Journal of Computational and Applied Mathematics 230 (2009) 69–82

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Properties of a family of generalized NCP-functions and a derivative free algorithm for complementarity problems

Sheng-Long Hu [a], Zheng-Hai Huang [a,*], Jein-Shan Chen [b,1]

[a] Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, PR China

[b] Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

ARTICLE INFO

Article history:
Received 7 September 2008
Received in revised form 24 October 2008

MSC:
90C33
90C56
65K10

Keywords:
Complementarity problem
NCP-function
Merit function
Derivative free algorithm

ABSTRACT

In this paper, we propose a new family of NCP-functions and the corresponding merit functions, which are the generalization of some popular NCP-functions and the related merit functions. We show that the new NCP-functions and the corresponding merit functions possess a system of favorite properties. Specially, we show that the new NCP-functions are strongly semismooth, Lipschitz continuous, and continuously differentiable; and that the corresponding merit functions have SC^1 property (i.e., they are continuously differentiable and their gradients are semismooth) and LC^1 property (i.e., they are continuously differentiable and their gradients are Lipschitz continuous) under suitable assumptions. Based on the new NCP-functions and the corresponding merit functions, we investigate a derivative free algorithm for the nonlinear complementarity problem and discuss its global convergence. Some preliminary numerical results are reported.

1. Introduction

In the last few decades, researchers have put a lot of their energy and attention on the complementarity problem due to its various applications in operation research, economics, and engineering (see, for example, [10,13,23]). The nonlinear complementarity problem (NCP) is to find a point $x \in \Re^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^{\mathrm{T}}F(x) = 0, \tag{1.1}$$

where $F : \Re^n \to \Re^n$ is a continuously differentiable mapping with $F := (F_1, F_2, \ldots, F_n)^{\mathrm{T}}$.

Many solution methods have been developed to solve NCP (1.1), for example, [3,5,13–17,19,23,27,28]. For more details, please refer to the excellent monograph [9]. One of the most popular methods is to reformulate the NCP (1.1) as an unconstrained optimization problem and then to solve the reformulated problem by the unconstrained optimization technique. This kind of method is called the merit function method, where the merit function is generally constructed by some NCP-function.

Definition 1.1. A function $\phi : \Re^2 \to \Re$ is called an NCP-function [2,18,25,26], if it satisfies

$$\phi(a, b) = 0 \iff a \geq 0,\ b \geq 0, \qquad ab = 0.$$

Furthermore, if $\phi(a, b) \geq 0$ for all $(a, b) \in \Re^2$ then the NCP-function ϕ is called a nonnegative NCP-function. In addition, if a function $\Psi : \Re^n \to \Re$ is nonnegative and $\Psi(x) = 0$ if and only if x solves the NCP, then Ψ is called a merit function for the NCP.

* Corresponding author. Tel.: +86 22 27402850; fax: +86 22 27403425.
E-mail addresses: huangzhenghai@tju.edu.cn (Z.-H. Huang), jschen@math.ntnu.edu.tw (J.-S. Chen).

1 Member of Mathematics Division, National Center for Theoretical Sciences, Taipei Office.

0377-0427/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2008.10.056

If ϕ is an NCP-function, then it is easy to see that the function $\Psi : \Re^n \to \Re$ defined by $\Psi(x) := \sum_{i=1}^n \phi^2(x_i, F_i(x))$ is a merit function for the NCP. Thus, finding a solution of the NCP is equivalent to finding a global minimum of the unconstrained minimization $\min_{x\in\Re^n} \Psi(x)$ with optimal value 0.

Many NCP-functions have been proposed in the literature. Among them, the FB function is one of the most popular NCP-functions, which is defined by

$$\phi(a, b) := \sqrt{a^2 + b^2} - a - b, \quad \forall (a, b) \in \Re^2.$$

One of the main generalization of FB function was given in [18]:

$$\phi_\theta(a, b) := \sqrt{(a - b)^2 + \theta ab} - a - b, \quad \theta \in (0, 4), \forall (a, b) \in \Re^2. \tag{1.2}$$

Another main generalization was given in [5]:

$$\phi_p(a, b) := \sqrt[p]{|a|^p + |b|^p} - a - b, \quad p \in (1, \infty), \forall (a, b) \in \Re^2. \tag{1.3}$$

It has been proved in [3–6,18,22] that the functions ϕ_θ given in (1.2) and ϕ_p given in (1.3) possess a system of favorite properties, such as, strong semismoothness, Lipschitz continuity, and continuous differentiability. It has also been proved that the corresponding merit functions of ϕ_θ and ϕ_p have SC^1 property (i.e., they are continuously differentiable and their gradients are semismooth) and LC^1 property (i.e., they are continuously differentiable and their gradients are Lipschitz continuous) under suitable assumptions.

Motivated by [18,5], we introduce in this paper the following functions:

$$\phi_{\theta p}(a, b) := \sqrt[p]{\theta(|a|^p + |b|^p) + (1 - \theta)|a - b|^p} - a - b, \quad p > 1, \theta \in (0, 1], (a, b) \in \Re^2 \tag{1.4}$$

and

$$\Psi_{\theta p}(x) := \frac{1}{2} \sum_{i=1}^n \phi_{\theta p}^2(x_i, F_i(x)). \tag{1.5}$$

Is the function $\phi_{\theta p}$ an NCP-function? If it is, do the functions given by (1.4) *and* (1.5) *have the same properties as those known functions mentioned above? Furthermore, how is the numerical behavior of the merit function methods based on the functions defined by* (1.4) and (1.5)?

In this paper, we will answer the questions mentioned above partly. Firstly, we show that the function $\phi_{\theta p}$ defined by (1.4) is an NCP-function; and discuss some favorite properties of the NCP-function (1.4) and its nonnegative NCP-function, including strong semismoothness, Lipschitz continuity, and continuous differentiability. Since the function $\phi_{\theta p}$ defined by (1.4) is an NCP-function, it follows that the function $\Psi_{\theta p}$ defined by (1.5) is a merit function associated to the NCP-function $\phi_{\theta p}$. We also show that the merit function $\Psi_{\theta p}$ has SC^1 property and LC^1 property. Secondly, we investigate a derivative free method based on the functions defined by (1.4) and (1.5) and show its global convergence. (Note: usually the nonsmooth Newton method is faster than the derivative free method for solving NCPs. However, the derivative free algorithm may overcome the case where strong conditions are sometimes needed to guarantee that the Jacobian of the merit function is nonsingular or very expensive to compute.) Thirdly, we report the preliminary numerical results for test problems from MCPLIB. The preliminary numerical results show, on the average, that the algorithm works better when $\theta = 1$ (according to the FB-type function), $\theta = 0.9$ and $\theta = 0.25$, and when $p = 1.1$ or $p = 2$ or $p = 20$ generally.

The rest of this paper is organized as follows: Various properties of the new NCP-function (1.4) and the nonnegative NCP-function associated to (1.4) are established in the next section. In Section 3, some properties of the merit function defined by (1.5) are analyzed. In Section 4, we investigate a derivative free algorithm for the NCP and show its global convergence. Some preliminary numerical results are reported in Section 5 and final conclusions are given in the last section.

Throughout this paper, unless stated otherwise, all vectors are column vectors, the subscript T denotes transpose, $\Re^n$ denotes the space of n-dimensional real column vectors, and $\Re^n_+$ (respectively, $\Re^n_{++}$) denotes the nonnegative (respectively, positive) orthant in $\Re^n$. For any vectors $u, v \in \Re^n$, we write $(u^T, v^T)^T$ as (u, v) for simplicity. For $x \in \Re^n$, we use $x \geq 0$ (respectively, $x > 0$) to mean $x \in \Re^n_+$ (respectively, $x \in \Re^n_{++}$). We use ":=" to mean "be defined as". We denote by $\|u\|$ the 2-norm of u and $\|u\|_p$ the p-norm with $p > 1$. We use ∇F to denote the gradient of F (while $\frac{\partial F(x)}{\partial x_i}$ denotes to the i-th component of the gradient of F) and $\nabla^2 F$ to denote the second order derivative of F. We use $\alpha = o(\beta)$ (respectively, $\alpha = O(\beta)$) to mean $\frac{\alpha}{\beta}$ tends to zero (respectively, bounded uniformly) as $\beta \to 0$.

2. Properties of the new NCP-function

In this section, we show that the function $\phi_{\theta p}$ defined by (1.4) is an NCP-function, and discuss its properties which are similar to those obtained in [3,5] for the function ϕ_p defined by (1.3). We also study a nonnegative NCP-function associated with $\phi_{\theta p}$, and discuss its properties. In addition, we discuss the semismooth-related properties due to its importance in semismooth and smooth analysis [8,10,15,16,20,24].

For convenience, we define

$$\eta_{\theta p}(a,b) := \sqrt[p]{\theta(|a|^p + |b|^p) + (1-\theta)|a-b|^p}, \quad p > 1, \theta \in (0,1], (a,b) \in \Re^2. \tag{2.1}$$

The proofs of the following propositions are trivial, we omit their proofs here.

Proposition 2.1. *The function $\phi_{\theta p}$ defined by (1.4) is an NCP-function.*

Proposition 2.2. *The function $\eta_{\theta p}$ defined by (2.1) is a norm on $\Re^2$ for all $p > 1, \theta \in (0,1]$.*

Now, we briefly introduce the concept of semismoothness, which was originally introduced in [20] for functionals and was extended to vector valued functions in [24]. A locally Lipschitz function $F : \Re^n \to \Re^m$, which has the generalized Jacobian $\partial F(x)$ in the sense of Clarke [8], is said to be semismooth (or strongly semismooth) at $x \in \Re^n$, if F is directionally differentiable at x and

$$F(x+h) - F(x) - Vh = o(\|h\|) \quad (\text{or} = O(\|h\|^2))$$

holds for any $V \in \partial F(x+h)$.

Proposition 2.3. *Let $\phi_{\theta p}$ be defined by (1.4), then for all $\theta \in (0,1]$ and $p > 1$,*

(i) *$\phi_{\theta p}$ is sub-additive, i.e., $\phi_{\theta p}((a,b)+(c,d)) \le \phi_{\theta p}(a,b) + \phi_{\theta p}(c,d)$ for all $(a,b),(c,d) \in \Re^2$;*
(ii) *$\phi_{\theta p}$ is positive homogenous, i.e., $\phi_{\theta p}(\alpha(a,b)) = \alpha\phi_{\theta p}(a,b)$ for all $(a,b) \in \Re^2$ and $\alpha > 0$;*
(iii) *$\phi_{\theta p}$ is a convex function on $\Re^2$, i.e., $\phi_{\theta p}(\alpha(a,b)+(1-\alpha)(c,d)) \le \alpha\phi_{\theta p}(a,b)+(1-\alpha)\phi_{\theta p}(c,d)$ for all $(a,b),(c,d) \in \Re^2$ and $\alpha \in [0,1]$;*
(iv) *$\phi_{\theta p}$ is Lipschitz continuous on $\Re^2$;*
(v) *$\phi_{\theta p}$ is continuously differentiable on $\Re^2 \setminus \{(0,0)\}$;*
(vi) *$\phi_{\theta p}$ is strongly semismooth on $\Re^2$.*

Proof. By using $\phi_{\theta p}((a,b)) = \eta_{\theta p}(a,b) - (a+b)$ and Proposition 2.2, we can obtain that the results (i), (ii), and (iii) hold.

Consider the result (iv). Since $\eta_{\theta p}$ is a norm on $\Re^2$ from Proposition 2.2 and any two norms in finite dimensional space are equivalent, it follows that there exists a positive constant κ such that

$$\eta_{\theta p}(a,b) \le \kappa\|(a,b)\|, \quad \forall (a,b) \in \Re^2,$$

where $\|\cdot\|$ represents the Euclidean norm on $\Re^2$. Hence, for all $(a,b),(c,d) \in \Re^2$,

$$\begin{aligned} |\phi_{\theta p}(a,b) - \phi_{\theta p}(c,d)| &= |\eta_{\theta p}(a,b) - (a+b) - \eta_{\theta p}(c,d) + (c+d)| \\ &\le |\eta_{\theta p}(a,b) - \eta_{\theta p}(c,d)| + |a-c| + |b-d| \\ &\le \eta_{\theta p}(a-c,b-d) + \sqrt{2}\|(a-c,b-d)\| \\ &\le \kappa\|(a-c,b-d)\| + \sqrt{2}\|(a-c,b-d)\| \\ &= (\kappa+\sqrt{2})\|(a-c,b-d)\|. \end{aligned}$$

Hence, $\phi_{\theta p}$ is Lipschitz continuous with Lipschitz constant $\kappa + \sqrt{2}$, i.e., the result (iv) holds.

Consider the result (v). If $(a,b) \ne (0,0)$, then $\eta_{\theta p}(a,b) \ne 0$ by Proposition 2.2. By a direct calculation, we get

$$\frac{\partial\phi_{\theta p}(a,b)}{\partial a} = \frac{\theta\,\mathrm{sgn}(a)|a|^{p-1} + (1-\theta)\mathrm{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}(a,b)^{p-1}} - 1; \tag{2.2}$$

$$\frac{\partial\phi_{\theta p}(a,b)}{\partial b} = \frac{\theta\,\mathrm{sgn}(b)|b|^{p-1} - (1-\theta)\mathrm{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}(a,b)^{p-1}} - 1, \tag{2.3}$$

where $\mathrm{sgn}(\cdot)$ is the symbol function. It is easy to see from (2.2) and (2.3) that the result (v) holds.

Consider the result (vi). Since $\phi_{\theta p}$ is a convex function by the result (iii), we get that it is a semismooth function. Noticing that $\phi_{\theta p}$ is continuously differentiable except $(0,0)$, it is sufficient to prove that it is strongly semismooth at $(0,0)$. For any $(h,k) \in \Re^2 \setminus \{(0,0)\}$, $\phi_{\theta p}$ is differentiable at (h,k), and hence, $\nabla\phi_{\theta p}(h,k) = \left(\frac{\partial\phi_{\theta p}(h,k)}{\partial a}, \frac{\partial\phi_{\theta p}(h,k)}{\partial b}\right)^{\mathrm{T}}$. So,

$$\begin{aligned} &\phi_{\theta p}((0,0)+(h,k)) - \phi_{\theta p}(0,0) - \left(\frac{\partial\phi_{\theta p}(h,k)}{\partial a}, \frac{\partial\phi_{\theta p}(h,k)}{\partial b}\right)\begin{pmatrix} h \\ k \end{pmatrix} \\ &\quad = \sqrt[p]{\theta(|h|^p + |k|^p) + (1-\theta)|h-k|^p} - (h+k) \\ &\qquad - \left(\frac{\mathrm{sgn}(h)|h|^{p-1} + \mathrm{sgn}(h-k)|h-k|^{p-1}}{\eta_{\theta p}(h,k)^{p-1}} - 1\right)h - \left(\frac{\mathrm{sgn}(k)|k|^{p-1} - \mathrm{sgn}(h-k)|h-k|^{p-1}}{\eta_{\theta p}(h,k)^{p-1}} - 1\right)k \end{aligned}$$

$$
\begin{aligned}
&= \sqrt[p]{\theta(|h|^p+|k|^p)+(1-\theta)|h-k|^p} - \frac{\mathrm{sgn}(h)|h|^{p-1}h+\mathrm{sgn}(k)|k|^{p-1}k+\mathrm{sgn}(h-k)|h-k|^{p-1}(h-k)}{\eta_{\theta p}(h,k)^{p-1}} \\
&= \sqrt[p]{\theta(|h|^p+|k|^p)+(1-\theta)|h-k|^p} - \frac{|h|^p+|k|^p+|h-k|^p}{\eta_{\theta p}(h,k)^{p-1}} \\
&= \eta_{\theta p}(h,k) - \frac{|h|^p+|k|^p+|h-k|^p}{\eta_{\theta p}(h,k)^{p-1}} \\
&= \frac{\eta_{\theta p}(h,k)^p-(|h|^p+|k|^p+|h-k|^p)}{\eta_{\theta p}(h,k)^{p-1}} \\
&= 0 \\
&= O(\|(h,k)\|^2).
\end{aligned}
$$

Thus, we obtain that $\phi_{\theta p}$ is strongly semismooth.

We complete the proof. □

Proposition 2.4. *Let $\phi_{\theta p}$ be defined by (1.4) and $\{(a^k, b^k)\} \subseteq \Re^2$. Then, $|\phi_{\theta p}(a^k, b^k)| \to \infty$ if one of the following conditions is satisfied.*

(i) $a^k \to -\infty$; (ii) $b^k \to -\infty$; (iii) $a^k \to \infty$ *and* $b^k \to \infty$.

Proof. (i) Suppose that $a^k \to -\infty$. If $\{b^k\}$ is bounded from above, then the result holds trivially. When $b^k \to \infty$, we have $-a^k > 0$ and $b^k > 0$ for all k sufficiently large, and hence,

$$\sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k-b^k|^p} - b^k \geq \sqrt[p]{\theta|b^k|^p+(1-\theta)|b^k|^p} - b^k = 0.$$

This, together with $-a^k \to \infty$ and the definition of $\phi_{\theta p}$, implies that the result holds.

(ii) For the case of $b^k \to -\infty$, a similar analysis yields the result of the proposition.

(iii) Suppose that $a^k \to \infty$ and $b^k \to \infty$. Since $p > 1$ and $\theta \in (0, 1]$, we have $(1-\theta)|a^k - b^k|^p \leq (1-\theta)(|a^k|^p + |b^k|^p)$ for all sufficiently large k. Thus, for all sufficiently large k,

$$\sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k-b^k|^p} \leq \sqrt[p]{|a^k|^p+|b^k|^p},$$

and hence,

$$(a^k+b^k) - \sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k-b^k|^p} \geq (a^k+b^k) - \sqrt[p]{|a^k|^p+|b^k|^p}.$$

By [5, Lemma 3.1] we know that $(a^k + b^k) - \sqrt[p]{|a^k|^p + |b^k|^p} \to \infty$ as $k \to \infty$ when the condition (iii) is satisfied. Thus, we obtain that

$$|\phi_{\theta p}(a^k, b^k)| = (a^k+b^k) - \sqrt[p]{\theta(|a^k|^p+|b^k|^p)+(1-\theta)|a^k-b^k|^p} \to \infty$$

as $k \to \infty$, which completes the proof. □

Now, we define a nonnegative function, associated with the function $\phi_{\theta p}$, as follows.

$$\psi_{\theta p}(a, b) := \frac{1}{2}\phi_{\theta p}^2(a, b),\ p > 1,\ \theta \in (0, 1],\ (a, b) \in \Re^2. \tag{2.4}$$

Proposition 2.5. *Let $\psi_{\theta p}$ be defined by (2.4), then for all $\theta \in (0, 1]$ and $p > 1$,*

(i) *$\psi_{\theta p}$ is an NCP-function;*
(ii) *$\psi_{\theta p}(a, b) \geq 0$ for all $(a, b) \in \Re^2$;*
(iii) *$\psi_{\theta p}$ is continuously differentiable on $\Re^2$;*
(iv) *$\psi_{\theta p}$ is strongly semismooth on $\Re^2$;*
(v) *$\frac{\partial\psi_{\theta p}(a,b)}{\partial a} \cdot \frac{\partial\psi_{\theta p}(a,b)}{\partial b} \geq 0$ for all $(a, b) \in \Re^2$, where the equality holds if and only if $\phi_{\theta p}(a, b) = 0$;*
(vi) *$\frac{\partial\psi_{\theta p}(a,b)}{\partial a} = 0 \iff \frac{\partial\psi_{\theta p}(a,b)}{\partial b} = 0 \iff \phi_{\theta p}(a, b) = 0$.*

Proof. By the definition of $\psi_{\theta p}$, it is easy to see that the results (i) and (ii) hold.

Consider the result (iii). By using Proposition 2.3 and the definition of $\psi_{\theta p}$, it is sufficient to prove that $\psi_{\theta p}$ is differentiable at $(0, 0)$ and the gradient is continuous at $(0, 0)$. In fact, for all $(a, b) \in \Re^2 \setminus \{(0, 0)\}$, we have,

$$
\begin{aligned}
|\phi_{\theta p}(a,b)| &= \left|\sqrt[p]{\theta(|a|^p+|b|^p)+(1-\theta)|a-b|^p} - a - b\right| \\
&\leq \left|\sqrt[p]{\theta|a|^p} + \sqrt[p]{\theta|b|^p} + \sqrt[p]{(1-\theta)|a-b|^p}\right| + |a| + |b| \\
&\leq |a| + |b| + |a-b| + |a| + |b| \\
&\leq 3(|a|+|b|),
\end{aligned}
$$

where the second inequality follows from $p > 1$ and the third inequality follows from $\theta \in (0, 1]$. Hence,

$$\psi_{\theta p}(a, b) - \psi_{\theta p}(0, 0) = \frac{1}{2}\phi^2_{\theta p}(a, b) \le \frac{1}{2}\,(3(|a| + |b|))^2 \le O(|a|^2 + |b|^2).$$

Thus, similar to that of [7, Proposition 1], we can get that $\psi_{\theta p}$ is differentiable at $(0, 0)$ with $\nabla\psi_{\theta p}(0, 0) = (0, 0)^{\mathrm{T}}$. Now, we prove that for all $(a, b) \in \Re^2 \setminus \{(0, 0)\}$,

$$\left|\frac{\theta \operatorname{sgn}(a)|a|^{p-1} + (1-\theta)\operatorname{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}(a, b)^{p-1}}\right| \le 1, \tag{2.5}$$

$$\left|\frac{\theta \operatorname{sgn}(b)|b|^{p-1} - (1-\theta)\operatorname{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}(a, b)^{p-1}}\right| \le 1. \tag{2.6}$$

In fact,

$$\begin{aligned}
&\left|\frac{\theta \operatorname{sgn}(a)|a|^{p-1} + (1-\theta)\operatorname{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}(a, b)^{p-1}}\right| \le \frac{\theta|a|^{p-1} + (1-\theta)|a-b|^{p-1}}{\eta_{\theta p}(a, b)^{p-1}}\\
&= \frac{\theta^{\frac{1}{p}}|\theta^{\frac{1}{p}}a|^{p-1} + (1-\theta)^{\frac{1}{p}}|(1-\theta)^{\frac{1}{p}}(a-b)|^{p-1}}{\eta_{\theta p}(a, b)^{p-1}}\\
&\le \frac{\left(\left(\theta^{\frac{1}{p}}\right)^p + \left((1-\theta)^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}}\left(\left(\left|\theta^{\frac{1}{p}}a\right|^{p-1}\right)^{\frac{p}{p-1}} + \left(\left|(1-\theta)^{\frac{1}{p}}(a-b)\right|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}}{\eta_{\theta p}(a, b)^{p-1}}\\
&= \frac{(\theta + (1-\theta))(x^p + z^p)^{\frac{p-1}{p}}}{\eta_{\theta p}(a, b)^{p-1}}\\
&= \frac{(x^p + z^p)^{\frac{p-1}{p}}}{(x^p + y^p + z^p)^{\frac{p-1}{p}}}\\
&= \left(\frac{x^p + z^p}{x^p + y^p + z^p}\right)^{\frac{p-1}{p}}\\
&\le 1,
\end{aligned}$$

where $x := |\theta^{\frac{1}{p}}a|^p$, $y := |\theta^{\frac{1}{p}}b|^p$, $z := |(1-\theta)^{\frac{1}{p}}(a-b)|^p$; the first inequality follows from the triangle inequality; the second inequality follows from the well-known Hölder inequality; the second equality follows from the definitions of x and z; the third equality follows from the definitions of $\eta_{\theta p}(a, b)$, x, y and z; and the third inequality follows from the fact that x, y and z are all nonnegative. So, (2.5) holds. Similar analysis will derive that (2.6) holds.

Thus, it follows from (2.5) and (2.6) that both $\frac{\partial\phi_{\theta p}(a,b)}{\partial a}$ and $\frac{\partial\phi_{\theta p}(a,b)}{\partial b}$ are uniformly bounded. Since $\phi_{\theta p}(a, b) \to 0$ as $(a, b) \to (0, 0)$, we get the desired result.

Consider the result (iv). Since the composition of strongly semismooth function is also strongly semismooth (see [11, Theorem 19]), by Proposition 2.3(vi) and the definition of $\psi_{\theta p}$ we obtain that the desired result holds.

Consider the result (v). It is obvious that $\frac{\partial\psi_{\theta p}(a,b)}{\partial a} = 0$ when $(a, b) = (0, 0)$. Now, suppose that $(a, b) \ne (0, 0)$. Since

$$\frac{\partial\psi_{\theta p}(a, b)}{\partial a}\cdot\frac{\partial\psi_{\theta p}(a, b)}{\partial b} = \frac{\partial\phi_{\theta p}(a, b)}{\partial a}\cdot\frac{\partial\phi_{\theta p}(a, b)}{\partial b}\cdot\phi_{\theta p}(a, b)^2, \tag{2.7}$$

by (2.2), (2.3), (2.5) and (2.6), we obtain that $\frac{\partial\phi_{\theta p}(a,b)}{\partial a} \le 0$ and $\frac{\partial\phi_{\theta p}(a,b)}{\partial b} \le 0$ for all $(a, b) \in \Re^2$, that is, the first result of (v) holds. In addition, from (2.7) it is obvious that the sufficient condition of the second result of (v) holds. Now, we suppose that $\frac{\partial\psi_{\theta p}(a,b)}{\partial a}\cdot\frac{\partial\psi_{\theta p}(a,b)}{\partial b} = 0$. Then, it is sufficient to prove that $\phi_{\theta p}(a, b) = 0$ when $\frac{\partial\phi_{\theta p}(a,b)}{\partial a}\cdot\frac{\partial\phi_{\theta p}(a,b)}{\partial b} = 0$. Suppose that $\frac{\partial\phi_{\theta p}(a,b)}{\partial a} = 0$ without loss of generality. From the proof of (iii) in this proposition, it is easy to see that it must be $y = 0$, and hence, $b = 0$. After a simple symbol discussion for (2.2), we may get $a \ge 0$. Hence $\phi_{\theta p}(a, b) = 0$ by Proposition 2.1. So, the result (v) holds.

Consider the result (vi). Since

$$\frac{\partial\psi_{\theta p}(a, b)}{\partial a} = \frac{\partial\phi_{\theta p}(a, b)}{\partial a}\phi_{\theta p}(a, b), \qquad \frac{\partial\psi_{\theta p}(a, b)}{\partial b} = \frac{\partial\phi_{\theta p}(a, b)}{\partial b}\phi_{\theta p}(a, b),$$

the result (vi) is immediately satisfied from the above analysis.

We complete the proof. □

Lemma 2.1 *([21, Theorem 3.3.5]). If $f : D \subseteq \Re^n \to \Re^m$ has a second derivative at each point of a convex set $D_0 \subseteq D$, then $\|\nabla f(y) - \nabla f(x)\| \le \sup_{0\le t\le 1} \|\nabla^2 f(x + t(y - x))\| \cdot \|y - x\|$.*

Theorem 2.1. *The gradient function of the function $\psi_{\theta p}$ defined by (2.4) with $p \ge 2, \theta \in (0, 1]$ is Lipschitz continuous, that is, there exists a positive constant L such that*

$$\|\nabla\psi_{\theta p}(a, b) - \nabla\psi_{\theta p}(c, d)\| \le L\|(a, b) - (c, d)\| \tag{2.8}$$

holds for all $(a, b), (c, d) \in \Re^2$.

Proof. It follows from the definition of $\psi_{\theta p}$ and the proof of Proposition 2.5(iii) that $\nabla\psi_{\theta p}(a, b) = \nabla\phi_{\theta p}(a, b)\phi_{\theta p}(a, b)$ when $(a, b) \neq (0, 0)$, and $\nabla\psi_{\theta p}(0, 0) = (0, 0)^T$. From Proposition 2.5(iii) we know that $\psi_{\theta p}$ is continuously differentiable. The proof is divided into the following three cases.

Case 1. If $(a, b) = (c, d) = (0, 0)$, it follows from Proposition 2.5 that $\nabla\psi_{\theta p}(0, 0) = (0, 0)$, and hence, (2.8) holds for all positive number L.

Case 2. Consider the case that one of (a, b) and (c, d) is $(0, 0)$, but not all. We assume that $(a, b) \neq (0, 0)$ and $(c, d) = (0, 0)$ without loss of generality. Then,

$$\begin{aligned}
\|\nabla\psi_{\theta p}(a, b) - \nabla\psi_{\theta p}(c, d)\| &= \|\nabla\psi_{\theta p}(a, b) - (0, 0)\| \\
&= \|\nabla\phi_{\theta p}(a, b)\phi_{\theta p}(a, b) - (0, 0)\| \\
&= \|\nabla\phi_{\theta p}(a, b)\|\phi_{\theta p}(a, b) \\
&= \|\nabla\phi_{\theta p}(a, b)\||\phi_{\theta p}(a, b) - \phi_{\theta p}(0, 0)| \\
&\le L\|(a, b) - (0, 0)\|,
\end{aligned}$$

where the inequality follows from the fact that $\{\|\nabla\phi_{\theta p}(a, b)\|\}$ is uniformly bounded on $\Re^2$ (which can be obtained from the proof of Proposition 2.5(iii)) and $\phi_{\theta p}$ is Lipschitz continuous on $\Re^2$ given in Proposition 2.3(iv). Hence, (2.8) holds for some positive constant L.

Case 3. If both (a, b) and (c, d) are not $(0, 0)$, we will use Lemma 2.1 to prove (2.8) holds for this case. For simplicity, we denote

$$\begin{aligned}
\hat h_1 &:= \frac{\theta \operatorname{sgn}(a)|a|^{p-1} + (1-\theta)\operatorname{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}^{p-1}(a, b)}; \\
\hat h_2 &:= \frac{\theta \operatorname{sgn}(b)|b|^{p-1} - (1-\theta)\operatorname{sgn}(a-b)|a-b|^{p-1}}{\eta_{\theta p}^{p-1}(a, b)}; \\
\hat a_1 &:= (\theta|a|^{p-2} + (1-\theta)|a-b|^{p-2})\eta_{\theta p}^{p}(a, b); \\
\hat a_2 &:= -\hat h_1^2 \eta_{\theta p}^{2p-2}(a, b); \\
\hat b_1 &:= -(1-\theta)|a-b|^{p-2}\eta_{\theta p}^{p}(a, b); \\
\hat b_2 &:= -\hat h_1 \hat h_2 \eta_{\theta p}^{2p-2}(a, b); \\
\hat c_1 &:= (\theta|b|^{p-2} + (1-\theta)|a-b|^{p-2})\eta_{\theta p}^{p}(a, b); \\
\hat c_2 &:= -\hat h_2^2 \eta_{\theta p}^{2p-2}(a, b).
\end{aligned}$$

When $(a, b) \neq (0, 0)$, by a direct calculation, we have

$$\begin{aligned}
\frac{\partial^2\psi_{\theta p}(a, b)}{\partial a^2} &= (\hat h_1 - 1)^2 + (p-1)\frac{\hat a_1 + \hat a_2}{\eta_{\theta p}^{2p-1}(a, b)}(\eta_{\theta p}(a, b) - (a+b)); \\
\frac{\partial^2\psi_{\theta p}(a, b)}{\partial a\partial b} &= (\hat h_1 - 1)(\hat h_2 - 1) + (p-1)\frac{\hat b_1 + \hat b_2}{\eta_{\theta p}^{2p-1}(a, b)}(\eta_{\theta p}(a, b) - (a+b)); \\
\frac{\partial^2\psi_{\theta p}(a, b)}{\partial b^2} &= (\hat h_2 - 1)^2 + (p-1)\frac{\hat c_1 + \hat c_2}{\eta_{\theta p}^{2p-1}(a, b)}(\eta_{\theta p}(a, b) - (a+b)); \\
\frac{\partial^2\psi_{\theta p}(a, b)}{\partial b\partial a} &= \frac{\partial^2\psi_{\theta p}(a, b)}{\partial a\partial b},
\end{aligned}$$

where the last equality follows from the fact that $\frac{\partial^2\psi_{\theta p}(a,b)}{\partial a\partial b}$ and $\frac{\partial^2\psi_{\theta p}(a,b)}{\partial b\partial a}$ are continuous when $(a, b) \neq (0, 0)$. Since $p \ge 2$ and $\eta_{\theta p}(\cdot, \cdot)$ is a norm on $\Re^2$ by Proposition 2.2, it is easy to verify that

$$|a+b| \le |a| + |b| \le \sqrt[p]{|a|^p + |b|^p} + \sqrt[p]{|a|^p + |b|^p} = 2\|(a, b)\|_p \le 2\kappa^*\eta_{\theta p}(a, b),$$

where $\kappa^* > 0$ is a constant depending on θ and p.

$$\begin{aligned}
\frac{\hat{a}_1}{\eta_{\theta p}^{2p-2}(a,b)} &= \frac{\theta|a|^{p-2} + (1-\theta)|a-b|^{p-2}}{\eta_{\theta p}^{p-2}(a,b)} \\
&= \frac{\theta|a|^{p-2}}{\eta_{\theta p}^{p-2}(a,b)} + \frac{(1-\theta)|a-b|^{p-2}}{\eta_{\theta p}^{p-2}(a,b)} \\
&\le \theta^{\frac{2}{p}} + (1-\theta)^{\frac{2}{p}} \\
&\le 2.
\end{aligned}$$

Similarly, we have

$$\frac{|\hat{b}_1|}{\eta_{\theta p}^{2p-2}(a,b)} \le 1; \qquad \frac{\hat{c}_1}{\eta_{\theta p}^{2p-2}(a,b)} \le 2.$$

These, together with the results $|\hat{h}_1| \le 1$ and $|\hat{h}_2| \le 1$ given in Proposition 2.5, yield

$$\frac{|\hat{a}_2|}{\eta_{\theta p}^{2p-2}(a,b)} \le 1; \qquad \frac{|\hat{b}_2|}{\eta_{\theta p}^{2p-2}(a,b)} \le 1; \qquad \frac{|\hat{c}_2|}{\eta_{\theta p}^{2p-2}(a,b)} \le 1.$$

Thus,

$$\begin{aligned}
\left|\frac{\partial^2 \psi_{\theta p}(a,b)}{\partial a^2}\right| &= \left|(\hat{h}_1 - 1)^2 + (p-1)\frac{\hat{a}_1 + \hat{a}_2}{\eta_{\theta p}^{2p-1}(a,b)}(\eta_{\theta p}(a,b) - (a+b))\right| \\
&\le |(\hat{h}_1 - 1)^2| + (p-1)\left(\left|\frac{\hat{a}_1 + \hat{a}_2}{\eta_{\theta p}^{2p-1}(a,b)}\eta_{\theta p}(a,b)\right| + \left|\frac{\hat{a}_1 + \hat{a}_2}{\eta_{\theta p}^{2p-1}(a,b)}(a+b)\right|\right) \\
&\le 4 + (1+2\kappa^*)(p-1)\left(\frac{\hat{a}_1}{\eta_{\theta p}^{2p-2}(a,b)} + \frac{|\hat{a}_2|}{\eta_{\theta p}^{2p-2}(a,b)}\right) \\
&\le 4 + 3(1+2\kappa^*)(p-1); \\
\left|\frac{\partial^2 \psi_{\theta p}(a,b)}{\partial a \partial b}\right| &= \left|(\hat{h}_1 - 1)(\hat{h}_2 - 1) + (p-1)\frac{\hat{a}_1 + \hat{a}_2}{\eta_{\theta p}^{2p-1}(a,b)}(\eta_{\theta p}(a,b) - (a+b))\right| \\
&\le |(\hat{h}_1 - 1)(\hat{h}_2 - 1)| + (p-1)\left(\left|\frac{\hat{b}_1 + \hat{b}_2}{\eta_{\theta p}^{2p-1}(a,b)}\eta_{\theta p}(a,b)\right| + \left|\frac{\hat{b}_1 + \hat{b}_2}{\eta_{\theta p}^{2p-1}(a,b)}(a+b)\right|\right) \\
&\le 4 + (1+2\kappa^*)(p-1)\left(\frac{|\hat{b}_1|}{\eta_{\theta p}^{2p-2}(a,b)} + \frac{|\hat{b}_2|}{\eta_{\theta p}^{2p-2}(a,b)}\right) \\
&\le 4 + 2(1+2\kappa^*)(p-1); \\
\left|\frac{\partial^2 \psi_{\theta p}(a,b)}{\partial b^2}\right| &= \left|(\hat{h}_2 - 1)^2 + (p-1)\frac{\hat{c}_1 + \hat{c}_2}{\eta_{\theta p}^{2p-1}(a,b)}(\eta_{\theta p}(a,b) - (a+b))\right| \\
&\le |(\hat{h}_2 - 1)^2| + (p-1)\left(\left|\frac{\hat{c}_1 + \hat{c}_2}{\eta_{\theta p}^{2p-1}(a,b)}\eta_{\theta p}(a,b)\right| + \left|\frac{\hat{c}_1 + \hat{c}_2}{\eta_{\theta p}^{2p-1}(a,b)}(a+b)\right|\right) \\
&\le 4 + (1+2\kappa^*)(p-1)\left(\frac{\hat{c}_1}{\eta_{\theta p}^{2p-2}(a,b)} + \frac{|\hat{c}_2|}{\eta_{\theta p}^{2p-2}(a,b)}\right) \\
&\le 4 + 3(1+2\kappa^*)(p-1).
\end{aligned}$$

Hence, there exists a positive constant L such that (2.8) holds by Lemma 2.1.

Combining Cases 1–3, we complete the proof. □

Remark 2.1. It should be noted that $\nabla\psi_{\theta p}$ is not Lipschitz continuous for all $\theta \in (0, 1]$ when $p \in (1, 2)$. In fact, if we fixed $\theta = 1$. For $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$, we have

$$\begin{aligned}
\|\nabla\psi_{1p}(a,b) - \nabla\psi_{1p}(c,d)\| &= \|\nabla\phi_{1p}(a,b)\phi_{1p}(a,b) - \nabla\phi_{1p}(c,d)\phi_{1p}(c,d)\| \\
&\ge \left|\frac{\operatorname{sgn}(a)|a|^{p-1}}{\|(a,b)\|_p^{p-1}}\phi_{1p}(a,b) - \frac{\operatorname{sgn}(c)|c|^{p-1}}{\|(c,d)\|_p^{p-1}}\phi_{1p}(c,d) + \phi_{1p}(c,d) - \phi_{1p}(a,b)\right|
\end{aligned}$$

$$\geq \left| \frac{\operatorname{sgn}(a)|a|^{p-1}}{\|(a,b)\|_p^{p-1}} \phi_{1p}(a,b) - \frac{\operatorname{sgn}(c)|c|^{p-1}}{\|(c,d)\|_p^{p-1}} \phi_{1p}(c,d) \right| - |\phi_{1p}(c,d) - \phi_{1p}(a,b)|$$
$$\geq \left| \frac{\operatorname{sgn}(a)|a|^{p-1}}{\|(a,b)\|_p^{p-1}} \phi_{1p}(a,b) - \frac{\operatorname{sgn}(c)|c|^{p-1}}{\|(c,d)\|_p^{p-1}} \phi_{1p}(c,d) \right| - (\kappa + \sqrt{2})\|(c,d) - (a,b)\|,$$

where $\kappa + \sqrt{2}$ is given in Proposition 2.3(iv). If we let $(a, b) = (1, -n)$, $(c, d) = (-1, -n)$ with $n \in (1, \infty)$, we have

$$\begin{aligned}
\left| \frac{\operatorname{sgn}(a)|a|^{p-1}}{\|(a,b)\|_p^{p-1}} \phi_{1p}(a,b) - \frac{\operatorname{sgn}(c)|c|^{p-1}}{\|(c,d)\|_p^{p-1}} \phi_{1p}(c,d) \right| &= \frac{\sqrt[p]{1+n^p} + (n-1)}{(1+n^p)^{(p-1)/p}} + \frac{\sqrt[p]{1+n^p} + (n+1)}{(1+n^p)^{(p-1)/p}} \\
&= 2\frac{\sqrt[p]{1+n^p} + n}{(1+n^p)^{(p-1)/p}} \\
&\geq \frac{4n}{(1+n^p)^{(p-1)/p}} \\
&= \frac{4n^{2-p}n^{p-1}}{(1+n^p)^{(p-1)/p}} \\
&= \frac{4n^{2-p}}{(1+(1/n)^p)^{(p-1)/p}} \\
&\geq n^{2-p},
\end{aligned}$$

where the first and the second inequalities follow from $2 > p > 1$ and $n > 1$. Since $\|(a, b) - (c, d)\| = 2$ and $n \in (1, \infty)$, form the above inequalities it is easy to verify that $\nabla\psi_{1p}$ is not Lipschitz continuous.

3. Properties of merit function

In this section, we consider the merit function for the NCP defined by (1.5), and then discuss its several important properties. These properties provide the theoretical basis for the algorithm we discuss in the next section. In addition, we also discuss the semismooth-related properties of the merit function.

Define

$$\Phi_{\theta p}(x) := \begin{pmatrix} \phi_{\theta p}(x_1, F_1(x)) \\ \cdots \\ \phi_{\theta p}(x_n, F_n(x)) \end{pmatrix}. \tag{3.1}$$

Then, the merit function defined by (1.5) can be written as

$$\Psi_{\theta p}(x) = \frac{1}{2}\|\Phi_{\theta p}(x)\|^2 = \sum_{i=1}^{n} \psi_{\theta p}(x_i, F_i(x)). \tag{3.2}$$

Proposition 3.1. (i) *The function $\psi_{\theta p}$ defined by (2.4) with $p \geq 2$, $\theta \in (0, 1]$ is an SC^1 function. Hence, if every F_i is an SC^1 function, then the function $\Psi_{\theta p}$ defined by (3.2) with $p \geq 2$, $\theta \in (0, 1]$ is also an SC^1 function.*

(ii) *If every F_i is an LC^1 function, then the function $\Phi_{\theta p}$ defined by (3.1) with $p > 1$, $\theta \in (0, 1]$ is strongly semismooth.*

(iii) *The function $\psi_{\theta p}$ defined by (2.4) with $p \geq 2$, $\theta \in (0, 1]$ is an LC^1 function. Hence, if every F_i is an LC^1 function, then the function $\Psi_{\theta p}$ defined by (3.2) with $p \geq 2$, $\theta \in (0, 1]$ is also an LC^1 function.*

Proof. (i) By Proposition 2.5, it is sufficient to prove that $\nabla\psi_{\theta p}$ is semismooth. It is obvious from the proof of Theorem 2.1 that $\nabla\psi_{\theta p}(a, b)$ is continuously differentiable when $(a, b) \neq (0, 0)$, so we only need to show the semismoothness of $\nabla\psi_{\theta p}(a, b)$ at $(0, 0)$. For any $(h_1, h_2) \in \Re^2 \setminus \{(0, 0)\}$, we know that $\nabla\psi_{\theta p}$ is differentiable at (h_1, h_2), and hence, we only need to show that

$$\nabla\psi_{\theta p}(h_1, h_2) - \nabla\psi_{\theta p}(0, 0) - \nabla^2\psi_{\theta p}(h_1, h_2) \cdot (h_1, h_2)^{\mathrm{T}} = o(\|(h_1, h_2)\|). \tag{3.3}$$

In fact, let $\hat{a}_1, \hat{a}_2, \hat{b}_1, \hat{b}_2, \hat{c}_1, \hat{c}_2$ be similarly defined as those in Theorem 2.1 with (a, b) being replaced by (h_1, h_2). Denote

$$\hat{h}_3 := (p-1)\frac{\hat{a}_1 + \hat{a}_2}{\eta_{\theta p}^{2p-1}(h_1, h_2)} \phi_{\theta p}(h_1, h_2);$$
$$\hat{h}_4 := (p-1)\frac{\hat{b}_1 + \hat{b}_2}{\eta_{\theta p}^{2p-1}(h_1, h_2)} \phi_{\theta p}(h_1, h_2);$$

$$\hat{h}_5 := (p-1)\frac{\hat{c}_1+\hat{c}_2}{\eta_{\theta p}^{2p-1}(h_1,h_2)}\phi_{\theta p}(h_1,h_2),$$

and

$$\begin{aligned}
m_1 &:= (\theta|h_1|^{p-2}+(1-\theta)|h_1-h_2|^{p-2})\eta_{\theta p}^{p}(h_1,h_2)h_1-\hat{h}_1^2\eta_{\theta p}^{2p-2}(h_1,h_2)h_1;\\
m_2 &:= (1-\theta)|h_1-h_2|^{p-2}\eta_{\theta p}^{p}(h_1,h_2)h_2+\hat{h}_1\hat{h}_2\eta_{\theta p}^{2p-2}(h_1,h_2)h_2;\\
m_3 &:= (\theta|h_1|^{p-2}+(1-\theta)|h_1-h_2|^{p-2})\eta_{\theta p}^{p}(h_1,h_2)h_1-(1-\theta)|h_1-h_2|^{p-2}\eta_{\theta p}^{p}(h_1,h_2)h_2;\\
m_4 &:= \hat{h}_1\hat{h}_2\eta_{\theta p}^{2p-2}(h_1,h_2)h_2+\hat{h}_1^2\eta_{\theta p}^{2p-2}(h_1,h_2)h_1;\\
m_5 &:= (\theta\mathrm{sgn}(h_1)|h_1|^{p-1}+(1-\theta)\mathrm{sgn}(h_1-h_2)|h_1-h_2|^{p-1})\eta_{\theta p}^{p}(h_1,h_2);\\
m_6 &:= \hat{h}_1\hat{h}_2\eta_{\theta p}^{2p-2}(h_1,h_2)h_2+\hat{h}_1^2\eta_{\theta p}^{2p-2}(h_1,h_2)h_1.
\end{aligned}$$

Then,

$$\begin{pmatrix}H_1\\H_2\end{pmatrix} := \begin{pmatrix}\hat{h}_1-1\\ \hat{h}_2-1\end{pmatrix}\cdot\phi_{\theta p}(h_1,h_2)-\begin{pmatrix}0\\0\end{pmatrix}-\begin{pmatrix}(\hat{h}_1-1)^2+\hat{h}_3 & (\hat{h}_1-1)(\hat{h}_2-1)+\hat{h}_4\\ (\hat{h}_1-1)(\hat{h}_2-1)+\hat{h}_4 & (\hat{h}_2-1)^2+\hat{h}_5\end{pmatrix}\cdot\begin{pmatrix}h_1\\h_2\end{pmatrix}$$

and hence,

$$\begin{aligned}
H_1 &= (\hat{h}_1-1)\phi_{\theta p}(h_1,h_2)-((\hat{h}_1-1)^2+\hat{h}_3)h_1-((\hat{h}_1-1)(\hat{h}_2-1)+\hat{h}_4)h_2\\
&= (\hat{h}_1-1)\phi_{\theta p}(h_1,h_2)-\hat{h}_3h_1-\hat{h}_4h_2-(\hat{h}_1-1)((\hat{h}_1-1)h_1+(\hat{h}_2-1)h_2)\\
&= (\hat{h}_1-1)\phi_{\theta p}(h_1,h_2)-\hat{h}_3h_1-\hat{h}_4h_2-(\hat{h}_1-1)\phi_{\theta p}(h_1,h_2)\\
&= -(p-1)\left(\frac{\hat{a}_1+\hat{a}_2}{\eta_{\theta p}^{2p-1}(h_1,h_2)}h_1+\frac{\hat{b}_1+\hat{b}_2}{\eta_{\theta p}^{2p-1}(h_1,h_2)}h_2\right)\phi_{\theta p}(h_1,h_2)\\
&= -(p-1)\phi_{\theta p}(h_1,h_2)\left(\frac{m_1-m_2}{\eta_{\theta p}^{2p-1}(h_1,h_2)}\right)\\
&= -(p-1)\phi_{\theta p}(h_1,h_2)\left(\frac{m_3-m_4}{\eta_{\theta p}^{2p-1}(h_1,h_2)}\right)\\
&= -(p-1)\phi_{\theta p}(h_1,h_2)\left(\frac{m_5-m_6}{\eta_{\theta p}^{2p-1}(h_1,h_2)}\right)\\
&= -(p-1)\phi_{\theta p}(h_1,h_2)\left(\hat{h}_1-\hat{h}_1\frac{\hat{h}_1h_1+\hat{h}_2h_2}{\eta_{\theta p}(h_1,h_2)}\right)\\
&= -(p-1)\phi_{\theta p}(h_1,h_2)(\hat{h}_1-\hat{h}_1)\\
&= 0,
\end{aligned}$$

where the third equality follows from $\hat{h}_1h_1+\hat{h}_2h_2=\eta_{\theta p}$ given in the proof of Proposition 2.3 and the definition of $\phi_{\theta p}$, the fourth equality follows from the definitions of $\hat{h}_3$, $\hat{h}_4$, the fifth equality follows from the definitions of $\hat{a}_1$, $\hat{a}_2$, $\hat{b}_1$, $\hat{b}_2$, and the eighth equality follows from $\hat{h}_1h_1+\hat{h}_2h_2=\eta_{\theta p}$ given in the proof of Proposition 2.3.

Similar analysis yields $H_2=0$. Thus, $\nabla\psi_{\theta p}$ is semismooth. Furthermore, $\psi_{\theta p}$ is an SC^1 function.

(ii) Since the LC^1 function is strongly semismooth and the composition of strongly semismooth function is also strongly semismooth, it follows from Proposition 2.3(vi) that the desired results holds.

(iii) By using the above results, it is easy that the result (iii) holds.

We complete the proof. □

Remark 3.1. The results of Proposition 3.1(i)(iii) do not hold when $p\in(1,2)$ for all $\theta\in(0,1]$ since $\nabla\psi_{\theta p}$ is not locally Lipschitz continuous in general. For example, let $(a,b)=\left(\frac{1}{n},-1\right)$ and $(c,d)=\left(-\frac{1}{n},-1\right)$, similar to Remark 2.1, we can obtain that $\nabla\psi_{\theta p}$ is not Lipschitz continuous in any neighborhood of $(0,-1)$.

Definition 3.1. Let $F:\Re^n\to\Re^n$.

- F is said to be monotone if $(x-y)^{\mathrm{T}}(F(x)-F(y))\geq 0$ for all $x,y\in\Re^n$.
- F is said to be strongly monotone with modulus $\mu>0$ if $(x-y)^{\mathrm{T}}(F(x)-F(y))\geq\mu\|x-y\|^2$ for all $x,y\in\Re^n$.

Table 1
GAP (10^{-3}), CPU (seconds).

Problem	θ	$p = 1.5$				$p = 2$				$p = 3$			
		GAP	NF	IT	CPU	GAP	NF	IT	CPU	GAP	NF	IT	CPU
	0.1	3.6	86137	10846	56.437	3.6	77365	9811	32.375	3.54	81163	10202	48.703
	0.25	4.21	82397	11081	53.453	4.29	85263	11374	35.468	4.27	85754	11358	61.844
	0.5	4.01	45466	6804	29.156	3.1	43966	6441	17.906	4.17	45353	6577	26.828
sppe(1)	0.75	4.59	35460	5946	23.219	3.61	43137	7015	17.687	3.62	43652	7080	25.688
	0.9	7.05	25116	4481	16.079	6.9	26851	4732	11.094	6.74	26639	4754	15.75
	1	5.13	21003	3863	13.547	7.27	23194	4385	9.641	7.54	24758	4671	14.657
	0.1	3.6	84178	10589	52.531	3.91	80241	10128	33.734	3.04	79550	9915	47.969
	0.25	3.74	81997	11004	51.562	3.72	84976	11299	36.891	4.33	87541	11510	51.313
	0.5	4.05	44719	6674	28.75	2.95	41147	6051	18.156	3	44550	6442	26.047
sppe(2)	0.75	3.32	36050	6008	22.859	3.74	43372	7040	17.954	5.98	44847	7236	27.016
	0.9	7	24939	4430	16.188	4.05	27135	4752	11.25	6.51	27843	4905	16.766
	1	5.62	19257	3612	12.156	7.58	23910	4449	10.094	5.08	24432	4574	14.453
	0.1	1.74	1720	389	1.375	1.65	1745	400	0.782	1.64	1636	380	0.844
	0.25	1.8	1473	378	0.797	1.81	1599	422	0.719	1.89	1623	422	0.828
nash(1^+)	0.5	2.08	1755	489	0.922	2.16	1900	533	0.875	2.19	1429	398	0.766
	0.75	2.63	831	297	0.469	2.59	994	352	0.453	2.49	1010	347	0.531
	0.9	2.82	556	262	0.312	3.07	649	259	0.313	3.06	754	296	0.422
	1	3.02	484	256	0.312	2.92	672	315	0.453	3.36	769	336	0.453
	0.1	1.74	1732	392	1	1.65	1763	406	0.938	1.64	1647	383	0.922
	0.25	1.8	1157	307	0.625	1.81	1564	405	0.703	1.89	1653	432	0.875
nash(2^+)	0.5	2.08	1744	482	0.921	2.16	1858	509	0.843	2.19	1374	371	0.703
	0.75	2.63	800	276	0.438	2.59	941	335	0.438	2.49	1068	366	0.578
	0.9	2.82	561	255	0.312	3.06	703	275	0.36	3.08	626	242	0.344
	1	3	421	224	0.266	2.92	643	304	0.313	3.36	828	352	0.438
	0.1	9.9	87	41	0.031	9.36	87	41	0.015	9.34	87	41	0.031
	0.25	9.68	250	135	0.094	9.52	283	153	0.078	9.66	283	153	0.078
cycle	0.5	7.39	41	30	0.016	9.52	82	65	0.031	9.3	82	65	0.031
	0.75	8.65	11	10	0	5.88	11	10	0	9.63	8	7	0
	0.9	8.7	5	4	0	3.31	6	5	0.016	4.94	6	5	0
	1	5.19	8	6	0	0.00829	5	4	0	0.279	4	3	0
	0.1	1.21	221	108	0.141	1.25	182	88	0.094	1.25	182	88	0.094
	0.25	1.21	282	154	0.156	1.25	269	147	0.109	1.24	269	147	0.25
explcp	0.5	0.599	13	5	0	0.089	13	5	0.016	0.0185	13	5	0
	0.75	5.15	19	13	0.015	4.8	19	12	0	6.16	18	11	0
	0.9	0.39	10	7	0.016	0.393	12	7	0.015	2.84	16	8	0.016
	1	0.653	14	11	0	0.0723	9	6	0	0.00263	7	4	0

- F is said to be a P_0-function if $\max_{1\le j\le n, x_i\ne y_i}(x_i - y_i)(F_i(x) - F_i(y)) \ge 0$ for all $x, y \in \Re^n$ and $x \ne y$.
- F is said to be a uniform P-function with modulus $\mu > 0$ if $\max_{1\le j\le n}(x_i - y_i)(F_i(x) - F_i(y)) \ge \mu\|x - y\|^2$ for all $x, y \in \Re^n$.

Proposition 3.2. *Let $\Psi_{\theta p} : \Re^n \to \Re$ be defined by (3.2) with $p > 1, \theta \in (0, 1]$. Then $\Psi_{\theta p}(x) \ge 0$ for all $x \in \Re^n$ and $\Psi_{\theta p}(x) = 0$ if and only if x solves the NCP (1.1). Moreover, suppose that the solution set of the NCP (1.1) is nonempty, then x is a global minimizer of $\Psi_{\theta p}$ if and only if x solves the NCP (1.1).*

Proof. The result follows from Proposition 2.5 immediately. □

Proposition 3.3. *Let $\Psi_{\theta p} : \Re^n \to \Re$ be defined by (3.2) with $p > 1, \theta \in (0, 1]$. Suppose that F is either a monotone function or a P_0-function, then every stationary point of $\Psi_{\theta p}$ is a global minima of $\min_{x\in\Re^n} \Psi_{\theta p}(x)$; and therefore solves the NCP (1.1).*

Proof. By using Proposition 2.5 and [5, Lemma 2.1], the proof of the proposition is similar to the one given in [5, Proposition 3.4]. We omit it here. □

Proposition 3.4. *Let $\Psi_{\theta p}$ be defined by (3.2) with $\theta \in (0, 1]$ and $p > 1$. Suppose that F is either a strongly monotone function or a uniform P-function. Then the level sets*

$$\mathcal{L}(\Psi_{\theta p}, \gamma) := \{x \in \Re^n | \Psi_{\theta p}(x) \le \gamma\}$$

are bounded for all $\gamma \in \Re$.

Proof. Using Proposition 2.4, the proof is similar to the one given in [5, Proposition 3.5]. We omit it here. □

4. A derivative free algorithm

In this section, we study a derivative free algorithm for complementarity problems based on the new family of NCP-functions and the related merit functions. In addition, we prove the global convergence of the algorithm.

Table 2
GAP (10^{-3}), CPU (seconds).

Problem	θ	$p = 1.5$				$p = 2$				$p = 3$			
		GAP	NF	IT	CPU	GAP	NF	IT	CPU	GAP	NF	IT	CPU
	0.1	0.194	2275	390	1.032	0.199	2015	335	0.687	0.199	1825	305	0.64
	0.25	0.205	1581	291	0.578	0.229	2810	497	0.922	0.218	2643	463	0.937
gafni(1^+)	0.5	0.266	2305	444	0.843	0.266	3441	644	1.109	0.264	3699	687	1.312
	0.75	0.298	1132	250	0.422	0.312	1900	425	0.641	0.316	2005	430	0.734
	0.9	0.334	1084	301	0.406	0.302	1107	273	0.36	0.337	1446	338	0.531
	1	0.354	665	194	0.25	0.391	1058	290	0.359	0.393	1312	337	0.485
	0.1	0.194	1953	326	0.75	0.199	2187	366	0.829	0.199	1865	312	0.671
	0.25	0.205	1508	272	0.547	0.229	2497	430	0.812	0.218	2528	436	0.922
gafni(2^+)	0.5	0.266	2029	382	0.734	0.266	3243	604	1.125	0.264	3413	629	1.218
	0.75	0.298	1425	314	0.515	0.312	1887	419	0.625	0.316	2070	443	0.75
	0.9	0.334	973	254	0.359	0.302	878	211	0.297	0.337	1020	239	0.375
	1	0.354	716	210	0.265	0.391	698	180	0.235	0.393	1092	272	0.391
	0.1	0.194	1560	254	0.625	0.199	1801	295	0.75	0.199	1576	259	0.61
	0.25	0.205	1478	265	0.563	0.229	2684	468	0.859	0.218	2560	442	0.906
gafni(3^+)	0.5	0.266	2003	377	0.735	0.266	3247	605	1.047	0.264	3651	677	1.297
	0.75	0.298	1462	326	0.546	0.312	1918	429	0.64	0.316	2061	441	0.75
	0.9	0.334	989	261	0.375	0.302	987	235	0.313	0.337	1114	266	0.406
	1	0.354	636	186	0.234	0.391	956	251	0.328	0.393	1227	303	0.453
	0.1	1.7	3634	593	1.265	1.66	1860	307	0.516	1.62	1873	314	0.547
	0.25	1.42	1441	249	0.438	1.82	755	128	0.203	1.37	743	126	0.219
josephy(1)	0.5	2.72	457	85	0.14	2.05	1382	266	0.375	2.03	1273	238	0.375
	0.5	2.45	1447	335	0.453	2.24	1000	231	0.281	2.43	968	218	0.297
	0.75	2.86	649	176	0.203	2.85	907	224	0.25	2.94	1166	288	0.344
	1	2.77	1077	296	0.328	3.17	1260	344	0.343	3	710	191	0.203
	0.1	1.66	1324	210	0.453	1.67	1540	245	0.563	1.62	1519	242	0.453
	0.25	1.83	634	108	0.187	1.72	729	123	0.188	1.85	921	154	0.266
josephy(2)	0.5	2.16	713	135	0.219	2.05	1151	213	0.313	2.1	1193	221	0.359
	0.75	2.45	1279	282	0.406	2.24	1025	224	0.282	2.29	539	114	0.156
	0.9	2.86	645	166	0.203	2.85	949	237	0.265	2.9	1068	265	0.313
	1	2.77	1132	314	0.359	3.17	1306	347	0.375	3	643	168	0.203
	0.1	1.7	3826	662	1.375	1.66	1931	350	0.547	1.62	1956	365	0.609
	0.25	1.42	1538	297	0.468	1.79	1387	273	0.421	1.83	907	186	0.282
josephy(3)	0.5	2.15	669	131	0.219	2.05	1157	220	0.328	2.13	1318	252	0.422
	0.75	2.45	1617	419	0.562	2.24	1078	270	0.328	2.43	1255	318	0.438
	0.9	2.86	722	212	0.234	2.85	992	254	0.282	2.94	1284	326	0.391
	1	2.77	1170	335	0.36	3.17	1281	353	0.359	3	753	214	0.234

Algorithm 4.1 (*A Derivative Free Algorithm*).

Step 0 *Given* $p > 1, \theta \in (0, 1]$ *and* $x_0 \in \Re^n$. *Choose* $\sigma, \rho, \gamma \in (0, 1)$. *Set* $k := 0$.
Step 1 *If* $\Psi_{\theta p}(x_k) = 0$, *stop, otherwise go to step 2.*
Step 2 *Find the smallest nonnegative integer* m_k *such that*

$$\Psi_{\theta p}(x_k + \rho^{m_k} d_k(\gamma^{m_k})) \le (1 - \sigma\rho^{2m_k})\Psi_{\theta p}(x_k), \tag{4.1}$$

where $d_k(\gamma^{m_k}) := -\frac{\partial\Psi_{\theta p}(x_k,F(x_k))}{\partial b} - \gamma^{m_k}\frac{\partial\Psi_{\theta p}(x_k,F(x_k))}{\partial a}$.

Step 3 *Set* $x_{k+1} := x_k + \rho^{m_k} d_k(\gamma^{m_k})$, $k := k + 1$ *and go to Step 1.*

Proposition 4.1. *Let* $x_k \in \Re^n$ *and* F *be a monotone function. Then the search direction defined in Algorithm 4.1 satisfies the descent condition* $\nabla\Psi_{\theta p}(x_k)^{\mathrm{T}} d_k < 0$ *as long as* x_k *is not a solution of the NCP (1.1). Moreover, if* F *is strongly monotone with modulus* $\mu > 0$, *then* $\nabla\Psi_{\theta p}(x_k)^{\mathrm{T}} d_k < -\mu\|d_k\|^2$.

Proof. Using Proposition 2.5, the proof is similar to the one given in [5, Lemma 4.1]. □

Proposition 4.2. *Suppose that* F *is strongly monotone. Then the sequence* $\{x_k\}$ *generated by Algorithm 4.1 has at least one accumulation point and any accumulation point is a solution of the NCP (1.1).*

Proof. We only need to show that if $\{x_k\}$ has an accumulation point, then the corresponding $\{d_k\}$ has also an accumulation point. In fact, under this condition, $\{x_k\}$ is bounded by Propositions 3.4 and 4.1. Without loss of generality, we could assume $x_k \to x_*$. So, $\{\frac{\partial\Psi_{\theta p}(x_k,F(x_k))}{\partial b}\}$ and $\{\frac{\partial\Psi_{\theta p}(x_k,F(x_k))}{\partial a}\}$ are bounded since $\Psi_{\theta p}$ is continuously differentiable. This together with the fact $\gamma \in (0, 1)$ gives that the direction sequence $\{d_k\}$ is bounded. The rest of the proof are similar to those given in [5, Proposition 4.1] by using Propositions 3.4 and 4.1. □

Table 3
GAP (10^{-3}), CPU (seconds).

Problem	θ	$p = 1.5$				$p = 2$				$p = 3$			
		GAP	NF	IT	CPU	GAP	NF	IT	CPU	GAP	NF	IT	CPU
	0.1	1.62	1419	224	0.484	1.64	1463	231	0.407	1.64	1431	226	0.437
	0.25	1.42	1400	234	0.453	1.71	730	122	0.282	1.84	1005	168	0.328
josephy(4)	0.5	1.49	448	84	0.14	2.05	1088	201	0.328	2.05	1173	218	0.391
	0.75	2.45	1403	305	0.468	2.24	816	176	0.234	2.43	1016	225	0.312
	0.9	2.86	664	175	0.203	2.86	896	216	0.25	2.9	1161	286	0.343
	1	2.77	1141	323	0.375	3.17	1193	312	0.344	4.99	203	52	0.062
	0.1	1.64	1131	179	0.391	1.61	1131	179	0.344	1.62	1055	167	0.328
	0.25	1.42	1113	186	0.453	1.66	519	87	0.156	1.5	280	47	0.094
josephy(5)	0.5	1.87	394	73	0.125	2.05	1045	194	0.282	2.12	1184	221	0.391
	0.75	2.45	1283	282	0.438	2.24	693	148	0.219	2.29	391	83	0.125
	0.9	2.86	622	160	0.203	2.85	933	231	0.266	2.9	1033	256	0.312
	1	2.77	1091	301	0.343	3.17	1272	339	0.359	3	536	139	0.156
	0.1	1.63	1311	209	0.438	1.66	1871	299	0.657	1.62	1807	289	0.578
	0.25	1.42	1394	233	0.422	1.8	1009	169	0.281	1.85	949	159	0.282
josephy(6)	0.5	2.16	679	129	0.203	2.13	1065	197	0.297	2.15	1207	226	0.359
	0.75	2.45	1275	280	0.406	2.24	893	195	0.25	2.43	1035	225	0.313
	0.9	2.86	667	176	0.219	2.85	945	236	0.266	2.94	1261	309	0.391
	1	2.77	1081	297	0.328	3.17	1249	336	0.36	3	682	180	0.203
	0.1	3.7	4456	773	1.594	3.75	7289	1251	2.062	3.74	9265	1590	2.813
	0.25	3.84	1269	228	0.375	3.85	1240	223	0.328	3.98	1571	285	0.469
kojshin(1)	0.5	4.08	2640	542	0.844	4.41	1703	350	0.485	4.7	2753	557	0.938
	0.75	4.98	1301	313	0.453	5.28	884	211	0.282	5.67	963	227	0.297
	0.9	6.34	1199	339	0.375	6.37	1007	268	0.281	6.19	634	166	0.188
	1	7.07	738	227	0.235	6.5	634	204	0.187	6.47	457	138	0.141
	0.1	3.69	1952	340	1.312	3.75	7327	1257	2.125		*		
	0.25	3.93	16786	918	5.109	3.86	1226	219	0.375	3.86	1148	204	0.375
	0.5		*			4.41	1411	287	0.422	4.66	2544	510	0.906
kojshin(2)	0.75		*			5.28	861	196	0.282	5.67	890	199	0.313
	0.9	6.34	1139	308	0.438	6.37	1033	273	0.328	6.18	675	162	0.234
	1	7.07	764	221	0.281	6.5	2291	292	0.64	6.41	393	113	0.14
	0.1	3.66	4767	864	1.703	3.75	7579	1347	2.094	3.74	9653	1707	2.922
	0.25	3.84	1460	287	0.437	3.86	1619	317	0.438	3.98	1613	312	0.469
kojshin(3)	0.5	4.08	2552	522	0.782	4.41	1558	318	0.422	4.6	2608	525	0.782
	0.75	4.98	1494	399	0.5	5.28	1134	308	0.328	5.67	1151	299	0.343
	0.9	6.34	1213	345	0.39	6.37	1071	291	0.297	6.19	653	174	0.187
	1	7.07	802	249	0.265	6.5	623	194	0.187	6.24	501	155	0.156

5. Numerical results

In this section, we implement Algorithm 4.1 for complementarity problems from MCPLIB in MATLAB 7.3 in order to see the numerical behavior of Algorithm 4.1. All numerical experiments are done at a PC with CPU of 2.4 GHz and RAM of 256 MB. Throughout our computational experiments, we adopt the following as the stopping rules, which were also used in [5].

- $\Psi_{\theta p}(x_k) \leq 10^{-5}$ and $d \leq 5.0 \times 10^{-3}$; or
- $\Psi_{\theta p}(x_k) \leq 3.0 \times 10^{-7}$ and $d \leq 3.0 \times 10^{-2}$; or
- $\Psi_{\theta p}(x_k) \leq 3.0 \times 10^{-6}$ and $d \leq 10^{-2}$,

where d represents the dual gap of the underlying optimization problem. We also terminate the algorithm if the step length is less than 10^{-10} or the number of iteration is lager than 5×10^6 or $\Psi_{\theta p}(x_k) \leq 10^{-10}$ or $d \leq 10^{-10}$. We use the nonmonotone line search scheme described in [12] instead of the standard monotone line search, i.e., we compute the smallest nonnegative integer h such that

$$\Psi_{\theta p}(x_k + \rho^h d_k) \leq C_k - \sigma \rho^{2h} \Psi_{\theta p}(x_k),$$

where

$$C_k = \max_{i=k-m_k,\ldots,k} \Psi_{\theta p}(x_i) \quad \text{and} \quad m_k = \begin{cases} 0 & \text{if } k \leq s, \\ \min\{m_{k-1}+1, \hat{m}\} & \text{otherwise.} \end{cases}$$

Throughout the experiments, the parameters we used are: $\hat{m} = 5, s = 5, \rho = 0.6, \sigma = 0.5$ and $\gamma = 0.8$. In order to improve the numerical results, we scale some problems, i.e., divide the function F in (1.1) by 20, in our numerical implement. It is easy to verify that such a modification does not destroy any results we obtained earlier.

We test problems in MCPLIB [1] for two purposes, one is to investigate the numerical behavior of these optimization problems for different $\theta \in (0.1, 1]$ when p varies from 1.1 to 3; and another is to see the relationship between the numerical behavior of the test problems and the parameter p for fixed $\theta \in (0, 1]$. The numerical results are listed in Tables 1–4,

Table 4
$\theta = 0.25$, GAP (10^{-3}), CPU (seconds).

Problem	p	GAP	NF	IT	CPU
	1.1	9.99	232090	22185	117.47
	1.5	9.99	281267	26644	139.52
	2	9.99	307795	28853	114.23
bertsekas	4	9.99	293914	27663	147.5
	10	9.99	265398	25347	135.36
	20	9.97	240184	23133	125.67
billups	1.1–20	5.47e−016	69	1	0.016
	1.1	3.24	169	117	0.204
	1.5	7.21	342	233	0.344
	2	3.18	401	272	0.281
tobin(1^+)	4	9.44	396	268	0.406
	10	9.55	320	218	0.36
	20	9.58	256	178	0.281
	1.1	8.97	290	196	0.312
	1.5	9.7	572	383	0.64
	2	9.47	680	454	0.468
tobin(2^+)	4	9.85	661	443	0.75
	10	7.42	475	321	0.5
	20	1.24	353	241	0.359
	1.1	0.189	1679	300	0.687
	1.5	0.205	1581	291	0.578
	2	0.229	2810	497	1
gafni(1^+)	4	0.21	3223	581	1.156
	10	0.212	1562	293	0.562
	20	0.214	1287	237	0.469
	1.1	0.189	1707	300	0.688
	1.5	0.205	1508	272	1.046
	2	0.229	2497	430	0.844
gafni(2^+)	4	0.21	2953	519	1.078
	10	0.212	1555	287	0.563
	20	0.214	1260	230	0.469
	1.1	0.189	1608	286	0.656
	1.5	0.205	1478	265	0.625
	2	0.229	2684	468	0.844
gafni(3^+)	4	0.21	2923	512	1.063
	10	0.212	1734	326	0.625
	20	0.214	1241	227	0.453
	1.1	0.913	477	163	0.157
	1.5	1.15	292	120	0.11
	2	1.78	233	108	0.078
mathinum(1^+)	4	1.53	237	111	0.094
	10		*		
	20		*		
	1.1	0.859	143	87	0.063
	1.5	3.52	98	69	0.031
	2	3.92	93	67	0.032
mathinum(1^+)	4	3.94	105	70	0.047
	10	3.41	129	81	0.063
	20	1.59	190	99	0.078
	1.1	0.55	120	84	0.063
	1.5	0.74	91	68	0.046
	2	0.911	78	62	0.032
mathinum(3^+)	4	0.491	90	68	0.047
	10	1.27	122	82	0.047
	20	2.65	252	113	0.093

respectively. However, we only listed $\theta = 0.1, 0.25, 0.5, 0.75, 0.9, 1$, $p = 1.5, 2, 3$ and $\theta = 0.25$, $p = 1.1, 1.5, 2, 4, 10, 20$ in Tables, respectively, for simplicity. Among these Tables, **Problem** denotes the problem of MCPLIB tested; **GAP** denotes the final dual gap of the underlying problem when the algorithm terminates; **NF** denotes the number of function value computation; **IT** denotes the number of iteration; **CPU** denotes the cpu time when the algorithm terminates; $*$ denotes the algorithm fails to get an optimizer; and $+$ denotes the underlying problem is scaled. Some interesting phenomenon in the process of numerical experiments are summarized as follows:

- From Tables 1–4 we may see that Algorithm 4.1 works well for the tested problem in MCPLIB [1]. The numerical results listed in Tables 1–4 are comparable to those given in [5];
- From Tables 1–3 we may see that not all the best numerical results of the algorithm appear in the case of $\theta = 1$ for all tested problems with any p. It shows that for all p, on the average, Algorithm 4.1 works better when θ closes $\theta = 1$, $\theta = 0.9$ and $\theta = 0.25$;

- From Table 4 we may see that for $\theta = 0.25$, the best numerical results appear in the case of $p = 1.1$ or $p = 2$ or $p = 20$.

6. Conclusions

In this paper, we proposed a new NCP-function which is a generalization of the one proposed in [5]. The latter includes the well-known FB function as a special case. We also introduced the corresponding merit function of the new NCP-function. The new NCP-function and the corresponding merit function enjoy the same properties as those given in [5], such as strong semismoothness, Lipschitz continuity, continuous differentiability, SC^1 property, LC^1 property, etc. A derivative free algorithm based on the new NCP-function and the new merit function for complementarity problems was discussed, and some preliminary numerical results for test problems from MCPLIB were reported. As a further research topic, it is worth investigating whether or not this class of NCP-functions can be generalized to the case of second-order cones or positive semidefinite matrix cones or symmetric cones. Another issue to be studied is to compare the numerical results of the derivative free algorithm with other methods when the proposed generalized NCP-function or the corresponding merit function is used.

Acknowledgements

The authors are very grateful to the referee for his/her valuable comments and suggestions on the paper, which have considerably improved the paper. The authors are also very grateful to Dr. Shaohua Pan (School of Mathematical Sciences, South China University of Technology, P.R. China) for her help on implementing the algorithm discussed in this paper.

The second author's work is partially supported by the National Natural Science Foundation of China (Grant No. 10571134 and No. 10871144) and the Natural Science Foundation of Tianjin (Grant No. 07JCYBJC05200).

The third author's work is partially supported by National Science Council of Taiwan.

References

[1] S.C. Billups, S.P. Dirkse, M.C. Soares, A comparison of algorithms for large scale mixed complementarity problems, Comput. Optim. Appl. 7 (1977) 3–25.
[2] B. Chen, X. Chen, C. Kanzow, A penalized Fischer–Burmeister NCP-function, Math. Program. 88 (2000) 211–216.
[3] J.S. Chen, The semismooth-related properties of a merit function and a descent method for the nonlinear complementarity problem, J. Global Optim. 36 (2006) 565–580.
[4] J.S. Chen, On some NCP-function based on the generalized Fischer–Burmeister function, Asia-Pac. J. Oper. Res. 24 (2007) 401–420.
[5] J.S. Chen, S.H. Pan, A family of NCP functions and a desent method for the nonlinear complementarity problem, Comput. Optim. Appl. 40 (2008) 389–404.
[6] J.S. Chen, S.H. Pan, A regularization semismooth Newton method based on the generalized Fischer–Burmeister function for P_0-NCPs, J. Comput. Appl. Math. 220 (2008) 464–479.
[7] J.S. Chen, P. Tseng, An unconstrained smooth minimization reformulation of the second-order cone complementarity problem, Math. Program. 104 (2005) 293–372.
[8] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[9] F. Facchinei, J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer Verlag, New York, 2003.
[10] M.C. Ferris, J.S. Pang, Engineering and economic applications of complementarity problems, SIAM J. Rev. 39 (1997) 669–713.
[11] A. Fischer, Solution of monotone complementarity problems with Lipschitzian functions, Math. Program. 76 (1997) 513–532.
[12] L. Grippo, F. Lampariello, S. Lucidi, A nonmonotone line search technique for Newton's method, SIAM. J. Numer. Anal. 23 (1986) 707–716.
[13] P.T. Harker, J.S. Pang, Finite dimensional variational inequality and nonlinear complementarity problem: A survey of theory, algorithms and applications, Math. Program. 48 (1990) 161–220.
[14] Z.H. Huang, The global linear and local quadratic convergence of a non-interior continuation algorithm for the LCP, IMA J. Numer. Anal. 25 (2005) 670–684.
[15] Z.H. Huang, W.Z. Gu, A smoothing-type algorithm for solving linear complementarity problems with strong convergence properties, Appl. Math. Optim. 57 (2008) 17–29.
[16] Z.H. Huang, L. Qi, D. Sun, Sub-quadratic convergence of a smoothing Newton algorithm for the P_0- and monotone LCP, Math. Program. 99 (2004) 423–441.
[17] H.Y. Jiang, M. Fukushima, L. Qi, et al., A trust region method for solving generalized complementarity problems, SIAM. J. Optim. 8 (1998) 140–157.
[18] C. Kanzow, H. Kleinmichel, A new class of semismooth Newton method for nonlinear complementarity problems, Comput. Optim. Appl. 11 (1998) 227–251.
[19] C. Kanzow, N. Yamashita, M. Fukushima, New NCP-functions and their properties, J. Optim. Theory Appl. 94 (1997) 115–135.
[20] R. Mifflin, Semismooth and semiconvex functions in constrained optimization, SIAM J. Control Optim. 15 (1977) 957–972.
[21] J.M. Ortega, W. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, SIAM, Philadelphia, 2000.
[22] S.H. Pan, J.S. Chen, A semismooth Newton method for the SOCCP based on one-parametric class of SOC complementarity functions, Comput. Optim. Appl. (2008), in press (doi:10.1007/s10589-008-9166-9).
[23] J.S. Pang, Complementarity problems, in: R. Horst, P. Pardalos (Eds.), in: Handbook of Global Optimization, Kluwer Academic Publishers, Boston. Massachusetts, 1994, pp. 271–338.
[24] L. Qi, J. Sun, A nonsmooth version of Newton's method, Math. Program. 58 (1993) 353–367.
[25] D. Sun, L. Qi, On NCP-functions, Comput. Optim. Appl. 13 (1999) 201–220.
[26] P. Tseng, Growth behaviour of a class of merit functions for the nonlinear complementarity problem, J. Optim. Theory Appl. 89 (1996) 17–37.
[27] N. Yamashita, M. Fukushima, On stationary points of the implicit Lagrangian for nonlinear complementarity problems, J. Optim. Theory Appl. 84 (1995) 653–663.
[28] N. Yamashita, M. Fukushima, Modified Newton methods for solving a semismooth reformulation of monotone complementarity problems, Math. Program. 76 (1997) 469–491.