar{x}$. Since $\left\{\frac{F(x^k)-F(\bar{x})}{\|x^k-\bar{x}\|}\right\}$ is bounded by (6) and $\left\{\frac{x^k-\bar{x}}{\|x^k-\bar{x}\|}\right\}$ is also bounded, there exists a subsequence, $\xi \in \mathbb{R}^m$, and $d \in \mathbb{R}^n$ such that

$$\frac{F(x^{k_j}) - F(\bar{x})}{\|x^{k_j} - \bar{x}\|} \to \xi \quad \text{and} \quad \frac{x_j^k - \bar{x}}{\|x_j^k - \bar{x}\|} \to d.$$

Now, take a matrix $A \in T_F(x) = \mathbb{R}^{m \times n}$ such that $\xi = Ad$. Note that such matrix always exists because $A$ has $mn$ variables. Hence,

$$\frac{F(x^{k_j}) - F(\bar{x}) - A(x^{k_j} - \bar{x})}{\|x^{k_j} - \bar{x}\|} \to \xi - Ad = 0,$$

i.e.,

$$F(x^{k_j}) - F(\bar{x}) - A(x^{k_j} - \bar{x}) = o(\|x^{k_j} - \bar{x}\|).$$

Remark 2.1(i) says that any superset of an $H$-differential is also an $H$-differential. This indicates that if a function $g$ is $H$-differentiable at $x$ with $T_g(x)$, then $g$ is also $H$-differentiable at $x$ with $\widetilde{T}_g(x)$ whenever $T_g(x) \subseteq \widetilde{T}_g(x)$. However, for an arbitrary function $g$, it is not guaranteed to become an $H$-differentiable function by simply taking a larger set. To see this, we present below that there exists a function that is not $H$-differentiable even if $T_g(x)$ takes the whole space. For example, consider the function

$$g(t) = |t|^p \quad \text{with} \quad p \in (0, 1).$$

Let $d = 1$ and $t_k \downarrow 0$. Then, $(g(t_k) - g(0))/t_k = |t_k|^{p-1} \to +\infty$, since $p < 1$. Hence, $T_g(0) = \emptyset$, which implies $g$ is not $H$-differentiable at 0. Indeed, $g(t) = |t|^p$ is not calm at 0.

## 3 Calmness

This section is devoted to properties of calmness. First, we explore some basic properties about composite function and then establish the calmness relation between $f^{\mathcal{L}_\theta}$ and $f$.

For mappings $F, G : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^l \to \mathbb{R}^n$, we define $(F \cdot G)(x) := F(x)^T G(x)$, $(F \circ S)(x) := F(S(x))$, and $(y \circ F)(x) := y^T F(x) = \sum_{i=1}^m y_i F_i(x)$ where $y = (y_1, \cdots, y_m)^T \in \mathbb{R}^m$ and $F_i$ is the component function of $F$. Inspired by [21, Chapter 9], we obtain the following results.

**Proposition 3.1** *Given three mappings $F, G : \mathbb{R}^n \to \mathbb{R}^m$, $S : \mathbb{R}^l \to \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$, and $\bar{z} \in \mathbb{R}^l$. Suppose that $S$ is continuous at $\bar{z}$. Then,*

*(a)* $\operatorname{cam}(\beta F)(\bar{x}) = |\beta| \operatorname{cam}(F)(\bar{x})$ *for all* $\beta \in \mathbb{R}$;
*(b)* $\operatorname{cam}(F+G)(\bar{x}) \le \operatorname{cam}(F)(\bar{x}) + \operatorname{cam}(G)(\bar{x})$;
*(c)* $\operatorname{cam}(F \cdot G)(\bar{x}) \le \operatorname{cam}(F)(\bar{x}) \operatorname{cam}(G)(\bar{x}) + \|F(\bar{x})\| \operatorname{cam}(G)(\bar{x}) + \|G(\bar{x})\| \operatorname{cam}(F)(\bar{x})$;
*(d)* $\operatorname{cam}(F \circ S)(\bar{z}) \le \operatorname{cam}(F)(S(\bar{z})) \operatorname{cam}(S)(\bar{z})$;
*(e)* *If* $\operatorname{cam}(F)(\bar{x}) < +\infty$, *then* $\operatorname{cam}(F)(\bar{x}) = \max_{y \in \mathbb{B}} (y \circ F)(\bar{x}) = \max_{\|y\|=1} (y \circ F)(\bar{x})$;
*(f)* $\operatorname{cam}(F_i)(\bar{x}) \le \operatorname{cam}(F)(\bar{x})$ *for* $i = 1, \cdots, m$ *and*

$$\operatorname{cam}(F)(\bar{x}) \le \|(\operatorname{cam}(F_1)(\bar{x}), \operatorname{cam}(F_2)(\bar{x}), \cdots, \operatorname{cam}(F_m)(\bar{x}))\| . \tag{12}$$

*Proof* (a) The result follows from (7) because

$$\operatorname{cam}(\beta F)(\bar{x}) = \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \frac{\|\beta F(x) - \beta F(\bar{x})\|}{\|x - \bar{x}\|} = |\beta| \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} = |\beta| \operatorname{cam}(F)(\bar{x}).$$

(b) The result follows from

$$\begin{aligned}
\operatorname{cam}(F+G)(\bar{x}) &= \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \frac{\|(F+G)(x) - (F+G)(\bar{x})\|}{\|x - \bar{x}\|} \\
&\le \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \left\{ \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} + \frac{\|G(x) - G(\bar{x})\|}{\|x - \bar{x}\|} \right\} \\
&\le \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} + \limsup_{\substack{x \to \bar{x} \\ x \ne \bar{x}}} \frac{\|G(x) - G(\bar{x})\|}{\|x - \bar{x}\|} \\
&= \operatorname{cam}(F)(\bar{x}) + \operatorname{cam}(G)(\bar{x}).
\end{aligned}$$

(c) It is trivial if $F$ is not calm at $\bar{x}$, since $\operatorname{cam}(F)(\bar{x}) = +\infty$ in this case. If $F$ is calm at $\bar{x}$, then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|F(x) - F(\bar{x})\| \le (\operatorname{cam}(F)(\bar{x}) + \epsilon) \|x - \bar{x}\|, \qquad \forall x \in \mathbb{B}(\bar{x}, \delta).$$

For any $x \in \mathbb{B}(\bar{x}, \hat{\delta})$ with $\hat{\delta} := \min\{\delta, 1\}$, we have

$$\begin{aligned}
\|F(x)\| &\le \|F(x) - F(\bar{x})\| + \|F(\bar{x})\| \le (\operatorname{cam}(F)(\bar{x}) + \epsilon) \|x - \bar{x}\| + \|F(\bar{x})\| \\
&\le \operatorname{cam}(F)(\bar{x}) + \|F(\bar{x})\| + \epsilon,
\end{aligned}$$

which means that $F$ is locally bounded at $\bar{x}$. Thus,

$$\begin{aligned}\operatorname{cam}(F\cdot G)(\bar{x}) &= \limsup_{\substack{x\to\bar{x}\\ x\neq\bar{x}}} \frac{\|(F\cdot G)(x)-(F\cdot G)(\bar{x})\|}{\|x-\bar{x}\|}\\ &= \limsup_{\substack{x\to\bar{x}\\ x\neq\bar{x}}} \frac{\|F(x)^T G(x)-F(\bar{x})^T G(\bar{x})\|}{\|x-\bar{x}\|}\\ &\leq \limsup_{\substack{x\to\bar{x}\\ x\neq\bar{x}}} \left\{\frac{\|F(x)^T G(x)-F(x)^T G(\bar{x})\|}{\|x-\bar{x}\|}+\frac{\|F(x)^T G(\bar{x})-F(\bar{x})^T G(\bar{x})\|}{\|x-\bar{x}\|}\right\}\\ &\leq \limsup_{\substack{x\to\bar{x}\\ x\neq\bar{x}}} \left\{\frac{\|F(x)\|\|G(x)-G(\bar{x})\|}{\|x-\bar{x}\|}+\frac{\|G(\bar{x})\|\|F(x)-F(\bar{x})\|}{\|x-\bar{x}\|}\right\}\\ &\leq (\operatorname{cam}(F)(\bar{x})+\|F(\bar{x})\|+\epsilon)\ \operatorname{cam}(G)(\bar{x})+\|G(\bar{x})\|\operatorname{cam}(F)(\bar{x}).\end{aligned}$$

Since $\epsilon > 0$ can be taken sufficiently small, the desired result follows.
(d) Notice that

$$\begin{aligned}\operatorname{cam}(F\circ S)(\bar{z}) &= \limsup_{\substack{z\to\bar{z}\\ z\neq\bar{z}}} \frac{\|(F\circ S)(z)-(F\circ S)(\bar{z})\|}{\|z-\bar{z}\|}\\ &= \limsup_{\substack{z\to\bar{z}\\ z\neq\bar{z}}} \frac{\|F(S(z))-F(S(\bar{z}))\|}{\|S(z)-S(\bar{z})\|}\frac{\|S(z)-S(\bar{z})\|}{\|z-\bar{z}\|}\\ &\leq \limsup_{\substack{z\to\bar{z}\\ z\neq\bar{z}}} \frac{\|F(S(z))-F(S(\bar{z}))\|}{\|S(z)-S(\bar{z})\|}\limsup_{\substack{z\to\bar{z}\\ z\neq\bar{z}}}\frac{\|S(z)-S(\bar{z})\|}{\|z-\bar{z}\|}\\ &\leq \operatorname{cam}(F)(S(\bar{z}))\operatorname{cam}(S)(\bar{z}).\end{aligned}$$

(e) Given $y \in \mathbb{B}$, for the linear mapping $y : \mathbb{R}^n \to \mathbb{R}$ defined as $y(v) := \langle y, v\rangle$, it is clear that $\operatorname{cam}(y)(v) = \|y\|$ for all $v \in \mathbb{R}^n$, since

$$\begin{aligned}\operatorname{cam}(y)(v) &= \limsup_{\substack{u\to v\\ u\neq v}} \frac{y(u)-y(v)}{\|u-v\|}\\ &= \limsup_{\substack{u\to v\\ u\neq v}} \frac{\langle y, u-v\rangle}{\|u-v\|}\\ &= \|y\|,\end{aligned}$$

where the "limsup" can be attained by taking $u = v + ty$ with $t > 0$. Hence

$$\begin{aligned}\operatorname{cam}(y \circ F)(\bar{x}) &= \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{(y \circ F)(x) - (y \circ F)(\bar{x})}{\|x - \bar{x}\|} \\ &= \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{\langle y, F(x) - F(\bar{x})\rangle}{\|x - \bar{x}\|} \\ &\leq \|y\| \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} \\ &= \|y\| \operatorname{cam}(F)(\bar{x}) \\ &\leq \operatorname{cam}(F)(\bar{x}),\end{aligned}$$

where the last step is due to the fact $\|y\| \leq 1$ since $y \in \mathbb{B}$. Hence,

$$\operatorname{cam}(F)(\bar{x}) \geq \max_{y \in \mathbb{B}} \operatorname{cam}(y \circ F)(\bar{x}). \tag{13}$$

Conversely,

$$\operatorname{cam}(F)(\bar{x}) = \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} = \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{\|F(x^k) - F(\bar{x})\|}{\|x^k - \bar{x}\|} = \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{\langle y^k, F(x^k) - F(\bar{x})\rangle}{\|x^k - \bar{x}\|} \tag{14}$$

where the last step comes from the fact

$$\|F(x^k) - F(\bar{x})\| = \max_{y \in \mathbb{B}} \langle y, F(x^k) - F(\bar{x})\rangle = \langle y^k, F(x^k) - F(\bar{x})\rangle$$

for some $y^k$ with $\|y^k\| = 1$. Since $\{y^k\}$ is bounded, we assume $y^k$ converges to $\bar{y}$ with $\|\bar{y}\| = 1$. Thus, it follows from (14) that

$$\begin{aligned}\operatorname{cam}(F)(\bar{x}) &= \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{\langle \bar{y}, F(x^k) - F(\bar{x})\rangle}{\|x^k - \bar{x}\|} + \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{\langle y^k - \bar{y}, F(x^k) - F(\bar{x})\rangle}{\|x^k - \bar{x}\|} \\ &= \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{\langle \bar{y}, F(x^k) - F(\bar{x})\rangle}{\|x^k - \bar{x}\|} \\ &= \lim_{\substack{x^k \to \bar{x} \\ x^k \neq \bar{x}}} \frac{(\bar{y} \circ F)(x^k) - (\bar{y} \circ F)(\bar{x})}{\|x^k - \bar{x}\|} \\ &\leq \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{(\bar{y} \circ F)(x) - (\bar{y} \circ F)(\bar{x})}{\|x - \bar{x}\|} \\ &= \operatorname{cam}(\bar{y} \circ F)(\bar{x}) \leq \max_{\|y\|=1} \operatorname{cam}(y \circ F)(\bar{x}) \leq \max_{y \in \mathbb{B}} \operatorname{cam}(y \circ F)(\bar{x}),\end{aligned}$$

where the second equality is due to the fact $\operatorname{cam}(F)(\bar{x}) < +\infty$ and $y^k \to \bar{y}$. This together with (13) yields the desired result.

(f) Notice that

$$\begin{aligned}\text{cam}(F_i)(\bar{x}) &= \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{|F_i(x) - F_i(\bar{x})|}{\|x - \bar{x}\|} \leq \limsup_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{\|F(x) - F(\bar{x})\|}{\|x - \bar{x}\|} \\ &= \text{cam}(F)(\bar{x}), \qquad \forall i = 1, \ldots, m.\end{aligned}$$

The relation (12) holds trivially if $F_i$ is not calm at $\bar{x}$ for some $i = 1, \ldots, m$. Hence we need to show that (12) holds when $F_i$ is calm at $\bar{x}$ for all $i = 1, \ldots, m$. In this case, for any $\epsilon > 0$ we get from (8) that

$$|F_i(x) - F_i(\bar{x})| \leq (\text{cam}(F_i)(\bar{x}) + \epsilon)\, \|x - \bar{x}\|, \qquad \forall x \in \mathbb{B}(\bar{x}, \delta_i), \;\; i = 1, \ldots, m.$$

For $x \in \mathbb{B}(\bar{x}, \delta)$ with $\delta := \min\{\delta_1, \ldots, \delta_m\}$, we have

$$\begin{aligned}\|F(x) - F(\bar{x})\| &= \left( \sum_{i=1}^{m} |F_i(x) - F_i(\bar{x})|^2 \right)^{\frac{1}{2}} \\ &\leq \left( \sum_{i=1}^{m} (\text{cam}(F_i)(\bar{x}) + \epsilon)^2 \right)^{\frac{1}{2}} \|x - \bar{x}\|.\end{aligned}$$

Hence

$$\text{cam}(F)(\bar{x}) \leq \|(\text{cam}(F_1)(\bar{x}) + \epsilon, \text{cam}(F_2)(\bar{x}) + \epsilon, \cdots, \text{cam}(F_m)(\bar{x}) + \epsilon)\|\,.$$

Since $\epsilon > 0$ can be taken sufficiently small, the desired result follows. □

Note that Prop. 3.1(a) means that $\text{cam}(F)(\bar{x})$ is positive homogeneous on $F$, and Prop. 3.1(b) indicates $\text{cam}(F)(\bar{x})$ is sublinear on $F$. These two facts imply that $\text{cam}(F)(\bar{x})$ is convex in $F$. As mentioned above, we know that $F$ is calm at $\bar{x}$ if and only if $\text{cam}(F)(\bar{x}) < +\infty$. Hence, the above results further indicate the following statements.

*Remark 3.1* (a) If $F$ and $G$ are calm at $\bar{x}$, then $F + G$ and $\beta F$ for $\beta \in \mathbb{R}$ are calm at $\bar{x}$, i.e., the set of all functions being calm at $\bar{x}$ constitutes a linear subspace.
(b) If $F$ and $G$ are calm at $\bar{x}$, then $F \cdot G$ is calm at $\bar{x}$.
(c) $F$ is calm if and only if $F_i$ is calm for $i = 1, 2, \cdots, m$.
(d) If $F$ is calm at $S(\bar{z})$ and $S$ is calm at $\bar{z}$, then $F \circ S$ is calm at $\bar{z}$.

The relation of calmness and calm modulus between $f^{\mathcal{L}_\theta}$ and $f$ are given below.

**Theorem 3.1** *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and $f^{\mathcal{L}_\theta}$ be defined as in (4). Suppose $x$ has spectral factorization given as in (1–3). Then,*

*(a) $f^{\mathcal{L}_\theta}$ is calm at $x$ if and only if $f$ is calm at $\lambda_i(x)$ for $i = 1, 2$. Moreover, if $f^{\mathcal{L}_\theta}$ is calm at $x$, then*

$$\text{cam}(f)(\lambda_i(x)) \leq \text{cam}(f^{\mathcal{L}_\theta})(x), \qquad \forall\, i = 1, 2;$$

*if $f$ is calm at $\lambda_i(x)$ for $i = 1, 2$, then*

$$\text{cam}(f^{\mathcal{L}_\theta})(x) \leq \frac{\sqrt{2}\max\{\tan\theta, \cot\theta\}(\tan\theta + \cot\theta + 2)}{\tan\theta + \cot\theta}\, \text{cam}(f)(x_1), \tag{15}$$

*when* $x_2 = 0$*; otherwise*

$$
\begin{aligned}
&\mathrm{cam}(f^{\mathcal{L}_\theta})(x)\\
&\quad\le \frac{\sqrt{2}\max\{\tan\theta, \cot\theta\}}{\tan\theta + \cot\theta}\{(1+\tan\theta)\mathrm{cam}(f)(\lambda_1(x)) + (1+\cot\theta)\mathrm{cam}(f)(\lambda_2(x))\}\\
&\qquad + \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{\lambda_2(x) - \lambda_1(x)}
\end{aligned}
\tag{16}
$$

*(b)* $f^{\mathcal{L}_\theta}$ *is calm over* $\mathbb{R}^n$ *if and only if* $f$ *is calm over* $\mathbb{R}$.

*Proof* (a) "⇒" Suppose that $f^{\mathcal{L}_\theta}$ is calm at $x$. To proceed the arguments, we discuss two cases.

**Case 1** $x_2 = 0$. Note that

$$
\begin{aligned}
\mathrm{cam}(f)(x_1) &= \limsup_{t\to 0} \frac{|f(x_1+t) - f(x_1)|}{|t|}\\
&= \limsup_{t\to 0} \frac{\|f^{\mathcal{L}_\theta}(x+te) - f^{\mathcal{L}_\theta}(x)\|}{|t|}\\
&\le \mathrm{cam}(f^{\mathcal{L}_\theta})(x).
\end{aligned}
$$

This says that $f$ is calm at $\lambda_i(x) = x_1$ with $\mathrm{cam}(f)(\lambda_i(x)) \le \mathrm{cam}(f^{\mathcal{L}_\theta})(x)$ for $i = 1, 2$.

**Case 2** $x_2 \neq 0$. Let $y = x + tu_x^{(1)}$ for $t \in (\lambda_1(x) - \lambda_2(x), \lambda_2(x) - \lambda_1(x))$. Then, $\lambda_1(y) = \lambda_1(x) + t$, $\lambda_2(y) = \lambda_2(x)$, $u_y^{(i)} = u_x^{(i)}$ for $i = 1, 2$. Note that

$$
\|y - x\| = |t|\|u_x^{(1)}\| \ \text{ and } \ \left\|f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x)\right\| = |f(\lambda_1(x)+t) - f(\lambda_1(x))|\cdot\|u_x^{(1)}\|.
$$

Hence

$$
\begin{aligned}
\mathrm{cam}(f)(\lambda_1(x)) &= \limsup_{t\to 0} \frac{|f(\lambda_1(x)+t) - f(\lambda_1(x))|}{|t|}\\
&= \limsup_{\substack{y = x + tu_x^{(1)}\\ t\to 0}} \frac{\|f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x)\|}{\|y - x\|}\\
&\le \mathrm{cam}(f^{\mathcal{L}_\theta})(x).
\end{aligned}
$$

Hence, $f$ is calm at $\lambda_1(x)$ with $\mathrm{cam}(f)(\lambda_1(x)) \le \mathrm{cam}(f^{\mathcal{L}_\theta})(x)$. By following the same arguments, we readily obtain the calmness of $f$ at $\lambda_2(x)$ with $\mathrm{cam}(f)(\lambda_2(x)) \le \mathrm{cam}(f^{\mathcal{L}_\theta})(x)$.
"⇐" Suppose that $f$ is calm at $\lambda_i(x)$ for $i = 1, 2$. Consider the following two cases.

**Case 1** $x_2 \neq 0$. Let $\phi(z_2) = \bar{z}_2 = \frac{z_2}{\|z_2\|}$ for $z_2 \neq 0$. Since $x_2 \neq 0$, then $\phi(z_2)$ is continuously differentiable near $x_2$ with $\nabla\phi(z_2) = \frac{1}{\|z_2\|}(I - \bar{z}_2\bar{z}_2^T)$. According to [10, Lemma 1]

$$
I - \bar{z}_2\bar{z}_2^T = (u_1, \ldots, u_{n-2})\mathrm{diag}[1, 1, \ldots, 1](u_1, \ldots, u_{n-2})^T
$$

where $\{u_1, \ldots, u_{n-2}\}$ is any orthonormal set of vectors that spans the subspace of $\mathbb{R}^{n-1}$ orthogonal to $\bar{z}_2$. This implies $\|\nabla\phi(z_2)\| = \frac{1}{\|z_2\|}\|I - \bar{z}_2\bar{z}_2^T\| = \frac{1}{\|z_2\|}$. For any given $\epsilon \in$

$(0, \|x_2\|)$, we have $\|z_2\| \geq \|x_2\| - \epsilon$ as $z_2$ sufficiently close $x_2$, and hence $\|\nabla\phi(z_2)\| \leq 1/(\|x_2\| - \epsilon)$. Thus, as $y$ is sufficiently close to $x$ we have

$$\begin{aligned} \|\phi(y_2) - \phi(x_2)\| &= \left\| \int_0^1 \nabla\phi\left(x_2 + t(y_2 - x_2)\right)(y_2 - x_2)dt \right\| \\ &\leq \int_0^1 \|\nabla\phi\left(x_2 + t(y_2 - x_2)\right)\| \, \|y_2 - x_2\| dt \\ &\leq \frac{1}{\|x_2\| - \epsilon} \|y_2 - x_2\| \\ &\leq \frac{1}{\|x_2\| - \epsilon} \|y - x\|. \end{aligned} \tag{17}$$

Then, it follows from (5) that

$$\begin{aligned} &\|f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x)\| \\ &= \frac{1}{\tan\theta + \cot\theta} \left\| \begin{bmatrix} \tan\theta\,[f(\lambda_1(y)) - f(\lambda_1(x))] + \cot\theta\,[f(\lambda_2(y)) - f(\lambda_2(x))] \\ ([f(\lambda_1(x)) - f(\lambda_1(y))] + [f(\lambda_2(y)) - f(\lambda_2(x))])\,\phi(y_2) \\ +[f(\lambda_2(x)) - f(\lambda_1(x))]\,(\phi(y_2) - \phi(x_2)) \end{bmatrix} \right\| \\ &\leq \frac{1}{\tan\theta + \cot\theta} \left\| \begin{bmatrix} \tan\theta\,[f(\lambda_1(y)) - f(\lambda_1(x))] + \cot\theta\,[f(\lambda_2(y)) - f(\lambda_2(x))] \\ ([f(\lambda_1(x)) - f(\lambda_1(y))] + [f(\lambda_2(y)) - f(\lambda_2(x))])\,\phi(y_2) \end{bmatrix} \right\| \\ &\quad + \frac{1}{\tan\theta + \cot\theta} |f(\lambda_2(x)) - f(\lambda_1(x))| \|\phi(y_2) - \phi(x_2)\| \qquad (18) \\ &\leq \frac{1}{\tan\theta + \cot\theta} \{|\tan\theta\,[f(\lambda_1(y)) - f(\lambda_1(x))] + \cot\theta\,[f(\lambda_2(y)) - f(\lambda_2(x))]| \\ &\quad + |[f(\lambda_1(x)) - f(\lambda_1(y))] + [f(\lambda_2(y)) - f(\lambda_2(x))]|\} \\ &\quad + \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{(\tan\theta + \cot\theta)(\|x_2\| - \epsilon)} \|y - x\| \\ &\leq \frac{1}{\tan\theta + \cot\theta} \{(1 + \tan\theta)\,|f(\lambda_1(y)) - f(\lambda_1(x))| + (1 + \cot\theta)|\,f(\lambda_2(y)) - f(\lambda_2(x))\,|\} \\ &\quad + \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{\lambda_2(x) - \lambda_1(x)} \frac{\|x_2\|}{\|x_2\| - \epsilon} \|y - x\|, \end{aligned}$$

where the second inequality is due to (17) and the last step comes from the fact

$$\begin{aligned} \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{(\tan\theta + \cot\theta)(\|x_2\| - \epsilon)} &= \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{(\tan\theta + \cot\theta)\|x_2\|} \frac{\|x_2\|}{\|x_2\| - \epsilon} \\ &= \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{\lambda_2(x) - \lambda_1(x)} \frac{\|x_2\|}{\|x_2\| - \epsilon} \end{aligned}$$

due to $\lambda_2(x) - \lambda_1(x) = (\tan\theta + \cot\theta)\|x_2\|$. Hence

$$\begin{aligned} &\frac{f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x)}{\|y - x\|} \\ &\quad \leq \frac{1}{\tan\theta + \cot\theta} \\ &\qquad \times \left\{ (1 + \tan\theta) \frac{|f(\lambda_1(y)) - f(\lambda_1(x))|}{\|y - x\|} + (1 + \cot\theta) \frac{|f(\lambda_2(y)) - f(\lambda_2(x))|}{\|y - x\|} \right\} \\ &\qquad + \frac{|f(\lambda_2(x)) - f(\lambda_1(x))|}{\lambda_2(x) - \lambda_1(x)} \frac{\|x_2\|}{\|x_2\| - \epsilon} \end{aligned}$$

$$
\begin{aligned}
&\le \frac{\sqrt{2}\max\{\tan\theta,\cot\theta\}}{\tan\theta+\cot\theta} \\
&\quad\times\left\{(1+\tan\theta)\frac{|f(\lambda_1(y))-f(\lambda_1(x))|}{|\lambda_1(y)-\lambda_1(x)|}+(1+\cot\theta)\frac{|f(\lambda_2(y))-f(\lambda_2(x))|}{|\lambda_2(y)-\lambda_2(x)|}\right\} \\
&\quad+\frac{|f(\lambda_2(x))-f(\lambda_1(x))|}{\lambda_2(x)-\lambda_1(x)}\frac{\|x_2\|}{\|x_2\|-\epsilon},
\end{aligned}
\tag{19}
$$

where the last step comes from

$$
\begin{aligned}
\frac{|f(\lambda_i(y))-f(\lambda_i(x))|}{\|y-x\|} &= \frac{|f(\lambda_i(y))-f(\lambda_i(x))|}{|\lambda_i(y)-\lambda_i(x)|}\frac{|\lambda_i(y)-\lambda_i(x)|}{\|y-x\|} \\
&\le \sqrt{2}\max\{\tan\theta,\cot\theta\}\frac{|f(\lambda_i(y))-f(\lambda_i(x))|}{|\lambda_i(y)-\lambda_i(x)|}
\end{aligned}
$$

because $\|\lambda_i(y)-\lambda_i(x)\| \le \sqrt{2}\max\{\tan\theta,\cot\theta\}\|y-x\|$ for $i=1,2$ by [28]. Taking limsup on both sides of (19) and using the fact that $\epsilon>0$ can be sufficiently small, it follows that $f^{\mathcal{L}_\theta}$ is calm at $x$ with the upper bound of $\mathrm{cam}(f^{\mathcal{L}_\theta})(x)$ given as in (16).

**Case 2** $x_2=0$. In this case, take $\bar{x}_2=\bar{y}_2$ (i.e., $\phi(x_2)=\phi(y_2)$). Following the similar argument as (18) and (19), we have

$$
\begin{aligned}
&\frac{\|f^{\mathcal{L}_\theta}(y)-f^{\mathcal{L}_\theta}(x)\|}{\|y-x\|} \\
&\le \frac{\sqrt{2}\max\{\tan\theta,\cot\theta\}}{\tan\theta+\cot\theta}\{(1+\tan\theta)\mathrm{cam}(f)(\lambda_1(x))+(1+\cot\theta)\mathrm{cam}(f)(\lambda_2(x))\} \\
&= \frac{\sqrt{2}\max\{\tan\theta,\cot\theta\}(\tan\theta+\cot\theta+2)}{\tan\theta+\cot\theta}\mathrm{cam}(f)(x_1),
\end{aligned}
$$

where the last step follows from the fact $\mathrm{cam}(f)(\lambda_i)=\mathrm{cam}(f)(x_1)$ since $\lambda_i(x)=x_1$ for $i=1,2$. Hence, $f^{\mathcal{L}_\theta}$ is calm at $x$ with the upper bound of $\mathrm{cam}(f^{\mathcal{L}_\theta})(x)$ given as in (15).
(b) This is an immediate consequence of part (a). □

## 4 *H*-differentiability

In this section, we answer the question about whether, as like the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, and ($\rho$-order) semismoothness (see [5,28]), the $H$-differentiability of $f^{\mathcal{L}_\theta}$ can be inherited by that of $f$ and vise versa? In addition, whether there exists some relationship between $T_f$ and $T_{f^{\mathcal{L}_\theta}}$? The following theorem provides an affirmative answer.

**Theorem 4.1** *Let $f:\mathbb{R}\to\mathbb{R}$ be a real-valued function and $f^{\mathcal{L}_\theta}$ be defined as in (4). Suppose $x$ has spectral factorization given as in (1–3). Then, the following hold.*

*(a) If $f$ is $H$-differentiable at $\lambda_i(x)$ with $T_f(\lambda_i(x))$ as the $H$-differential for $i=1,2$, then $f^{\mathcal{L}_\theta}$ is $H$-differentiable at $x$ with*

$$
T_{f^{\mathcal{L}_\theta}}(x) = \left\{\left[\begin{array}{cc} \frac{1}{1+\cot^2\theta}a_1+\frac{1}{1+\tan^2\theta}a_2 & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1+\frac{\tan\theta}{1+\tan^2\theta}a_2\right)w^T \\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1+\frac{\tan\theta}{1+\tan^2\theta}a_2\right)w & \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1+\frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)I \end{array}\right] \middle| \begin{array}{c} a_i\in T_f(x_1) \\ i=1,2 \\ \|w\|=1 \end{array}\right\}
\tag{20}
$$

*when* $x_2 = 0$*; otherwise*

$$T_{f^{\mathcal{L}_\theta}}(x) = \left\{ \left[ \begin{array}{cc} \frac{a_1}{1+\cot^2\theta} + \frac{a_2}{1+\tan^2\theta} & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2^T \\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2 & \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1 + \frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2\bar{x}_2^T \\ & +\frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x)-\lambda_1(x)}\left(I - \bar{x}_2\bar{x}_2^T\right) \end{array} \right] \middle| \begin{array}{c} a_i \in T_f(\lambda_i(x)) \\ i = 1,2 \end{array} \right\} \tag{21}$$

*(b) If* $f^{\mathcal{L}_\theta}$ *is H-differentiable at* $x$ *with* $T_{f^{\mathcal{L}_\theta}}(x)$ *as the H-differential, then* $f$ *is H-differentiable at* $\lambda_i(x)$ *with*

$$T_f(\lambda_i(x)) = \left\{ \frac{1}{\|u_x^{(i)}\|^2}(u_x^{(i)})^T A u_x^{(i)} \;\middle|\; A \in T_{f^{\mathcal{L}_\theta}}(x) \right\}, \quad i = 1,2$$

*when* $x_2 \neq 0$*; otherwise*

$$T_f(\lambda_i(x)) = \left\{ e^T A e \mid A \in T_{f^{\mathcal{L}_\theta}}(x) \right\}, \quad i = 1,2.$$

*Proof* (a) Let $t_k \downarrow 0$ and $d^k \to d$ with $\|d^k\| = 1$. We proceed the arguments by discussing two cases.

**Case 1** For $x_2 = 0$, we know $x + t_k d^k = (x_1 + t_k d_1^k, t_k d_2^k)^T$. Hence,

$$\begin{aligned} & f^{\mathcal{L}_\theta}(x + t_k d^k) - f^{\mathcal{L}_\theta}(x) \\ &= \left[ \begin{array}{c} \frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)}{1+\tan^2\theta} \\ \left(-\frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right)\bar{d}_2^k \end{array} \right] \\ &\quad - \begin{bmatrix} f(x_1) \\ 0 \end{bmatrix} \\ &= \left[ \begin{array}{c} \frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)}{1+\tan^2\theta} - f(x_1) \\ \left(-\frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right)\bar{d}_2^k \end{array} \right]. \end{aligned}$$

For $\left(f^{\mathcal{L}_\theta}(x + t_k d_k) - f^{\mathcal{L}_\theta}(x)\right)/t_k$, the first component is

$$\begin{aligned} & \frac{1}{t_k}\left[ \frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)}{1+\tan^2\theta} - f(x_1) \right] \\ &= \frac{1}{t_k}\left[ \frac{f(x_1 + t_k(d_1^k - \|d_2^k\|\cot\theta)) - f(x_1)}{1+\cot^2\theta} + \frac{f(x_1 + t_k(d_1^k + \|d_2^k\|\tan\theta)) - f(x_1)}{1+\tan^2\theta} \right] \\ &\twoheadrightarrow \frac{a_1(d_1 - \|d_2\|\cot\theta)}{1+\cot^2\theta} + \frac{a_2(d_1 + \|d_2\|\tan\theta)}{1+\tan^2\theta} \quad \text{as } t_k \downarrow 0, \end{aligned} \tag{22}$$

since $f$ is $H$-differentiable at $\lambda_i(x) = x_1$ for $i = 1, 2$ and $a_1, a_2 \in T_f(x_1)$. For the second component, similar to (22) (as $t_k \downarrow 0$), we have

$$\frac{1}{t_k}\left[-\frac{f(x_1 + t_k d_1^k - t_k\|d_2^k\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \frac{f(x_1 + t_k d_1^k + t_k\|d_2^k\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right]\bar{d}_2^k$$
$$\twoheadrightarrow \left[-\frac{\cot\theta}{1+\cot^2\theta}a_1(d_1 - \|d_2\|\cot\theta) + \frac{\tan\theta}{1+\tan^2\theta}a_2(d_1 + \|d_2\|\tan\theta)\right]\bar{d}_2,$$

where $\bar{d}_2^k$ converges to $\bar{d}_2$ if $d_2 \neq 0$, and converges to some $w \in \mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $d_2 = 0$, and in the latter case we can take $\bar{d}_2 = w$. The above two limits show that

$$\begin{aligned}
&\frac{f^{\mathcal{L}_\theta}(x + t_k d^k) - f^{\mathcal{L}_\theta}(x)}{t_k} \\
&\twoheadrightarrow \frac{1}{1+\cot^2\theta}a_1(d_1 - \|d_2\|\cot\theta)\begin{bmatrix}1 & 0\\ 0 & \cot\theta\end{bmatrix}\begin{bmatrix}1\\ -\bar{d}_2\end{bmatrix} \\
&\quad + \frac{1}{1+\tan^2\theta}a_2(d_1 + \|d_2\|\tan\theta)\begin{bmatrix}1 & 0\\ 0 & \tan\theta\end{bmatrix}\begin{bmatrix}1\\ \bar{d}_2\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{1+\cot^2\theta}a_1(d_1 - \|d_2\|\cot\theta) + \frac{1}{1+\tan^2\theta}a_2(d_1 + \|d_2\|\tan\theta)\\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1(d_1 - \|d_2\|\cot\theta) + \frac{\tan\theta}{1+\tan^2\theta}a_2(d_1 + \|d_2\|\tan\theta)\right)\bar{d}_2\end{bmatrix} \\
&= \begin{bmatrix}\left(\frac{1}{1+\cot^2\theta}a_1 + \frac{1}{1+\tan^2\theta}a_2\right)d_1 + \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\|d_2\|\\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)d_1\bar{d}_2 + \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1 + \frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)d_2\end{bmatrix} \\
&= \begin{bmatrix}\frac{1}{1+\cot^2\theta}a_1 + \frac{1}{1+\tan^2\theta}a_2 & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{d}_2^T\\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{d}_2 & \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1 + \frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)I\end{bmatrix}d,
\end{aligned} \tag{23}$$

where we used the fact that $\|d_2\| = \bar{d}_2^T d_2$, which is true even if $d_2 = 0$.

**Case 2** For $x_2 \neq 0$, then $x + t_k d^k = (x_1 + t_k d_1^k, x_2 + t_k d_2^k)^T$. Hence,

$$\begin{aligned}
&f^{\mathcal{L}_\theta}(x + t_k d^k) - f^{\mathcal{L}_\theta}(x) \\
&= \begin{bmatrix}\frac{f(x_1+t_kd_1^k-\|x_2+t_kd_2^k\|\cot\theta)}{1+\cot^2\theta} + \frac{f(x_1+t_kd_1^k+\|x_2+t_kd_2^k\|\tan\theta)}{1+\tan^2\theta}\\ \left(-\frac{f(x_1+t_kd_1^k-\|x_2+t_kd_2^k\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \frac{f(x_1+t_kd_1^k+\|x_2+t_kd_2^k\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right)\frac{x_2+t_kd_2^k}{\|x_2+t_kd_2^k\|}\end{bmatrix} \\
&\quad - \begin{bmatrix}\frac{f(x_1-\|x_2\|\cot\theta)}{1+\cot^2\theta} + \frac{f(x_1+\|x_2\|\tan\theta)}{1+\tan^2\theta}\\ \left(-\frac{f(x_1-\|x_2\|\cot\theta)\cot\theta}{1+\cot^2\theta} + \frac{f(x_1+\|x_2\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right)\frac{x_2}{\|x_2\|}\end{bmatrix}.
\end{aligned}$$

Since $x_2 \neq 0$, we know $\|x_2\|$ is continuously differentiable and

$$\|x_2 + t_k d_2^k\| = \|x_2\| + t_k \bar{x}_2^T d_2^k + o(t_k).$$

The first component of $\big(f^{\mathcal{L}_\theta}(x+t_kd^k)-f^{\mathcal{L}_\theta}(x)\big)/t_k$ is (when $t_k\downarrow 0$)

$$\begin{aligned}
&\frac{1}{t_k}\left[\frac{f(x_1+t_kd_1^k-\|x_2+t_kd_2^k\|\cot\theta)}{1+\cot^2\theta}+\frac{f(x_1+t_kd_1^k+\|x_2+t_kd_2^k\|\tan\theta)}{1+\tan^2\theta}\right.\\
&\quad\left.-\frac{f(x_1-\|x_2\|\cot\theta)}{1+\cot^2\theta}-\frac{f(x_1+\|x_2\|\tan\theta)}{1+\tan^2\theta}\right]\\
&=\frac{1}{t_k}\left[\frac{f(x_1+t_kd_1^k-\|x_2+t_kd_2^k\|\cot\theta)-f(x_1-\|x_2\|\cot\theta)}{1+\cot^2\theta}\right.\\
&\quad\left.+\frac{f(x_1+t_kd_1^k+\|x_2+t_kd_2^k\|\tan\theta)-f(x_1+\|x_2\|\tan\theta)}{1+\tan^2\theta}\right]\\
&=\frac{1}{t_k}\left[\frac{f\left(x_1+t_kd_1^k-\|x_2\|\cot\theta-t_k\bar{x}_2^Td_2^k\cot\theta+o(t_k)\right)-f(x_1-\|x_2\|\cot\theta)}{1+\cot^2\theta}\right.\\
&\quad\left.+\frac{f\left(x_1+t_kd_1^k+\|x_2\|\tan\theta+t_k\bar{x}_2^Td_2^k\tan\theta+o(t_k)\right)-f(x_1+\|x_2\|\tan\theta)}{1+\tan^2\theta}\right]\\
&\twoheadrightarrow\frac{a_1(d_1-\bar{x}_2^Td_2\cot\theta)}{1+\cot^2\theta}+\frac{a_2(d_1+\bar{x}_2^Td_2\tan\theta)}{1+\tan^2\theta},
\end{aligned}$$

where $a_i\in T_f(\lambda_i(x))$ for $i=1,2$.

The second component of $\big(f^{\mathcal{L}_\theta}(x+t_kd^k)-f^{\mathcal{L}_\theta}(x)\big)/t_k$ is (when $t_k\downarrow 0$)

$$\begin{aligned}
&\frac{1}{t_k}\left\{\left[-\frac{f(x_1+t_kd_1^k-\|x_2+t_kd_2^k\|\cot\theta)\cot\theta}{1+\cot^2\theta}+\frac{f(x_1+t_kd_1^k+\|x_2+t_kd_2^k\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right]\frac{x_2+t_kd_2^k}{\|x_2+t_kd_2^k\|}\right.\\
&\quad\left.-\left[-\frac{f(x_1-\|x_2\|\cot\theta)\cot\theta}{1+\cot^2\theta}+\frac{f(x_1+\|x_2\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right]\frac{x_2}{\|x_2\|}\right\}\\
&\twoheadrightarrow\left[-\frac{\cot\theta}{1+\cot^2\theta}a_1\left(d_1-\bar{x}_2^Td_2\cot\theta\right)+\frac{\tan\theta}{1+\tan^2\theta}a_2\left(d_1+\bar{x}_2^Td_2\tan\theta\right)\right]\bar{x}_2\\
&\quad+\left[-\frac{f(x_1-\|x_2\|\cot\theta)\cot\theta}{1+\cot^2\theta}+\frac{f(x_1+\|x_2\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right]\left[\frac{1}{\|x_2\|}\left(I-\bar{x}_2\bar{x}_2^T\right)\right]d_2.
\end{aligned}$$

The above two limits show that

$$\begin{aligned}
&\frac{f^{\mathcal{L}_\theta}(x+t_kd^k)-f^{\mathcal{L}_\theta}(x)}{t_k}\\
&\twoheadrightarrow\begin{bmatrix}\frac{a_1}{1+\cot^2\theta}+\frac{a_2}{1+\tan^2\theta} & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1+\frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2^T\\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1+\frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2 & \begin{array}{c}\left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1+\frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)\bar{x}_2\bar{x}_2^T\\ +\frac{f(\lambda_2(x))-f(\lambda_1(x))}{\lambda_2(x)-\lambda_1(x)}\left(I-\bar{x}_2\bar{x}_2^T\right)\end{array}\end{bmatrix}d.
\end{aligned}$$

(b) Let $\xi_k\to 0$. Again, we discuss two cases.

**Case 1** For $x_2\neq 0$, we have $x+\xi_ku_x^1=(\lambda_1(x)+\xi_k)u_x^{(1)}+\lambda_2(x)u_x^{(2)}$ as $k$ sufficiently large, since $\xi_k\to 0$. Because $f^{\mathcal{L}_\theta}$ is $H$-differentiable at $x$, there exists $A\in T_{f^{\mathcal{L}_\theta}}(x)$ (by taking a subsequence if necessary) such that

$$f^{\mathcal{L}_\theta}(x+\xi_ku_x^{(1)})-f^{\mathcal{L}_\theta}(x)-A\xi_ku_x^1=o(|\xi_k|),$$

which implies

$$\left\langle f^{\mathcal{L}_\theta}(x+\xi_ku_x^{(1)})-f^{\mathcal{L}_\theta}(x)-A\xi_ku_x^{(1)},u_x^{(1)}\right\rangle=o(|\xi_k|).$$

Hence

$$\left\langle [f(\lambda_1(x)+\xi_k)-f(\lambda_1(x))]\, u_x^{(1)} - A\xi_k u_x^{(1)}, u_x^{(1)} \right\rangle = o(|\xi_k|),$$

i.e.,

$$[f(\lambda_1(x)+\xi_k)-f(\lambda_1(x))]\, \|u_x^{(1)}\|^2 - \xi_k (u_x^{(1)})^T A u_x^{(1)} = o(|\xi_k|).$$

Thus,

$$f(\lambda_1(x)+\xi_k)-f(\lambda_1(x)) - \frac{1}{\|u_x^{(1)}\|^2}(u_x^{(1)})^T A u_x^{(1)} \xi_k = o(|\xi_k|),$$

which means $\frac{1}{\|u_x^{(1)}\|^2}(u_x^{(1)})^T A u_x^{(1)} \in T_f(\lambda_1(x))$. Similarly, we can also conclude

$$\frac{1}{\|u_x^{(2)}\|^2}(u_x^{(2)})^T A u_x^{(2)} \in T_f(\lambda_2(x)).$$

**Case 2** For $x_2 = 0$, the result follows immediately from the same arguments by taking $u_x^{(1)} = e$. □

We know that if a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ is Fréchet differentiable at $\bar{x}$, then $T_F(\bar{x})$ is singleton with $T_F(\bar{x}) = \{\nabla F(\bar{x})\}$. It should be pointed out that the converse statement may fail in general, i.e., if $F$ is $H$-differentiable at $\bar{x}$ and $T_F(\bar{x})$ is singleton, then $F$ is not necessarily Fréchet differentiable at $\bar{x}$. Of course, these two concepts can be equivalent under some particular case, which is illustrated below.

**Theorem 4.2** *A function $g : \mathbb{R} \to \mathbb{R}$ is Fréchet differentiable at $\gamma \in \mathbb{R}$ if and only if $g$ is $H$-differentiable at $\gamma \in \mathbb{R}$ with $T_g(\gamma)$ being a singleton set.*

*Proof* The necessity is clear. To show the sufficiency, let $T_g(\gamma) = \{\eta\}$. Consider

$$\limsup_{t\downarrow 0} \frac{g(\gamma+t)-g(\gamma)}{t} = \lim_{t_k\downarrow 0} \frac{g(\gamma+t_k)-g(\gamma)}{t_k} = \eta$$

where the last step follows from (10) with $d^k = d = 1$. Similarly,

$$\liminf_{t\downarrow 0} \frac{g(\gamma+t)-g(\gamma)}{t} = \lim_{t_k\downarrow 0} \frac{g(\gamma+t_k)-g(\gamma)}{t_k} = \eta.$$

Hence

$$g'_+(\gamma) = \lim_{t\downarrow 0} \frac{g(\gamma+t)-g(\gamma)}{t} = \eta.$$

On the other hand,

$$\begin{aligned}\limsup_{t\uparrow 0} \frac{g(\gamma+t)-g(\gamma)}{t} &= \lim_{t_k\uparrow 0} \frac{g(\gamma+t_k)-g(\gamma)}{t_k} = -\lim_{-t_k\downarrow 0} \frac{g(\gamma-(-t_k))-g(\gamma)}{-t_k} \\ &= -\lim_{\substack{t'_k:=-t_k \\ t'_k\downarrow 0}} \frac{g(\gamma-t'_k)-g(\gamma)}{t'_k} = -\eta(-1) = \eta\end{aligned}$$

where the last step follows from (10) with $d^k = d = -1$. Similarly, we have

$$\liminf_{t\uparrow 0} \frac{g(\gamma+t)-g(\gamma)}{t} = \eta.$$

Hence

$$g'_{-}(\gamma) = \lim_{t \uparrow 0} \frac{g(\gamma + t) - g(\gamma)}{t} = \eta.$$

Since the left and right derivative of $g$ at $\gamma$ are the same, then $g$ is differentiable at $\gamma$. □

The foregoing result shows that if $T_f(\lambda_i(x))$ for $i = 1, 2$ is singleton, then $f$ is differentiable at $\lambda_i(x)$, which in turn implies that $f^{\mathcal{L}_\theta}$ is also differentiable at $x$ with $T_{f^{\mathcal{L}_\theta}}(x) = \{\nabla f^{\mathcal{L}_\theta}(x)\}$; see [5, Theorem 3.3] and [28, Theorem 2.3] for the relation of $f$ and $f^{\mathcal{L}_\theta}$ on differentiability and the exact formula of $\nabla f^{\mathcal{L}_\theta}$.

In [4, Theorem 2.4], it is pointed out that at $x = (x_1, 0)$ if $T_f(x_1)$ is not a singleton set, i.e., $a_1 \neq a_2$ in (20), then $T_{f^{\mathcal{L}_\theta}}$ is not a singleton, i.e., in this case $f^{\mathcal{L}_\theta}$ cannot be $H$-differentiable if requiring that $T_{f^{\mathcal{L}_\theta}}$ just have a one element. There leaves an interesting and important question: whether $f^{\mathcal{L}_\theta}$ can be $H$-differentiable by taking larger set. It is indeed the main contribution of Theorem 4.1. The answer is affirmative. This is clear from the formula of $T_{f^{\mathcal{L}_\theta}}(x)$ given in (20) due to the multi-choice of $w$ with $\|w\| = 1$. In other words, Theorem 4.1 in this paper improves the result of [4, Theorem 2.4] when $\mathcal{L}_\theta$ reduces to SOC case (one direction cannot be guaranteed in [4, Theorem 2.4] due to the aforementioned reason).

## 5 CCCP under $H$-differentiability

The application of $H$-differentiability to nonlinear complementarity problem, symmetric cone complementarity problem, and variational inequalities have been studied in [4,22,24,25]. In this section, we further study the circular cone complementarity problem under $H$-differentiability. More precisely, the circular cone complementarity problem (CCCP) is to find $x \in \mathbb{R}^n$ such that

$$F(x) \in \mathcal{L}_\theta \quad G(x) \in \mathcal{L}_\theta^* \quad \langle F(x), G(x) \rangle = 0. \tag{24}$$

CCCP is a type of nonsymmetric cone complementarity problem and includes the second-order cone complementarity (SOCCP) problem as a special case ($\theta = 45°$). The CCCP is introduced in [19], where some merit functions related with natural residual (NR) functions are presented. Here we study the merit function associated with the Fisher-Burmeister (FB) complementarity function.

In the framework of second-order cone, the Fisher-Burmeister (FB) complementarity function is defined by

$$\phi_{\mathrm{FB}}(x; y) := (x^2 + y^2)^{\frac{1}{2}} - x - y,$$

where $x^2 := x \circ x$ means the Jordan product associated with second-order cone, i.e., for any $x, y \in \mathbb{R}^n$,

$$x \circ y = \begin{bmatrix} \langle x, y \rangle \\ x_1 y_2 + y_1 x_2 \end{bmatrix}.$$

Let

$$f(x) := \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right), \tag{25}$$

where

$$\widetilde{F}(x) := AF(x), \quad \widetilde{G}(x) := A^{-1}G(x), \quad \psi_{\mathrm{FB}}(x, y) := \frac{1}{2}\|\phi_{\mathrm{FB}}(x, y)\|^2.$$

**Lemma 5.1** *If $F$ and $G$ are H-differentiable at $x$, then*

(i) *$\widetilde{F}$ and $\widetilde{G}$ are H-differentiable at $x$ with*

$$T_{\widetilde{F}}(x) = AT_F(x) \text{ and } T_{\widetilde{G}}(x) = A^{-1}T_G(x).$$

(ii) *The function $f$ is H-differentiable at $x$ with*

$$\begin{aligned} &T_f(x) \\ &= \Big\{\nabla_x \psi_{\mathrm{FB}}\big(AF(x), A^{-1}G(x)\big)^T AM \\ &\qquad + \nabla_y \psi_{\mathrm{FB}}\big(AF(x), A^{-1}G(x)\big)^T A^{-1}N | M \in T_F(x), N \in T_G(x)\Big\}. \end{aligned}$$

*Proof* The result (i) follows from the definition of $H$-differentiability. The result (ii) comes from the fact that $\psi_{\mathrm{FB}}$ is continuous differentiable [8, Proposition 2]. □

**Theorem 5.1** *$x$ is a solution of CCCP if and only if $\psi_{\mathrm{FB}}\big(AF(x), A^{-1}G(x)\big) = 0$.*

*Proof* Notice that

$$\begin{aligned} &F(x) \in \mathcal{L}_\theta \quad G(x) \in \mathcal{L}_\theta^* \quad \langle F(x), G(x)\rangle = 0, \\ &\iff F(x) \in A^{-1}\mathcal{K} \quad G(x) \in \mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta} = A\mathcal{K} \quad \langle AF(x), A^{-1}G(x)\rangle = 0, \\ &\iff \widetilde{F}(x) \in \mathcal{K} \quad \widetilde{G}(x) \in \mathcal{K} \quad \langle \widetilde{F}(x), \widetilde{G}(x)\rangle = 0. \end{aligned}$$

Hence $x$ is a solution of CCCP if and only if $x$ is a solution of SOCCP($\widetilde{F}$, $\widetilde{G}$). Thus

$$\psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right) = 0,$$

which is the desired result. □

The above result shows that CCCP (24) can be expressed as an unconstrained minimization problem:

$$\min_{x\in\mathbb{R}^n} f(x) \tag{26}$$

where $f$ is defined as in (25).

When applying (26) to solve CCCP, we must answer the following questions: (i) under which conditions, the stationary point of (26) is a solution of CCCP; (ii) how to find the descend direction at non-stationary point.

**Theorem 5.2** *Suppose that $F$ and $G$ are H-differentiable and the H-differentials of $F$ and $G$ satisfy one of the following conditions*

(i) *for every $x \in \mathbb{R}^n$, $M \in T_F(x)$, $N \in T_G(x)$, $(AM, -A^{-1}N)$ is column monotone, i.e.,*

$$u^T AM - v^T A^{-1}N = 0 \Longrightarrow \langle u, v\rangle \geq 0, \quad \forall u, v \in \mathbb{R}^n. \tag{27}$$

(ii) *for every $x \in \mathbb{R}^n$, $M \in T_F(x)$, $N \in T_G(x)$, $M$ (or $N$) is invertible and $NM^{-1}$ (or $MN^{-1}$) is positive semidefinite.*

*Then the following statements are equivalent:*

*(a)* *$x$ is a solution of CCCP;*
*(b)* $0 \in T_f(x)$.

*Proof* The proof technique is adopted from [4]. If $x$ solves the CCCP, then $\psi_{\mathrm{FB}}\big(AF(x), A^{-1}G(x)\big) = 0$ by Theorem 5.1. According to [20, Proposition 3.3], we have

$$\nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = 0,$$

which together with Lemma 5.1 yields $0 \in T_f(x)$.

Conversely, if $0 \in T_f(x)$, then according to the formula of $T_f$ given in Lemma 5.1, there exists $M \in T_F(x)$ and $N \in T_G(x)$ such that

$$\nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)^T AM + \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)^T A^{-1}N = 0. \tag{28}$$

Case (i). Since $(AM, -A^{-1}N)$ is column monotone, then

$$\langle \nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\rangle \le 0,$$

i.e.,

$$\langle \nabla_x \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right)\rangle \le 0.$$

This together with [20, Proposition 3.3] yields

$$\langle \nabla_x \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right)\rangle = 0,$$

and hence

$$\psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right) = 0.$$

Case (ii). It only consider the case of $M$ being invertible, since the case of $N$ being invertible is similar. It follows from (28) that

$$\nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) + A^{-1}(M^{-1})^T N^T A^{-1} \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = 0.$$

Since $NM^{-1}$ is positive semidefinite, then

$$\begin{aligned}
0 &\le \left\langle \nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle \\
&= \left\langle -A^{-1}(M^{-1})^T N^T A^{-1} \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle \\
&= -\left\langle A^{-1} \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), NM^{-1}A^{-1} \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle \\
&\le 0.
\end{aligned}$$

Hence

$$\left\langle \nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle = 0,$$

i.e.,

$$\left\langle \nabla_x \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right)\right\rangle = 0.$$

Hence

$$\psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) = \psi_{\mathrm{FB}}\left(\widetilde{F}(x), \widetilde{G}(x)\right) = 0.$$

Thus, $x$ is a solution of CCCP by Theorem 5.1. □

The above result shows that under some conditions $x$ is a solution to the CCCP if and only if $x$ is a stationary point of $f$. For a non-stationary point, a descent direction is proposed as below.

**Theorem 5.3** *Suppose $F$ and $G$ are $H$-differentiable and the $H$-differentials of $F$ and $G$ satisfy (27). If $0 \notin T_f(x)$ then one of the following holds. In particular, if*

(i) *there exists a invertible matrix $M \in T_F(x)$, then*

$$d_{\mathrm{FB}}(x) := -M^{-1}A^{-1}\nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)$$

(ii) *there exists a invertible matrix $N \in T_G(x)$ , then*

$$d_{\mathrm{FB}}(x) := -N^{-1}A\nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)$$

*is a descent direction of $f$ at $x$.*

*Proof* Consider the case of $M$ being invertible. Since $0 \notin T_f(x)$, $\psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right) \neq 0$ by Theorems 5.1 and 5.2. It then follows from [20, Proposition 3.3] that

$$\left\langle \nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle > 0. \tag{29}$$

According to the expression in Lemma 5.1 (ii), we have

$$\begin{aligned}
&\left(\nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)^T AM + \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)^T A^{-1}N\right) d_{\mathrm{FB}}(x)\\
&= -\left\langle \nabla_x \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), \nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle\\
&\quad -\left\langle A^{-1}\nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right), NM^{-1}A^{-1}\nabla_y \psi_{\mathrm{FB}}\left(AF(x), A^{-1}G(x)\right)\right\rangle\\
&< 0,
\end{aligned}$$

where the last step is due to (29) and the fact that $NM^{-1}$ is positive semidefinite by (27). This completes the proof. □

The above results further enrich those given in [4] and [19]; for example, a merit function associated with FB function is given and the case of $M$ being invertible is also considered.

## 6 Final remarks

In this paper, the exact formula of calmness modulus and $H$-differential are established between $f^{\mathcal{L}_\theta}$ and $f$. In addition, we also study a merit function approach to solve the CCCP under $H$-differentiability. In other words, the results of this paper have an important application for complementarity problems. For example, characterize the $P_0$- and $P$-properties via $H$-differentials; every local minimizer or a stationary point of the merit function corresponding to the Fisher-Burmeister complementarity function of an $H$-differentiable function is a solution of the corresponding nonlinear complementarity problem [7,18].

In addition, second-order cone complementarity problem (SOCCP) is the extension of nonlinear complementarity problem to second-order cone settings. At present, the merit function for SOCCP is restricted in Lipschitz functions. To study the merit functions and characterizing $P_0$ and $P$-properties for SOCCP in the non-Lipschitz settings, the first target is to give the exact formula of $H$-differentiability. This is the main contribution of this paper. Indeed, we have established the corresponding result in the more general setting of circular cone. Nonetheless, there leave two rather important and interesting topics which we will keep an eye on them in our future research. We outline them as below.

1. Is it possible to establish the exact estimate or obtain more lower upper bound for $\mathrm{cam}(f^{\mathcal{L}_\theta})$?

2. Note that (23) can be rewritten as

$$\begin{bmatrix} \frac{1}{1+\cot^2\theta}a_1 + \frac{1}{1+\tan^2\theta}a_2 & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{d}_2^T \\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)\bar{d}_2 & \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1 + \frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)\bar{d}_2\bar{d}_2^T \end{bmatrix} d$$

where we use the fact that $d_2 = \bar{d}_2\bar{d}_2^T d_2$, which also holds for $d_2 = 0$. Hence, for $x$ with $x_2 = 0$, we have

$$T_{f^{\mathcal{L}_\theta}}(x) = \left\{ \begin{bmatrix} \frac{1}{1+\cot^2\theta}a_1 + \frac{1}{1+\tan^2\theta}a_2 & \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)w^T \\ \left(-\frac{\cot\theta}{1+\cot^2\theta}a_1 + \frac{\tan\theta}{1+\tan^2\theta}a_2\right)w & \left(\frac{\cot^2\theta}{1+\cot^2\theta}a_1 + \frac{\tan^2\theta}{1+\tan^2\theta}a_2\right)ww^T \end{bmatrix} \middle| \begin{array}{c} a_i \in T_f(x_1) \\ i = 1, 2 \\ \|w\| = 1 \end{array} \right\} \tag{30}$$

This is not contradicting to the results given in Theorem 4.1, because the expression of $H$-differential is not unique according to the Definition 2.2 or Remark 2.1(i). Note that the expression (30) is consistent with (21) if defining $0/0 := 0$. However, as $f$ is differentiable at $x_1$, then we know $f^{\mathcal{L}_\theta}$ is also differentiable at $x$ with $T_{f^{\mathcal{L}_\theta}}(x) = f'(x_1)I$. In this case (30) takes the form $f'(x_1)\begin{bmatrix} 1 & 0 \\ 0 & ww^T \end{bmatrix}$, while (20) takes $f'(x_1)I$. Due to this consideration, we adopt (20) as the $H$-differentials. This yields a question: does there exist other expression for $T_{f^{\mathcal{L}_\theta}}$ and can we establish their relationship?

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