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A Note on the Paper "The Algebraic Structure of the Arbitrary-Order Cone"

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Abstract In this short paper, we look into a conclusion drawn by Alzalg (J Optim Theory Appl 169:32–49, 2016). We think the conclusion drawn in the paper is incorrect by pointing out three things. First, we provide a counterexample that the proposed inner product does not satisfy bilinearity. Secondly, we offer an argument why a pth-order cone cannot be self-dual under any reasonable inner product structure on $\mathbb{R}^n$. Thirdly, even under the assumption that all elements operator commute, the inner product becomes an official inner product and the arbitrary-order cone can be shown as a symmetric cone, we think this condition is still unreasonable and very stringent so that the result can only be applied to very few cases.

Keywords pth-order cone · Second-order cone · Inner product · Jordan algebras

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1 Introduction

In the recent paper [1], Alzalg claims that the arbitrary-order cone is a symmetric cone for any order greater than or equal to 1, that is, the cone is homogeneous and self-dual. Ito and Lourenço [2] showed that the pth-order cone in dimension $n \geq 3$ are not homogeneous unless $p = 2$. In this short note, we show that the arbitrary-order cone is not self-dual for any order other than 2. In particular, we provide a counterexample indicating that the inner product defined in [1] is indeed not an inner product because it does not satisfy the bilinearity. We offer an argument why a pth-order cone cannot be self-dual under any reasonable inner product structure on $\mathbb{R}^n$. In addition, Alzalg assumes that all elements operator commute in order to show that the arbitrary-order cone is a symmetric cone. We think this condition is unreasonable and very stringent by elaborating the reason and counterexample. To sum up, we think the conclusion drawn in the paper [1] is very limited.

2 A Type of Inner Product

Alzalg first in [1] defines an inner product as follows:

$$\langle x, y \rangle_p := \frac{1}{2} x^{\mathrm{T}} \left(J_p(x) + J_p(y) \right) y, \tag{1}$$

where $x, y \in \mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$, $p \in [1, \infty]$, and $J_p(\cdot)$ is the matrix given by

$$J_p(x) := \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\|\bar{x}\|_p^2}{\|\bar{x}\|_2^2} I_{n-1} \end{bmatrix}, & \text{if } \bar{x} \neq 0, \\ I_n, & \text{if } \bar{x} = 0 \end{cases}$$

with $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Alzalg argues that $\langle x, y \rangle_p$ defined as in (1) is a new inner product on $\mathbb{R}^n$, and the pth-order cone becomes symmetric under this new product. The pth-order cone in $\mathbb{R}^n$ is defined as

$$\mathcal{P}_p := \left\{ x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} : \, x_1 \geq \|\bar{x}\|_p \right\}.$$

Unfortunately, the functional $\langle x, y \rangle_p$ fails to be *bilinear*, i.e., the functional $\langle x, y \rangle_p$ is not an inner product on $\mathbb{R}^n$. We provide the following counterexample.

Example 2.1 For any $v = (v_1, \bar{v}) \in \mathbb{R} \times \mathbb{R}^2$ and $p \geq 1$, we denote $\ell_v(p) := \frac{\|\bar{v}\|_p^2}{\|\bar{v}\|_2^2}$. Consider $x = (0, 1, 0)$, $y = (0, 0, 1)$ and $z = (0, 1, 1)$ in $\mathbb{R}^3$. Then, it follows from (1) that $\ell_x(p) = \ell_y(p)$. Moreover, we have

$$\langle x, z\rangle_p = \langle y, z\rangle_p = \frac{1}{2}\big(\ell_x(p) + \ell_z(p)\big),$$
$$\langle x + y, z\rangle_p = \langle z, z\rangle_p = \frac{1}{2}\big(\ell_z(p) + \ell_z(p)\big) = \ell_z(p).$$

Clearly $\langle x + y, z\rangle_p = \langle x, z\rangle_p + \langle y, z\rangle_p$ if and only if $\ell_x(p) = \ell_z(p)$. However, $\ell_x(p) = 1$ for any $p \geq 1$, and $\ell_z(p) = 2^{(2/p)-1}$, which equals 1 only when $p = 2$. Hence, $\langle x, y\rangle_p$ is never a bilinear form on $\mathbb{R}^n$ except for $p = 2$. In other words, $\langle x, y\rangle_p$ is not an inner product on $\mathbb{R}^n$ when $p \neq 2$.

3 Why Cannot the pth-Order Cone $\mathcal{P}_p$ be Self-Dual When $p \neq 2$?

In this section, we give a reason why the pth-order cone $\mathcal{P}_p$ cannot be self-dual under any (reasonable) inner product on $\mathbb{R}^n$ when $p \neq 2$. By a "reasonable inner product" on $\mathbb{R}^n$ we mean that the standard basis $\{e_1, e_2, \ldots, e_n\}$ of $\mathbb{R}^n$ is an *orthogonal* frame under the inner product $\langle\langle\cdot,\cdot\rangle\rangle$, that is,

$$\langle\langle e_i, e_j\rangle\rangle = 0 \quad \text{whenever} \quad 1 \leq i,\ j \leq n,\ i \neq j. \tag{2}$$

This inner product can be written as

$$\langle\langle x, y\rangle\rangle = x^{\mathrm{T}} M y, \quad x, y \in \mathbb{R}^n, \tag{3}$$

where M is a *diagonal* matrix $M = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_i > 0$ for all $i = 1, 2, \ldots, n$. This is a reasonable assumption on the inner products to make, since we will almost never take arbitrary frames in $\mathbb{R}^n$ to define a pth-order cone. Also we note that the "inner product" $\langle\cdot,\cdot\rangle_p$ appeared in [1] satisfies the property (2).

The following proposition can be proved by standard arguments in convex analysis; see [3].

Proposition 3.1 *If $\mathcal{P}_p$ is self-dual under the inner product $\langle\langle\cdot,\cdot\rangle\rangle$, then for each nonzero vector $x \in \partial\mathcal{P}_p$ there is a nonzero vector $x' \in \partial\mathcal{P}_p$ such that $\langle\langle x, x'\rangle\rangle = 0$.*

So now let us assume that $\mathcal{P}_p$ in $\mathbb{R}^n$ is self-dual under the inner product (3). For each $k = 2, 3, \ldots, n$, we consider the inner product between $u_k = e_1 + e_k$ and $v_k = e_1 - e_k$ to see that

$$0 \leq \langle\langle u_k, v_k\rangle\rangle = \lambda_1 - \lambda_k,$$

that is, $\lambda_1 \geq \lambda_k$ for all $k = 2, 3, \ldots, n$. On the other hand, if we apply Proposition 3.1 to $x = u_k$, then there is a nonzero vector $x' = (x_1, x_2, \ldots, x_n) \in \partial\mathcal{P}_p$ such that

$$0 = \langle\langle u_k, x'\rangle\rangle = \lambda_1 x_1 + \lambda_k x_k.$$

Since $\lambda_1 \geq \lambda_k > 0$ and $|x_k| \leq x_1$, it is necessary that $x_k = -x_1$ and $\lambda_1 = \lambda_k$; this holds for all $k = 2, 3, \ldots, n$. Therefore, M must be a positive multiple of the identity matrix, and the inner product $\langle\langle \cdot, \cdot \rangle\rangle$ is a scalar multiple of the standard Euclidean inner product on $\mathbb{R}^n$. Under this structure, the dual cone of $\mathcal{P}_p$ is $\mathcal{P}_q$, where $1/p + 1/q = 1$ (cf. [4]); hence, $\mathcal{P}_p$ can be self-dual only when $p = 2$ (or the trivial case $n = 2$). In any case, a pth-order cone cannot be self-dual under any reasonable inner product structure on $\mathbb{R}^n$ unless $p = 2$ or $n = 2$.

4 Operator Commutativity

In order to show the arbitrary-order cone being a symmetric cone, Alzalg further assumes (on page 36 in [1]) that

> Without loss of generality, throughout this paper, we assume that all elements operator commute. If elements do not operator commute, we can scale the underlying optimization problem so that the scaled elements operator commute [5].

We argue that the condition that all elements operator commute is too harsh and not very useful. Recall that for any x, y in a Euclidean Jordan algebra $\mathbb{R}^n$, if the elements x and y operator commute, it implies that x and y have the same spectral decomposition (see [6]), i.e.,

$$x = \lambda_1(x)e^1 + \lambda_2(x)e^2, \quad y = \mu_1(y)e^1 + \mu_2(y)e^2,$$

where $\{e^1, e^2\}$ are a Jordan frame in $\mathbb{R}^n$. Indeed, under this condition, the considered pth-order cone in $\mathbb{R}^n$ becomes the pth-order cone in the subspace generated by e^1 and e^2 in $\mathbb{R}^n$. In other words, the pth-order cone is restricted to a set which behaves similarly to the second-order cone because under transformation $J_p(x)$, the pth-order cone can be recast as the second-order cone. It means the assumption leads to a very special subcase, and this is not what we want for real pth-order cone.

From the above-quoted paragraph, it seems that Alzalg thought that all elements can be made operator commute after rescaling. However, as noted by one of the reviewers, the only Euclidean Jordan algebra $\mathcal{J}$ where all elements operator commute is $\mathbb{R}^n$ (with the usual inner product and componentwise Jordan product). This can be seen by noting that any two primitive idempotents operator commute in this algebra $\mathcal{J}$ and hence (via simultaneous spectral decompositions) are either identical or orthogonal. This implies that there is only one Jordan frame. (If there is a primitive idempotent outside a given Jordan frame, then it is orthogonal to all the elements of the Jordan frame and hence orthogonal to the unit element.) Via the spectral decomposition theorem, the algebra becomes $\mathbb{R}^n$.

In fact, based on the matrix $J_p(x)$ in [1], we can establish the relationship between the pth-order cone and second-order cone. We first define the matrix $\bar{J}_p(x)$ for any $x = (x_1, \bar{x}) \in \mathbb{R}^n$ as bellow:

$$\bar{J}_p(x) := \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\|\bar{x}\|_2}{\|\bar{x}\|_p} I_{n-1} \end{bmatrix}, & \text{if } \bar{x} \neq 0, \\ I_n, & \text{if } \bar{x} = 0. \end{cases}$$

Theorem 4.1 *For any* $x = (x_1, \bar{x}) \in \mathbb{R}^n$ *and* $p \geq 1$, *we have*

(a) $x \in \mathcal{P}_2 \implies \bar{J}_p(x)x \in \mathcal{P}_p$;
(b) $x \in \mathcal{P}_p \implies [\bar{J}_p(x)]^{-1}x \in \mathcal{P}_2$.

Proof For $\bar{x} = 0$, the results of (a) and (b) are obvious. Thus, we only consider the case $\bar{x} \neq 0$.

(a) For any $x \in \mathcal{P}_2$, we have $\|\bar{x}\|_2 \leq x_1$. Moreover, we know that $\bar{J}_p(x)x = (x_1, \frac{\|\bar{x}\|_2}{\|\bar{x}\|_p}\bar{x}) \in \mathbb{R}^n$. Hence, it follows that $\left\|(\|\bar{x}\|_2/\|\bar{x}\|_p) \cdot \bar{x}\right\|_p = \|\bar{x}\|_2 \leq x_1$, i.e., $\bar{J}_p(x)x \in \mathcal{P}_p$.
(b) With the similar arguments, for any $x \in \mathcal{P}_p$, we obtain that $\|\bar{x}\|_p \leq x_1$. Note that $\left[\bar{J}_p(x)\right]^{-1} x = \left(x_1, \frac{\|\bar{x}\|_p}{\|\bar{x}\|_2}\bar{x}\right) \in \mathbb{R}^n$. Therefore, we have $\left\|(\|\bar{x}\|_p/\|\bar{x}\|_2) \cdot \bar{x}\right\|_2 = \|\bar{x}\|_p \leq x_1$, which says $\left[\bar{J}_p(x)\right]^{-1} x \in \mathcal{P}_2$. The proof is complete. □

5 Conclusions

In this short paper, we show that the conclusion that the arbitrary-order cone is a symmetric cone for any order greater than or equal to 1 drawn in a recent paper by Alzalg is invalid. First, we provide a counterexample to show that the inner product proposed therein does not satisfy bilinearity. Secondly, we offer an argument why a pth-order cone cannot be self-dual under any reasonable inner product structure on $\mathbb{R}^n$. Thirdly, we show that the assumption of all elements operator commute is unreasonable and very stringent so that the result can only be applied to very few cases.

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