Numerical Functional Analysis and Optimization, 32(5):507–523, 2011

ISSN: 0163-0563 print/1532-2467 online
DOI: 10.1080/01630563.2011.563893

ON THE LORENTZ CONE COMPLEMENTARITY PROBLEMS IN INFINITE-DIMENSIONAL REAL HILBERT SPACE

Xin-He Miao[1] **and Jein-Shan Chen**[2]

[1]*Department of Mathematics, School of Science, Tianjin University, Tianjin, P. R. China*
[2]*Department of Mathematics, National Taiwan Normal University, Taipei, Taiwan*

□ *In this article, we consider the Lorentz cone complementarity problems in infinite-dimensional real Hilbert space. We establish several results that are standard and important when dealing with complementarity problems. These include proving the same growth of the Fishcher–Burmeister merit function and the natural residual merit function, investigating property of bounded level sets under mild conditions via different merit functions, and providing global error bounds through the proposed merit functions. Such results are helpful for further designing solution methods for the Lorentz cone complementarity problems in Hilbert space.*

Keywords Error bound; FB-function; Lorentz cone; Merit function; NR-function; R_{02}-property.

AMS Subject Classification 90C33; 26B65.

1. MOTIVATION AND INTRODUCTION

Recently there has been much attention on symmetric cone optimization, see [4, 13, 14, 20, 21], and references therein. The symmetric cone $\mathcal{K}$ is intimately related to Euclidean Jordan algebra since it provides an essential toolbox for the analysis. In addition, the symmetric cone has special structure in Euclidean Jordan algebra $(\mathbb{V}, \circ, \langle\cdot,\cdot\rangle)$ [8, 12], namely, $\mathcal{K} = \{x^2 = x \circ x \mid x \in \mathbb{V}\}$. It is natural to ask what will happen if we go further beyond Euclidean Jordan algebra. In fact, it is known that the class of Euclidean Jordan algebras belongs to the class of JB-algebras [27]. More

Received 26 October 2010; Revised and Accepted 14 February 2011.
J.-S. C. is a member of Mathematics Division, National Center for Theoretical Sciences, Taipei Office. This author's work is partially supported by National Science Council of Taiwan.
Address correspondence to Jein-Shan Chen, Department of Mathematics, National Taiwan Normal University, 88, Section 4, Ting Chou Road, Taipei 11677, Taiwan; E-mail: jschen@math.ntnu.edu.tw

specifically, a finite-dimensional JB-algebra coincides with a Euclidean Jordan algebra. There is a subclass of JB-algebras called JB-algebra of finite rank which attracts our attention because every JB-algebra of finite rank is direct sums of spin factors and Euclidean Jordan algebras. What is a spin factor? Indeed, a spin factor has form of $\mathbb{R} \oplus \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. In view of this, we realize that Hilbert space is the very basic structure when we go beyond a Euclidean Jordan algebra. This is the main motivation why we consider the complementarity problems in Hilbert space. We will focus on real Hilbert space for the sake of convenience and reality.

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, and write the norm induced by $\langle \cdot, \cdot \rangle$ as $\| \cdot \|$. In general, the set of squared elements in $\mathcal{H}$ is no longer self-dual. We will define a Lorentz cone denoted by Ω which is self-dual in next section. Then, given a bounded continuous function $F : \mathcal{H} \to \mathcal{H}$, we will focus on the Lorentz cone complementarity problem (*CP* for short) which is to find an element $z \in \mathcal{H}$ such that

$$z \in \Omega, \quad w = F(z) \in \Omega, \quad \text{and} \quad \langle z, w \rangle = 0. \tag{1}$$

Such a problem is a natural extension of symmetric cone complementarity problems (SCCPs) in Euclidean Jordan algebras. In the finite-dimensional space, a well-known approach for solving the SCCPs is merit function method, which reformulates the SCCPs as a global minimization over Euclidean Jordan algebras via a certain merit function [1, 2, 5, 6, 9, 13, 19, 25, 26]. For this approach, it aims to find a smooth function $\Phi : \mathbb{V} \times \mathbb{V} \to \mathbb{R}_+$ such that

$$\Phi(x, y) = 0 \Longleftrightarrow x \in \mathcal{K}, \quad y \in \mathcal{K} \quad \text{and} \quad \langle x, y \rangle = 0,$$

where $\mathcal{K}$ is the symmetric cone in $\mathbb{V}$. Then, the SCCPs can be expressed as an unconstrained smooth minimization problem:

$$\min_{x \in \mathbb{V}} \Psi(x) := \Phi(x, F(x)),$$

we call such a Ψ a merit function for the SCCPs. It is well known that the complementarity function associated with the symmetric cone plays a key role in the development of merit function methods. For the approach to be effective, the choice of the complementarity function is crucial. Recently, merit function method was extended to solve the Lorentz cone complementarity problems in the setting of infinite-dimensional real Hilbert space (see [7, 22, 29]).

In finite-dimensional space, two popular symmetric cone complementarity functions are the Fischer-Burmeister (FB) symmetric cone complementarity function $\phi_{\rm FB}$ and the natural residual (NR) symmetric cone complementarity function $\phi_{\rm NR}$. Moreover, some properties of these two complementarity functions were studied. For example, the globally Lipschitzian continuty [23], strongly semismooth property [2], the global Lipschitz continuous gradients [18], etc.

In real Hilbert space $\mathscr{H}$, the Fischer–Burmeister (FB) function was introduced in [7, 29] and defined as

$$\phi_{\rm FB}(z, w) := (z^2 + w^2)^{1/2} - (z + w) \quad \forall z, w \in \mathscr{H}, \tag{2}$$

where z^2 and $z^{1/2}$ will be explained in next section. Let z_+ denote the metric projection $\Pi_\Omega(z)$ of $z \in \mathscr{H}$ onto the Lorentz cone Ω. Then, the NR complementarity function in infinite-dimensional real Hilbert space is given as follows

$$\phi_{\rm NR}(z, w) := z - (z - w)_+ \quad \forall z, w \in \mathscr{H}.$$

When $\mathscr{H} = \mathbb{R}$, for these two complementarity functions, Tseng [25] proved the following important inequality:

$$(2 - \sqrt{2})\|\phi_{\rm NR}(a, b)\| \le \|\phi_{\rm FB}(a, b)\| \le (2 + \sqrt{2})\|\phi_{\rm NR}(a, b)\|. \tag{3}$$

Recently, Bi et al. [1] extended this important inequality to the setting of symmetric cones. Along this direction, we generalize inequality (3) to the setting of Hilbert space. Next, we come to merit function approach for solving Lorentz cone complementarity problems in Hilbert space. To this end, we define $\Phi_{\rm FB} : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ as

$$\Phi_{\rm FB}(z, y) := \frac{1}{2}\|\phi_{\rm FB}(z, y)\|^2.$$

Then, solving problem (1) is equivalent to solving the following unconstrained smooth minimization problem:

$$\min_{z \in \mathscr{H}} \Psi_{\rm FB}(z) := \Phi_{\rm FB}(z, F(z)) = \frac{1}{2}\|\phi_{\rm FB}(z, F(z))\|^2, \tag{4}$$

where Ψ is called a *merit function* associated with Ω in $\mathscr{H}$. In finite-dimensional space, Bi et al. [1] have established the global error bound property of the FB merit function for the SCCPs. There is another kind of merit function which was also widely studied [3, 14, 28] in the setting of finite-dimensional space. It is a slight modification of the merit function studied by Yamashita and Fukushima, that is $\Psi_\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Psi_\alpha(x) := \Phi_\alpha(x, F(x)) = \frac{\alpha}{2}\|(x \circ F(x))_+\|^2 + \Phi_{\rm FB}(x, F(x)), \tag{5}$$

where $\alpha \geq 0$. When $\alpha = 0$, Ψ_α reduces to FB merit function. When $\alpha = 1$, Chen [3] established the global error bound property of the merit function Ψ_α for the SOCCPs; Liu et al. [14] obtained the global error bound property of the merit function Ψ_α for the SCCPs. In this article, we also generalize Ψ_α to the setting of Hilbert space and explore similar results. Property of bounded level sets will be discussed as well. In particular, it only takes F being R_{02}-function to guarantee property of bounded level set for Ψ_α. However, it needs much more stronger conditions for $\Psi_{\rm FB}$ to hold such property. All the results established in this article are standard and important when dealing with complementarity problems, in particular, they are helpful for further designing solution methods for problem (1).

2. PRELIMINARIES

In this section, we briefly introduce some basic concepts in real (infinite-dimensional or finite-dimensional) Hilbert space $\mathcal{H}$, and review some basic materials. These concepts and materials play important roles in subsequent analysis. More details and related results can be found in [7, 16, 17, 29].

Let $\mathcal{H}$ be a infinite-dimensional real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and e be an unit vector in $\mathcal{H}$ (i.e., $\|e\| = \sqrt{\langle e, e \rangle} = 1$). In [7], Chiang, Pan, and Chen considered the following closed convex cone

$$\Omega(e, r) = \{z \in \mathcal{H} \mid \langle z, e \rangle \geq r\|z\|\},$$

where $r \in \mathbb{R}$ and $e \in \mathcal{H}$ with $0 < r < 1$ and $\|e\| = 1$. Define

$$\langle e \rangle^\perp := \{x \in \mathcal{H} \mid \langle x, e \rangle = 0\},$$

that is, $\langle e \rangle^\perp$ is the orthogonal complementarity space of e in $\mathcal{H}$. Since $\mathcal{H}$ is a Hilbert space, for any element $z \in \mathcal{H}$, there are $x \in \langle e \rangle^\perp$ and $\lambda \in \mathbb{R}$ such that $z = x + \lambda e$ (in fact, $\lambda = \langle z, e \rangle$). With this, it can be verified that

$$\Omega(e, r) = \{z \in \mathcal{H} \mid \langle z, e \rangle \geq r\|z\|\} = \left\{ z = x + \lambda e \mid \lambda \geq \frac{r}{\sqrt{1 - r^2}} \|x\| \right\}.$$

Hence, for the closed convex cone $\Omega(e, r)$, when $r = \frac{1}{\sqrt{2}}$, we can see

$$\Omega\left(e, \frac{1}{\sqrt{2}}\right) = \Omega^*\left(e, \frac{1}{\sqrt{2}}\right),$$

where $\Omega^*\big(e, \frac{1}{\sqrt{2}}\big)$ is the dual cone of $\Omega\big(e, \frac{1}{\sqrt{2}}\big)$, that is, $\Omega^*\big(e, \frac{1}{\sqrt{2}}\big) := \{z \in \mathcal{H} \mid \langle z, w \rangle \geq 0,\ \forall w \in \Omega(e, \frac{1}{\sqrt{2}})\}$. This illustrates that $\Omega\big(e, \frac{1}{\sqrt{2}}\big)$ is a self-dual closed convex cone. In particular, if $\mathscr{H} = \mathbb{R}^n$ and $e = (1,0) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$, the set $\Omega(e, \frac{1}{\sqrt{2}})$ coincides with the Lorentz cone (also called second-order cone) $\mathscr{K}^n$ in $\mathbb{R}^n$. In view of this, $\Omega(e, \frac{1}{\sqrt{2}})$ is called the Lorentz cone (or second-order cone) in $\mathscr{H}$. Throughout this article, for the sake of simplicity, we denote $\Omega := \Omega(e, \frac{1}{\sqrt{2}})$. For any $z \in \mathscr{H}$, we write $z \succeq 0 (z \succ 0)$ when $z \in \Omega(z \in \operatorname{int}(\Omega))$, and $z \preceq 0 (z \prec 0)$ denote $-z \in \Omega(-z \in \operatorname{int}(\Omega))$.

Now, we come to the Jordan product in $\mathscr{H}$ associated with the Lorentz cone Ω. For any elements $z, w \in \mathscr{H}$ with $z = x + \lambda e$ and $w = y + \mu e$, where $x, y \in \langle e \rangle^\perp$ and $\lambda, \mu \in \mathbb{R}$, the *Jordan product* of z and w is defined by

$$z \circ w = \mu x + \lambda y + \langle z, w \rangle e.$$

Therefore, z^2 means $z \circ z$ for any $z \in \mathscr{H}$. From the definition of Jordan product and direct computation, it is not hard to prove the following properties.

Lemma 2.1.

(a) *For any $z = x + \lambda e \in \mathscr{H}$ with $x \in \langle e \rangle^\perp$ and $\lambda \in \mathbb{R}$, there have $z^2 = 2\lambda x + \|z\|^2 e \in \Omega$, and $\langle z, z \rangle = \langle z^2, e \rangle = \|z\|^2$.*
(b) *$z \circ w = w \circ z$ and $z \circ e = z$ for any $z, w \in \mathscr{H}$.*
(c) *$(z + w) \circ v = z \circ v + w \circ v$ for all $z, w, v \in \mathscr{H}$.*
(d) *$\langle z, w \circ v \rangle = \langle w, z \circ v \rangle = \langle v, z \circ w \rangle$ for all $z, w, v \in \mathscr{H}$.*
(e) *If $z = x + \lambda e \in \Omega$, there exists a unique element $z^{1/2} \in \Omega$ such that $z = (z^{1/2}) \circ (z^{1/2}) = (z^{1/2})^2$. Here*

$$z^{1/2} = \begin{cases} 0 & \text{if } z = 0, \\ \dfrac{x}{2\tau} + \tau e & \text{otherwise,} \quad \text{where } \tau = \sqrt{\dfrac{\lambda + \sqrt{\lambda^2 - \|x\|^2}}{2}}. \end{cases}$$

(f) *For any $z = x + \lambda e \in \mathscr{H}$, if $\lambda^2 - \|x\|^2 \neq 0$, then z is invertible with respect to the Jordan product, that is, there is a unique element $z^{-1} \in \mathscr{H}$ such that $z \circ z^{-1} = e$, where*

$$z^{-1} = \frac{1}{\lambda^2 - \|x\|^2}(-x + \lambda e).$$

Moreover, $z \in \operatorname{int}(\Omega)$ if and only if $z^{-1} \in \operatorname{int}(\Omega)$.

For any $z \in \mathscr{H}$, z can be expressed as $z = x + \lambda e$ where $x \in \langle e \rangle^\perp$ and $\lambda \in \mathbb{R}$. It is also easy to verify that z can be decomposed as

$$\begin{aligned} z = x + \lambda e &= (\lambda + \|x\|) e_1(z) + (\lambda - \|x\|) e_2(z) \\ &:= \lambda_1(z) e_1(z) + \lambda_2(z) e_2(z), \end{aligned}$$

where $e_i(z) = \frac{1}{2}(e + (-1)^{i+1}\frac{x}{\|x\|})$ for $i = 1, 2$ when $x \neq 0$, and $e_i(z) = \frac{1}{2}(e + (-1)^{i+1}\tilde{x})$ for $i = 1, 2$ when $x = 0$ ($\tilde{x}$ is any element in $\langle e\rangle^\perp$ satisfying $\|\tilde{x}\| = 1$). Here, $\lambda_1(z)$, $\lambda_2(z)$ and $e_1(z)$, $e_2(z)$ are called the spectral values and the associated spectral vectors of z, respectively. In addition, if $z \in \Omega$, we have

$$z^{1/2} = \sqrt{\lambda_1(z)}e_1(z) + \sqrt{\lambda_2(z)}e_2(z).$$

Clearly, when $x \neq 0$, the factorizations of z and $z^{1/2}$ are unique. Note that $\{e_1(z), e_2(z)\}$ is called a Jordan frame in the real Hilbert space $\mathscr{H}$.

Associated with every $z \in \mathscr{H}$, we define a linear transformation $L_z : \mathscr{H} \to \mathscr{H}$ as follows

$$L_z(w) := z \circ w \quad \text{for any } w \in \mathscr{H}.$$

L_z is called the Lyapunov transformation from $\mathscr{H}$ to $\mathscr{H}$. It can be seen that $L_z \in \mathscr{L}(\mathscr{H})$, where $\mathscr{L}(\mathscr{H})$ denotes the Banach space of all continuous linear transformation from $\mathscr{H}$ to $\mathscr{H}$.

For a convex cone Ω in $\mathscr{H}$, let Π_Ω denote the *metric projection* onto Ω [10]. For an $z \in \mathscr{H}$, $z_+ := \Pi_\Omega(z)$ if and only if $z_+ \in \Omega$ and $\|z - z_+\| \leq \|z - w\|$ for all $w \in \Omega$. This is also equivalent to $\langle z - z_+, w - z_+\rangle \leq 0$ for any $w \in \Omega$. Since Ω is convex cone, we have z_+ is unique. Similarly, z_- means $\Pi_\Omega(-z)$. Then, we have the following results.

Lemma 2.2. *Let $z = x + \lambda e \in \mathscr{H}$. Then the following results hold.*

(a) *If $z \succeq 0$, $z_+ = z$ and $z_- = 0$.*
(b) *If $z \preceq 0$, $z_+ = 0$ and $z_- = z$.*
(c) *If $z \notin \Omega$ and $-z \notin \Omega$, there have*

$$z_+ = \frac{\|x\| + \lambda}{2\|x\|}x + \frac{\|x\| + \lambda}{2}e = \max\{\lambda + \|x\|, 0\}e_1(z)$$

and

$$z_- = \frac{\lambda - \|x\|}{2\|x\|}x + \frac{\|x\| - \lambda}{2}e = \max\{\|x\| - \lambda, 0\}e_2(z).$$

(d) *For any $z \in \mathscr{H}$, we have $z = z_+ - z_-$ and $\|z\|^2 = \|z_+\|^2 + \|z_-\|^2$.*
(e) *For any $z \in \mathscr{H}$ and $w \in \Omega$, we have $\langle z, w\rangle \leq \langle z_+, w\rangle$ and $\|(z + w)_+\| \geq \|z_+\|$.*
(f) *For any $z \in \Omega$ and $w \in \mathscr{H}$ with $z^2 - w^2 \in \Omega$, we have $z - w \in \Omega$.*

Proof. These are well known results for projection and convex cones. □

Lemma 2.3 [22, Lemma 3.3]. *Let $\mathscr{H}$ be a real Hilbert space and ϕ_{FB}, Ψ_{FB} be given as in (2) and (4), respectively. Then, for any $z, w \in \mathscr{H}$, we have*

$$4\Phi_{\mathrm{FB}}(z,w) \geq 2\|\phi_{\mathrm{FB}}(z,w)_+\|^2 \geq \|(-z)_+\|^2 + \|(-w)_+\|^2.$$

To close this section, we talk about other concepts that play important roles in analysis of boundedness of level sets.

Definition 2.1. Let $\mathscr{H}$ be a real Hilbert space. For a bounded continuous mapping $F : \mathscr{H} \to \mathscr{H}$,

(a) F has uniform P^*-property if there exists $\rho > 0$ such that

$$\max_{i=1,2}\langle (z-w)\circ (F(z)-F(w)), e_i(w)\rangle \geq \rho\|z-w\|^2 \quad \forall z,w \in \mathscr{H}$$

where $e_i(w)$ for $i = 1, 2$ are the spectral vectors of w;

(b) F is called an R_{01}-function if for any sequence $\{z^k\}$ such that

$$\|z^k\| \to \infty, \quad \frac{(-z^k)_+}{\|z^k\|} \to 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \to 0 \quad \text{as } k \to \infty,$$

we have

$$\liminf_{k\to\infty} \frac{\langle z^k, F(z^k)\rangle}{\|z^k\|^2} > 0;$$

(c) F is called an R_{02}-function if for any sequence $\{z^k\}$ such that

$$\|z^k\| \to \infty, \quad \frac{(-z^k)_+}{\|z^k\|} \to 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \to 0 \quad \text{as } k \to \infty,$$

we have

$$\liminf_{k\to\infty} \frac{\lambda_1(z^k \circ F(z^k))}{\|z^k\|^2} > 0.$$

The above concepts are taken from [3, 14] in the setting of finite-dimensional space. Every R_{01}-function is R_{02}-function. In fact, R_{02}-function is equivalent to R_0-property defined in [24, Definition 3.2]. Besides, we recall the concept of the strong monotonicity for a bounded continuous function $F : \mathscr{H} \to \mathscr{H}$. That is, we say that F is strongly monotone with modulus $\mu > 0$ if for any $z, w \in \mathscr{H}$, there exist a constant $\mu > 0$ such that

$$\langle z-w, F(z)-F(w)\rangle \geq \mu\|z-w\|^2.$$

3. THE SAME GROWTH OF FB AND NR MERIT FUNCTION

In this section, we will give our first main result of this article. To prove it, the following lemmas will be needed.

Lemma 3.1. *Let $\mathcal{H}$ be a real Hilbert space. For $z \in \mathcal{H}$ with $z = x + \lambda e$, define $|z| := z_+ + z_-$, where z_+ and z_- are the same as in Lemma 2.2. Then, we have*

(a) $\langle z_+, z_- \rangle = 0$ *and* $z_+ \circ z_- = 0$;
(b) $|z|^2 = |z^2|$, $\||z|\| = \|z\|$ *and* $|z| = 2z_+ - z = z + 2z_-$.

Proof. The proof is obtained by straightforward calculation. thus it is omitted. □

Lemma 3.2. *Let $\mathcal{H}$ be a real Hilbert space. For any $z, w, u \in \mathcal{H}$,*

(a) *If $z \succeq 0, w \succeq 0$ and $z \succeq w$, we have $z^{1/2} \succeq w^{1/2}$;*
(b) *If $z \succeq 0$ and $2z^2 = w^2 + u^2$, we have $z \succeq \frac{1}{2}(w + u)$.*

Proof. (a) Let $p = z^{1/2}$, $q = w^{1/2}$ and $r = p - q = x + \lambda e$. To prove $z^{1/2} \succeq w^{1/2}$, we need to verity that $\lambda \geq \|x\|$. Note that $0 \preceq z - w = p^2 - (p - r)^2 = 2p \circ r - r^2$. Let $e_2 = \frac{1}{2}\big(e - \frac{x}{\|x\|}\big)$ when $x \neq 0$, and $e_2 = \frac{1}{2}(e - \tilde{x})$ ($\tilde{x}$ is any element in $\langle e \rangle^\perp$ satisfying $\|\tilde{x}\| = 1$) when $x = 0$. Then, it follows that

$$\begin{aligned} 0 &\leq \langle 2p \circ r - r^2, e_2 \rangle \\ &= \langle 2p \circ r, e_2 \rangle - \langle r^2, e_2 \rangle \\ &= \langle 2p, r \circ e_2 \rangle - (\lambda - \|x\|)^2 \langle e_2, e_2 \rangle \\ &= (\lambda - \|x\|)\langle 2p, e_2 \rangle - \frac{(\lambda - \|x\|)^2}{2}, \end{aligned}$$

which implies $\frac{(\lambda - \|x\|)^2}{2} \leq (\lambda - \|x\|)\langle 2p, e_2 \rangle$. This, together with $p \succeq 0$ and $e_2 \succeq 0$, yields $\lambda - \|x\| \geq 0$. The desired result follows.

(b) The arguments are similar to those for [1, Lemma 3.1]. For completeness, we present them as follows. Since $w^2 + u^2 - 2w \circ u = (w - u)^2 \succeq 0$, together with $2z^2 = w^2 + u^2$, this implies that

$$z^2 = \frac{1}{2}(w^2 + u^2) \succeq \frac{1}{4}(w^2 + u^2) + \frac{1}{2} w \circ u = \frac{1}{4}(w + u)^2.$$

From part (a) and $z \succeq 0$, we have $z \succeq \frac{1}{2}|w + u| \succeq \frac{1}{2}(w + u)$. □

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space. For any $z, w \in \mathcal{H}$, there holds*

$$(2 - \sqrt{2})\|\phi_{\mathrm{NR}}(z, w)\| \leq \|\phi_{\mathrm{FB}}(z, w)\| \leq (2 + \sqrt{2})\|\phi_{\mathrm{NR}}(z, w)\|. \qquad (6)$$

Proof. Fix any $z, w \in \mathcal{H}$. If $z^2 + w^2 = 0$, we have $z = w = 0$. Hence, the desired result is obvious. If $z^2 + w^2 \neq 0$, by Lemma 3.1, we obtain

$$\phi_{\mathrm{NR}}(z, w) = z - (z - w)_+ = \frac{1}{2}[(z + w) - |z - w|].$$

Furthermore, we know

$$\begin{aligned} \phi_{\mathrm{FB}}(z, w) &= z + w - (z^2 + w^2)^{1/2} \\ &= 2(z - (z - w)_+) + 2(z - w)_+ - (z - w) - (z^2 + w^2)^{1/2} \\ &= 2\phi_{\mathrm{NR}}(z, w) + |z - w| - (z^2 + w^2)^{1/2}. \end{aligned}$$

Let $p(z, w) = |z - w| - (z^2 + w^2)^{1/2}$. In view of triangle inequality, to prove inequality (6), it suffices to verity that

$$\|p(z, w)\| \leq \sqrt{2}\|\phi_{\mathrm{NR}}(z, w)\|. \tag{7}$$

To see this desired result, applying Lemma 3.2 gives

$$|z - w| + (z + w) \preceq 2(z^2 + w^2)^{1/2}, \tag{8}$$

Hence, we have

$$\begin{aligned} &2\|p(z, w)\|^2 - 4\|\phi_{\mathrm{NR}}(z, w)\|^2 \\ &\quad = 2\left(\|z - w\|^2 + \|(z^2 + w^2)^{1/2}\|^2 - 2\left\langle |z - w|, (z^2 + w^2)^{1/2}\right\rangle\right) \\ &\qquad - \|z + w\|^2 - \||z - w|\|^2 + 2\langle |z - w|, z + w\rangle \\ &\quad = 2\|z - w\|^2 - 2\langle |z - w|, 2(z^2 + w^2)^{1/2} - (z + w)\rangle \\ &\quad = \left\langle |z - w|, |z - w| + (z + w) - 2(z^2 + w^2)^{1/2}\right\rangle \leq 0, \end{aligned}$$

where the inequality holds is due to that $|z - w| \succeq 0$ and (8). This implies that inequality (7) is true. Then, the proof is complete. □

4. ERROR BOUND OF MERIT FUNCTIONS

Error bound is an important concept that indicates how close an arbitrary point is to the solution set of Lorentz cone complementarity problem (1). Thus, an error bound may be used to provide stopping criterion for an iterative method. As below, for FB merit function Ψ_{FB} and merit function Ψ_α, we draw conclusions about the error bounds for the solution of infinite-dimensional Lorentz cone complementarity problem (1), respectively.

For FB merit function Ψ_{FB}, we obtain a result about the error bounds for the solution of problem (1), which is an extension of [13, Theorem 6.3] in SCCP case.

Proposition 4.1. *Suppose that F is strongly monotone with modulus $\mu > 0$ and is Lipschitz continuous with constant l. Then,*

$$\frac{1}{(\sqrt{2}+1)(2+l)}\sqrt{\Psi_{\mathrm{FB}}(z)} \le \|z - z^*\| \le \frac{l+1}{\mu(\sqrt{2}-1)}\sqrt{\Psi_{\mathrm{FB}}(z)},$$

Proof. Fix any $z \in \mathscr{H}$, let $N(z) = \phi_{\mathrm{NR}}(z, F(z))$. Applying Theorem 3.1, we have

$$(3 - 2\sqrt{2})\|N(z)\|^2 \le \Psi_{\mathrm{FB}}(z) \le (3 + 2\sqrt{2})\|N(z)\|^2. \tag{9}$$

Since $N(z) = \phi_{\mathrm{NR}}(z, F(z)) = z - (z - F(z))_+$, it follows that $F(z) - N(z) = (z - F(z))_- \succeq 0$, $z - N(z) = (z - F(z))_+ \succeq 0$ and $\langle F(z) - N(z), z - N(z)\rangle = 0$. Because z^* is the unique solution of problem (1), we have

$$\begin{aligned}
0 &\ge \langle F(z) - N(z), z - N(z)\rangle - \langle F(z) - N(z), z^* - N(z^*)\rangle \\
&\quad - \langle F(z^*) - N(z^*), z - N(z)\rangle \\
&= \langle F(z) - F(z^*) - (N(z) - N(z^*)), z - z^* - (N(z) - N(z^*))\rangle \\
&\ge \langle F(z) - F(z^*), z - z^*\rangle - \langle F(z) - F(z^*), N(z) - N(z^*)\rangle \\
&\quad - \langle z - z^*, N(z) - N(z^*)\rangle \\
&\ge \mu\|z - z^*\|^2 - l\|z - z^*\|\|N(z)\| - \|z - z^*\|\|N(z)\|,
\end{aligned}$$

where the last inequality is due to the strong monotonicity and the Lipschitz continuity of F. Thus, we obtain

$$\|z - z^*\| \le \frac{l+1}{\mu}\|N(z)\|. \tag{10}$$

On the other hand,

$$\begin{aligned}
\|N(z)\| &= \|z - (z - F(z))_+ - (z^* - (z^* - F(z^*))_+)\| \\
&\le \|z - z^*\| + \|z - z^* - (F(z) - F(z^*))\| \\
&\le 2\|z - z^*\| + \|F(z) - F(z^*)\| \\
&\le (2 + l)\|z - z^*\|.
\end{aligned} \tag{11}$$

Combining (9), (10), and (11) leads to

$$\frac{1}{(\sqrt{2}+1)(2+l)}\sqrt{\Psi_{\mathrm{FB}}(z)} \le \|z - z^*\| \le \frac{l+1}{\mu(\sqrt{2}-1)}\sqrt{\Psi_{\mathrm{FB}}(z)}$$

which is the desired result. □

From Proposition 4.1, we see that if we use FB merit function Ψ_{FB} to provide error bound for the solution of problem (1), we need the conditions of F being strongly monotone and Lipschitz continuous. However, to provide error bound for the solution of (1) via Ψ_α merit function, we may weaken the aforementioned conditions to uniform P^*-property. We will prove this in Proposition 4.2. For this purpose, we present two technical lemmas first.

Lemma 4.1 [15, Lemma 2.2]. *Let $z = x + \lambda e, w = y + \mu e \in \mathscr{H}$ with $x, y \in \langle e \rangle^\perp$ and $\lambda, \mu \in \mathbb{R}$. The following two conditions are equivalent:*

(a) $z \succeq 0,\ w \succeq 0,$ *and* $\langle z, w \rangle = 0$;
(b) $z \succeq 0,\ w \succeq 0,$ *and* $z \circ w = 0$.

In each case, z and w operator commute.

Lemma 4.2 [15, Lemma 2.4]. *For any $z, w \in \Omega$, if z and w operator commute, then $z \circ w \in \Omega$.*

Proposition 4.2. *Let Ψ_α be defined as in (5) and $\alpha > 0$. Suppose that F has uniform P^*-property and the CP (1) has a solution z^*. Then, there exists a scalar $\tau > 0$ such that*

$$\tau \|z - z^*\|^2 \leq \|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \quad \forall z \in \mathscr{H}. \tag{12}$$

Moreover,

$$\|z - z^*\| \leq \theta \Psi_\alpha(z)^{\frac{1}{4}}, \quad \forall z \in \mathscr{H},$$

where θ is a positive constant.

Proof. Since F has the uniform P^*-property, there exists $\rho > 0$ such that

$$\rho \|z - z^*\|^2 \leq \max_{i=1,2} \langle (z - z^*) \circ (F(z) - F(z^*)), e_i(z^*) \rangle, \tag{13}$$

where $\{e_i(z^*) | i = 1, 2\}$ is the Jordan frame about z^* in $\mathscr{H}$. From z^* being a solution of (1), it follows that

$$(z - z^*) \circ (F(z) - F(z^*)) = z \circ F(z) - z \circ F(z^*) - z^* \circ F(z).$$

Note that $F(z^*) \in \Omega$, $z^* \in \Omega$ and $e_i(z^*) \in \Omega$. By Lemmas 4.1, 4.2, and 2.2(e), we have

$$\langle -z, F(z^*) \circ e_i(z^*) \rangle \leq \langle (-z)_+, F(z^*) \circ e_i(z^*) \rangle$$

and

$$\langle -F(z), z^* \circ e_i(z^*)\rangle \leq \langle (-F(z))_+, z^* \circ e_i(z^*)\rangle.$$

Moreover, from Lemma 4.1, we have that z^* and $F(z^*)$ share the same Jordan frame. Let

$$z^* = \lambda_1(z^*)e_1(z^*) + \lambda_2(z^*)e_2(z^*)$$

and

$$F(z^*) = \lambda_1(F(z^*))e_1(z^*) + \lambda_2(F(z^*))e_2(z^*).$$

Then, it follows that

$$\begin{aligned}
&\langle (z - z^*) \circ (F(z) - F(z^*)), e_i(z^*)\rangle \\
&\quad = \langle z \circ F(z), e_i(z^*)\rangle + \langle -z \circ F(z^*), e_i(z^*)\rangle + \langle -z^* \circ F(z), e_i(z^*)\rangle \\
&\quad = \langle z \circ F(z), e_i(z^*)\rangle + \langle -z, F(z^*) \circ e_i(z^*)\rangle + \langle -F(z), z^* \circ e_i(z^*)\rangle \\
&\quad \leq \langle (z \circ F(z))_+, e_i(z^*)\rangle + \langle (-z)_+, F(z^*) \circ e_i(z^*)\rangle + \langle (-F(z))_+, z^* \circ e_i(z^*)\rangle \\
&\quad \leq \|(z \circ F(z))_+\| \|e_i(z^*)\| + \|(-z)_+\| \|F(z^*) \circ e_i(z^*)\| \\
&\qquad + \|(-F(z))_+\| \|z^* \circ e_i(z^*)\| \\
&\quad \leq \frac{1}{\sqrt{2}}[\|(z \circ F(z))_+\| + \lambda_i(F(z^*))\|(-z)_+\| + \lambda_i(z^*)\|(-F(z))_+\|] \\
&\quad \leq \max\left\{\frac{1}{\sqrt{2}}, \frac{\lambda_i(F(z^*))}{\sqrt{2}}, \frac{\lambda_i(z^*)}{\sqrt{2}}\right\}[\|(z \circ F(z))_+\| + \|(-z)_+\| + \|(-F(z))_+\|] \\
&\quad \leq \max\left\{\frac{1}{\sqrt{2}}, \frac{\lambda_2(F(z^*))}{\sqrt{2}}, \frac{\lambda_1(z^*)}{\sqrt{2}}\right\}[\|(z \circ F(z))_+\| + \|(-z)_+\| + \|(-F(z))_+\|],
\end{aligned}$$

where the first inequality is from Lemma 2.2(e), and the last inequality is from the facts that $\lambda_i(F(z^*)) \geq 0$, $\lambda_1(z^*) \geq \lambda_2(z^*) \geq 0$ and $z^* \circ F(z^*) = 0$, i.e., $\lambda_i(F(z^*)) \cdot \lambda_i(z^*) = 0$ for $i = 1, 2$. Define

$$\tau := \frac{\rho}{\max\left\{\frac{1}{\sqrt{2}}, \frac{\lambda_2(F(z^*))}{\sqrt{2}}, \frac{\lambda_1(z^*)}{\sqrt{2}}\right\}}.$$

This together (12) and (13) yields

$$\tau\|z - z^*\|^2 \leq \|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \quad \forall z \in \mathscr{H}.$$

Now, we come to the second part of the proposition. We know that

$$\Psi_\alpha(z) = \frac{\alpha}{2}\|(z \circ F(z))_+\|^2 + \Psi_{\mathrm{FB}}(z),$$

which says $\|(z \circ F(z))_+\| \leq \sqrt{\frac{2}{\alpha}}\Psi_\alpha(z)^{1/2}$. Moreover, from Lemma 2.3, it follows that

$$\begin{aligned}\|(-F(z))_+\| + \|(-z)_+\| &\leq \sqrt{2}(\|(-F(z))_+\|^2 + \|(-z)_+\|^2)^{\frac{1}{2}} \\ &\leq 2\sqrt{2}\Psi_{\mathrm{FB}}(z)^{1/2} \\ &\leq 2\sqrt{2}\Psi_\alpha(z)^{1/2}.\end{aligned}$$

Therefore,

$$\|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \leq \left(\sqrt{\frac{2}{\alpha}} + 2\sqrt{2}\right)\Psi_\alpha(z)^{1/2}.$$

Letting $\theta = \sqrt{\frac{\sqrt{2}(2\sqrt{\alpha}+1)}{\tau\sqrt{\alpha}}}$ shows the second part result. □

The condition $\alpha > 0$ is necessary in Proposition 4.2. From the proof, we see that $\theta \to \infty$ as $\alpha \to 0$. Thus, $\alpha \neq 0$ is necessary. In fact, when $\alpha = 0$, we have $\Psi_\alpha = \Psi_{\mathrm{FB}}$. This explains that for the FB merit function Ψ_{FB}, we can not reach our desired result under the same conditions in Proposition 4.2.

5. BOUNDEDNESS OF LEVEL SETS

The boundedness of level sets of a merit function is also important since it is a necessary condition to ensures that the sequence generated by a descent method has at least one accumulation point. In this section, we will show the boundedness of level sets of Ψ_{FB} and Ψ_α under different conditions. The relation between such conditions will be discussed as well.

Lemma 5.1. *For any $z^k, w^k \in \mathcal{H}$, let $\lambda_2(z^k) \leq \lambda_1(z^k)$ and $\mu_2(w^k) \leq \mu_1(w^k)$ denote the spectral values of z^k and w^k, respectively. Then,*

(a) *if $\lambda_2(z^k) \to -\infty$ or $\mu_2(w^k) \to -\infty$, we have $\Psi_i \to \infty$ for $i = 1, 2$,*

$$\text{where } \Psi_1(z) := \frac{1}{2}(\|(-z^k)_+\|^2 + \|(-w^k)_+\|^2) \quad \textit{and}$$
$$\Psi_2(z) := \frac{1}{2}\|\phi_{\mathrm{FB}}(z^k, w^k)_+\|^2.$$

(b) *if $\{\lambda_2(z^k)\}$ and $\{\mu_2(w^k)\}$ are bounded below, $\lambda_1(z^k) \to \infty$, $\mu_1(w^k) \to \infty$ and $\frac{z^k}{\|z^k\|} \circ \frac{w^k}{\|w^k\|} \not\to 0$ as $k \to \infty$, then $\Phi_{\mathrm{FB}}(z^k, w^k) \to \infty$.*

Proof. (a) See [22, Lemma 4.4].

(b) Suppose that $\{\phi_{\mathrm{FB}}(z^k, w^k)\}$ is bounded. Define $u^k := ((z^k)^2 + (w^k)^2)^{1/2}$ for each k. From the definition of ϕ_{FB}, we have $z^k + w^k = u^k - \phi_{\mathrm{FB}}(z^k, w^k)$ for each k. Squaring two sides of the above equality leads to

$$2z^k \circ w^k = -2u^k \circ \phi_{\mathrm{FB}}(z^k, w^k) + \phi_{\mathrm{FB}}^2(z^k, w^k).$$

Since $\|z^k\| \geq \frac{1}{\sqrt{2}}\lambda_1(z^k)$ and $\|w^k\| \geq \frac{1}{\sqrt{2}}\mu_1(w^k)$, by the conditions of this lemma, we have

$$\lim_{k\to\infty} \frac{u^k}{\|z^k\|\|w^k\|} = \lim_{k\to\infty} \left[\frac{(z^k)^2}{\|z^k\|^2\|w^k\|^2} + \frac{(w^k)^2}{\|z^k\|^2\|w^k\|^2}\right]^{1/2} = 0.$$

This together with the boundedness of $\{\phi_{\mathrm{FB}}(z^k, w^k)\}$ implies

$$\lim_{k\to\infty} \frac{2z^k \circ w^k}{\|z^k\|\|w^k\|} = \lim_{k\to\infty} \frac{-2u^k \circ \phi_{\mathrm{FB}}(z^k, w^k) + \phi_{\mathrm{FB}}^2(z^k, w^k)}{\|z^k\|\|w^k\|} = 0.$$

Thus, $\lim_{k\to\infty} \frac{z^k}{\|z^k\|} \circ \frac{w^k}{\|w^k\|} = 0$, which contradicts the given assumption. Hence, the conclusion is proved. □

Remark. The condition of Lemma 5.1(b) was discussed in Lemma 4.2 of [11] and Lemma 4.1 of [3] in finite-dimensional space as well.

Condition A ([18]). For any sequence $\{z^k\} \subseteq \mathscr{H}$ satisfying $\|z^k\| \to \infty$, if $\{\lambda_2(z^k)\}$ and $\lambda_2(F(z^k))$ are bounded below, and $\lambda_1(z^k) \to \infty$, $\lambda_1(F(z^k)) \to \infty$ as $k \to \infty$. Then

$$\limsup_{k\to\infty} \left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|} \right\rangle > 0.$$

Using Lemma 5.1 and similar arguments as in [3, Proposition 4.2], we may obtain the following propositions, which are the boundedness of level sets for Ψ_{FB} and Ψ_α, respectively.

Proposition 5.1. *Let Ψ_{FB} be given as in* (4). *Assume that F is a strongly monotone function with modulus $\mu > 0$ and satisfies condition A. Then, the level set*

$$\mathscr{L}(\gamma) := \{z \in H| \ \ \Psi_{\mathrm{FB}}(z) \leq \gamma\}$$

is bounded for all $\gamma \geq 0$.

Proof. We prove this result by contradiction. Suppose there exists an unbounded sequence $\{z^k\} \subseteq \mathscr{L}(\gamma)$ for some $\gamma \geq 0$. We claim that the sequence of the smallest spectral values of z^k and $F(z^k)$ are bounded below. If not, by Lemma 5.1, we have $\Psi_i(z^k) \to \infty$ for $i = 1, 2$, which implies $\Psi_{\mathrm{FB}}(z^k) \to \infty$. This contradicts $\{z^k\} \subseteq \mathscr{L}(\gamma)$. On the other hand, by the strongly monotone property of F, there exists a constant $\mu > 0$ such that

$$\begin{aligned}\mu\|z^k\|^2 &\leq \langle z^k, F(z^k) - F(0)\rangle \\ &\leq \|z^k\|\|F(z^k) - F(0)\| \leq \|z^k\|(\|F(z^k)\| + \|F(0)\|).\end{aligned}$$

This implies that $\mu\|z^k\| \leq \|F(z^k)\| + \|F(0)\|$. It follows from unboundedness of $\{z^k\}$ and boundedness of F that $\{F(z^k)\}$ is unbounded, which says that $\lambda_1(z^k) \to \infty$ and $\lambda_1(F(z^k)) \to \infty$ as $k \to \infty$. By condition A, it gives

$$\limsup_{k\to\infty}\left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|}\right\rangle > 0,$$

which implies

$$\limsup_{k\to\infty}\lambda_1\left[\frac{z^k}{\|z^k\|} \circ \frac{F(z^k)}{\|F(z^k)\|}\right] > 0.$$

From this, we have $\frac{z^k}{\|z^k\|} \circ \frac{F(z^k)}{\|F(z^k)\|} \not\to 0$. Together with Lemma 5.1(b), we obtain $\Psi_{\mathrm{FB}}(z^k) = \Phi_{\mathrm{FB}}(z^k, F(z^k)) \to \infty$. Hence, this is a contradiction to $z^k \subseteq \mathscr{L}(\gamma)$. The proof is complete. □

In fact, the condition A and the strong monotonicity of F in Proposition 5.1 can be replaced by the Lipschitz continuity of F and R_{01}-function. This can be verified as below. Since the sequence of the smaller spectral values of z^k and $F(z^k)$ are bounded below, we have $(-z^k)_+$ and $(-F(z^k))_+$ are bounded above. For any sequence z^k satisfying $\|z^k\| \to \infty$, by the definition of R_{01}-function, we have

$$\liminf_{k\to\infty}\frac{\langle z^k, F(z^k)\rangle}{\|z^k\|^2} > 0. \tag{14}$$

This implies that

$$\begin{aligned}\limsup_{k\to\infty}\left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|}\right\rangle &\geq \liminf_{k\to\infty}\left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|}\right\rangle \\ &= \liminf_{k\to\infty}\frac{\langle z^k, F(z^k)\rangle}{\|z^k\|^2}\frac{\|z^k\|}{\|F(z^k)\|} > 0,\end{aligned}$$

where the last inequality is due to (14) and the Lipschitz continuity of F.

Next, we show the bounded level sets for Ψ_α. As will be seen, it requires F being R_{02}-function to guarantee this property. In view of the above remark and the fact that every R_{01}-function is an R_{02}-function, we see this condition is weaker that in Proposition 5.1, although their proofs are similar.

Proposition 5.2. *Let Ψ_α be given as in* (5). *Suppose that F is an R_{02}-function. Then, the level sets*

$$\mathscr{L}(\gamma) := \{z \in \mathscr{H} \mid \Psi_\alpha(z) \leq \gamma\}.$$

is bounded for all $\gamma \geq 0$.

Proof. We prove this result by contradiction again. Suppose there exists an unbounded sequence $\{z^k\} \subseteq \mathscr{L}(\gamma)$ for some $\gamma \geq 0$. We claim that the sequence of the smaller spectral values of z^k and $F(z^k)$ are bounded below. In fact, if not, by Lemma 5.1, we have $\Psi_i(z^k) \to \infty$ for $i = 1, 2$, which says $\Psi_\alpha(z^k) \to \infty$. This contradicts $\{z^k\} \subseteq \mathscr{L}(\gamma)$. Therefore, $\{(-z^k)_+\}$ and $\{(-F(z^k))_+\}$ are bounded above. Thus, for any sequence $\{z^k\}$ satisfying $\|z^k\| \to \infty$, we have

$$\frac{(-z^k)_+}{\|z^k\|} \to 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \to 0.$$

By the definition of R_{02}-function, we have

$$\liminf_{k\to\infty} \frac{\lambda_1(z^k \circ F(z^k))}{\|z^k\|^2} > 0.$$

This implies that $\lambda_1(z^k \circ F(z^k)) \to \infty$. Hence, $\|(z^k \circ F(z^k))_+\| \to \infty$. This together with definition of Ψ_α, which leads to $\Psi_\alpha(z^k) \to \infty$. This contradicts $\{z^k\} \subseteq \mathscr{L}(\gamma)$. Therefore, the desired result is proved. $\square$

6. CONCLUDING REMARKS

In this article, we have studied the Lorentz cone complementarity problems in real Hilbert space for which we prove the same growth of FB and NR merit functions, provide a global error bound for the solutions via two kinds of merit functions, and discuss the property of the bounded level sets of these two kinds of merit functions under different conditions. Such results will be helpful and useful for further designing solution methods for solving problem (1).

REFERENCES

1. S.-J. Bi, S.-H. Pan, and J.-S. Chen (2010). The same growth of FB and NR symmetric cone complementarity functions. To appear in *Optimization Letters.* DOI: 10.1007/s11590-010-0257-z2011
2. Y.-L. Chang, S.-H. Pan, and J.-S. Chen (2009). Strong semismoothness of FB SC complementarity function. Submitted.

3. J.-S. Chen (2007). Conditions for error bounds and bounded Level sets of some merit functions for the second-order cone complementarity problem. *Journal of Optimization Theory and Applications* 135:459–473.
4. J.-S. Chen and S.H. Pan (2010). An entropy-like proximal algorithm and the exponential multiplier method for symmetric cone programming. *Computational Optimization and Applications* 47:477–499.
5. J.-S. Chen and P. Tseng (2005). An unconstrained smooth minimization reformation of the second-order cone complementarity problem. *Mathematical Programming* 104:293–327.
6. X. Chen and H.-D. Qi (2006). Cartesian *P*-property and its applications to the semidefinite linear complementarity problem. *Mathematical Programming* 106:177–201.
7. Y. Chiang, S.H. Pan, and J.-S. Chen (2010). A merit function method for infinite-dimensional SOCPs. Submitted.
8. J. Faraut and A. Korányi (1994). *Analysis on Symmetric Cones*. Oxford Mathematical Monographs Oxford University Press, New York.
9. M. Fukushima, Z.-Q. Lou, and P. Tseng (2002). Smoothing functions for second-order cone complementarity problems. *SIAM Journal on Optimization* 12:436–460.
10. M.S. Gowda, R. Sznajder, and J. Tao (2004). Some *P*-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra and Its Applications* 393:203–232.
11. Z.-H. Huang and T. Ni (2010). Smoothing algorithms for complementarity problems over symmetric cones. *Computational Optimization and Applications* 45(3):557–579.
12. M. Koecher (1999). *The Minnesota Notes on Jordan Algebras and Their Applications*. Edited and annotated by A. Brieg and S. Walcher, Springer, Berlin.
13. L.-C. Kong, L. Tuncel, and N.-H. Xiu (2009). Vector-valued implicit Larangian for symmetric cone complementarity problems. *Asia-Pacific Journal of Operational Research* 26:199–233.
14. Y.-J. Liu, Z.-W. Zhang, and Y.-H. Wang (2006). Some properties of a class of merit functions for symmetric cone complementarity problems. *Asia Pacific Journal of Operational Research* 23:473–495.
15. X.-H. Miao and Z.-H. Huang (2011). The column-sufficiency and row-sufficiency of the linear transformation on Hilbert spaces. *Journal of Global Optimization* 49(1):109–123.
16. X.-H. Miao, Z.-H. Huang, and J.-Y. Han (2010). Some w-unique and w-P properties for linear transformations on Hilbert spaces. *Acta Mathematicae Applicatae Sinica (English Series)* 26:23–32.
17. X.-H. Miao and Z.-H. Huang (2011). GUS-property for Lorentz cone linear complementarity problems on Hilbert spaces. To appear in *Science in China (Series A)*.
18. S.-H. Pan and J.-S. Chen (2009). A one-parametric class of merit functions for the symmtric cone complementarity problem. *Journal of Mathematical Analysis and Applications* 355:195–215.
19. S.-H. Pan and J.-S. Chen (2009). A damped Gauss-Newton method for the second-order cone complementarity problem. *Applied Mathematics and Optimization* 59:293–318.
20. S.-H. Pan, Y.-L. Chang, and J.-S. Chen (2010). Stationary point conditions for the FB merit function associated with symmetric cones. *Operations Research Letters* 30:372–377.
21. S. Schmieta and F. Alizadeh (2003). Extensions of primal-dual interior point algorithms to symmetric cones. *Mathematical Programming* 96:409–438.
22. J.-H. Sun and J.-S. Chen (2010). Two classes of merit functions for infinite-dimensional second order complimentary problems. *Numerical Functional Analysis and Optimization* 31(4):387–413.
23. D. Sun and J. Sun (2008). Löwner's operator and spectral functions on Euclidean Jordan algebras. *Mathematics of Operations Research* 33:421–445.
24. J. Tao and M.S. Gowda (2005). Some P-properties for the nonlinear transformations on Euclidean Jordan algebra. *Mathematics of Operations Research* 30:985–1004.
25. P. Tseng (1996). Growth behavior of a class of merit functions for the nonlinear complementarity problems. *Journal of Optimization Theory and Applications* 89:17–37.
26. P. Tseng (1998). Merit function for semidefinite complementarity problems. *Mathematical Programming* 83:159–185.
27. H. Upmeier (1985). *Symmetric Banach Manifolds and Jordan C^*-Algebras*. North Holland Mathematics Studies, New York.
28. K. Yamada, N. Yamashita, and M. Fukushima (2000). A new derivative-free descent method for the nonlinear complementarity problem. In: *Nonlinear Optimization and Related Topics*. (G. Di Pillo and F. Giannessi, eds.) Kluwer Academic Publisher, Netherlands, pp. 463–489.
29. C.-Y. Yang, Y.-L. Chang, and J.-S. Chen (2010). Analysis of nonsmooth vector-valued functions associated with infinite-dimensional second-order cones. Submitted.