CHARACTERIZATIONS OF SOLUTION SETS FOR TWO NONSYMMETRIC CONE PROGRAMS

MING-YEN LI, CHING-YU YANG, XIN-HE MIAO, AND JEIN-SHAN CHEN

Abstract. This paper is devoted to the characterizations of solution sets for a general cone-constrained convex programming problems. In particular, when the cone reduces to two specific and nonsymmetric cones, that is, the power cone and the exponential cone, we demonstrate that the conclusion holds by exploiting the structures of those two cones.

1. Introduction

In this paper, we consider the following general cone-constrained convex programming problem:

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & -g(x) \in \mathcal{K} \\ & x \in C, \end{array} \tag{1.1}$$

where C is a closed convex set in $\mathbb{R}^n$, $\mathcal{K}$ is a closed convex cone in $\mathbb{R}^r$, $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, and $g : \mathbb{R}^n \to \mathbb{R}^r$ is a continuous $\mathcal{K}$-convex mapping, i.e., for every $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, there holds

$$tg(x) + (1-t)g(y) - g(tx + (1-t)y) \in \mathcal{K}.$$

It is known that constrained optimization problems including cone-constrained problems arise in a variety of scientific and engineering applications [11, 12, 18]. For constrained optimization problems, an important issue is the characterization of solution sets. This is because the characterizations and properties of solution sets is fundamental and crucial for understanding of the behavior of solution methods for solving optimization problems, see [4, 10, 15, 17, 19, 20, 23]. In 1988, Mangasarian [19] considered characterizations of the solution set of a differentiable convex programming problem. Later, Burke and Ferris [4] extended the results given in [19] to the setting of nondifferentiable convex programming. Moreover, for problem (1.1), when the function f is pseudolinear, $g = 0$, and the set $C = \{x \in \mathbb{R}^n \mid Ax = b\}$, Jeyakumar et al. [17] described the characterization of the solution set of so-called pseudolinear programs. In addition, for cone-constrained convex programming problems, Jeyakumar et al. [15] also provided the characterization of the solution set in terms of subgradients and Lagrange multipliers. Following the topic on

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the characterization of the solution set in [15], Miao and Chen [20] further considered a type of cone-constrained convex programming problem and simplified the corresponding results in [15]. In particular, when the cone reduces to three specific cones i.e., p-order cone [2, 24], L^p cone [12], and circular cone [25], the obtained conclusions can be achieved by exploiting the special structures of those three cones.

The main purpose of this paper is to describe the characterization of the solution set of problem (1.1), which is a generalization of the problem in [20]. Moreover, when the cone $\mathcal{K}$ reduces to two types of convex cones, i.e., the power cone $\mathcal{K}^{\alpha}_{m,n}$ and the exponential cone $\mathcal{K}_e$ (see Section 2 for details), we may obtain characterizations of the solution sets via exploiting the special structures of these two convex cones. Why do we focus on these two cones? There are two main reasons. The first one is because that these two non-symmetric cones appear in a lot of practical applications such as location problems and geometric programming [6, 13, 21, 22]. The second reason is indeed more important. More specifically, through appropriate transformations (for example, α-representation and extended α-representation defined in [6]), plenty of non-symmetric cones can be generated from the power cone $\mathcal{K}_\alpha$ and the exponential cone $\mathcal{K}_e$. In other words, these two cones are the cores of many non-symmetric cones in real world applications.

Toward the end of this section, we say a few words about notations which will be used in this paper. Throughout this paper, $\mathbb{R}$ denotes the space of real numbers, $\mathbb{R}_+$ denotes the set consisting of the nonnegative reals, and $\mathbb{R}^n$ means the n-dimensional real vector space endowed with the inner product $\langle \cdot, \cdot \rangle$. Moreover, we use $\|x\|$ to denote the Euclidean norm of x which induced by the inner product $\langle \cdot\cdot \rangle$, i.e., $\|x\| = \sqrt{\langle x, x \rangle}$. For any a set $\Omega \subseteq \mathbb{R}^n$, int Ω denotes the interior of Ω and bd Ω denotes the boundary of Ω. For any a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote $\partial f(x)$ the subdifferential of the function f at $x \in \mathbb{R}^n$.

2. Preliminaries

In this section, we briefly recall some background materials and useful results, which will be extensively used in subsequent analysis. More details can be found in [3, 7, 14, 11].

We start with the definition of the subdifferential of a function $f : \mathbb{R}^n \to \mathbb{R}$. The subdifferential of the function f at x is defined as

$$\partial f(x) := \{\xi \in \mathbb{R}^n \mid f(y) - f(x) \geq \langle \xi, y - x \rangle, \ \forall y \in \mathbb{R}^n\}.$$

If Ω is a convex set in $\mathbb{R}^n$, the normal cone $\mathcal{N}_\Omega(x)$ of the set Ω at $x \in \Omega$ is defined by

$$\mathcal{N}_\Omega(x) := \{\xi \in \mathbb{R}^n \mid \langle \xi, y - x \rangle \leq 0, \ \forall y \in \Omega\}.$$

When the convex set Ω corresponds to $\Omega = \{x \in \mathbb{R}^n \mid Ax = b\}$ with A being a $m \times n$ matrix, it is easy to verify that for any $x \in \Omega$, the normal cone $\mathcal{N}_\Omega(x)$ of the set Ω at x is written as

$$\mathcal{N}_\Omega(x) = \left\{A^T y \mid y \in \mathbb{R}^m\right\}.$$

For the problem (1.1), we know the function $g : \mathbb{R}^n \to \mathbb{R}^r$ is continuous $\mathcal{K}$-convex, which implies that the set $\{x \in \mathbb{R}^n \,|\, -g(x) \in \mathcal{K}\}$ is convex. Thus, it follows from the convexity of f that the problem (1.1) is a convex optimization problem. Let $\mathcal{F}$ and $\mathcal{S}$ be the feasible region and the solution set of the problem (1.1), respectively, that is,

$$\mathcal{F} := \{x \in C \,|\, -g(x) \in \mathcal{K}\} \quad \text{and} \quad \mathcal{S} := \{x \in \mathcal{F} \,|\, f(x) \leq f(y),\ \forall y \in \mathcal{F}\}.$$

According to the optimality conditions of the convex optimization problems, if the problem (1.1) satisfies the Slater condition [16], i.e., there exists $\bar{x} \in C$ with $-g(\bar{x}) \in \operatorname{int}\mathcal{K}$, it is known that $a \in \mathcal{S}$ if and only if the element a satisfies the KKT conditions, i.e., $a \in \mathcal{F}$ and there exists a Lagrange multiplier $\lambda_a \in \mathbb{R}^r$ such that

$$0 \in \partial f(a) + \partial(\lambda_a^T g)(a) + \mathcal{N}_C(a), \quad \lambda_a \in \mathcal{K}^* \ \text{ and } \ \lambda_a^T g(a) = 0, \tag{2.1}$$

where $\mathcal{K}^*$ denotes the dual cone of $\mathcal{K}$ given by

$$\mathcal{K}^* = \{z \in \mathbb{R}^r \,|\, \langle z, x\rangle \geq 0,\ \forall x \in \mathcal{K}\}.$$

In this paper, we always assume that the solution set $\mathcal{S}$ of the problem (1.1) is nonempty. From the above analysis, for $a \in \mathcal{S}$, there exists the corresponding Lagrange multiplier λ_a such that (a, λ_a) satisfies the KKT conditions (2.1). For convenience, we employ the Lagrange function $L_a(\cdot, \lambda_a) : \mathbb{R}^n \to \mathbb{R}$ associated with a defined by

$$L_a(x, \lambda_a) := f(x) + \lambda_a^T g(x) \quad \text{for all } x \in \mathbb{R}^n.$$

Then, the KKT conditions (2.1) can be reformulated into the form of

$$0 \in \partial L_a(a, \lambda_a) + \mathcal{N}_C(a), \quad \lambda_a \in \mathcal{K}^* \ \text{ and } \ \lambda_a^T g(a) = 0.$$

To close this section, we review the concepts of two specific closed convex cones, the explicit expressions of these two cones and their dual cones.

(1) power cone, see [6, 13]. It is a generalization of second-order cone (SOC) and defined as bellow:

$$\mathcal{K}^\alpha_{m,n} := \left\{ (x, z) \in \mathbb{R}^m_+ \times \mathbb{R}^n \,\middle|\, \|z\| \leq \prod_{i=1}^m x_i^{\alpha_i} \right\}$$

where $\alpha_i > 0$ and $\sum_{i=1}^m \alpha_i = 1$, $x = (x_1, \cdots, x_m)^T \in \mathbb{R}^m_+$, $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$. Indeed, $\mathcal{K}^\alpha_{m,n}$ is a solid (i.e., $\operatorname{int}\mathcal{K}^\alpha_{m,n} \neq \emptyset$), closed and convex cone, and its dual cone is given by

$$(\mathcal{K}^\alpha_{m,n})^* = \left\{ (\lambda, y) \in \mathbb{R}^m_+ \times \mathbb{R}^n \,\middle|\, \|y\| \leq \prod_{i=1}^m \left(\frac{\lambda_i}{\alpha_i}\right)^{\alpha_i} \right\}$$

where $\lambda = (\lambda_1, \cdots, \lambda_m)^T \in \mathbb{R}^m_+$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$. From the expression of the dual cone $(\mathcal{K}^\alpha_{m,n})^*$, we see that the dual cone $(\mathcal{K}^\alpha_{m,n})^*$ is also a solid, closed and convex cone. When $m = 1$, we note that the power cone is just second-order cone $\mathcal{K}^{n+1}$ [1, 5, 8, 9] defined as follows:

$$\mathcal{K}^{n+1} = \left\{ (x_1, z) \in \mathbb{R}_+ \times \mathbb{R}^n \,\middle|\, \|z\| \leq x_1 \right\}.$$

Hence, the power cone $\mathcal{K}^{\alpha}_{m,n}$ includes second-order cone $\mathcal{K}^{n+1}$ as a special case with $m=1$. In addition, from the expression of the power cone $\mathcal{K}^{\alpha}_{m,n}$ and its dual cone $(\mathcal{K}^{\alpha}_{m,n})^*$, it is not hard to verify that the boundary of the power cone $\mathcal{K}^{\alpha}_{m,n}$ and its dual cone $(\mathcal{K}^{\alpha}_{m,n})^*$ can be respectively expressed as follows:

$$\begin{aligned}
\mathrm{bd}\,\mathcal{K}^{\alpha}_{m,n} &= \left\{ (x,z) \in \mathbb{R}^m_+ \times \mathbb{R}^n \,\middle|\, \|z\| = \prod_{i=1}^m x_i^{\alpha_i} \right\}, \\
\mathrm{bd}\,(\mathcal{K}^{\alpha}_{m,n})^* &= \left\{ (\lambda,y) \in \mathbb{R}^m_+ \times \mathbb{R}^n \,\middle|\, \|y\| = \prod_{i=1}^m (\frac{\lambda_i}{\alpha_i})^{\alpha_i} \right\}.
\end{aligned}$$

In order to have further understanding of $\mathcal{K}^{\alpha}_{m,n}$, the pictures of four different cones $\mathcal{K}^{\alpha}_{m,n}$ in $\mathbb{R}^m_+ \times \mathbb{R}^n$ and their dual cones are depicted in Figure 1 and Figure 2, respectively.

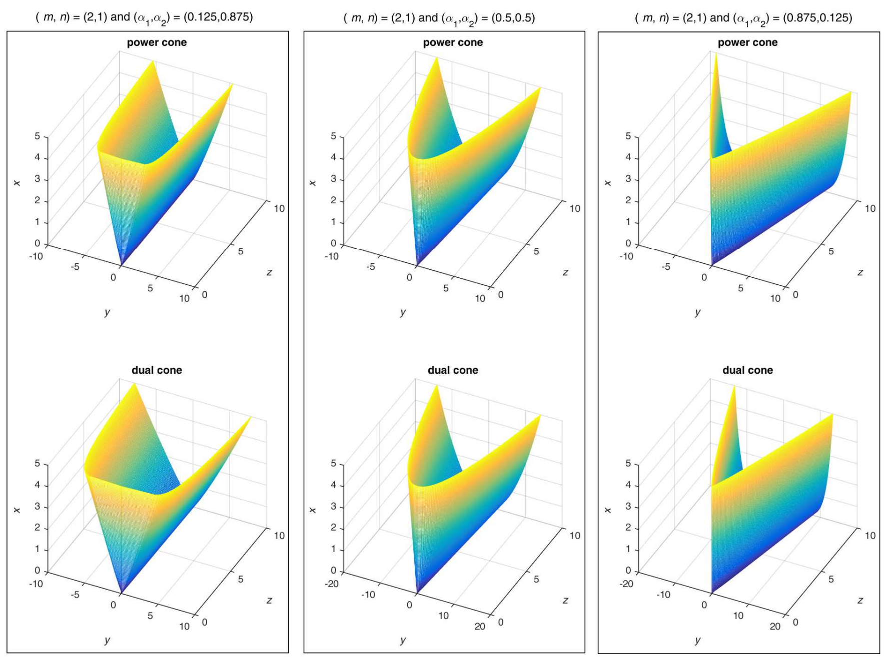

FIGURE 1. The 3-dimensional power cones and its dual cones with $m=2, n=1$ and different α_1, α_2

(2) exponential cone, see [6, 22]. The exponential cone is a cone in 3-dimensional Euclidean space $\mathbb{R}^3$, which is defined as bellow:

$$\mathcal{K}_e := \mathrm{cl}\left\{ (x_1,x_2,x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \le x_3,\ x_2 > 0 \right\}.$$

In fact, the exponential cone is also the union of two sets, i.e.,

$$\mathcal{K}_e := \left\{ (x_1,x_2,x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \le x_3,\ x_2 > 0 \right\} \cup \left\{ (x_1,0,x_3)^T \,\middle|\, x_1 \le 0,\ x_3 \ge 0 \right\}.$$

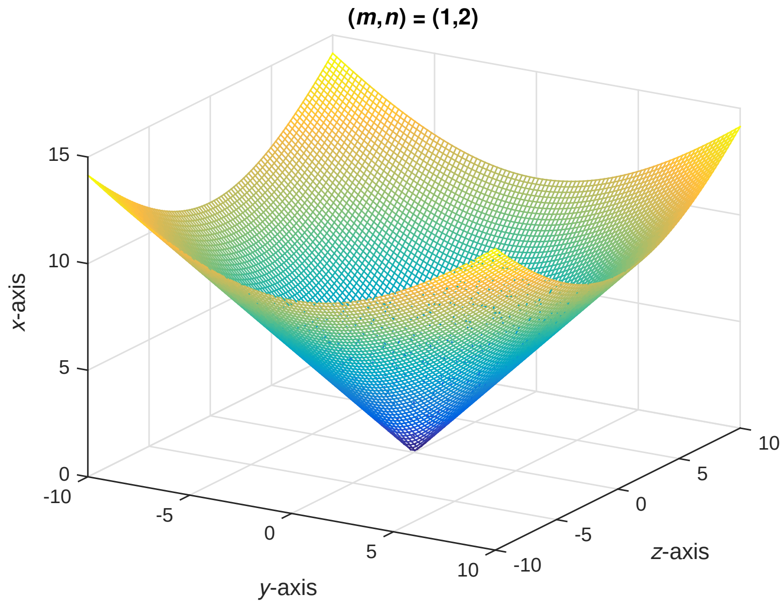

FIGURE 2. The 3-dimensional power cone with $m = 1, n = 2$, i.e., second-order cone

As shown in [6], the exponential cone $\mathcal{K}_e$ is a closed convex cone, and its dual cone $\mathcal{K}_e^*$ is given by

$$\mathcal{K}_e^* = \mathrm{cl}\left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3,\ y_1 < 0\right\}.$$

In a similar manner, the dual cone is also expressed as the union of the two corresponding sets, i.e.,

$$\mathcal{K}_e^* := \left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3,\ y_1 < 0\right\} \cup \left\{(0, y_2, y_3)^T \,\middle|\, y_2 \geq 0,\ y_3 \geq 0\right\}.$$

Note that the dual cone $\mathcal{K}_e^*$ is also a closed convex cone. The pictures of the exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$ are depicted in Figure 3 and Figure 4, respectively. Moreover, in view of the expressions of exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$ (or alternatively from Figure 3 and Figure 4, respectively), it is easy to verify that the boundary of exponential cone and its dual cone can be respectively expressed as follows:

$$\begin{aligned}
\mathrm{bd}\,\mathcal{K}_e &:= \left\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} = x_3,\ x_2 > 0\right\} \cup \left\{(x_1, 0, x_3)^T \,\middle|\, x_1 \leq 0,\ x_3 \geq 0\right\}, \\
\mathrm{bd}\,\mathcal{K}_e^* &:= \left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} = e y_3,\ y_1 < 0\right\} \cup \left\{(0, y_2, y_3)^T \,\middle|\, y_2 \geq 0,\ y_3 \geq 0\right\}.
\end{aligned}$$

3. CHARACTERIZATIONS OF SOLUTION SET

In this section, we describe the characterization of the solution set $\mathcal{S}$ for the problem (1.1) in terms of Lagrange multipliers and subgradients. Moreover, when

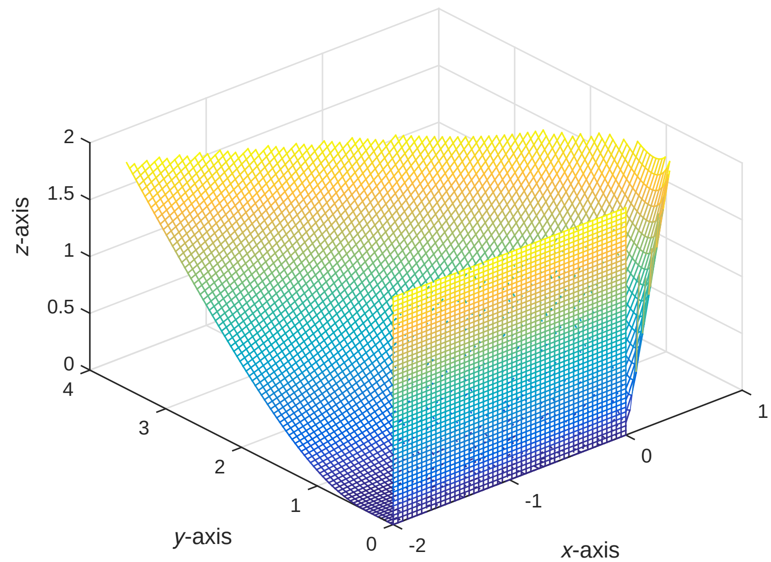

Figure 3. The exponential cone

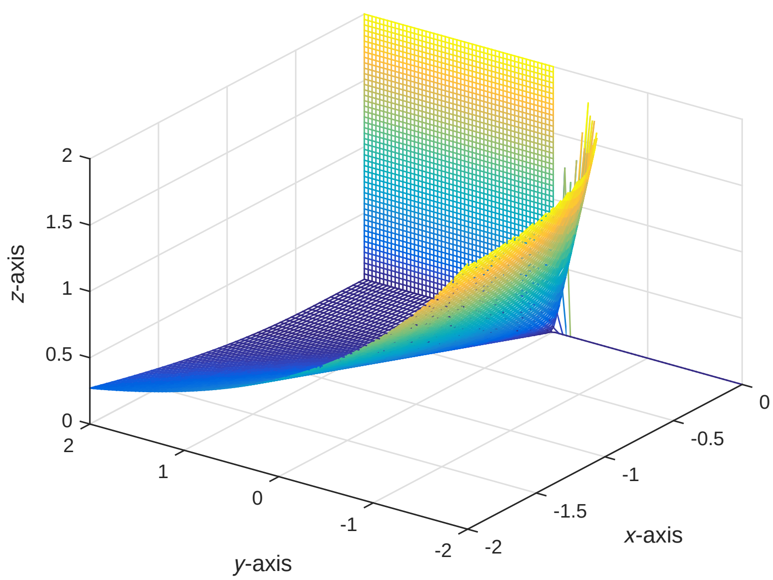

Figure 4. The dual cone of exponential cone

the cone $\mathcal{K}$ reduces to two specific cones, i.e., the power cone and the exponential cone, we can establish the same conclusions by exploiting the structures of the two types of specific cones.

Theorem 3.1. *For the problem (1.1), let $a \in \mathcal{S}$. Suppose that the corresponding Lagrange multiplier $\lambda_a \in \mathbb{R}^r$ satisfies the conditions:*

$$0 \in \partial L_a(a, \lambda_a) + \mathcal{N}_C(a), \quad \lambda_a \in \mathcal{K}^* \quad \text{and} \quad \lambda_a^T g(a) = 0. \tag{3.1}$$

Then, the following hold.

- **(a):** *If $\lambda_a = 0$, then for every $x \in \mathcal{S}$, there exists $\xi \in \mathcal{N}_C(a)$ such that*
$$-\xi \in \partial f(x).$$
- **(b):** *If $\lambda_a \neq 0$, then for every $x \in \mathcal{S}$ and $g(x) \neq 0$, we have*
$$-g(x) \in \operatorname{bd} \mathcal{K}, \quad \lambda_a \in \operatorname{bd} \mathcal{K}^* \quad \text{and} \quad \lambda_a^T g(x) = 0.$$

Proof. (a) For $\lambda_a = 0$, from the conditions (3.1), there exists $\xi \in \mathcal{N}_C(a)$ such that $-\xi \in \partial L_a(a, \lambda_a)$. By the definitions of the subdifferential and the Lagrange function, it follows that for any $y \in \mathbb{R}^n$, there has

$$\begin{aligned}
(-\xi)^T(y-a) &\leq L_a(y, \lambda_a) - L_a(a, \lambda_a) \\
&= f(y) + \lambda_a^T g(y) - f(a) - \lambda_a^T g(a) \\
&= f(y) - f(a).
\end{aligned}$$

This means $-\xi \in \partial f(a)$. Moreover, it follows from $\xi \in \mathcal{N}_C(a)$ that $(-\xi)^T(x-a) \geq 0$ for every $x \in \mathcal{S}$. This together with the properties of convex function yields

$$\begin{aligned}
f(y) - f(x) &= f(y) - f(a) \\
&\geq (-\xi)^T(y-a) \\
&= (-\xi)^T(y-x) + (-\xi)^T(x-a) \\
&\geq (-\xi)^T(y-x)
\end{aligned}$$

for every $x \in \mathcal{S}$ and any $y \in \mathbb{R}^n$, which says that $-\xi \in \partial f(x)$ for every $x \in \mathcal{S}$.

(b) For $\lambda_a \neq 0$, from the conditions (3.1), i.e.,

$$0 \in \partial L_a(a, \lambda_a) + \mathcal{N}_C(a), \quad \lambda_a \in \mathcal{K}^* \quad \text{and} \quad \lambda_a^T g(a) = 0,$$

there exists $\xi \in \mathcal{N}_C(a)$ such that $-\xi \in \partial L_a(a, \lambda_a)$. Then, for every $x \in \mathcal{S}$, we have

$$\begin{aligned}
f(x) + \lambda_a^T g(x) &= L_a(x, \lambda_a) \\
&\geq L_a(a, \lambda_a) + (-\xi)^T(x-a) \geq L_a(a, \lambda_a) \\
&= f(a) + \lambda_a^T g(a),
\end{aligned}$$

where the second inequality holds since $(-\xi)^T(x-a) \geq 0$ for $\xi \in \mathcal{N}_C(a)$. Now, using $x, a \in \mathcal{S}$ and $\lambda_a^T g(a) = 0$, we obtain that $\lambda_a^T g(x) \geq 0$ for every $x \in \mathcal{S}$. On the other hand, because $\lambda_a \in \mathcal{K}^*$ and $-g(x) \in \mathcal{K}$ for every $x \in \mathcal{S}$, this gives $\lambda_a^T(-g(x)) \geq 0$, which says $\lambda_a^T g(x) \leq 0$. Hence, we conclude that $\lambda_a^T g(x) = 0$ for every $x \in \mathcal{S}$.

Next, we show that $\lambda_a \in \operatorname{bd} \mathcal{K}^*$ and $-g(x) \in \operatorname{bd} \mathcal{K}$ for every $x \in \mathcal{S}$ and $g(x) \neq 0$. Here, we only prove $-g(x) \in \operatorname{bd} \mathcal{K}$ because with the same arguments, the conclusion of $\lambda_a \in \operatorname{bd} \mathcal{K}^*$ can be drawn. Now, we prove $-g(x) \in \operatorname{bd} \mathcal{K}$ by contradiction. Suppose that $-g(x) \in \operatorname{int} \mathcal{K}$. Then, there is a $\epsilon > 0$ such that $B(-g(x), \epsilon) \subseteq \mathcal{K}$ where B is open ball with radius ϵ. This implies that for any $y \in \mathbb{R}^r$, there exists $\alpha > 0$ such that

$$-g(x) + \alpha y \in B(-g(x), \epsilon) \subseteq \mathcal{K}.$$

Moreover, since $\lambda_a \in \mathcal{K}^*$, we know that

$$\lambda_a^T(-g(x) + \alpha y) = -\lambda_a^T g(x) + \alpha \lambda_a^T y \geq 0.$$

Hence, it follows from $\lambda_a^T g(x) = 0$ for every $x \in \mathcal{S}$ that $\alpha \lambda_a^T y \geq 0$. By the arbitrariness of $y \in \mathbb{R}^r$, we obtain that $\lambda_a = 0$, which contradicts the condition $\lambda_a \neq 0$. Thus, $-g(x) \in \operatorname{bd} \mathcal{K}$. Then, the proof is complete. □

Next, we demonstrate that Theorem 3.1 in the settings of power cone and exponential cone can be achieved as well by using the structures of power cone and exponential cone, respectively. To this end, for the problem (1.1),

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & -g(x) \in \mathcal{K} \\ & x \in C, \end{array}$$

we consider the cases of $\mathcal{K} = \mathcal{K}^{\alpha}_{m,r-m}$ and $\mathcal{K} = \mathcal{K}_e$ respectively. Under each case, the problem (1.1) becomes a specific power cone or exponential cone constrained convex programming problem. To proceed, we need the following technical lemmas.

Lemma 3.2. [Weighted AM-GM inequality]. *For any $n \in \mathbb{N}$, suppose that $\xi_i \geq 0$ and $w_i > 0$ for $i = 1, \cdots, n$. Let $w = \sum_{j=1}^n w_j$. Then,*

$$\left(\prod_{j=1}^n \xi_j^{w_j} \right)^{\frac{1}{w}} \leq \frac{1}{w} \sum_{j=1}^n w_j \xi_j$$

with the equality holding if and only if $\xi_1 = \xi_2 = \cdots = \xi_n$.

Proof. This is a well-known inequality, please refer to [14] for a proof. □

Lemma 3.3. *Suppose that $a_i \geq 0$, $b_i \geq 0$ and $p_i > 0$ for $i = 1, 2, \cdots, n$, where $\sum_{i=1}^n p_i = 1$. Then, we have*

$$\sum_{i=1}^n a_i b_i \geq \prod_{i=1}^n \left(\frac{a_i b_i}{p_i} \right)^{p_i}.$$

Proof. For any $i = 1, 2, \cdots, n$, by $a_i \geq 0$, $b_i \geq 0$ and $p_i > 0$ with $\sum_{i=1}^n p_i = 1$, let $y_i = \frac{a_i b_i}{p_i}$ $(i = 1, \cdots, n)$. It is clear that $y_i \geq 0$ for any $i = 1, \cdots, n$. Then, from Lemma 3.2, we have

$$\begin{aligned} a_1 b_1 + a_2 b_2 + \cdots + a_n b_n &= p_1 y_1 + p_2 y_2 + \cdots + p_n y_n \\ &\geq y_1^{p_1} \cdots y_n^{p_n} \\ &= \left(\frac{a_1 b_1}{p_1} \right)^{p_1} \left(\frac{a_2 b_2}{p_2} \right)^{p_2} \cdots \left(\frac{a_n b_n}{p_n} \right)^{p_n}. \end{aligned}$$

This means $\sum_{i=1}^n a_i b_i \geq \Pi_{i=1}^n (\frac{a_i b_i}{p_i})^{p_i}$, which is the desired result. □

Lemma 3.4. *Let $h(t) = e^{t-1} - t$ on $\mathbb{R}$. Then, we have $h(t) \geq 0$ for all $t \in \mathbb{R}$.*

Proof. Since $h(t) = e^{t-1} - t$, we have $h'(t) = e^{t-1} - 1$. Thus, it follows that

$$h'(t) = e^{t-1} - 1 > 0, \quad \forall t > 1 \quad \text{and} \quad h'(t) = e^{t-1} - 1 < 0, \quad \forall t < 1.$$

This indicates that the function h is strictly increasing on $(1, \infty)$, and h is strictly decreasing on $(-\infty, 1)$. Thus, for any $t \in \mathbb{R}$, we have $h(t) \geq h(1) = 0$, which is the desired result. $\square$

Theorem 3.5. *For the problem (1.1), let $\mathcal{K} = \mathcal{K}^{\alpha}_{m,r-m}$ and $a \in \mathcal{S}$. Suppose that the corresponding Lagrange multiplier λ_a satisfies the conditions as Theorem 3.1, i.e.,*

$$0 \in \partial L_a(a, \lambda_a) + \mathcal{N}_C(a), \quad \lambda_a \in (\mathcal{K}^{\alpha}_{m,r-m})^* \quad \textit{and} \quad \lambda_a^T g(a) = 0.$$

If $\lambda_a \neq 0$, then for each $x \in \mathcal{S}$ and $g(x) \neq 0$, there have

$$-g(x) \in \operatorname{bd} \mathcal{K}^{\alpha}_{m,r-m}, \quad \lambda_a \in \operatorname{bd} (\mathcal{K}^{\alpha}_{m,r-m})^* \quad \textit{and} \quad \lambda_a^T g(x) = 0.$$

Proof. From the proof of Theorem 3.1, we know that $\lambda_a^T g(x) = 0$ for all $x \in \mathcal{S}$. Then, it remains to show that $-g(x) \in \operatorname{bd} \mathcal{K}^{\alpha}_{m,r-m}$ and $\lambda_a \in \operatorname{bd} (\mathcal{K}^{\alpha}_{m,r-m})^*$. For convenience, we denote $0 \neq -g(x) := (x, z) \in \mathcal{K}^{\alpha}_{m,r-m}$ and $0 \neq \lambda_a := (\lambda, y) \in (\mathcal{K}^{\alpha}_{m,r-m})^*$ with $m < r$. By the expressions of the power cone $\mathcal{K}^{\alpha}_{m,r-m}$ and its dual cone $(\mathcal{K}^{\alpha}_{m,r-m})^*$, it follows that

$$\|z\| \leq \prod_{i=1}^{m} x_i^{\alpha_i} \quad \text{and} \quad \|y\| \leq \prod_{i=1}^{m} \left(\frac{\lambda_i}{\alpha_i}\right)^{\alpha_i}$$

with $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Then, from $\lambda_a^T g(x) = 0$, we have

$$\begin{aligned}
0 &= \lambda^\top(-x) + y^\top(-z) \\
&\leq -\sum_{i=1}^{m} \lambda_i x_i + \|y\|\|z\| \\
&\leq -\sum_{i=1}^{m} \left(\frac{\lambda_i x_i}{\alpha_i}\right)^{\alpha_i} + \left[\prod_{i=1}^{m} \left(\frac{\lambda_i}{\alpha_i}\right)^{\alpha_i}\right] \left[\prod_{i=1}^{m} x_i^{\alpha_i}\right] \\
&\leq 0
\end{aligned}$$

where the first inequality holds due to the Cauchy-Schwarz inequality, and the last inequality holds due to Lemma 3.3. This implies that

$$\|z\| = \prod_{i=1}^{m} x_i^{\alpha_i} \quad \text{and} \quad \|y\| = \prod_{i=1}^{m} \left(\frac{\lambda_i}{\alpha_i}\right)^{\alpha_i}.$$

Hence, we conclude that

$$-g(x) \in \operatorname{bd} \mathcal{K}^{\alpha}_{m,r-m}, \quad \lambda_a \in \operatorname{bd} (\mathcal{K}^{\alpha}_{m,r-m})^* \quad \text{and} \quad \lambda_a^T g(x) = 0$$

and the proof is complete. $\square$

Theorem 3.6. *For the problem (1.1), let $\mathcal{K} = \mathcal{K}_e$ and $a \in \mathcal{S}$. Suppose that the corresponding Lagrange multiplier λ_a satisfies the conditions as Theorem 3.1, i.e.,*

$$0 \in \partial L_a(a, \lambda_a) + \mathcal{N}_C(a), \quad \lambda_a \in \mathcal{K}_e^* \ \ and \ \ \lambda_a^T g(a) = 0.$$

If $\lambda_a \neq 0$, then for each $x \in \mathcal{S}$ and $g(x) \neq 0$, there have

$$-g(x) \in \operatorname{bd} \mathcal{K}_e, \quad \lambda_a \in \operatorname{bd} \mathcal{K}_e^* \ \ and \ \ \lambda_a^T g(x) = 0.$$

Proof. Using the same arguments as the proof of Theorem 3.5 and applying Theorem 3.1, it is clear that $\lambda_a^T g(x) = 0$ for all $x \in \mathcal{S}$. Then it remains to show that $-g(x) \in \operatorname{bd} \mathcal{K}_e$ and $\lambda_a \in \operatorname{bd} \mathcal{K}_e^*$. Suppose that $0 \neq -g(x) := (x_1, x_2, x_3)^T \in \mathcal{K}_e$ and $0 \neq \lambda_a = (y_1, y_2, y_3)^T \in \mathcal{K}_e^*$. For convenience, we denote

$$\begin{aligned} A &:= \left\{ (x_1, x_2, x_3)^T \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \leq x_3, x_2 > 0 \right\}, \qquad B := \left\{ (x_1, 0, x_3)^T \,\middle|\, x_1 \leq 0, x_3 \geq 0 \right\}, \\ M &:= \left\{ (y_1, y_2, y_3)^T \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3, y_1 < 0 \right\}, \quad N := \left\{ (0, y_2, y_3)^T \,\middle|\, y_2 \geq 0, y_3 \geq 0 \right\}. \end{aligned}$$

Then, using the expressions of exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$, i.e.,

$$\begin{aligned} \mathcal{K}_e &= \left\{ (x_1, x_2, x_3)^T \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \leq x_3, x_2 > 0 \right\} \cup \left\{ (x_1, 0, x_3)^T \,\middle|\, x_1 \leq 0, x_3 \geq 0 \right\} \\ \mathcal{K}_e^* &= \left\{ (y_1, y_2, y_3)^T \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3, y_1 < 0 \right\} \cup \left\{ (0, y_2, y_3)^T \,\middle|\, y_2 \geq 0, y_3 \geq 0 \right\}, \end{aligned}$$

we have $\mathcal{K}_e = A \cup B$ and $\mathcal{K}_e^* = M \cup N$. To proceed the proof, we need to discuss four cases.

Cases 1. When $-g(x) \in A$, $\lambda_a \in M$, we have $x_2 e^{\frac{x_1}{x_2}} \leq x_3$ with $x_2 > 0$ and $-y_1 e^{\frac{y_2}{y_1}} \leq e y_3$ with $y_1 < 0$. This together with $\lambda_a^T g(x) = 0$ for all $x \in \mathcal{S}$ yields

$$\begin{aligned} 0 &= x_1 y_1 + x_2 y_2 + x_3 y_3 \\ &= -y_1 x_2 \left(\frac{x_1 y_1}{-y_1 x_2} + \frac{x_2 y_2}{-y_1 x_2} + \frac{x_3 y_3}{-y_1 x_2} \right) \\ &= -y_1 x_2 \left(\frac{x_1}{-x_2} + \frac{y_2}{-y_1} + (\frac{x_3}{x_2})(\frac{y_3}{-y_1}) \right) \\ &\geq -y_1 x_2 \left(-(\frac{x_1}{x_2} + \frac{y_2}{y_1}) + e^{\frac{x_1}{x_2}} e^{\frac{y_2}{y_1} - 1} \right) \\ &= -y_1 x_2 \left(-(\frac{x_1}{x_2} + \frac{y_2}{y_1}) + e^{\frac{x_1}{x_2} + \frac{y_2}{y_1} - 1} \right) \\ &\geq 0, \end{aligned}$$

where the last inequality is due to Lemma 3.4. Then, it follows that $\frac{x_3}{x_2} = e^{\frac{x_1}{x_2}}$ and $\frac{y_3}{-y_1} = e^{\frac{y_2}{y_1} - 1}$, i.e., $x_2 e^{\frac{x_1}{x_2}} = x_3$ and $-y_1 e^{\frac{y_2}{y_1}} = e y_3$, which says $-g(x) \in \operatorname{bd} A$ and $\lambda_a \in \operatorname{bd} M$. Thus, $-g(x) \in \operatorname{bd} \mathcal{K}_e$ and $\lambda_a \in \operatorname{bd} \mathcal{K}_e^*$.

Cases 2. When $-g(x) \in A$, $\lambda_a \in N$, we have $x_2 e^{\frac{x_1}{x_2}} \leq x_3$ with $x_2 > 0$, and $y_1 = 0$ with $y_2 \geq 0$ and $y_3 \geq 0$. Hence, it follows from $\lambda_a^T g(x) = 0$ for all $x \in \mathcal{S}$ that $0 = x_2 y_2 + x_3 y_3$. Because $x_2 > 0$, $y_2 \geq 0$, $y_3 \geq 0$ and $x_3 > 0$, we obtain that $y_2 = y_3 = 0$, i.e., $\lambda_a = (y_1, y_2, y_3)^T = (0, 0, 0)^T$, which contradicts $\lambda_a \neq 0$. This says that the subcase does not occur.

Cases 3. When $-g(x) \in B$, $\lambda_a \in M$, we have $x_1 \leq 0$, $x_3 \geq 0$, $x_2 = 0$ and $-y_1 e^{\frac{y_2}{y_1}} \leq ey_3$ with $y_1 < 0$. Because $\lambda_a^T g(x) = 0$ for all $x \in \mathcal{S}$, this implies $0 = x_1 y_1 + x_3 y_3$. Then, it follows from $x_1 \leq 0$, $x_3 \geq 0$, $y_1 < 0$ and $y_3 > 0$ that $x_1 = x_3 = 0$, i.e., $-g(x) = 0$. This contradicts $-g(x) \neq 0$. Hence, this subcase does not also occur.

Cases 4. When $-g(x) \in B$, $\lambda_a \in N$, in light of the expression of exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$, it is clear that $-g(x) \in \operatorname{bd} \mathcal{K}_e$ and $\lambda_a \in \operatorname{bd} \mathcal{K}_e^*$.

From the above discussions in all cases, we prove that

$$-g(x) \in \operatorname{bd} \mathcal{K}_e, \quad \lambda_a \in \operatorname{bd} \mathcal{K}_e^* \quad \text{and} \quad \lambda_a^T g(x) = 0.$$

Thus, the proof is complete. $\square$

Example 3.7. For $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, consider the nonlinear convex programming problem:

$$\begin{array}{ll} \min & f(x) = x_1^2 + x_2^2 + x_3^2 \\ \text{s.t.} & -g(x) = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \end{pmatrix} \in \mathcal{K}_e, \end{array}$$

where $\mathcal{K}_e$ is the exponential cone.

Let $\mathcal{F}$ and $\mathcal{S}$ be the feasible set and the solution set of this problem, respectively. It follows from $-g(x) = (-x_1, -x_2, -x_3)^T \in \mathcal{K}_e$ that $-x_2 e^{\frac{x_1}{x_2}} \leq -x_3$ with $-x_2 > 0$, or $-x_1 \leq 0, -x_3 \geq 0$ and $x_2 = 0$, which yields the feasible set

$$\mathcal{F} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \,|\, x_2 e^{\frac{x_1}{x_2}} \geq x_3,\ x_2 < 0 \right\} \cup \left\{ (x_1, 0, x_3)^T \,\middle|\, x_1 \geq 0,\ x_3 \leq 0 \right\}.$$

Noting that for any $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, we have

$$f(x) = x_1^2 + x_2^2 + x_3^2 \geq 0.$$

Thus, it is not hard to verify that $\bar{x} = (0, 0, 0)^T \in \mathbb{R}^3$ is a solution to the considered problem, i.e, $\bar{x} \in \mathcal{S}$. Moreover, for any $\bar{x} \neq x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, there has

$$\partial f(x) = 2(x_1, x_2, x_3)^T \neq 0.$$

In light of this, for the solution $\bar{x} \in \mathcal{S}$, it is easy to see that the corresponding Lagrange multiplier $\lambda_{\bar{x}} = (0, 0, 0)^T \in \mathcal{K}_e^*$ and $0 \in \partial L_{\bar{x}}(\bar{x}, \lambda_{\bar{x}}) = \partial f(\bar{x})$. All the above leads to

$$(0, 0, 0)^T \in \partial f(x) \iff x_1 = 0,\ x_2 = 0,\ x_3 = 0.$$

Therefore, we conclude that the solution set $\mathcal{S}$ can be expressed as

$$\mathcal{S} = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \,|\, x_1 = 0, x_2 = 0, x_3 = 0 \right\}.$$

Example 3.8. For $x = (x_1, x_2)^T \in \mathbb{R}^2$, consider the nonlinear convex programming problem:

$$\begin{array}{ll} \min & f(x) = \sqrt{u^2(x_1) + v^2(x_2)} + v(x_2) \\ \text{s.t.} & -g(x) = \left(\begin{array}{c} -v(x_2) \\ u(x_1) \end{array} \right) \in \mathcal{K}^{\alpha}_{1,1}, \end{array}$$

where $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ are both differentiable and $\alpha = 1$.

Let $\mathcal{F}$ and $\mathcal{S}$ be the feasible set and the solution set of this problem, respectively. Because $-g(x) = (-v(x_2), u(x_1))^T \in \mathcal{K}^{\alpha}_{1,1}$, we have $0 \leq |u(x_1)| \leq -v(x_2)$, which implies that the feasible set

$$\mathcal{F} = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \,|\, v(x_2) \leq -|u(x_1)| \leq 0 \right\}.$$

Noting that for any $x = (x_1, x_2)^T \in \mathbb{R}^2$, we have

$$f(x) = \sqrt{u^2(x_1) + v^2(x_2)} + v(x_2) \geq |v(x_2)| + v(x_2) \geq 0.$$

Thus, it is easy to check that $\bar{x} = (\bar{x}_1, \bar{x}_2)^T \in \mathbb{R}^2$ satisfying $u(\bar{x}_1) = 0$ and $v(\bar{x}_2) = 0$ is a solution of the considered problem, i.e, $\bar{x} \in \mathcal{S}$. Since for any $\bar{x} \neq x = (x_1, x_2)^T \in \mathbb{R}^2$ with $u(x_1) \neq 0$ or $v(x_2) \neq 0$, it can be computed that

$$\begin{aligned} \partial f(x) &= \left\{ \left(\frac{u(x_1)}{\sqrt{u^2(x_1) + v^2(x_2)}} u'(x_1), \frac{v(x_2)}{\sqrt{u^2(x_1) + v^2(x_2)}} v'(x_2) + v'(x_2) \right)^T \right\} \\ &= \left\{ \left[\begin{array}{cc} u'(x_1) & 0 \\ 0 & v'(x_2) \end{array} \right] \left(\begin{array}{c} \frac{u(x_1)}{\sqrt{u^2(x_1)+v^2(x_2)}} \\ \frac{v(x_2)}{\sqrt{u^2(x_1)+v^2(x_2)}} + 1 \end{array} \right) \right\}. \end{aligned}$$

Moreover, it can be verified that

$$\partial f(\bar{x}) = \left\{ \left[\begin{array}{cc} u'(\bar{x}_1) & 0 \\ 0 & v'(\bar{x}_2) \end{array} \right] \left\{ \left(\begin{array}{c} 0 \\ 1 \end{array} \right) + \mathcal{B} \right\} \right\},$$

where $\mathcal{B}$ denotes the closed unit ball in $\mathbb{R}^2$. Besides, for the solution $\bar{x} \in \mathcal{S}$, it is easy to see that if $u'(\bar{x}_1) \neq 0$ and $v'(\bar{x}_2) \neq 0$, the corresponding Lagrange multiplier $\lambda_{\bar{x}} = (0, 0)^T \in (\mathcal{K}^{\alpha}_{1,1})^*$, and $(0, 0)^T \in \partial L_{\bar{x}}(\bar{x}, \lambda_{\bar{x}}) = \partial f(\bar{x})$. With this, it follows that if $u'(x_1) \neq 0$ and $v'(x_2) \neq 0$,

$$(0, 0)^T \in \partial f(x) \iff u(x_1) = 0, \; v(x_2) \leq 0.$$

Therefore, we conclude that the solution set $\mathcal{S}$ may be expressed as

$$\mathcal{S} = \left\{ (x_1, x_2)^T \in \mathbb{R}^2 \,|\, u(x_1) = 0, v(x_2) \leq 0 \right\}.$$

In fact, when the convex set C reduces the special convex set $C := \{x \in \mathbb{R}^n \,|\, Ax = b\}$, where the matrix A is a $m \times n$ matrix, we see that Theorem 3.1 reduces to [20, Theorem 3.1]. This says that the considered problem in this paper includes the problem in [20] as a special case, which is presented the following corollary.

Corollary 3.9. [20, Theorem 3.1] *For the problem (1.1), let* $C := \{x \in \mathbb{R}^n \mid Ax = b\}$ *and* $a \in \mathcal{S}$*. Suppose that the corresponding Lagrange multiplier* $\lambda_a \in \mathbb{R}^r$ *satisfies the conditions:*

$$0 \in \partial L_a(a, \lambda_a) + \{A^T y \mid y \in \mathbb{R}^m\}, \quad \lambda_a \in \mathcal{K}^* \quad \text{and} \quad \lambda_a^T g(a) = 0.$$

Then, the following hold.

(a): *If* $\lambda_a = 0$*, then for each* $x \in S$*, there exists* $y \in \mathbb{R}^m$ *such that*

$$-A^T y \in \partial f(x).$$

(b): *If* $\lambda_a \neq 0$*, then for each* $x \in S$ *and* $g(x) \neq 0$*, there have*

$$-g(x) \in \partial\mathcal{K}, \quad \lambda_a \in \partial\mathcal{K}^* \quad \text{and} \quad \lambda_a^T g(x) = 0.$$

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