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THE VARIATIONAL GEOMETRY, PROJECTION EXPRESSION AND DECOMPOSITION ASSOCIATED WITH ELLIPSOIDAL CONES

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Abstract. Non-symmetric cones have long been mysterious to optimization researchers because of no unified analysis technique to handle these cones. Nonetheless, by looking into symmetric cones and non-symmetric cones, it is still possible to find relations between these kinds of cones. This paper tries an attempt to this aspect and focuses on an important class of convex cones, the *ellipsoidal* cone. There are two main reasons for it. The ellipsoidal cone not only includes the well known second-order cone, circular cone and elliptic cone as special cases, but also it can be converted to a second-order cone by a transformation and vice versa. With respect to the ellipsoidal cone, we characterize its dual cone, variational geometry, the projection mapping, and the decompositions. We believe these results may provide a fundamental approach on tackling with other unfamiliar non-symmetric cone optimization problems.

1. Introduction

During the past decades, symmetric cones associated with the Euclidean space $\mathbb{R}^n$, including nonnegative octant $\mathbb{R}^n_+$ and second-order cone $\mathcal{K}^n$, have been extensively studied from different views [1, 7, 8, 9, 10, 14]. With the developments of modern optimization, more and more non-symmetric cones appears in plenty of applications. However, due to the lack of a unified technical tool like the Euclidean Jordan Algebra (EJA) for symmetric cones, it seems no systematic study on non-symmetric cones. Until now, only a small group of them have been investigated thoroughly such as circular cone [6, 21] and p-order cone [3, 13, 20]. In this paper, we focus on another interesting type of non-symmetric cone, the *ellipsoidal cone*, which not only contains a few well known convex cones but also forms a bridge between symmetric cones and non-symmetric cones.

Before the formal discussion, we recall some definitions that will be used in the sequel. A set $\mathcal{K} \subseteq \mathbb{R}^n$ is a *cone* if $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$. In addition, suppose that $\mathcal{K}$ is closed, convex, pointed (i.e., $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$) and has a nonempty interior, we call $\mathcal{K}$ a *proper* cone. Let $\mathcal{S}^n$ be the collection of all real symmetric matrices in the n dimensional matrix space $\mathbb{R}^{n\times n}$. The proper cone $\mathcal{K}$ is called *ellipsoidal*, denoted by $\mathcal{K}_{\mathcal{E}}$, if there exists a nonsingular matrix $Q \in \mathcal{S}^n$ with exact one negative

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eigenvalue $\lambda_n \in \mathbb{R}$ and corresponding eigenvector $u_n \in \mathbb{R}^n$ such that

$$\mathcal{K}_{\mathcal{E}} := \{x \in \mathbb{R}^n \,|\, x^TQx \leq 0, \text{ and } u_n^Tx \geq 0\}, \tag{1.1}$$

where the matrix Q admits the orthogonal decomposition $Q = \sum_{i=1}^n \lambda_i u_i u_i^T$ with eigen-pairs (λ_i, u_i) for $i = 1, 2, \ldots, n$ satisfying the conditions

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > 0 > \lambda_n \text{ and } u_i^Tu_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.2}$$

Example 1.1. Let

$$Q = \begin{bmatrix} \frac{1}{2} & -\sqrt{2} & -\frac{1}{2} \\ -\sqrt{2} & 0 & -\sqrt{2} \\ -\frac{1}{2} & -\sqrt{2} & \frac{1}{2} \end{bmatrix} \in \mathcal{S}^3 \quad \text{and} \quad u_n = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{bmatrix} \in \mathbb{R}^3.$$

Figure 1 shows an ellipsoidal cone in $\mathbb{R}^3$ generated by these parameters Q and u_n.

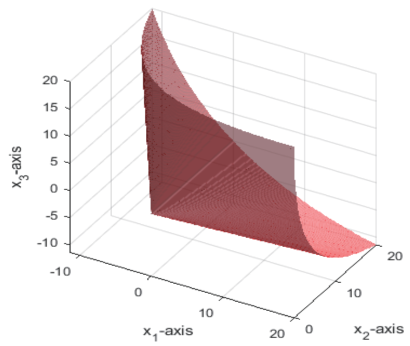

FIGURE 1. The graph of a 3-dimensional ellipsoidal cone.

The history of the ellipsoidal cone dates back to Stern and Wolkowicz's research [18] on characterizing conditions for the spectrum of a given matrix $A \in \mathbb{R}^{n\times n}$ under the existence of an ellipsoidal cone. After that, they also provide an equivalent description on exponential nonnegativity for the second-order cone [19], which is related to the solution set of a linear autonomous system $\dot{\xi} = A\xi$ and further applied to modelling rendezvous of the multiple agents system and measuring dispersion in directional datasets, see [4, 17] for more details.

On the other hand, the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$ includes the second-order cone, circular cone and elliptic cone as special cases. To see this, we verify them as below.

Example 1.2. **(a)** Second-order cone [7, 8]:

$$\mathcal{K}^n := \left\{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}\| \leq x_n\right\},$$

where $\|\bar{x}\|$ stands for the Euclidean norm of $\bar{x} \in \mathbb{R}^{n-1}$. Clearly, $\mathcal{K}^n$ is an ellipsoidal cone with

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad u_n = e_n,$$

where I_{n-1} denotes the identity matrix of order $n-1$ and e_n is the n-th column vector of I_n.

(b) Circular cone [6, 21]:

$$\mathcal{L}_\theta := \left\{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}\| \leq x_n \tan\theta\right\}, \text{ where } \theta \in (0, \frac{\pi}{2}).$$

It is not hard to see that the circular cone $\mathcal{L}_\theta$ is also a special case of ellipsoidal cone with

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix} \quad \text{and} \quad u_n = e_n.$$

(c) Elliptic cone [2]:

$$\mathcal{K}^n_M := \left\{(\bar{x}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}\| \leq x_n\right\},$$

where M is any nonsingular matrix of order $n-1$. Obviously, the elliptic cone $\mathcal{K}^n_M$ can be viewed as an ellipsoidal cone by letting

$$Q = \begin{bmatrix} M^T M & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad u_n = e_n.$$

Remark 1.3. We elaborate more about the aforementioned convex cones. In fact, there hold the relations [1] as follows:

$$\mathcal{K}^n \subseteq \mathcal{L}_\theta \subseteq \mathcal{K}^n_M \subseteq \mathcal{K}_\mathcal{E} \subseteq \mathbb{R}^n.$$

Hence, the ellipsoidal cone is a natural generalization of the second-order cone, circular cone and elliptic cone, see Figure 2 for illustration.

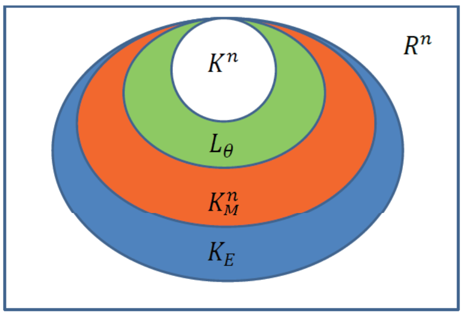

FIGURE 2. The relations among $\mathcal{K}^n$, $\mathcal{L}_\theta$, $\mathcal{K}^n_M$, and $\mathcal{K}_\mathcal{E}$.

Unlike symmetric cone optimization, there is no unified framework for dealing with non-symmetric cone optimization. The experience and techniques for non-symmetric cone optimization are very limited. The paper aims to find a way which can help understanding more about non-symmetric cones. With this goal, we focus on the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ given as in (1.1). There are two main reasons for it. The first reason is that the ellipsoidal cone includes the well known second-order cone,

[1] The first inclusion comes from [21], the second one is established in [2].

circular cone and elliptic cone as special cases, as mentioned above. The second reason is indeed more important, through a transformation, the ellipsoidal cone and the second-order cone can be converted to each other, see Theorem 2.1 in Section 2 for more details. This is a key which may open a new vision because it connects symmetric cones and non-symmetric cones together. In order to pave a way to its corresponding non-symmetric cone optimization, we explore the interior and boundary sets, the dual cone, variational geometry including the tangent cone and the normal cone, the projection mapping and the decompositions with respect to the ellipsoidal cone. We believe these contexts will provide some fundamental bricks to build a systematic optimization theory related to the ellipsoidal cone. Moreover, with the connection (see Theorem 2.1 in Section 2) to second-order cone, some analysis techniques may be carried to the territory of mysterious non-symmetric cones. In other words, the links between these two types of cones may provide a new perspective view on how to deal with unfamiliar non-symmetric cones thoroughly, which is an important contribution to the development of non-symmetric cone optimization.

The remainder of this paper is organized as follows. In Section 2, we develop the theory on the dual of the ellipsoidal cone. In Section 3, we proceed with the study on its variational geometry including the tangent cone and the normal cone. As a byproduct, the explicit expressions of its interior and boundary sets are also established. Sections 4 and 5 are devoted to discovering a detailed exposition of the projection mapping and the decompositions with respect to the ellipsoidal cone, respectively. Finally, we have some concluding remarks and say a few words about future directions in Section 6.

2. The dual of the ellipsoidal cone

In this section, we develop the theory regarding the dual of the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$, which is denoted by $\mathcal{K}_{\mathcal{E}}^*$, in other words,

$$\mathcal{K}_{\mathcal{E}}^* := \{y \in \mathbb{R}^n \mid \langle x, y\rangle \geq 0,\ \forall x \in \mathcal{K}_{\mathcal{E}}\},$$

where $\langle \cdot, \cdot \rangle$ stands for the standard Euclidean inner product defined on $\mathbb{R}^n$. In what follows, we write the matrices $U \in \mathbb{R}^{n\times n}$ and $\Lambda \in \mathcal{S}^n$ to respectively represent

$$U := \begin{bmatrix} u_1 \ u_2 \ \dots \ u_n \end{bmatrix}, \quad \Lambda := \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n). \tag{2.1}$$

The orthogonal decomposition of Q given as in (1.2) implies

$$Q = U\Lambda U^T, \quad \text{and} \quad U^TU = UU^T = I_n.$$

For any given vector $x \in \mathcal{K}_{\mathcal{E}}$, due to the orthogonal property of the sets $\{u_i\}_{i=1}^n$, there exists a vector $\alpha := [\alpha_1, \alpha_2, \dots, \alpha_n]^T \in \mathbb{R}^n$ such that

$$x = U\alpha, \quad x^TQx = \alpha^TU^TQU\alpha = \alpha^T\Lambda\alpha = \sum_{i=1}^n \lambda_i\alpha_i^2, \quad u_n^Tx = u_n^T\left(\sum_{i=1}^n \alpha_i u_i\right) = \alpha_n.$$

The set $\mathcal{K}_{\mathcal{E}}$ can be rewritten as the form $U\Delta_\alpha$ with

$$\Delta_\alpha := \left\{\alpha \in \mathbb{R}^n \,\Big|\, \sum_{i=1}^n \lambda_i\alpha_i^2 \leq 0,\ \alpha_n \geq 0\right\}. \tag{2.2}$$

If we take $\lambda_i = 1$ for $i = 1, 2, \ldots, n-1$ and $\lambda_n = -1$, then the set Δ_α reduces to second-order cone

$$\mathcal{K}^n := \left\{ \alpha \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^{n-1} \alpha_i^2 \leq \alpha_n^2,\ \alpha_n \geq 0 \right\}. \tag{2.3}$$

For any $\alpha \in \Delta_\alpha$, in light of the relation (1.2) for $\{\lambda_i\}_{i=1}^n$, we have

$$\begin{aligned}
\alpha \in \Delta_\alpha \quad &\Longleftrightarrow \quad \sum_{i=1}^{n} \lambda_i \alpha_i^2 \leq 0 \ \text{ and } \ \alpha_n \geq 0 \\
&\Longleftrightarrow \quad \sum_{i=1}^{n-1} \left(\lambda_i^{1/2}\alpha_i\right)^2 \leq \left((-\lambda_n)^{1/2}\alpha_n\right)^2 \ \text{ and } \ (-\lambda_n)^{1/2}\alpha_n \geq 0 \\
&\Longleftrightarrow \quad \left(\lambda_1^{1/2}\alpha_1, \lambda_2^{1/2}\alpha_2, \ldots, \lambda_{n-1}^{1/2}\alpha_{n-1}, (-\lambda_n)^{1/2}\alpha_n\right)^T \in \mathcal{K}^n \\
&\Longleftrightarrow \quad \alpha \in D\mathcal{K}^n
\end{aligned}$$

where D is a $n \times n$ diagonal matrix in the form of

$$D := \operatorname{diag}\left((\lambda_1)^{-1/2}, (\lambda_2)^{-1/2}, \ldots, (\lambda_{n-1})^{-1/2}, (-\lambda_n)^{-1/2}\right). \tag{2.4}$$

Thus, the relation between Δ_α and $\mathcal{K}^n$ is described as $\Delta_\alpha = D\mathcal{K}^n$, which implies

$$\mathcal{K}_\mathcal{E} = U\Delta_\alpha = UD\mathcal{K}^n = T\mathcal{K}^n, \quad \text{where } T := UD. \tag{2.5}$$

It is clear that the matrix $T \in \mathbb{R}^{n\times n}$ is nonsingular. The relation (2.5) between the ellipsoidal cone $\mathcal{K}_\mathcal{E}$ and the second-order cone $\mathcal{K}^n$ is depicted in Figure 3.

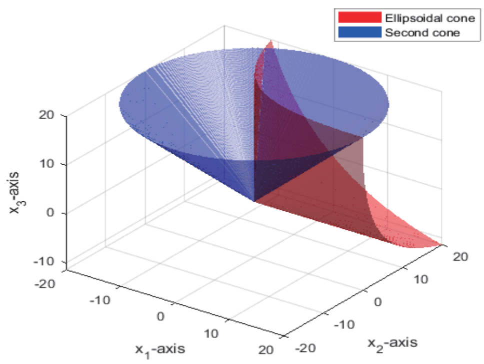

FIGURE 3. The graphs of a 3-dimensional ellipsoidal cone and a 3-dimensional second-order cone.

In fact, similar idea has been used in [18, Proposition 2.3] and [19, Lemma 2.2]. According to the relation (2.5), we can derive

$$\begin{aligned}
\mathcal{K}_\mathcal{E}^* &= \{y \in \mathbb{R}^n \,|\, \langle x, y\rangle \geq 0,\ \forall x \in \mathcal{K}_\mathcal{E}\} \\
&= \{y \in \mathbb{R}^n \,|\, \langle Tz, y\rangle \geq 0,\ \forall z \in \mathcal{K}^n\} \\
&= \left\{y \in \mathbb{R}^n \,|\, T^T y \in (\mathcal{K}^n)^* = \mathcal{K}^n\right\} \\
&= (T^T)^{-1}\mathcal{K}^n,
\end{aligned}$$

where

$$\begin{aligned} T^T &= (UD)^T = D^T U^T = DU^{-1} = D^2 D^{-1} U^{-1} = D^2 T^{-1}, \\ (T^T)^{-1} &= \left(D^2 T^{-1}\right)^{-1} = T D^{-2} = U D^{-1}. \end{aligned}$$

In addition, by denoting $|\Lambda| := \mathrm{diag}(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|) \in \mathcal{S}^n$, which means $|\Lambda| = D^{-2}$, then $(T^T)^{-1} = T|\Lambda|$ and the dual cone $\mathcal{K}_{\mathcal{E}}^*$ can be further expressed as

$$\mathcal{K}_{\mathcal{E}}^* = (T^T)^{-1}\mathcal{K}^n = UD^{-1}\mathcal{K}^n = T|\Lambda|\mathcal{K}^n, \tag{2.6}$$

which is displayed in Figure 4.

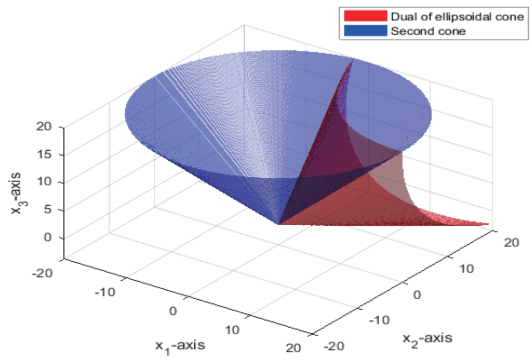

FIGURE 4. The graphs of the dual of a 3-dimensional ellipsoidal cone and a 3-dimensional second-order cone.

Likewise, we deduce the double dual $\mathcal{K}_{\mathcal{E}}^{**}$ of the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$ as follows:

$$\begin{aligned} \mathcal{K}_{\mathcal{E}}^{**} &= \{x \in \mathbb{R}^n \,|\, \langle x, y\rangle \geq 0, \ \forall y \in \mathcal{K}_{\mathcal{E}}^*\} \\ &= \{y \in \mathbb{R}^n \,|\, \langle x, T|\Lambda| z\rangle \geq 0, \ \forall z \in \mathcal{K}^n\} \\ &= \left\{y \in \mathbb{R}^n : (T|\Lambda|)^T x \in (\mathcal{K}^n)^* = \mathcal{K}^n\right\} \\ &= \left((T|\Lambda|)^T\right)^{-1}\mathcal{K}^n, \end{aligned}$$

where

$$\left((T|\Lambda|)^T\right)^{-1} = \left(|\Lambda| T^T\right)^{-1} = (T^T)^{-1}|\Lambda|^{-1} = T|\Lambda||\Lambda|^{-1} = T.$$

This implies that the connection between $\mathcal{K}_{\mathcal{E}}$ and its double dual $\mathcal{K}_{\mathcal{E}}^{**}$ is

$$\mathcal{K}_{\mathcal{E}}^{**} = T\mathcal{K}^n = \mathcal{K}_{\mathcal{E}},$$

see Figure 5 for illustration.

To sum up these discussions, we state the relations among the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$, its dual cones $\mathcal{K}_{\mathcal{E}}^*$, $\mathcal{K}_{\mathcal{E}}^{**}$, and the second-order cone $\mathcal{K}^n$ in the following theorem.

Theorem 2.1. *Let $\mathcal{K}_{\mathcal{E}}$ and $\mathcal{K}^n$ be defined as in (1.1) and (2.3), respectively. Then, we have*

(a) $\mathcal{K}_{\mathcal{E}} = T\mathcal{K}^n$ *and* $\mathcal{K}^n = T^{-1}\mathcal{K}_{\mathcal{E}}$;
(b) $\mathcal{K}_{\mathcal{E}}^* = T|\Lambda|\mathcal{K}^n$ *and* $\mathcal{K}^n = |\Lambda|^{-1}T^{-1}\mathcal{K}_{\mathcal{E}}^*$;
(c) $\mathcal{K}_{\mathcal{E}}^* = T|\Lambda|T^{-1}\mathcal{K}_{\mathcal{E}}$ *and* $\mathcal{K}_{\mathcal{E}}^{**} = \mathcal{K}_{\mathcal{E}}$.

The next theorem presents an explicit description of the dual cone $\mathcal{K}_{\mathcal{E}}^*$.

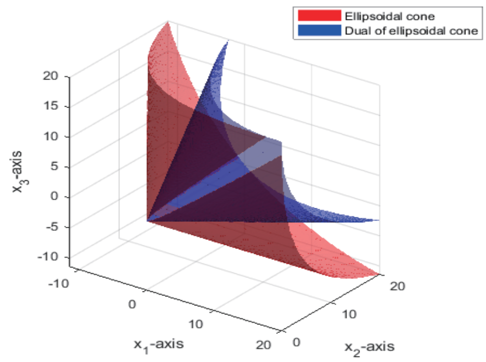

FIGURE 5. The graphs of a 3-dimensional ellipsoidal cone and its dual.

Theorem 2.2. *Let $\mathcal{K}_\mathcal{E}$ be an ellipsoidal cone defined as in (1.1). The dual cone $\mathcal{K}_\mathcal{E}^*$ is equivalently expressed by*

$$\mathcal{K}_\mathcal{E}^* = \left\{ y \in \mathbb{R}^n \,|\, y^T Q^{-1} y \le 0, \; u_n^T y \ge 0 \right\}. \tag{2.7}$$

Proof. In view of (2.6), it suffices to show that the set of the right-hand side in (2.7) is equal to the set $UD^{-1}\mathcal{K}^n$, where the matrix U is defined as in (2.1). Using (1.2) for $\{\lambda_i\}_{i=1}^n$, for any given $y \in \mathbb{R}^n$ there exists a vector $\beta \in \mathbb{R}^n$ such that $y = U\beta$, which yields

$$\begin{aligned}
& y^T Q^{-1} y \le 0 \text{ and } u_n^T y \ge 0 \\
\iff & \sum_{i=1}^{n} \lambda_i^{-1} \beta_i^2 \le 0 \text{ and } \beta_n \ge 0 \\
\iff & \sum_{i=1}^{n-1} \left(\lambda_i^{-1/2}\beta_i\right)^2 \le \left((-\lambda_n)^{-1/2}\beta_n\right)^2 \text{ and } (-\lambda_n)^{-1/2}\beta_n \ge 0 \\
\iff & \left(\lambda_1^{-1/2}\beta_1, \lambda_2^{-1/2}\beta_2, \ldots, \lambda_{n-1}^{-1/2}\beta_{n-1}, (-\lambda_n)^{-1/2}\beta_n\right)^T \in \mathcal{K}^n \\
\iff & \beta \in D^{-1}\mathcal{K}^n.
\end{aligned}$$

Then, the desired result follows. $\square$

As a byproduct, we denote $\mathcal{K}_\mathcal{E}^\circ$ the polar of the ellipsoidal cone $\mathcal{K}_\mathcal{E}$. Applying $\mathcal{K}_\mathcal{E}^\circ = -\mathcal{K}_\mathcal{E}^*$ and (2.7), the exact form of the polar cone $\mathcal{K}_\mathcal{E}^\circ$ is given by

$$\mathcal{K}_\mathcal{E}^\circ := \left\{ y \in \mathbb{R}^n \,|\, y^T Q^{-1} y \le 0, \; u_n^T y \le 0 \right\}. \tag{2.8}$$

Remark 2.3. By applying (2.7), the duals of the circular cone and the elliptic cone, denoted by $\mathcal{L}_\theta^*$ and $(\mathcal{K}_M^n)^*$ respectively, can be characterized as

$$\mathcal{L}_\theta^* := \left\{ (\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{y}_{n-1}\| \le y_n \cot\theta \right\} = \mathcal{L}_{\frac{\pi}{2}-\theta}, \text{ with } \theta \in (0, \frac{\pi}{2}),$$

$$(\mathcal{K}_M^n)^* := \left\{ (\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\middle|\, \left\| \left(M^{-1}\right)^T \bar{y}_{n-1} \right\| \le y_n \right\} = \mathcal{K}^n_{(M^{-1})^T},$$

where $\bar{y}_{n-1} := (y_1, y_2, \ldots, y_{n-1})^T \in \mathbb{R}^{n-1}$. Same arguments can be applied to the polar cone $\mathcal{K}_\mathcal{E}^\circ$ given as in (2.8). In other words, the polar of the circular cone $\mathcal{L}_\theta^\circ$

and the elliptic cone $(\mathcal{K}^n_M)^\circ$ are described as

$$\mathcal{L}^\circ_\theta := \left\{(\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{y}_{n-1}\| \leq -y_n \cot\theta\right\},$$
$$(\mathcal{K}^n_M)^\circ := \left\{(\bar{y}_{n-1}, y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\middle|\, \left\|\left(M^{-1}\right)^T \bar{y}_{n-1}\right\| \leq -y_n\right\}.$$

3. The variational geometry of the ellipsoidal cone

In this section, we pay attention to the variational geometry of the ellipsoidal cone $\mathcal{K}_\mathcal{E}$, which includes the tangent cone $\mathcal{T}_{\mathcal{K}_\mathcal{E}}(x)$ and the normal cone $\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x)$. From the convexity of $\mathcal{K}_\mathcal{E}$ and the definitions of variational geometry in convex analysis [15], we have

$$\mathcal{T}_{\mathcal{K}_\mathcal{E}}(x) := \{d \in \mathbb{R}^n \,|\, \exists\, t_n \downarrow 0, \mathrm{dist}(x + t_n d, \mathcal{K}_\mathcal{E}) = o(t_n)\},$$
$$\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x) := \{v \in \mathbb{R}^n \,|\, \langle v, d\rangle \leq 0,\ \forall d \in \mathcal{T}_{\mathcal{K}_\mathcal{E}}(x)\},$$

where $\mathrm{dist}(x, S)$ denotes the distance from $x \in \mathbb{R}^n$ to the set S, that is,

$$\mathrm{dist}(x, S) := \min_{y\in S} \|x - y\|.$$

The following theorem presents characterizations of $\mathcal{T}_{\mathcal{K}_\mathcal{E}}(x)$ and $\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x)$ in terms of those tangent cone and normal cone for the second-order cone $\mathcal{K}^n$.

Theorem 3.1. *Let $T \in \mathbb{R}^{n\times n}$ be a nonsingular matrix defined as in (2.5). For any $x \in \mathcal{K}_\mathcal{E}$, there exists a vector $\alpha = T^{-1}x \in \mathcal{K}^n$ such that*

$$\mathcal{T}_{\mathcal{K}_\mathcal{E}}(x) = T\,\mathcal{T}_{\mathcal{K}^n}(\alpha) \quad \text{and} \quad \mathcal{N}_{\mathcal{K}_\mathcal{E}}(x) = T|\Lambda|\,\mathcal{N}_{\mathcal{K}^n}(\alpha).$$

Proof. For any $d \in \mathbb{R}^n$, we denote $p := T^{-1}d$. Then, applying Theorem 2.1(a) yields

$$\begin{aligned}
\|T\|^{-1}\,\mathrm{dist}\,(x + t_n d, \mathcal{K}_\mathcal{E}) &= \|T\|^{-1}\,\mathrm{dist}\,(x + t_n d, T\mathcal{K}^n) \\
&= \|T\|^{-1} \min_{y\in\mathcal{K}^n} \|x + t_n d - Ty\| \\
&= \|T\|^{-1} \min_{y\in\mathcal{K}^n} \left\|T(T^{-1}x + t_n T^{-1}d - y)\right\| \\
&\leq \min_{y\in\mathcal{K}^n} \|\alpha + t_n p - y\| \\
&= \mathrm{dist}\,(\alpha + t_n p, \mathcal{K}^n) \\
&= \mathrm{dist}\left(\alpha + t_n p, T^{-1}\mathcal{K}_\mathcal{E}\right) \\
&\leq \|T^{-1}\| \min_{w\in\mathcal{K}_\mathcal{E}} \|x + t_n d - w\| \\
&= \|T^{-1}\|\,\mathrm{dist}\,(x + t_n d, \mathcal{K}_\mathcal{E})\,.
\end{aligned}$$

On the other hand, from definition, there exists $t_n \downarrow 0$ such that $\mathrm{dist}(x+t_n d, \mathcal{K}_\mathcal{E}) = o(t_n)$ if and only if $d \in \mathcal{T}_{\mathcal{K}_\mathcal{E}}(x)$. Thus, we know $\mathrm{dist}(\alpha + t_n p, \mathcal{K}^n) = o(t_n)$, which yields $T^{-1}d = p \in \mathcal{T}_{\mathcal{K}^n}(\alpha)$. The opposite inclusion can be achieved in the similar way. In summary, we have shown $\mathcal{T}_{\mathcal{K}_\mathcal{E}}(x) = T\mathcal{T}_{\mathcal{K}^n}(\alpha)$.

As for the part of $\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x)$, we have

$$\begin{aligned}
\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x) &= \{v \in \mathbb{R}^n \,|\, \langle v, d\rangle \leq 0,\ \forall d \in \mathcal{T}_{\mathcal{K}_\mathcal{E}}(x)\} \\
&= \{v \in \mathbb{R}^n \,|\, \langle v, Tp\rangle \leq 0,\ \forall p \in \mathcal{T}_{\mathcal{K}^n}(\alpha)\} \\
&= \left\{v \in \mathbb{R}^n \,|\, \langle T^T v, p\rangle \leq 0,\ \forall p \in \mathcal{T}_{\mathcal{K}^n}(\alpha)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left\{v \in \mathbb{R}^n \,|\, T^T v \in \mathcal{N}_{\mathcal{K}^n}(\alpha)\right\} \\
&= \left\{v \in \mathbb{R}^n \,|\, v \in \left(T^T\right)^{-1} \mathcal{N}_{\mathcal{K}^n}(\alpha)\right\}.
\end{aligned}
$$

Together with the fact $(T^T)^{-1} = T|\Lambda|$, it follows that $\mathcal{N}_{\mathcal{K}_\mathcal{E}}(x) = T|\Lambda|\mathcal{N}_{\mathcal{K}^n}(\alpha)$. □

For convenience, we also denote $\operatorname{int}\mathcal{K}_\mathcal{E}$ and $\operatorname{bd}\mathcal{K}_\mathcal{E}$ the interior and the boundary of the ellipsoidal cone $\mathcal{K}_\mathcal{E}$, respectively. Then, it follows from Theorem 2.1 and [15, Theorem 6.6] that

$$
\operatorname{int}\mathcal{K}_\mathcal{E} = T(\operatorname{int}\mathcal{K}^n) \quad \text{and} \quad \operatorname{bd}\mathcal{K}_\mathcal{E} = T(\operatorname{bd}\mathcal{K}^n). \tag{3.1}
$$

This together with the definition of $\mathcal{K}^n$ implies that

$$
\begin{aligned}
\operatorname{int}\mathcal{K}^n &:= \left\{\alpha \in \mathbb{R}^n \,|\, \alpha^T Q_n \alpha < 0,\ e_n^T \alpha > 0\right\}, \\
\operatorname{bd}\mathcal{K}^n &:= \left\{\alpha \in \mathbb{R}^n \,|\, \alpha^T Q_n \alpha = 0,\ e_n^T \alpha > 0\right\} \cup \{0\},
\end{aligned}
$$

where the matrix Q_n is given by

$$
Q_n := \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix} \in \mathcal{S}^n. \tag{3.2}
$$

For any given $x \in \mathcal{K}_\mathcal{E}$ and its corresponding vector $\alpha = T^{-1}x \in \mathcal{K}^n$, from (3.1), we obtain

$$
\begin{aligned}
\operatorname{int}\mathcal{K}_\mathcal{E} &= \left\{x \in \mathbb{R}^n \,\Big|\, \left(T^{-1}x\right)^T Q_n (T^{-1}x) < 0,\ e_n^T T^{-1} x > 0\right\}, \\
\operatorname{bd}\mathcal{K}_\mathcal{E} &= \left\{x \in \mathbb{R}^n \,\Big|\, \left(T^{-1}x\right)^T Q_n (T^{-1}x) = 0,\ e_n^T T^{-1} x > 0\right\} \cup \{0\}.
\end{aligned}
$$

Due to the definitions of T and Λ as in (2.1) and (2.4), we also have some useful transformations

$$
\begin{aligned}
&\left(T^{-1}\right)^T Q_n T^{-1} = \left(D^{-1}U^{-1}\right)^T Q_n D^{-1} U^{-1} = U D^{-1} Q_n D^{-1} U^T = U \Lambda U^T = Q, \\
&e_n^T T^{-1} = e_n^T (UD)^{-1} = e_n^T D^{-1} U^{-1} = e_n^T D^{-1} U^T = (-\lambda_n)^{1/2} u_n^T.
\end{aligned}
$$

With the above discussions, we provide the explicit expressions for $\operatorname{int}\mathcal{K}_\mathcal{E}$ and $\operatorname{bd}\mathcal{K}_\mathcal{E}$.

Theorem 3.2. *Let $\mathcal{K}_\mathcal{E}$ be an ellipsoidal cone defined as in (1.1). Then, the interior and the boundary of $\mathcal{K}_\mathcal{E}$ are respectively given by*

$$
\begin{aligned}
\mathit{int}\,\mathcal{K}_\mathcal{E} &= \left\{x \in \mathbb{R}^n \,|\, x^T Q x < 0,\ u_n^T x > 0\right\}, \\
\mathit{bd}\,\mathcal{K}_\mathcal{E} &= \left\{x \in \mathbb{R}^n \,|\, x^T Q x = 0,\ u_n^T x > 0\right\} \cup \{0\}.
\end{aligned}
$$

Remark 3.3. Similar to Remark 2.3, we conclude from Theorem 3.2 that the interior and the boundary of $\mathcal{L}_\theta$ and $\mathcal{K}_M^n$ are described by

$$
\begin{aligned}
\operatorname{int}\mathcal{L}_\theta &= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| < x_n \tan\theta\right\}, \\
\operatorname{bd}\mathcal{L}_\theta &= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| = x_n \tan\theta > 0\right\} \cup \{0\}, \\
\operatorname{int}\mathcal{K}_M^n &= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| < x_n\right\}, \\
\operatorname{bd}\mathcal{K}_M^n &= \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| = x_n > 0\right\} \cup \{0\},
\end{aligned}
$$

where $\bar{x}_{n-1} := (x_1, x_2, \ldots, x_{n-1})^T \in \mathbb{R}^{n-1}$.

To present the tangent cone and normal cone, we first recall their counterparts for second-order cone $\mathcal{K}^n$, which can be found in [5]:

$$\mathcal{T}_{\mathcal{K}^n}(\alpha) = \begin{cases} \mathbb{R}^n & \text{if} \quad \alpha \in \operatorname{int} \mathcal{K}^n, \\ \mathcal{K}^n & \text{if} \quad \alpha = 0, \\ \{p \in \mathbb{R}^n \mid p^T Q_n \alpha \leq 0\} & \text{if} \quad \alpha \in \operatorname{bd} \mathcal{K}^n \setminus \{0\}, \end{cases}$$

$$\mathcal{N}_{\mathcal{K}^n}(\alpha) = \begin{cases} \{0\} & \text{if} \quad \alpha \in \operatorname{int} \mathcal{K}^n, \\ -\mathcal{K}^n & \text{if} \quad \alpha = 0, \\ \mathbb{R}_+(Q_n \alpha) & \text{if} \quad \alpha \in \operatorname{bd} \mathcal{K}^n \setminus \{0\}, \end{cases}$$

where Q_n is defined as in (3.2) and $\mathbb{R}_+(Q_n\alpha)$ stands for the set $\{\eta Q_n \alpha \mid \eta \geq 0\}$.

Combining Theorem 3.1, Theorem 3.2 with the definitions of $\mathcal{T}_{\mathcal{K}^n}(\alpha)$ and $\mathcal{N}_{\mathcal{K}^n}(\alpha)$, we present the expressions of tangent cone and normal cone regarding $\mathcal{K}_{\mathcal{E}}$ as below.

Theorem 3.4. *For any given $x \in \mathbb{R}^n$, the tangent cone and normal cone with respect to the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$ at x are described by*

$$\mathcal{T}_{\mathcal{K}_{\mathcal{E}}}(x) = \begin{cases} \mathbb{R}^n & \text{if} \quad x \in int \mathcal{K}_{\mathcal{E}}, \\ \mathcal{K}_{\mathcal{E}} & \text{if} \quad x = 0, \\ \{d \in \mathbb{R}^n \mid d^T Q x \leq 0\} & \text{if} \quad x \in bd \mathcal{K}_{\mathcal{E}} \setminus \{0\}, \end{cases}$$

$$\mathcal{N}_{\mathcal{K}_{\mathcal{E}}}(x) = \begin{cases} \{0\} & \text{if} \quad x \in int \mathcal{K}_{\mathcal{E}}, \\ \mathcal{K}_{\mathcal{E}}^{\circ} & \text{if} \quad x = 0, \\ \mathbb{R}_+(Qx) & \text{if} \quad x \in bd \mathcal{K}_{\mathcal{E}} \setminus \{0\}, \end{cases}$$

where $\mathbb{R}_+(Qx) := \{\eta Q x \mid \eta \geq 0\}$.

Remark 3.5. We also present the following two special cases when $\mathcal{K}_{\mathcal{E}}$ reduces to $\mathcal{L}_\theta$ or $\mathcal{K}^n_M$. In fact, if take

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix} \quad \text{or} \quad Q = \begin{bmatrix} M^T M & 0 \\ 0 & -1 \end{bmatrix},$$

where M is any given nonsingular matrix of order $n-1$ as in Example 1.2(c). Then, the tangent cone and normal cone of $\mathcal{L}_\theta$ and $\mathcal{K}^n_M$ are respectively given by

$$\mathcal{T}_{\mathcal{L}_\theta}(x) = \begin{cases} \mathbb{R}^n & \text{if} \quad x \in \operatorname{int} \mathcal{L}_\theta, \\ \mathcal{L}_\theta & \text{if} \quad x = 0, \\ \Xi_{\mathcal{L}_\theta} & \text{if} \quad x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\}, \end{cases}$$

$$\mathcal{N}_{\mathcal{L}_\theta}(x) = \begin{cases} \{0\} & \text{if} \quad x \in \operatorname{int} \mathcal{L}_\theta, \\ \mathcal{L}_\theta^{\circ} & \text{if} \quad x = 0, \\ \mathbb{R}_+(\bar{x}, -x_n \tan^2\theta) & \text{if} \quad x \in \operatorname{bd} \mathcal{L}_\theta \setminus \{0\} \end{cases}$$

where $\Xi_{\mathcal{L}_\theta} := \left\{(\bar{d}_{n-1}, d_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \langle \bar{d}_{n-1}, \bar{x}_{n-1}\rangle - d_n x_n \tan^2\theta \leq 0\right\}$ and $\bar{d}_{n-1} := (d_1, d_2, \ldots, d_{n-1})^T \in \mathbb{R}^{n-1}$. Similarly, we also obtain

$$\mathcal{T}_{\mathcal{K}^n_M}(x) = \begin{cases} \mathbb{R}^n & \text{if} \quad x \in \operatorname{int} \mathcal{K}^n_M, \\ \mathcal{K}^n_M & \text{if} \quad x = 0, \\ \Xi_{\mathcal{K}^n_M} & \text{if} \quad x \in \operatorname{bd} \mathcal{K}^n_M \setminus \{0\}, \end{cases}$$

$$\mathcal{N}_{\mathcal{K}^n_M}(x) = \begin{cases} \{0\} & \text{if} \quad x \in \operatorname{int} \mathcal{K}^n_M, \\ (\mathcal{K}^n_M)^{\circ} & \text{if} \quad x = 0, \\ \mathbb{R}_+\left(M^T M \bar{x}_{n-1}, -x_n\right) & \text{if} \quad x \in \operatorname{bd} \mathcal{K}^n_M \setminus \{0\}, \end{cases}$$

where $\Xi_{\mathcal{K}^n_M} := \big\{(\bar{d}_{n-1}, d_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \langle M\bar{d}_{n-1}, M\bar{x}_{n-1}\rangle - d_n x_n \le 0\big\}$.

4. The projection onto the ellipsoidal cone

In this section, we focus on the projection of any vector $y \in \mathbb{R}^n$ onto the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$. In other words, the following minimization problem is considered:

$$\begin{array}{ll} \min & \frac{1}{2}\|x-y\|^2 \\ \text{s.t.} & x \in \mathcal{K}_{\mathcal{E}}. \end{array} \tag{4.1}$$

From the first-order optimality condition (e.g. [16, Theorem 6.12]), it is known that $0 \in x - y + \mathcal{N}_{\mathcal{K}_{\mathcal{E}}}(x)$, which implies

$$x = (I + \mathcal{N}_{\mathcal{K}_{\mathcal{E}}})^{-1}(y) := \Pi_{\mathcal{K}_{\mathcal{E}}}(y),$$

where $\Pi_{\mathcal{K}_{\mathcal{E}}}(y)$ denotes the projection of y onto $\mathcal{K}_{\mathcal{E}}$.

On the other hand, from the orthogonal property (2.1) for the set $\{u_i\}_{i=1}^n$, there exist $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}^n$ such that $x = U\alpha$ and $y = U\beta$. For simplicity, we write

$$\begin{aligned} &\Lambda := \operatorname{diag}(\bar{\Lambda}_{n-1}, \lambda_n) \in \mathcal{S}^n,\ \bar{\Lambda}_{n-1} := \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \in \mathcal{S}^{n-1}, \\ &\alpha := (\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\ \bar{\alpha}_{n-1} := (\alpha_1, \alpha_2, \ldots, \alpha_{n-1})^T \in \mathbb{R}^{n-1}, \\ &\beta := (\bar{\beta}_{n-1}, \beta_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\ \bar{\beta}_{n-1} := (\beta_1, \beta_2, \ldots, \beta_{n-1})^T \in \mathbb{R}^{n-1}. \end{aligned}$$

The problem (4.1) is equivalent to solving the elliptic optimization problem with respect to the variables $(\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, that is,

$$\begin{array}{ll} \min & \frac{1}{2}\left(\|\bar{\alpha}_{n-1} - \bar{\beta}_{n-1}\|^2 + (\alpha_n - \beta_n)^2\right) \\ \text{s.t.} & \left\|\bar{M}\bar{\alpha}_{n-1}\right\| \le \alpha_n, \end{array} \tag{4.2}$$

where $\bar{M}$ is a diagonal matrix of order $n-1$ in the form of

$$\bar{M} := \operatorname{diag}\left(\sqrt{\frac{\lambda_1}{(-\lambda_n)}}, \sqrt{\frac{\lambda_2}{(-\lambda_n)}}, \ldots, \sqrt{\frac{\lambda_{n-1}}{(-\lambda_n)}}\right). \tag{4.3}$$

It is easy to verify that the matrix $\bar{M}$ also satisfies the equation

$$\bar{\Lambda}_{n-1} + \lambda_n \bar{M}^T\bar{M} = 0. \tag{4.4}$$

Theorem 4.1. *Let $\mathcal{K}_{\mathcal{E}}$ be an ellipsoidal cone defined as in (1.1) and $y \in \mathbb{R}^n$. Then, the projection of y onto $\mathcal{K}_{\mathcal{E}}$ is given by*

$$\Pi_{\mathcal{K}_{\mathcal{E}}}(y) = \begin{cases} y & \text{if } y \in \mathcal{K}_{\mathcal{E}}, \\ 0 & \text{if } y \in \mathcal{K}_{\mathcal{E}}^\circ, \\ U\alpha & \text{otherwise}, \end{cases}$$

where the matrix $U \in \mathbb{R}^{n\times n}$ is defined as in (2.1), the vector $\alpha = (\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ is the optimal solution of (4.2) and has the following forms:

(a) *If $\beta_n = 0$, then*

$$\bar{\alpha}_{n-1} = \left(I_{n-1} + \bar{M}^T\bar{M}\right)^{-1}\bar{\beta}_{n-1}, \quad \alpha_n = \|\bar{M}\left(I_{n-1} + \bar{M}^T\bar{M}\right)^{-1}\bar{\beta}_{n-1}\|.$$

(b) *If $\beta_n \neq 0$, then*

$$\bar{\alpha}_{n-1} = \left(I_{n-1} - \eta_0 \lambda_n \bar{M}^T \bar{M}\right)^{-1} \bar{\beta}_{n-1}, \quad \alpha_n = \frac{\beta_n}{1 + \eta_0 \lambda_n},$$

where the matrix $\bar{M} \in \mathcal{S}^{n-1}$ is defined as in (4.3) and the scalar $\eta_0 \in \mathbb{R}$ satisfies the relations

$$\eta_0 \in \begin{cases} (0, -1/\lambda_n) & \text{if} \quad \beta_n > 0, \\ (-1/\lambda_n, +\infty) & \text{if} \quad \beta_n < 0 \end{cases} \quad \text{and} \quad \sum_{i=1}^{n} \frac{\lambda_i \beta_i^2}{(1 + \eta_0 \lambda_i)^2} = 0 \tag{4.5}$$

with $\beta \in \mathbb{R}^n$ lying outside of the set $\left\{\beta \in \mathbb{R}^n \mid \|\bar{M}\bar{\beta}_{n-1}\| \leq \beta_n\right\}$ and its polar $\left\{\beta \in \mathbb{R}^n \mid \|\bar{M}^{-1}\bar{\beta}_{n-1}\| \leq \beta_n\right\}$.

Proof. By checking the definition of $\mathcal{K}_{\mathcal{E}}$ or $\mathcal{K}_{\mathcal{E}}^{\circ}$, it is trivial to obtain the first two cases. It remains to discuss the case of $y \notin \mathcal{K}_{\mathcal{E}} \cup \mathcal{K}_{\mathcal{E}}^{\circ}$. From Theorem 3.4, there exists a scalar $\eta_0 > 0$ such that

$$x = \Pi_{\mathcal{K}_{\mathcal{E}}}(y) \in \mathrm{bd}\mathcal{K}_{\mathcal{E}} \setminus \{0\} \quad \text{and} \quad 0 = x - y + \eta_0 Q x. \tag{4.6}$$

We set $x = U\alpha$ and $y = U\beta$ as earlier, where $U \in \mathbb{R}^{n\times n}$ is defined as in (2.1). Then, the relations (4.6) are equivalent to the system with respect to the variables $\alpha \in \mathbb{R}^n$ and $\eta_0 \in \mathbb{R}$ as follows:

$$\begin{cases} \beta = (I + \eta_0 \Lambda)\alpha, \\ \alpha^T \Lambda \alpha = 0, \\ \alpha_n > 0, \ \eta_0 > 0. \end{cases} \tag{4.7}$$

It turns out that the system (4.7) can be rewritten in the following form

$$\begin{cases} \bar{\beta}_{n-1} = (I_{n-1} + \eta_0 \bar{\Lambda}_{n-1})\bar{\alpha}_{n-1}, \\ \beta_n = (1 + \eta_0 \lambda_n)\alpha_n, \\ \bar{\alpha}_{n-1}^T \bar{\Lambda}_{n-1} \bar{\alpha}_{n-1} + \lambda_n \alpha_n^2 = 0, \\ \alpha_n > 0, \ \eta_0 > 0. \end{cases} \tag{4.8}$$

Next, we proceed to show that the following two subcases hold for the system (4.8).
(a) If $\beta_n = 0$, then $\eta_0 = -\frac{1}{\lambda_n} > 0$. From the system (4.8) and the equation (4.4), we have

$$\bar{\alpha}_{n-1} = \left(I_{n-1} - \frac{1}{\lambda_n}\bar{\Lambda}_{n-1}\right)^{-1} \bar{\beta}_{n-1} = \left(I_{n-1} + \bar{M}^T \bar{M}\right)^{-1} \bar{\beta}_{n-1},$$

$$\alpha_n = \left(\frac{\bar{\alpha}_{n-1}^T \bar{\Lambda}_{n-1} \bar{\alpha}_{n-1}}{-\lambda_n}\right)^{1/2} = \left\|\bar{M}\left(I_{n-1} + \bar{M}^T \bar{M}\right)^{-1} \bar{\beta}_{n-1}\right\|.$$

(b) If $\beta_n \neq 0$, from the second and fourth relations in (4.8), we know $\eta_0 \neq -\frac{1}{\lambda_n}$. The first two relations in (4.8) and the equation (4.4) further imply that

$$\bar{\alpha}_{n-1} = \left(I_{n-1} + \eta_0 \bar{\Lambda}_{n-1}\right)^{-1} \bar{\beta}_{n-1} = \left(I_{n-1} - \eta_0 \lambda_n \bar{M}^T \bar{M}\right)^{-1} \bar{\beta}_{n-1}, \quad \alpha_n = \frac{\beta_n}{1 + \eta_0 \lambda_n},$$

where $\eta_0 \in \mathbb{R}$ satisfies the condition

$$\eta_0 \in \begin{cases} (0, -1/\lambda_n) & \text{if} \quad \beta_n > 0, \\ (-1/\lambda_n, +\infty) & \text{if} \quad \beta_n < 0. \end{cases}$$

In addition, we have

$$\begin{aligned}\bar{\alpha}_{n-1}^T\bar{\Lambda}_{n-1}\bar{\alpha}_{n-1}+\lambda_n\alpha_n^2 &= \bar{\beta}_{n-1}^T\left(I_{n-1}+\eta_0\bar{\Lambda}_{n-1}\right)^{-1}\bar{\Lambda}_{n-1}\left(I_{n-1}+\eta_0\bar{\Lambda}_{n-1}\right)^{-1}\bar{\beta}_{n-1}\\ &\quad+\lambda_n\left(\frac{\beta_n}{1+\eta_0\lambda_n}\right)^2\\ &=\sum_{i=1}^n\frac{\lambda_i\beta_i^2}{(1+\eta_0\lambda_i)^2}\end{aligned}$$

and the third relation in (4.8) reduces to the equation (4.5). Since $y\notin\mathcal{K}_{\mathcal{E}}\cup\mathcal{K}_{\mathcal{E}}^{\circ}$ and $y=U\beta$, from the definitions of $\mathcal{K}_{\mathcal{E}}$ and $\mathcal{K}_{\mathcal{E}}^{\circ}$, we obtain

$$\beta\notin\left\{\beta\in\mathbb{R}^n\mid\sum_{i=1}^n\lambda_i\beta_i^2\le 0,\ \beta_n\ge 0\right\}\cup\left\{\beta\in\mathbb{R}^n\mid\sum_{i=1}^n\lambda_i^{-1}\beta_i^2\le 0,\ \beta_n\le 0\right\},$$

which means that $\beta\notin\left\{\beta\in\mathbb{R}^n\mid\|\bar{M}\bar{\beta}_{n-1}\|\le\beta_n\right\}\cup\left\{\beta\in\mathbb{R}^n\mid\|\bar{M}^{-1}\bar{\beta}_{n-1}\|\le\beta_n\right\}$. $\square$

Remark 4.2. For the projection onto the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$, we emphasize that this projection is not yet an explicit expression because it is hard to solve the equation (4.5) with respect to the variable $\eta_0\in\mathbb{R}$ in general. However, under some special cases, the equation (4.5) has closed-form solutions. For example, if we set

$$U=I_n,\ \lambda_i=1(i=1,2,\ldots,n-1),\ \lambda_n=-1 \text{ or } \lambda_n=-\tan^2\theta,$$

which correspond to the cases of the second-order cone $\mathcal{K}^n$ and the circular cone $\mathcal{L}_\theta$. For more details about their projections, we refer the readers to [9, Proposition 3.3] and [21, Theorem 3.2].

5. The decompositions of the ellipsoidal cone

In this section, we try to express out the decompositions with respect to the ellipsoidal cone. Let $\mathcal{K}_{\bar{M}}^n$ be an elliptic cone with the matrix $\bar{M}$ defined as in (4.3), i.e.,

$$\mathcal{K}_{\bar{M}}^n:=\left\{(\bar{\alpha}_{n-1},\alpha_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid\|\bar{M}\bar{\alpha}_{n-1}\|\le\alpha_n\right\}. \tag{5.1}$$

According to [11, Remark 2.2], the dual cone of $\mathcal{K}_{\bar{M}}^n$ is defined by

$$(\mathcal{K}_{\bar{M}}^n)^*=\{(\bar{\beta}_{n-1},\beta_n)\in\mathbb{R}^{n-1}\times\mathbb{R}\mid\|\bar{M}^{-1}\bar{\beta}_{n-1}\|\le\beta_n\}=\mathcal{K}_{\bar{M}^{-1}}^n. \tag{5.2}$$

It is easy to see that the ellipsoidal cone $\mathcal{K}_{\mathcal{E}}$ and its dual cone $(\mathcal{K}_{\mathcal{E}})^*$ can be described in terms of $\mathcal{K}_{\bar{M}}^n$ and its dual cone $\mathcal{K}_{\bar{M}^{-1}}^n$.

Theorem 5.1. *Let $\mathcal{K}_{\mathcal{E}}$ be an ellipsoidal cone defined as in (1.1) and $\mathcal{K}_{\bar{M}}^n$ be an elliptic cone defined as in (5.1). Then, we have*

$$\mathcal{K}_{\mathcal{E}}=U\mathcal{K}_{\bar{M}}^n,\ \mathcal{K}_{\mathcal{E}}^*=U\mathcal{K}_{\bar{M}^{-1}}^n.$$

Proof. For any given $x\in\mathcal{K}_{\mathcal{E}}$, since $\{u_i\}_{i=1}^n$ are orthogonal to each other, there exists a vector $\alpha=(\bar{\alpha}_{n-1},\alpha_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ such that $x=U\alpha$. From the definition of $\mathcal{K}_{\mathcal{E}}$, we have

$$x\in\mathcal{K}_{\mathcal{E}}$$

$$
\begin{aligned}
&\iff x^TQx \le 0, u_n^Tx \ge 0\\
&\iff \alpha^T\Lambda\alpha \le 0, \alpha_n \ge 0\\
&\iff \bar{\alpha}_{n-1}^T\bar{\Lambda}_{n-1}\bar{\alpha}_{n-1} + \lambda_n\alpha_n^2 \le 0, \alpha_n \ge 0\\
&\iff \bar{\alpha}_{n-1}^T\bar{M}^T\bar{M}\bar{\alpha}_{n-1} \le \alpha_n^2, \alpha_n \ge 0\\
&\iff x = U\alpha, \alpha \in \mathcal{K}_{\bar{M}}^n,
\end{aligned}
$$

which implies the relation $\mathcal{K}_{\mathcal{E}} = U\mathcal{K}_{\bar{M}}^n$. One the other hand, for any given $y \in \mathcal{K}_{\mathcal{E}}^*$, due to the orthogonal property of $\{u_i\}_{i=1}^n$, there exists a vector $\beta = (\bar{\beta}_{n-1}, \beta_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that $y = U\beta$. It follows from above that

$$
\begin{aligned}
& y \in \mathcal{K}_{\mathcal{E}}^*\\
&\iff y^TQ^{-1}y \le 0, u_n^Ty \ge 0\\
&\iff \beta^T\Lambda^{-1}\beta \le 0, \beta_n \ge 0\\
&\iff \bar{\beta}_{n-1}^T\bar{\Lambda}_{n-1}^{-1}\bar{\beta}_{n-1} + \lambda_n^{-1}\beta_n^2 \le 0, \beta_n \ge 0\\
&\iff \sum_{i=1}^{n-1}\frac{(-\lambda_n)}{\lambda_i}\beta_i^2 \le \beta_n^2, \beta_n \ge 0\\
&\iff \bar{\beta}_{n-1}^T(\bar{M}^{-1})^T\bar{M}^{-1}\bar{\beta}_{n-1} \le \beta_n^2, \beta_n \ge 0\\
&\iff y = U\beta, \beta \in \mathcal{K}_{\bar{M}^{-1}}^n.
\end{aligned}
$$

Therefore, we obtain $\mathcal{K}_{\mathcal{E}}^* = U\mathcal{K}_{\bar{M}^{-1}}^n$. □

Inspired by recent studies on spectral factorization associated with p-order cone in [13, Theorem 2.3] or [12, Theorem 3.2], there exists one type of the decomposition for a point $(\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with respect to the elliptic cone $\mathcal{K}_{\bar{M}}^n$.

Type I:

$$
\begin{bmatrix} \bar{\alpha}_{n-1} \\ \alpha_n \end{bmatrix} = \begin{cases}
\begin{array}{l} \dfrac{\alpha_n + \|\bar{M}\bar{\alpha}_{n-1}\|}{2}\begin{bmatrix} \dfrac{\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix} \\ +\dfrac{\alpha_n - \|\bar{M}\bar{\alpha}_{n-1}\|}{2}\begin{bmatrix} \dfrac{-\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix} \end{array} & \text{if } \bar{\alpha}_{n-1} \neq 0, \\[4ex]
\dfrac{\alpha_n}{2}\begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} + \dfrac{\alpha_n}{2}\begin{bmatrix} \dfrac{-w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} & \text{if } \bar{\alpha}_{n-1} = 0,
\end{cases}
$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$. Focusing on the right-hand side of the Type I decomposition, we observe that the vectors

$$
\begin{bmatrix} \dfrac{\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{-\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{-w}{\|\bar{M}w\|} \\ 1 \end{bmatrix}
$$

all belong to the set $\mathcal{K}_{\bar{M}}^n$, which is different from the decomposition with respect to the circular cone $\mathcal{L}_\theta$ established in [21, Theorem 3.1], since its associated dual cone $\mathcal{L}_\theta^*$ is involved in.

In contrast to the Type I decomposition, through importing the information of its dual cone $\mathcal{K}^n_{\bar{M}^{-1}}$ defined as in (5.2), we present another type of decomposition for any given point $(\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ with respect to the elliptic cone $\mathcal{K}^n_{\bar{M}}$ and its dual cone $\mathcal{K}^n_{\bar{M}^{-1}}$.

Type II:

$$\begin{bmatrix} \bar{\alpha}_{n-1} \\ \alpha_n \end{bmatrix} = \begin{cases} \dfrac{\alpha_n + \|\bar{M}^{-1}\bar{\alpha}_{n-1}\|}{\|\bar{M}^{-1}\bar{\alpha}_{n-1}\| + \|\bar{M}\bar{\alpha}_{n-1}\|} \cdot \|\bar{M}\bar{\alpha}_{n-1}\| \cdot \begin{bmatrix} \dfrac{\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix} & \\ + \dfrac{\alpha_n - \|\bar{M}\bar{\alpha}_{n-1}\|}{\|\bar{M}^{-1}\bar{\alpha}_{n-1}\| + \|\bar{M}\bar{\alpha}_{n-1}\|} \cdot \|\bar{M}^{-1}\bar{\alpha}_{n-1}\| \cdot \begin{bmatrix} \dfrac{-\bar{\alpha}_{n-1}}{\|\bar{M}^{-1}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \bar{\alpha}_{n-1} \neq 0, \\ \dfrac{\alpha_n}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|} \cdot \|\bar{M}w\| \cdot \begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} & \\ + \dfrac{\alpha_n}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|} \cdot \|\bar{M}^{-1}w\| \cdot \begin{bmatrix} \dfrac{-w}{\|\bar{M}^{-1}w\|} \\ 1 \end{bmatrix} & \text{if } \bar{\alpha}_{n-1} = 0, \end{cases}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$. In contrast to the Type I decomposition, these vectors

$$\begin{bmatrix} \dfrac{\bar{\alpha}_{n-1}}{\|\bar{M}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix}$$

belong to the set $\mathcal{K}^n_{\bar{M}}$, whereas the vectors

$$\begin{bmatrix} \dfrac{-\bar{\alpha}_{n-1}}{\|\bar{M}^{-1}\bar{\alpha}_{n-1}\|} \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \dfrac{-w}{\|\bar{M}^{-1}w\|} \\ 1 \end{bmatrix}$$

belong to its dual cone $\mathcal{K}^n_{\bar{M}^{-1}}$.

The following theorem presents the decompositions regarding the ellipsoidal cone.

Theorem 5.2. *Let $\mathcal{K}_{\mathcal{E}}$ be an ellipsoidal cone defined as in (1.1) and $\mathcal{K}^*_{\mathcal{E}}$ be its dual cone defined as in (2.7). For any given $x \in \mathbb{R}^n$, it has two types of decompositions, namely Type I and Type II.*

Type I:

$$x = \begin{cases} s^{(1)}_{I_a}(x) \cdot v^{(1)}_{I_a}(x) + s^{(2)}_{I_a}(x) \cdot v^{(2)}_{I_a}(x) & \text{if } \bar{U}^T_{n-1}x \neq 0, \\ s^{(1)}_{I_b}(x) \cdot v^{(1)}_{I_b}(x) + s^{(2)}_{I_b}(x) \cdot v^{(2)}_{I_b}(x) & \text{if } \bar{U}^T_{n-1}x = 0, \end{cases}$$

where $s^{(1)}_{I_a}(x)$, $s^{(2)}_{I_a}(x)$, $s^{(1)}_{I_b}(x)$, $s^{(2)}_{I_b}(x)$ *and* $v^{(1)}_{I_a}(x)$, $v^{(2)}_{I_a}(x)$, $v^{(1)}_{I_b}(x)$, $v^{(2)}_{I_b}(x)$ *have the following expressions*

$$s^{(1)}_{I_a}(x) := u_n^T x + \|\bar{M}\bar{U}^T_{n-1}x\|, \quad v^{(1)}_{I_a}(x) := \frac{1}{2}\cdot\left(\frac{\bar{U}_{n-1}\bar{U}^T_{n-1}x}{\|\bar{M}\bar{U}^T_{n-1}x\|} + u_n\right) \in \mathcal{K}_{\mathcal{E}},$$

$$s^{(2)}_{I_a}(x) := u_n^T x - \|\bar{M}\bar{U}^T_{n-1}x\|, \quad v^{(2)}_{I_a}(x) := \frac{1}{2}\cdot\left(-\frac{\bar{U}_{n-1}\bar{U}^T_{n-1}x}{\|\bar{M}\bar{U}^T_{n-1}x\|} + u_n\right) \in \mathcal{K}_{\mathcal{E}},$$

$$s^{(1)}_{I_b}(x) := u_n^T x, \quad v^{(1)}_{I_b}(x) := \frac{1}{2}\cdot\left(\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|} + u_n\right) \in \mathcal{K}_{\mathcal{E}},$$

$$s^{(2)}_{I_b}(x) := u_n^T x, \quad v^{(2)}_{I_b}(x) := \frac{1}{2}\cdot\left(-\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|} + u_n\right) \in \mathcal{K}_{\mathcal{E}}$$

with any given nonzero vector $w \in \mathbb{R}^{n-1}$ *and a diagonal matrix* $\bar{M}$ *looks like*

$$\bar{M} = \left[\frac{\bar{U}^T_{n-1}(Q - \lambda_n u_n u_n^T)\bar{U}_{n-1}}{(-\lambda_n)}\right]^{1/2}. \tag{5.3}$$

Type II:

$$x = \begin{cases} s^{(1)}_{II_a}(x)\cdot v^{(1)}_{II_a}(x) + s^{(2)}_{II_a}(x)\cdot v^{(2)}_{II_a}(x) & \text{if } \bar{U}^T_{n-1}x \neq 0, \\ s^{(1)}_{II_b}(x)\cdot v^{(1)}_{II_b}(x) + s^{(2)}_{II_b}(x)\cdot v^{(2)}_{II_b}(x) & \text{if } \bar{U}^T_{n-1}x = 0, \end{cases}$$

where $s^{(1)}_{II_a}(x)$, $s^{(2)}_{II_a}(x)$, $s^{(1)}_{II_b}(x)$, $s^{(2)}_{II_b}(x)$ *and* $v^{(1)}_{II_a}$, $v^{(2)}_{II_a}$, $v^{(1)}_{II_b}$, $v^{(2)}_{II_b}$ *are defined as follows:*

$$s^{(1)}_{II_a}(x) := u_n^T x + \|\bar{M}^{-1}\bar{U}^T_{n-1}x\|, \quad v^{(1)}_{II_a}(x) := \frac{\bar{U}_{n-1}\bar{U}^T_{n-1}x + \|\bar{M}\bar{U}^T_{n-1}x\|\cdot u_n}{\|\bar{M}^{-1}\bar{U}^T_{n-1}x\| + \|\bar{M}\bar{U}^T_{n-1}x\|} \in \mathcal{K}_{\mathcal{E}},$$

$$s^{(2)}_{II_a}(x) := u_n^T x - \|\bar{M}\bar{U}^T_{n-1}x\|, \quad v^{(2)}_{II_a}(x) := \frac{-\bar{U}_{n-1}\bar{U}^T_{n-1}x + \|\bar{M}^{-1}\bar{U}^T_{n-1}x\|\cdot u_n}{\|\bar{M}^{-1}\bar{U}^T_{n-1}x\| + \|\bar{M}\bar{U}^T_{n-1}x\|} \in \mathcal{K}^*_{\mathcal{E}},$$

$$s^{(1)}_{II_b}(x) := u_n^T x, \quad v^{(1)}_{II_b}(x) := \frac{\bar{U}_{n-1}w + \|\bar{M}w\|\cdot u_n}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|} \in \mathcal{K}_{\mathcal{E}},$$

$$s^{(2)}_{II_b}(x) := u_n^T x, \quad v^{(2)}_{II_b}(x) := \frac{-\bar{U}_{n-1}w + \|\bar{M}^{-1}w\|\cdot u_n}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|} \in \mathcal{K}^*_{\mathcal{E}}.$$

Proof. For any given $x \in \mathbb{R}^n$, due to the orthogonal property of $\{u_i\}_{i=1}^n$, there exists a vector $\alpha = (\bar{\alpha}_{n-1}, \alpha_n) \in \mathbb{R}^{n-1}\times\mathbb{R}$ such that $x = U\alpha$, therefore we obtain $\bar{\alpha}_{n-1} = \bar{U}^T_{n-1}x$ and $\alpha_n = u_n^T x$. From Theorem 5.1 and the decomposition formulas of α with respect to $\mathcal{K}_{\bar{M}}$ and its dual cone $\mathcal{K}_{\bar{M}^{-1}}$, we know

$$s_{I_a}^{(1)}(x) := u_n^T x + \|\bar{M}\bar{U}_{n-1}^T x\|,\ s_{I_a}^{(2)}(x) := u_n^T x - \|\bar{M}\bar{U}_{n-1}^T x\|,\ s_{I_b}^{(1)}(x) = s_{I_b}^{(2)}(x) := u_n^T x,$$

$$v_{I_a}^{(1)}(x) := \frac{1}{2}\cdot U \begin{bmatrix} \dfrac{\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}},\ v_{I_a}^{(2)}(x) := \frac{1}{2}\cdot U \begin{bmatrix} -\dfrac{\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}},$$

$$v_{I_b}^{(1)}(x) := \frac{1}{2}\cdot U \begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}},\ v_{I_b}^{(2)}(x) := \frac{1}{2}\cdot U \begin{bmatrix} \dfrac{-w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}}$$

and

$$s_{II_a}^{(1)}(x) := u_n^T x + \|\bar{M}^{-1}\bar{U}_{n-1}^T x\|,\ s_{II_a}^{(2)}(x) := u_n^T x - \|\bar{M}\bar{U}_{n-1}^T x\|,\ s_{II_b}^{(1)}(x) = s_{II_b}^{(2)}(x) := u_n^T x,$$

$$v_{II_a}^{(1)}(x) := \frac{1}{\|\bar{M}^{-1}\bar{U}_{n-1}^T x\| + \|\bar{M}\bar{U}_{n-1}^T x\|}\cdot \|\bar{M}\bar{U}_{n-1}^T x\| \cdot U \begin{bmatrix} \dfrac{\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}},$$

$$v_{II_a}^{(2)}(x) := \frac{1}{\|\bar{M}^{-1}\bar{U}_{n-1}^T x\| + \|\bar{M}\bar{U}_{n-1}^T x\|}\cdot \|\bar{M}^{-1}\bar{U}_{n-1}^T x\| \cdot U \begin{bmatrix} \dfrac{-\bar{U}_{n-1}^T x}{\|\bar{M}^{-1}\bar{U}_{n-1}^T x\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}}^*,$$

$$v_{II_b}^{(1)}(x) := \frac{1}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|}\cdot \|\bar{M}w\| \cdot U \begin{bmatrix} \dfrac{w}{\|\bar{M}w\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}},$$

$$v_{II_b}^{(2)}(x) := \frac{1}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|}\cdot \|\bar{M}^{-1}w\| \cdot U \begin{bmatrix} \dfrac{-w}{\|\bar{M}^{-1}w\|} \\ 1 \end{bmatrix} \in \mathcal{K}_{\mathcal{E}}^*.$$

From the orthogonal decomposition of Q, the definition of $\bar{M}$ in (4.3) and the relation (4.4), we obtain

$$Q = U\Lambda U^T = \bar{U}_{n-1}\bar{\Lambda}_{n-1}\bar{U}_{n-1}^T + \lambda_n u_n u_n^T = \bar{U}_{n-1}(-\lambda_n)\bar{M}^T\bar{M}\bar{U}_{n-1}^T + \lambda_n u_n u_n^T,$$

and then the diagonal matrix $\bar{M}$ has an explicit expression as in (5.3). □

Corollary 5.3. *Let $x \in \mathbb{R}^n$ be the decompositions of Type I and Type II given as in Theorem 5.2. Then, the scalar parts of these decompositions satisfy the following relations*

(a) $s_{I_b}^{(1)}(x) = s_{I_b}^{(2)}(x) = s_{II_b}^{(1)}(x) = s_{II_b}^{(2)}(x) = u_n^T x$.

(b) $s_{I_a}^{(2)}(x) = s_{II_a}^{(2)}(x) = s_{I_b}^{(1)}(x) - \|\bar{M}\bar{U}_{n-1}^T x\|$.

(c) $s_{I_a}^{(1)}(x) - \|\bar{M}\bar{U}_{n-1}^T x\| = s_{II_a}^{(1)}(x) - \|\bar{M}^{-1}\bar{U}_{n-1}^T x\| = s_{I_b}^{(1)}(x)$.

(d) $s_{I_a}^{(1)}(x) + s_{I_a}^{(2)}(x) = 2s_{I_b}^{(1)}(x)$, $s_{I_a}^{(1)}(x) - s_{I_a}^{(2)}(x) = 2\|\bar{M}\bar{U}_{n-1}^T x\|$.

(e) $s_{II_a}^{(1)}(x) - s_{II_a}^{(2)}(x) = \|\bar{M}\bar{U}_{n-1}^T x\| + \|\bar{M}^{-1}\bar{U}_{n-1}^T x\|$.

Moreover, we also obtain

$$x \in \mathcal{K}_{\mathcal{E}} \Leftrightarrow s_{I_a}^{(2)}(x) = s_{II_a}^{(2)}(x) \geq 0.$$

On the other hand, some relations between these vector parts hold as follows:

$$v_{I_a}^{(1)}(x) \in mid\left\{\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|}, u_n\right\}, \qquad v_{I_a}^{(2)}(x) \in mid\left\{-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|}, u_n\right\},$$

$$v_{I_b}^{(1)}(x) \in mid\left\{\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|}, u_n\right\}, \qquad v_{I_b}^{(2)}(x) \in mid\left\{-\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|}, u_n\right\},$$

$$v_{II_a}^{(1)}(x) \in conv\left\{\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}^{-1}\bar{U}_{n-1}^T x\|}, u_n\right\}, \quad v_{II_a}^{(2)}(x) \in conv\left\{-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|}, u_n\right\},$$

$$v_{II_b}^{(1)}(x) \in conv\left\{\frac{\bar{U}_{n-1}w}{\|\bar{M}^{-1}w\|}, u_n\right\}, \qquad v_{II_b}^{(2)}(x) \in conv\left\{-\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|}, u_n\right\},$$

where $mid\{a,b\}$ *and* $conv\{a,b\}$ *stand for the midpoint and the line segment between* a *and* b*, respectively.*

Remark 5.4. From Corollary 5.3, the vectors $v_{I_a}^{(2)}(x)$ and $v_{II_a}^{(2)}(x)$ can be rewritten as

$$\tau \cdot \left(-\frac{\bar{U}_{n-1}\bar{U}_{n-1}^T x}{\|\bar{M}\bar{U}_{n-1}^T x\|}\right) + (1-\tau)\cdot u_n, \quad \text{where} \quad \tau = \begin{cases} \dfrac{1}{2} & \text{for} \quad v_{I_a}^{(2)}(x), \\[2ex] \dfrac{s_{II_b}^{(1)} - s_{II_a}^{(2)}}{s_{II_a}^{(1)} - s_{II_a}^{(2)}} & \text{for} \quad v_{II_a}^{(2)}(x). \end{cases}$$

Similarly, we can express the vectors $v_{I_b}^{(2)}(x)$ and $v_{II_b}^{(2)}(x)$ by

$$\eta \cdot \left(-\frac{\bar{U}_{n-1}w}{\|\bar{M}w\|}\right) + (1-\eta)\cdot u_n, \quad \text{where} \quad \eta = \begin{cases} \dfrac{1}{2} & \text{for} \quad v_{I_b}^{(2)}(x), \\[2ex] \dfrac{\|\bar{M}w\|}{\|\bar{M}^{-1}w\| + \|\bar{M}w\|} & \text{for} \quad v_{II_b}^{(2)}(x). \end{cases}$$

According to Theorem 5.2 and Figure 2, we investigate the decompositions with respect to some special cases of the ellipsoidal cone in the following examples.

Example 5.5. Consider the second-order cone

$$\mathcal{K}^n := \left\{(\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid \|\bar{x}\| \le x_n\right\}.$$

In this case, we know

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}, \ \bar{U}_{n-1} = \bar{E}_{n-1}, \ u_n = e_n, \ \bar{M} = I_{n-1}, \ \lambda_n = -1,$$

where $\bar{E}_{n-1} := [e_1, e_2, \ldots, e_{n-1}] \in \mathbb{R}^{n\times(n-1)}$. With respect to the second-order cone $\mathcal{K}^n$, the scalar parts and the vector parts in two types of decompositions are given by

Type I, Type II:

$$s^{(1)}_{I_a}(x) = s^{(1)}_{II_a}(x) := x_n + \|\bar{x}_{n-1}\|, \quad v^{(1)}_{I_a}(x) = v^{(1)}_{II_a}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}^n,$$

$$s^{(2)}_{I_a}(x) = s^{(2)}_{II_a}(x) := x_n - \|\bar{x}_{n-1}\|, \quad v^{(2)}_{I_a}(x) = v^{(2)}_{II_a}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}^n,$$

$$s^{(1)}_{I_b}(x) = s^{(1)}_{II_b}(x) := x_n, \qquad v^{(1)}_{I_b}(x) = v^{(1)}_{II_b}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{w}{\|w\|} \\ 1 \end{bmatrix} \in \mathcal{K}^n,$$

$$s^{(2)}_{I_b}(x) = s^{(2)}_{II_b}(x) := x_n, \qquad v^{(2)}_{I_b}(x) = v^{(2)}_{II_b}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{w}{\|w\|} \\ 1 \end{bmatrix} \in \mathcal{K}^n,$$

An interesting property about these decompositions is that they coincide with each other and reduces to the classical spectral decomposition of the second-order cone $\mathcal{K}^n$ as

$$x = \begin{cases} (x_n + \|\bar{x}_{n-1}\|) \cdot \dfrac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} + (x_n - \|\bar{x}_{n-1}\|) \cdot \dfrac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} \neq 0, \\ x_n \cdot \dfrac{1}{2}\begin{bmatrix} \dfrac{w}{\|w\|} \\ 1 \end{bmatrix} + x_n \cdot \dfrac{1}{2}\begin{bmatrix} -\dfrac{w}{\|w\|} \\ 1 \end{bmatrix} & \text{if } \bar{x}_{n-1} = 0, \end{cases}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$.

Example 5.6. Consider the circular cone

$$\mathcal{L}_\theta := \left\{ (\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|\bar{x}_{n-1}\| \le x_n \tan\theta \right\}$$

In this case, we know

$$Q = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -\tan^2\theta \end{bmatrix}, \; \bar{U}_{n-1} = \bar{E}_{n-1}, \; u_n = e_n, \; \bar{M} = I_{n-1}, \; \lambda_n = -\tan^2\theta.$$

The scalar parts and the vector parts in two types of decompositions with respect to the circular cone $\mathcal{L}_\theta$ are given by

Type I:

$$s_{I_a}^{(1)}(x) := x_n + \cot\theta\|\bar{x}_{n-1}\|, \quad v_{I_a}^{(1)}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

$$s_{I_a}^{(2)}(x) := x_n - \cot\theta\|\bar{x}_{n-1}\|, \quad v_{I_a}^{(2)}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

$$s_{I_b}^{(1)}(x) := x_n, \quad v_{I_b}^{(1)}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

$$s_{I_b}^{(2)}(x) := x_n, \quad v_{I_b}^{(2)}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

Type II:

$$s_{II_a}^{(1)}(x) := x_n + \tan\theta\|\bar{x}_{n-1}\|, \quad v_{I_a}^{(1)}(x) := \frac{\cot\theta}{\tan\theta+\cot\theta}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

$$s_{II_a}^{(2)}(x) := x_n - \cot\theta\|\bar{x}_{n-1}\|, \quad v_{I_a}^{(2)}(x) := \frac{\tan\theta}{\tan\theta+\cot\theta}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\tan\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta^*,$$

$$s_{II_b}^{(1)}(x) := x_n, \quad v_{I_b}^{(1)}(x) := \frac{\cot\theta}{\tan\theta+\cot\theta}\begin{bmatrix} \dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta,$$

$$s_{II_b}^{(2)}(x) := x_n, \quad v_{I_b}^{(2)}(x) := \frac{\tan\theta}{\tan\theta+\cot\theta}\begin{bmatrix} -\dfrac{w}{\tan\theta\|w\|} \\ 1 \end{bmatrix} \in \mathcal{L}_\theta^*.$$

Consequently, their decompositions have the following expressions:

Type I:

$$x = \begin{cases} \begin{aligned} &(x_n + \cot\theta\|\bar{x}_{n-1}\|)\cdot\frac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \\ &+(x_n - \cot\theta\|\bar{x}_{n-1}\|)\cdot\frac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \end{aligned} & \text{if } \ \bar{x}_{n-1} \neq 0, \\ x_n\cdot\dfrac{1}{2}\begin{bmatrix} \dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} + x_n\cdot\dfrac{1}{2}\begin{bmatrix} -\dfrac{w}{\cot\theta\|w\|} \\ 1 \end{bmatrix} & \text{if } \ \bar{x}_{n-1} = 0, \end{cases}$$

Type II:

$$x = \begin{cases} (x_n + \tan\theta\|\bar{x}_{n-1}\|) \cdot \dfrac{\cot\theta}{\tan\theta + \cot\theta} \begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\cot\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \\ +(x_n - \cot\theta\|\bar{x}_{n-1}\|) \cdot \dfrac{\tan\theta}{\tan\theta + \cot\theta} \begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\tan\theta\|\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if } \ \bar{x}_{n-1} \neq 0, \\ x_n \cdot \dfrac{\tan\theta}{\tan\theta + \cot\theta} \begin{bmatrix} -\dfrac{w}{\tan\theta\|w\|} \\ 1 \end{bmatrix} \\ +x_n \cdot \dfrac{\tan\theta}{\tan\theta + \cot\theta} \begin{bmatrix} -\dfrac{w}{\tan\theta\|w\|} \\ 1 \end{bmatrix} & \text{if } \ \bar{x}_{n-1} = 0, \end{cases}$$

where w is any given nonzero vector in $\mathbb{R}^{n-1}$.

Example 5.7. Consider the elliptic cone

$$\mathcal{K}_M^n := \left\{ (\bar{x}_{n-1}, x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, \|M\bar{x}_{n-1}\| \le x_n \right\}.$$

In this case, we know

$$Q = \begin{bmatrix} M^T M & 0 \\ 0 & -1 \end{bmatrix}, \ \bar{U}_{n-1} = \begin{bmatrix} \bar{U}_{n-1,n-1} \\ 0 \end{bmatrix}, \ u_n = e_n,$$
$$\bar{M} = (\bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1})^{1/2}, \ \lambda_n = -1,$$

where M is any nonsingular matrix of order $n-1$ and $\bar{U}_{n-1,n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$ is an orthogonal matrix satisfying the condition

$$\bar{U}_{n-1,n-1} \bar{M}^T \bar{M} \bar{U}_{n-1,n-1}^T = M^T M.$$

Therefore, we obtain

$$(\bar{M}\bar{U}_{n-1}^T x)^T \bar{M}\bar{U}_{n-1}^T x = \bar{x}_{n-1}^T M^T M \bar{x}_{n-1},$$
$$(\bar{M}^{-1}\bar{U}_{n-1}^T x)^T \bar{M}^{-1}\bar{U}_{n-1}^T x = \bar{x}_{n-1}^T M^T M \bar{x}_{n-1},$$

which show that $\|\bar{M}\bar{U}_{n-1}^T x\| = \|M\bar{x}_{n-1}\|$ and $\|\bar{M}^{-1}\bar{U}_{n-1}^T x\| = \|(M^{-1})^T \bar{x}_{n-1}\|$. If we set $w := \bar{U}_{n-1,n-1}^T \eta$, then $\eta \neq 0$ and $\bar{U}_{n-1} w = (\eta, 0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, since $w \neq 0$ and the orthogonal property of $\bar{U}_{n-1.n-1}$. Moreover, by simple calculation, we also obtain

$$(\bar{M}w)^T \bar{M}w = w^T \bar{U}_{n-1,n-1}^T M^T M \bar{U}_{n-1,n-1} w = \eta^T M^T M \eta,$$
$$(\bar{M}^{-1}w)^T \bar{M}^{-1}w = w^T \bar{U}_{n-1,n-1}^T M^{-1} (M^T)^{-1} \bar{U}_{n-1,n-1} w = \eta^T ((M^{-1})^T)^T (M^{-1})^T \eta,$$

therefore we have $\|\bar{M}w\| = \|M\eta\|$ and $\|\bar{M}^{-1}w\| = \|(M^{-1})^T\eta\|$. With respect to the elliptic cone $\mathcal{K}_M^n$, their scalar parts and vector parts are given by

Type I:

$$s_{I_a}^{(1)}(x) := x_n + \|M\bar{x}_{n-1}\|,\ v_{I_a}^{(1)}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

$$s_{I_a}^{(2)}(x) := x_n - \|M\bar{x}_{n-1}\|,\ v_{I_a}^{(2)}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

$$s_{I_b}^{(1)}(x) := x_n, \qquad v_{I_b}^{(1)}(x) := \frac{1}{2}\begin{bmatrix} \dfrac{\eta}{\|M\eta\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

$$s_{I_b}^{(2)}(x) := x_n, \qquad v_{I_b}^{(2)}(x) := \frac{1}{2}\begin{bmatrix} -\dfrac{\eta}{\|M\eta\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

Type II:

$$s_{II_a}^{(1)}(x) := x_n + \|(M^{-1})^T\bar{x}_{n-1}\|,\ s_{II_a}^{(2)}(x) := x_n - \|M\bar{x}_{n-1}\|,$$

$$v_{I_a}^{(1)}(x) := \frac{\|M\bar{x}_{n-1}\|}{\|(M^{-1})^T\bar{x}_{n-1}\| + \|M\bar{x}_{n-1}\|}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

$$v_{I_a}^{(2)}(x) := \frac{\|(M^{-1})^T\bar{x}_{n-1}\|}{\|(M^{-1})^T\bar{x}_{n-1}\| + \|M\bar{x}_{n-1}\|}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|(M^{-1})^T\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M^*,$$

$$s_{II_b}^{(1)}(x) := x_n,\ v_{I_b}^{(1)}(x) := \frac{\|M\eta\|}{\|(M^{-1})^T\eta\| + \|M\eta\|}\begin{bmatrix} \dfrac{\|\eta\|}{\|M\eta\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M,$$

$$s_{II_b}^{(2)}(x) := x_n,\ v_{I_b}^{(2)}(x) := \frac{\|(M^{-1})^T\eta\|}{\|(M^{-1})^T\eta\| + \|M\eta\|}\begin{bmatrix} -\dfrac{\eta}{\|(M^{-1})^T\eta\|} \\ 1 \end{bmatrix} \in \mathcal{K}_M^*.$$

The decompositions with respect to the elliptic cone $\mathcal{K}_M^n$ can be summarized as follows:

Type I:

$$x = \begin{cases} (x_n + \|M\bar{x}_{n-1}\|) \cdot \dfrac{1}{2}\begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} \\ +(x_n - \|M\bar{x}_{n-1}\|) \cdot \dfrac{1}{2}\begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \text{if} \quad \bar{x}_{n-1} \neq 0, \\ \\ x_n \cdot \dfrac{1}{2}\begin{bmatrix} \dfrac{\eta}{\|M\eta\|} \\ 1 \end{bmatrix} + x_n \cdot \dfrac{1}{2}\begin{bmatrix} -\dfrac{\eta}{\|M\eta\|} \\ 1 \end{bmatrix} & \text{if} \quad \bar{x}_{n-1} = 0, \end{cases}$$

Type II:

$$
x = \begin{cases}
(x_n + \|(M^{-1})^T \bar{x}_{n-1}\|) \cdot \\
\dfrac{\|M\bar{x}_{n-1}\|}{\|(M^{-1})^T \bar{x}_{n-1}\| + \|M\bar{x}_{n-1}\|} \begin{bmatrix} \dfrac{\bar{x}_{n-1}}{\|M\bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \\
+(x_n - \|M\bar{x}_{n-1}\|) \cdot & \text{if } \ \bar{x}_{n-1} \neq 0, \\
\dfrac{\|(M^{-1})^T \bar{x}_{n-1}\|}{\|(M^{-1})^T \bar{x}_{n-1}\| + \|M\bar{x}_{n-1}\|} \begin{bmatrix} -\dfrac{\bar{x}_{n-1}}{\|(M^{-1})^T \bar{x}_{n-1}\|} \\ 1 \end{bmatrix} & \\
\\
x_n \cdot \dfrac{\|M\eta\|}{\|(M^{-1})^T \eta\| + \|M\eta\|} \begin{bmatrix} \dfrac{\|\eta\|}{\|M\eta\|} \\ 1 \end{bmatrix} & \text{if } \ \bar{x}_{n-1} = 0, \\
+x_n \cdot \dfrac{\|(M^{-1})^T \eta\|}{\|(M^{-1})^T \eta\| + \|M\eta\|} \begin{bmatrix} -\dfrac{\eta}{\|(M^{-1})^T \eta\|} \\ 1 \end{bmatrix} &
\end{cases}
$$

where η is any given nonzero vector in $\mathbb{R}^{n-1}$.

6. Concluding remarks

In this paper, we have characterized some main properties of the ellipsoidal cone from optimization viewpoint. The reasons why we focus on this special cone are already explained earlier. There are two aspects that these results are helpful for its corresponding non-symmetric cone optimization. One is that the explicit expressions of its dual cone, tangent cone, and normal cone can be used to establish the first-order optimality condition of minimization problems related to the ellipsoidal cone. The other one is that the projection mapping and the decompositions with respect to the ellipsoidal cone are introduced, then its corresponding conic functions can be defined and studied. There are plenty of non-symmetric cones in real world. We believe the analysis techniques used in this paper will pave a way to tackling with other unfamiliar non-symmetric cone optimization problems. Among all lots of non-symmetric cones, is there a way to clarify them? This is another important direction which we are interested in.

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