

## On the Lorentz cone complementarity problems in infinite-dimensional real Hilbert space

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**Abstract** In this paper, we consider the Lorentz cone complementarity problems in infinite-dimensional real Hilbert space. We establish several results which are standard and important when dealing with complementarity problems. These include proving the same growth of the Fishcher-Burmeister merit function and the natural residual merit function, investigating property of bounded level sets under mild conditions via different merit functions, and providing global error bounds through the proposed merit functions. Such results are helpful for further designing solution methods for the Lorentz cone complementarity problems in Hilbert space.

**Key words.** Lorentz cone, FB-function, NR-function, merit function, error bound,  $R_{02}$ -property.

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# 1 Motivation and Introduction

Recently there has been much attention on symmetric cone optimization, see [4, 13, 14, 20, 21] and references therein. The symmetric cone  $\mathcal{K}$  is intimately related to Euclidean Jordan algebra since it provides an essential toolbox for the analysis. In addition, the symmetric cone has special structure in Euclidean Jordan algebra  $(\mathbb{V}, \circ, \langle \cdot, \cdot \rangle)$  [8, 12], namely,  $\mathcal{K} = \{x^2 = x \circ x \mid x \in \mathbb{V}\}$ . It is natural to ask what will happen if we go further beyond Euclidean Jordan algebra. In fact, it is known that the class of Euclidean Jordan algebras belongs to the class of JB-algebras [29]. More specifically, a finite-dimensional JB-algebra coincides with a Euclidean Jordan algebra. There is a subclass of JB-algebras called JB-algebra of finite rank which attracts our attention because every JB-algebra of finite rank is direct sums of spin factors and Euclidean Jordan algebras. What is a spin factor? Indeed, a spin factor has form of  $\mathbb{R} \oplus \mathcal{H}$  where  $\mathcal{H}$  is a Hilbert space. In view of this, we realize that Hilbert space is the very basic structure when we go beyond a Euclidean Jordan algebra. This is the main motivation why we consider the complementarity problems in Hilbert space. We will focus on real Hilbert space for the sake of convenience and reality.

Let  $\mathcal{H}$  be a real Hilbert space endowed with an inner product  $\langle \cdot, \cdot \rangle$ , and write the norm induced by  $\langle \cdot, \cdot \rangle$  as  $\|\cdot\|$ . In general, the set of squared elements in  $\mathcal{H}$  is no longer self-dual. We will define a Lorentz cone denoted by  $\Omega$  which is self-dual in next section. Then, given a bounded continuous function  $F : \mathcal{H} \rightarrow \mathcal{H}$ , we will focus on the Lorentz cone complementarity problem (*CP* for short) which is to find an element  $z \in \mathcal{H}$  such that

$$z \in \Omega, \quad w = F(z) \in \Omega, \quad \text{and} \quad \langle z, w \rangle = 0. \quad (1)$$

Such a problem is a natural extension of symmetric cone complementarity problems (SCCPs) in Euclidean Jordan algebras. In the finite-dimensional space, a well-known approach for solving the SCCPs is merit function method, which reformulates the SCCPs as a global minimization over Euclidean Jordan algebras via a certain merit function [1, 2, 5, 6, 9, 13, 19, 25, 26]. For this approach, it aims to find a smooth function  $\Phi : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}_+$  such that

$$\Phi(x, y) = 0 \iff x \in \mathcal{K}, \quad y \in \mathcal{K} \quad \text{and} \quad \langle x, y \rangle = 0,$$

where  $\mathcal{K}$  is the symmetric cone in  $\mathbb{V}$ . Then, the SCCPs can be expressed as an unconstrained smooth minimization problem:

$$\min_{x \in \mathbb{V}} \Psi(x) := \Phi(x, F(x)),$$

we call such a  $\Psi$  a merit function for the SCCPs. It is well-known that the complementarity function associated with the symmetric cone plays a key role in the development

of merit function methods. For the approach to be effective, the choice of the complementarity function is crucial. Recently, merit function method was extended to solve the Lorentz cone complementarity problems in the setting of infinite-dimensional real Hilbert space, see [7, 22, 27].

In finite-dimensional space, two popular symmetric cone complementarity functions are the Fischer-Burmeister (FB) symmetric cone complementarity function  $\phi_{\text{FB}}$  and the natural residual (NR) symmetric cone complementarity function  $\phi_{\text{NR}}$ . Moreover, some properties of these two complementarity functions were studied. For example, the globally Lipschitzian continuity [23], strongly semismooth property [2] and the global Lipschitz continuous gradients [18], etc.

In real Hilbert space  $\mathcal{H}$ , the Fischer–Burmeister (FB) function was introduced in [7, 27] and defined as

$$\phi_{\text{FB}}(z, w) := (z^2 + w^2)^{1/2} - (z + w) \quad \forall z, w \in \mathcal{H}, \quad (2)$$

where  $z^2$  and  $z^{1/2}$  will be explained in next section. Let  $z_+$  denote the metric projection  $\Pi_{\Omega}(z)$  of  $z \in \mathcal{H}$  onto the Lorentz cone  $\Omega$ . Then, the NR complementarity function in infinite-dimensional real Hilbert space is given as follows

$$\phi_{\text{NR}}(z, w) := z - (z - w)_+ \quad \forall z, w \in \mathcal{H}.$$

When  $\mathcal{H} = \mathbb{R}$ , for these two complementarity functions, Tseng [25] proved the following important inequality:

$$(2 - \sqrt{2})\|\phi_{\text{NR}}(a, b)\| \leq \|\phi_{\text{FB}}(a, b)\| \leq (2 + \sqrt{2})\|\phi_{\text{NR}}(a, b)\|. \quad (3)$$

Recently, Bi, Pan and Chen [1] extended this important inequality to the setting of symmetric cones. Along this direction, we generalize inequality (3) to the setting of Hilbert space. Next, we come to merit function approach for solving Lorentz cone complementarity problems in Hilbert space. To this end, we define  $\Phi_{\text{FB}} : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  as

$$\Phi_{\text{FB}}(z, y) := \frac{1}{2}\|\phi_{\text{FB}}(z, y)\|^2.$$

Then, solving problem (1) is equivalent to solving the following unconstrained smooth minimization problem:

$$\min_{z \in \mathcal{H}} \Psi_{\text{FB}}(z) := \Phi_{\text{FB}}(z, F(z)) = \frac{1}{2}\|\phi_{\text{FB}}(z, F(z))\|^2, \quad (4)$$

where  $\Psi$  is called a *merit function* associated with  $\Omega$  in  $\mathcal{H}$ . In finite-dimensional space, Bi, Pan and Chen [1] have established the global error bound property of the FB merit function for the SCCPs. There is another kind of merit function which was also widely studied ([3, 14, 28]) in the setting of finite-dimensional space. It is a slight modification

of the merit function studied by Yamashita and Fukushima, i.e.,  $\Psi_\alpha : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$\Psi_\alpha(x) := \Phi_\alpha(x, F(x)) = \frac{\alpha}{2} \|(x \circ F(x))_+\|^2 + \Phi_{\text{FB}}(x, F(x)), \quad (5)$$

where  $\alpha \geq 0$ . When  $\alpha = 0$ ,  $\Psi_\alpha$  reduces to FB merit function. When  $\alpha = 1$ , Chen [3] established the global error bound property of the merit function  $\Psi_\alpha$  for the SOCCPs; Liu, Zhang and Wang [14] obtained the global error bound property of the merit function  $\Psi_\alpha$  for the SCCPs. In this paper, we also generalize  $\Psi_\alpha$  to the setting of Hilbert space and explore similar results. Property of bounded level sets will be discussed as well. In particular, it only takes  $F$  being  $R_{02}$ -function to guarantee property of bounded level set for  $\Psi_\alpha$ . However, it needs much more stronger conditions for  $\Psi_{\text{FB}}$  to hold such property. All the results established in this paper are standard and important when dealing with complementarity problems, in particular, they are helpful for further designing solution methods for problem (1).

## 2 Preliminaries

In this section, we briefly introduce some basic concepts in real (infinite-dimensional or finite-dimensional) Hilbert space  $\mathcal{H}$ , and review some basic materials. These concepts and materials play important roles in subsequent analysis. More details and related results can be found in [7, 16, 17, 27].

Let  $\mathcal{H}$  be a infinite-dimensional real Hilbert space with the inner product  $\langle \cdot, \cdot \rangle$ , and  $e$  be an unit vector in  $\mathcal{H}$  (i.e.  $\|e\| = \sqrt{\langle e, e \rangle} = 1$ ). In [7], Chiang, Pan and Chen considered the following closed convex cone

$$\Omega(e, r) = \{z \in \mathcal{H} \mid \langle z, e \rangle \geq r\|z\|\},$$

where  $r \in \mathbb{R}$  and  $e \in \mathcal{H}$  with  $0 < r < 1$  and  $\|e\| = 1$ . Define

$$\langle e \rangle^\perp := \{x \in \mathcal{H} \mid \langle x, e \rangle = 0\},$$

i.e.,  $\langle e \rangle^\perp$  is the orthogonal complementarity space of  $e$  in  $\mathcal{H}$ . Since  $\mathcal{H}$  is a Hilbert space, for any element  $z \in \mathcal{H}$ , there are  $x \in \langle e \rangle^\perp$  and  $\lambda \in \mathbb{R}$  such that  $z = x + \lambda e$  (in fact,  $\lambda = \langle z, e \rangle$ ). With this, it can be verified that

$$\Omega(e, r) = \{z \in \mathcal{H} \mid \langle z, e \rangle \geq r\|z\|\} = \left\{ z = x + \lambda e \mid \lambda \geq \frac{r}{\sqrt{1-r^2}} \|x\| \right\}.$$

Hence, for the closed convex cone  $\Omega(e, r)$ , when  $r = \frac{1}{\sqrt{2}}$ , we can see

$$\Omega(e, \frac{1}{\sqrt{2}}) = \Omega^*(e, \frac{1}{\sqrt{2}}),$$

where  $\Omega^*(e, \frac{1}{\sqrt{2}})$  is the dual cone of  $\Omega(e, \frac{1}{\sqrt{2}})$ , i.e.,  $\Omega^*(e, \frac{1}{\sqrt{2}}) := \{z \in \mathcal{H} \mid \langle z, w \rangle \geq 0, \forall w \in \Omega(e, \frac{1}{\sqrt{2}})\}$ . This illustrates that  $\Omega(e, \frac{1}{\sqrt{2}})$  is a self-dual closed convex cone. In particular, if  $\mathcal{H} = \mathbb{R}^n$  and  $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{(n-1)}$ , the set  $\Omega(e, \frac{1}{\sqrt{2}})$  coincides with the Lorentz cone (also called second-order cone)  $\mathcal{K}^n$  in  $\mathbb{R}^n$ . In view of this,  $\Omega(e, \frac{1}{\sqrt{2}})$  is called the Lorentz cone (or second-order cone) in  $\mathcal{H}$ . Throughout this paper, for the sake of simplicity, we denote  $\Omega := \Omega(e, \frac{1}{\sqrt{2}})$ . For any  $z \in \mathcal{H}$ , we write  $z \succeq 0$  ( $z \succ 0$ ) when  $z \in \Omega$  ( $z \in \text{int}(\Omega)$ ), and  $z \preceq 0$  ( $z \prec 0$ ) denote  $-z \in \Omega$  ( $-z \in \text{int}(\Omega)$ ).

Now, we come to the Jordan product in  $\mathcal{H}$  associated with the Lorentz cone  $\Omega$ . For any elements  $z, w \in \mathcal{H}$  with  $z = x + \lambda e$  and  $w = y + \mu e$ , where  $x, y \in \langle e \rangle^\perp$  and  $\lambda, \mu \in \mathbb{R}$ , the *Jordan product* of  $z$  and  $w$  is defined by

$$z \circ w = \mu x + \lambda y + \langle z, w \rangle e.$$

Therefore,  $z^2$  means  $z \circ z$  for any  $z \in \mathcal{H}$ . From the definition of Jordan product and direct computation, it is not hard to prove the following properties.

**Lemma 2.1 (a)** *For any  $z = x + \lambda e \in \mathcal{H}$  with  $x \in \langle e \rangle^\perp$  and  $\lambda \in \mathbb{R}$ , there have  $z^2 = 2\lambda x + \|z\|^2 e \in \Omega$ , and  $\langle z, z \rangle = \langle z^2, e \rangle = \|z\|^2$ .*

(b)  $z \circ w = w \circ z$  and  $z \circ e = z$  for any  $z, w \in \mathcal{H}$ .

(c)  $(z + w) \circ v = z \circ v + w \circ v$  for all  $z, w, v \in \mathcal{H}$ .

(d)  $\langle z, w \circ v \rangle = \langle w, z \circ v \rangle = \langle v, z \circ w \rangle$  for all  $z, w, v \in \mathcal{H}$ .

(e) *If  $z = x + \lambda e \in \Omega$ , there exists a unique element  $z^{1/2} \in \Omega$  such that  $z = (z^{1/2}) \circ (z^{1/2}) = (z^{1/2})^2$ . Here*

$$z^{1/2} = \begin{cases} 0 & \text{if } z = 0, \\ \frac{x}{2\tau} + \tau e & \text{otherwise,} \end{cases} \quad \text{where } \tau = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - \|x\|^2}}{2}}.$$

(f) *For any  $z = x + \lambda e \in \mathcal{H}$ , if  $\lambda^2 - \|x\|^2 \neq 0$ , then  $z$  is invertible with respect to the Jordan product, i.e., there is a unique element  $z^{-1} \in \mathcal{H}$  such that  $z \circ z^{-1} = e$ , where*

$$z^{-1} = \frac{1}{\lambda^2 - \|x\|^2} (-x + \lambda e).$$

*Moreover,  $z \in \text{int}(\Omega)$  if and only if  $z^{-1} \in \text{int}(\Omega)$ .*

For any  $z \in \mathcal{H}$ ,  $z$  can be expressed as  $z = x + \lambda e$  where  $x \in \langle e \rangle^\perp$  and  $\lambda \in \mathbb{R}$ . It is also easy to verify that  $z$  can be decomposed as

$$\begin{aligned} z = x + \lambda e &= (\lambda + \|x\|)e_1(z) + (\lambda - \|x\|)e_2(z) \\ &:= \lambda_1(z)e_1(z) + \lambda_2(z)e_2(z), \end{aligned}$$

where  $e_i(z) = \frac{1}{2}(e + (-1)^{i+1} \frac{x}{\|x\|})$  for  $i = 1, 2$  when  $x \neq 0$ , and  $e_i(z) = \frac{1}{2}(e + (-1)^{i+1} \tilde{x})$  for  $i = 1, 2$  when  $x = 0$  ( $\tilde{x}$  is any element in  $\langle e \rangle^\perp$  satisfying  $\|\tilde{x}\| = 1$ ). Here  $\lambda_1(z)$ ,  $\lambda_2(z)$  and  $e_1(z)$ ,  $e_2(z)$  are called the spectral values and the associated spectral vectors of  $z$ , respectively. In addition, if  $z \in \Omega$ , we have

$$z^{1/2} = \sqrt{\lambda_1(z)}e_1(z) + \sqrt{\lambda_2(z)}e_2(z).$$

Clearly, when  $x \neq 0$ , the factorizations of  $z$  and  $z^{1/2}$  are unique. Note that  $\{e_1(z), e_2(z)\}$  is called a Jordan frame in the real Hilbert space  $\mathcal{H}$ .

Associated with every  $z \in \mathcal{H}$ , we define a linear transformation  $L_z : \mathcal{H} \rightarrow \mathcal{H}$  as follows

$$L_z(w) := z \circ w \quad \text{for any } w \in \mathcal{H}.$$

$L_z$  is called the Lyapunov transformation from  $\mathcal{H}$  to  $\mathcal{H}$ . It can be seen that  $L_z \in \mathcal{L}(\mathcal{H})$ , where  $\mathcal{L}(\mathcal{H})$  denotes the Banach space of all continuous linear transformation from  $\mathcal{H}$  to  $\mathcal{H}$ .

For a convex cone  $\Omega$  in  $\mathcal{H}$ , let  $\Pi_\Omega$  denote the *metric projection* onto  $\Omega$  [10]. For an  $z \in \mathcal{H}$ ,  $z_+ := \Pi_\Omega(z)$  if and only if  $z_+ \in \Omega$  and  $\|z - z_+\| \leq \|z - w\|$  for all  $w \in \Omega$ . This is also equivalent to  $\langle z - z_+, w - z_+ \rangle \leq 0$  for any  $w \in \Omega$ . Since  $\Omega$  is convex cone, we have  $z_+$  is unique. Similarly,  $z_-$  means  $\Pi_\Omega(-z)$ . Then, we have the following results.

**Lemma 2.2** *Let  $z = x + \lambda e \in \mathcal{H}$ . Then the following results hold.*

- (a) *If  $z \succeq 0$ ,  $z_+ = z$  and  $z_- = 0$ .*
- (b) *If  $z \preceq 0$ ,  $z_+ = 0$  and  $z_- = z$ .*
- (c) *If  $z \notin \Omega$  and  $-z \notin \Omega$ , there have*

$$z_+ = \frac{\|x\| + \lambda}{2\|x\|}x + \frac{\|x\| + \lambda}{2}e = \max\{\lambda + \|x\|, 0\}e_1(z)$$

and

$$z_- = \frac{\lambda - \|x\|}{2\|x\|}x + \frac{\|x\| - \lambda}{2}e = \max\{\|x\| - \lambda, 0\}e_2(z).$$

- (d) *For any  $z \in \mathcal{H}$ , we have  $z = z_+ - z_-$  and  $\|z\|^2 = \|z_+\|^2 + \|z_-\|^2$ .*
- (e) *For any  $z \in \mathcal{H}$  and  $w \in \Omega$ , we have  $\langle z, w \rangle \leq \langle z_+, w \rangle$  and  $\|(z + w)_+\| \geq \|z_+\|$ .*
- (f) *For any  $z \in \Omega$  and  $w \in \mathcal{H}$  with  $z^2 - w^2 \in \Omega$ , we have  $z - w \in \Omega$ .*

**Proof.** These are well-known results for projection and convex cones.  $\square$

**Lemma 2.3** [22, Lemma 3.3] *Let  $\mathcal{H}$  be a real Hilbert space and  $\phi_{\text{FB}}, \Psi_{\text{FB}}$  be given as in (2) and (4), respectively. Then, for any  $z, w \in \mathcal{H}$ , we have*

$$4\Phi_{\text{FB}}(z, w) \geq 2\|\phi_{\text{FB}}(z, w)_+\|^2 \geq \|(-z)_+\|^2 + \|(-w)_+\|^2.$$

To close this section, we talk about other concepts which play important roles in analysis of boundedness of level sets.

**Definition 2.1** *Let  $\mathcal{H}$  be a real Hilbert space. For a bounded continuous mapping  $F : \mathcal{H} \rightarrow \mathcal{H}$ ,*

(a)  *$F$  has uniform  $P^*$ -property if there exists  $\rho > 0$  such that*

$$\max_{i=1,2} \langle (z-w) \circ (F(z) - F(w)), e_i(w) \rangle \geq \rho \|z-w\|^2 \quad \forall z, w \in \mathcal{H}$$

*where  $e_i(w)$  for  $i = 1, 2$  are the spectral vectors of  $w$ ;*

(b)  *$F$  is called an  $R_{01}$ -function if for any sequence  $\{z^k\}$  such that*

$$\|z^k\| \rightarrow \infty, \quad \frac{(-z^k)_+}{\|z^k\|} \rightarrow 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

*we have*

$$\liminf_{k \rightarrow \infty} \frac{\langle z^k, F(z^k) \rangle}{\|z^k\|^2} > 0;$$

(c)  *$F$  is called an  $R_{02}$ -function if for any sequence  $\{z^k\}$  such that*

$$\|z^k\| \rightarrow \infty, \quad \frac{(-z^k)_+}{\|z^k\|} \rightarrow 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

*we have*

$$\liminf_{k \rightarrow \infty} \frac{\lambda_1(z^k \circ F(z^k))}{\|z^k\|^2} > 0.$$

The above concepts are taken from [3, 14] in the setting of finite-dimensional space. Every  $R_{01}$ -function is  $R_{02}$ -function. In fact,  $R_{02}$ -function is equivalent to  $R_0$ -property defined in [24, Definition 3.2]. Besides, we recall the concept of the strong monotonicity for a bounded continuous function  $F : \mathcal{H} \rightarrow \mathcal{H}$ . That is, we say that  $F$  is strongly monotone with modulus  $\mu > 0$  if for any  $z, w \in \mathcal{H}$ , there exist a constant  $\mu > 0$  such that

$$\langle z-w, F(z) - F(w) \rangle \geq \mu \|z-w\|^2.$$

### 3 The same growth of FB and NR merit function

In this section, we will give our first main result of this paper. To prove it, the following lemmas will be needed.

**Lemma 3.1** *Let  $\mathcal{H}$  be a real Hilbert space. For  $z \in \mathcal{H}$  with  $z = x + \lambda e$ , define  $|z| := z_+ + z_-$ , where  $z_+$  and  $z_-$  are the same as in Lemma 2.2. Then, we have*

- (a)  $\langle z_+, z_- \rangle = 0$  and  $z_+ \circ z_- = 0$ ;
- (b)  $|z|^2 = |z^2|$ ,  $\| |z| \| = \|z\|$  and  $|z| = 2z_+ - z = z + 2z_-$ .

**Proof.** The proof is obtained by straightforward calculation. thus it is omitted.  $\square$

**Lemma 3.2** *Let  $\mathcal{H}$  be a real Hilbert space. For any  $z, w, u \in \mathcal{H}$ ,*

- (a) *if  $z \succeq 0, w \succeq 0$  and  $z \succeq w$ , we have  $z^{1/2} \succeq w^{1/2}$ ;*
- (b) *if  $z \succeq 0$  and  $2z^2 = w^2 + u^2$ , we have  $z \succeq \frac{1}{2}(w + u)$ .*

**Proof.** (a) Let  $p = z^{1/2}$ ,  $q = w^{1/2}$  and  $r = p - q = x + \lambda e$ . To prove  $z^{1/2} \succeq w^{1/2}$ , we need to verify that  $\lambda \geq \|x\|$ . Note that  $0 \preceq z - w = p^2 - (p - r)^2 = 2p \circ r - r^2$ . Let  $e_2 = \frac{1}{2}(e - \frac{x}{\|x\|})$  when  $x \neq 0$ , and  $e_2 = \frac{1}{2}(e - \tilde{x})$  ( $\tilde{x}$  is any element in  $\langle e \rangle^\perp$  satisfying  $\|\tilde{x}\| = 1$ ) when  $x = 0$ . Then, it follows that

$$\begin{aligned} 0 &\leq \langle 2p \circ r - r^2, e_2 \rangle \\ &= \langle 2p \circ r, e_2 \rangle - \langle r^2, e_2 \rangle \\ &= \langle 2p, r \circ e_2 \rangle - (\lambda - \|x\|)^2 \langle e_2, e_2 \rangle \\ &= (\lambda - \|x\|) \langle 2p, e_2 \rangle - \frac{(\lambda - \|x\|)^2}{2}, \end{aligned}$$

which implies  $\frac{(\lambda - \|x\|)^2}{2} \leq (\lambda - \|x\|) \langle 2p, e_2 \rangle$ . This together with  $p \succeq 0$  and  $e_2 \succeq 0$  yields  $\lambda - \|x\| \geq 0$ . The desired result follows.

(b) The arguments are similar to those for [1, Lemma 3.1]. For completeness, we present them as follows. Since  $w^2 + u^2 - 2w \circ u = (w - u)^2 \succeq 0$ , together with  $2z^2 = w^2 + u^2$ , this implies that

$$z^2 = \frac{1}{2}(w^2 + u^2) \succeq \frac{1}{4}(w^2 + u^2) + \frac{1}{2}w \circ u = \frac{1}{4}(w + u)^2.$$

From part (a) and  $z \succeq 0$ , we have  $z \succeq \frac{1}{2}|w + u| \succeq \frac{1}{2}(w + u)$ .  $\square$



**Theorem 3.1** *Let  $\mathcal{H}$  be a real Hilbert space. For any  $z, w \in \mathcal{H}$ , there holds*

$$(2 - \sqrt{2})\|\phi_{\text{NR}}(z, w)\| \leq \|\phi_{\text{FB}}(z, w)\| \leq (2 + \sqrt{2})\|\phi_{\text{NR}}(z, w)\|. \quad (6)$$

**Proof.** Fix any  $z, w \in \mathcal{H}$ . If  $z^2 + w^2 = 0$ , we have  $z = w = 0$ . Hence, the desired result is obvious. If  $z^2 + w^2 \neq 0$ , by Lemma 3.1, we obtain

$$\phi_{\text{NR}}(z, w) = z - (z - w)_+ = \frac{1}{2}[(z + w) - |z - w|].$$

Furthermore, we know

$$\begin{aligned} \phi_{\text{FB}}(z, w) &= z + w - (z^2 + w^2)^{1/2} \\ &= 2(z - (z - w)_+) + 2(z - w)_+ - (z - w) - (z^2 + w^2)^{1/2} \\ &= 2\phi_{\text{NR}}(z, w) + |z - w| - (z^2 + w^2)^{1/2}. \end{aligned}$$

Let  $p(z, w) = |z - w| - (z^2 + w^2)^{1/2}$ . In view of triangle inequality, to prove inequality (6), it suffices to verify that

$$\|p(z, w)\| \leq \sqrt{2}\|\phi_{\text{NR}}(z, w)\|. \quad (7)$$

To see this desired result, applying Lemma 3.2 gives

$$|z - w| + (z + w) \preceq 2(z^2 + w^2)^{1/2}, \quad (8)$$

Hence, we have

$$\begin{aligned} &2\|p(z, w)\|^2 - 4\|\phi_{\text{NR}}(z, w)\|^2 \\ &= 2(\|z - w\|^2 + \|(z^2 + w^2)^{1/2}\|^2 - 2\langle |z - w|, (z^2 + w^2)^{1/2} \rangle) \\ &\quad - \|z + w\|^2 - \|z - w\|^2 + 2\langle |z - w|, z + w \rangle \\ &= 2\|z - w\|^2 - 2\langle |z - w|, 2(z^2 + w^2)^{1/2} - (z + w) \rangle \\ &= \langle |z - w|, |z - w| + (z + w) - 2(z^2 + w^2)^{1/2} \rangle \\ &\leq 0, \end{aligned}$$

where the inequality holds is due to that  $|z - w| \succeq 0$  and (8). This implies that inequality (7) is true. Then, the proof is complete.  $\square$

## 4 Error bound of merit functions

Error bound is an important concept that indicates how close an arbitrary point is to the solution set of Lorentz cone complementarity problem (1). Thus, an error bound may be used to provide stopping criterion for an iterative method. As below, for FB merit

function  $\Psi_{\text{FB}}$  and merit function  $\Psi_\alpha$ , we draw conclusions about the error bounds for the solution of infinite-dimensional Lorentz cone complementarity problem (1), respectively.

For FB merit function  $\Psi_{\text{FB}}$ , we obtain a result about the error bounds for the solution of problem (1), which is an extension of [13, Theorem 6.3] in SCCP case.

**Proposition 4.1** *Suppose that  $F$  is strongly monotone with modulus  $\mu > 0$  and is Lipschitz continuous with constant  $l$ . Then,*

$$\frac{1}{(\sqrt{2} + 1)(2 + l)} \sqrt{\Psi_{\text{FB}}(z)} \leq \|z - z^*\| \leq \frac{l + 1}{\mu(\sqrt{2} - 1)} \sqrt{\Psi_{\text{FB}}(z)},$$

**Proof.** Fix any  $z \in \mathcal{H}$ , let  $N(z) = \phi_{\text{NR}}(z, F(z))$ . Applying Theorem 3.1, we have

$$(3 - 2\sqrt{2})\|N(z)\|^2 \leq \Psi_{\text{FB}}(z) \leq (3 + 2\sqrt{2})\|N(z)\|^2. \quad (9)$$

Since  $N(z) = \phi_{\text{NR}}(z, F(z)) = z - (z - F(z))_+$ , it follows that  $F(z) - N(z) = (z - F(z))_- \succeq 0$ ,  $z - N(z) = (z - F(z))_+ \succeq 0$  and  $\langle F(z) - N(z), z - N(z) \rangle = 0$ . Because  $z^*$  is the unique solution of problem (1), we have

$$\begin{aligned} 0 &\geq \langle F(z) - N(z), z - N(z) \rangle - \langle F(z) - N(z), z^* - N(z^*) \rangle - \langle F(z^*) - N(z^*), z - N(z) \rangle \\ &= \langle F(z) - F(z^*) - (N(z) - N(z^*)), z - z^* - (N(z) - N(z^*)) \rangle \\ &\geq \langle F(z) - F(z^*), z - z^* \rangle - \langle F(z) - F(z^*), N(z) - N(z^*) \rangle - \langle z - z^*, N(z) - N(z^*) \rangle \\ &\geq \mu\|z - z^*\|^2 - l\|z - z^*\|\|N(z)\| - \|z - z^*\|\|N(z)\|, \end{aligned}$$

where the last inequality is due to the strong monotonicity and the Lipschitz continuity of  $F$ . Thus, we obtain

$$\|z - z^*\| \leq \frac{l + 1}{\mu} \|N(z)\|. \quad (10)$$

On the other hand,

$$\begin{aligned} \|N(z)\| &= \|z - (z - F(z))_+ - (z^* - (z^* - F(z^*))_+)\| \\ &\leq \|z - z^*\| + \|z - z^* - (F(z) - F(z^*))\| \\ &\leq 2\|z - z^*\| + \|F(z) - F(z^*)\| \\ &\leq (2 + l)\|z - z^*\|. \end{aligned} \quad (11)$$

Combining (9), (10) and (11) leads to

$$\frac{1}{(\sqrt{2} + 1)(2 + l)} \sqrt{\Psi_{\text{FB}}(z)} \leq \|z - z^*\| \leq \frac{l + 1}{\mu(\sqrt{2} - 1)} \sqrt{\Psi_{\text{FB}}(z)}$$

which is the desired result.  $\square$

From Proposition 4.1, we see that if we use FB merit function  $\Psi_{\text{FB}}$  to provide error bound for the solution of problem (1), we need the conditions of  $F$  being strongly

monotone and Lipschitz continuous. However, to provide error bound for the solution of (1) via  $\Psi_\alpha$  merit function, we may weaken the aforementioned conditions to uniform  $P^*$ -property. We will prove this in Proposition 4.2. For this purpose, we present two technical lemmas first.

**Lemma 4.1** [15, Lemma 2.2] *Let  $z = x + \lambda e, w = y + \mu e \in \mathcal{H}$  with  $x, y \in \langle e \rangle^\perp$  and  $\lambda, \mu \in \mathbb{R}$ . The following two conditions are equivalent:*

(a)  $z \succeq 0, \quad w \succeq 0, \quad \text{and} \quad \langle z, w \rangle = 0;$

(b)  $z \succeq 0, \quad w \succeq 0, \quad \text{and} \quad z \circ w = 0.$

*In each case,  $z$  and  $w$  operator commute.*

**Lemma 4.2** [15, Lemma 2.4] *For any  $z, w \in \Omega$ , if  $z$  and  $w$  operator commute, then  $z \circ w \in \Omega$ .*

**Proposition 4.2** *Let  $\Psi_\alpha$  be defined as in (5) and  $\alpha > 0$ . Suppose that  $F$  has uniform  $P^*$ -property and the CP (1) has a solution  $z^*$ . Then, there exists a scalar  $\tau > 0$  such that*

$$\tau \|z - z^*\|^2 \leq \|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \quad \forall z \in \mathcal{H}. \quad (12)$$

Moreover,

$$\|z - z^*\| \leq \theta \Psi_\alpha(z)^{\frac{1}{4}}, \quad \forall z \in \mathcal{H},$$

where  $\theta$  is a positive constant.

**Proof.** Since  $F$  has the uniform  $P^*$ -property, there exists  $\rho > 0$  such that

$$\rho \|z - z^*\|^2 \leq \max_{i=1,2} \langle (z - z^*) \circ (F(z) - F(z^*)), e_i(z^*) \rangle, \quad (13)$$

where  $\{e_i(z^*) | i = 1, 2\}$  is the Jordan frame about  $z^*$  in  $\mathcal{H}$ . From  $z^*$  being a solution of (1), it follows that

$$(z - z^*) \circ (F(z) - F(z^*)) = z \circ F(z) - z \circ F(z^*) - z^* \circ F(z).$$

Note that  $F(z^*) \in \Omega$ ,  $z^* \in \Omega$  and  $e_i(z^*) \in \Omega$ . By Lemma 4.1, Lemma 4.2 and Lemma 2.2(e), we have

$$\langle -z, F(z^*) \circ e_i(z^*) \rangle \leq \langle (-z)_+, F(z^*) \circ e_i(z^*) \rangle$$

and

$$\langle -F(z), z^* \circ e_i(z^*) \rangle \leq \langle (-F(z))_+, z^* \circ e_i(z^*) \rangle.$$

Moreover, from Lemma 4.1, we have that  $z^*$  and  $F(z^*)$  share the same Jordan frame. Let

$$z^* = \lambda_1(z^*)e_1(z^*) + \lambda_2(z^*)e_2(z^*)$$

and

$$F(z^*) = \lambda_1(F(z^*))e_1(z^*) + \lambda_2(F(z^*))e_2(z^*).$$

Then, it follows that

$$\begin{aligned} & \langle (z - z^*) \circ (F(z) - F(z^*)), e_i(z^*) \rangle \\ &= \langle z \circ F(z), e_i(z^*) \rangle + \langle -z \circ F(z^*), e_i(z^*) \rangle + \langle -z^* \circ F(z), e_i(z^*) \rangle \\ &= \langle z \circ F(z), e_i(z^*) \rangle + \langle -z, F(z^*) \circ e_i(z^*) \rangle + \langle -F(z), z^* \circ e_i(z^*) \rangle \\ &\leq \langle (z \circ F(z))_+, e_i(z^*) \rangle + \langle (-z)_+, F(z^*) \circ e_i(z^*) \rangle + \langle (-F(z))_+, z^* \circ e_i(z^*) \rangle \\ &\leq \|(z \circ F(z))_+\| \|e_i(z^*)\| + \|(-z)_+\| \|F(z^*) \circ e_i(z^*)\| + \|(-F(z))_+\| \|z^* \circ e_i(z^*)\| \\ &\leq \frac{1}{\sqrt{2}} [\|(z \circ F(z))_+\| + \lambda_i(F(z^*)) \|(-z)_+\| + \lambda_i(z^*) \|(-F(z))_+\|] \\ &\leq \max \left\{ \frac{1}{\sqrt{2}}, \frac{\lambda_i(F(z^*))}{\sqrt{2}}, \frac{\lambda_i(z^*)}{\sqrt{2}} \right\} [\|(z \circ F(z))_+\| + \|(-z)_+\| + \|(-F(z))_+\|] \\ &\leq \max \left\{ \frac{1}{\sqrt{2}}, \frac{\lambda_2(F(z^*))}{\sqrt{2}}, \frac{\lambda_1(z^*)}{\sqrt{2}} \right\} [\|(z \circ F(z))_+\| + \|(-z)_+\| + \|(-F(z))_+\|], \end{aligned}$$

where the first inequality is from Lemma 2.2(e), and the last inequality is from the facts that  $\lambda_i(F(z^*)) \geq 0$ ,  $\lambda_1(z^*) \geq \lambda_2(z^*) \geq 0$  and  $z^* \circ F(z^*) = 0$ , i.e.,  $\lambda_i(F(z^*)) \cdot \lambda_i(z^*) = 0$  for  $i = 1, 2$ . Define

$$\tau := \frac{\rho}{\max \left\{ \frac{1}{\sqrt{2}}, \frac{\lambda_2(F(z^*))}{\sqrt{2}}, \frac{\lambda_1(z^*)}{\sqrt{2}} \right\}}.$$

This together (12) and (13) yields

$$\tau \|z - z^*\|^2 \leq \|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \quad \forall z \in \mathcal{H}.$$

Now, we come to the second part of the proposition. We know that

$$\Psi_\alpha(z) = \frac{\alpha}{2} \|(z \circ F(z))_+\|^2 + \Psi_{\text{FB}}(z),$$

which says  $\|(z \circ F(z))_+\| \leq \sqrt{\frac{2}{\alpha}} \Psi_\alpha(z)^{1/2}$ . Moreover, from Lemma 2.3, it follows that

$$\begin{aligned} \|(-F(z))_+\| + \|(-z)_+\| &\leq \sqrt{2} (\|(-F(z))_+\|^2 + \|(-z)_+\|^2)^{\frac{1}{2}} \\ &\leq 2\sqrt{2} \Psi_{\text{FB}}(z)^{1/2} \\ &\leq 2\sqrt{2} \Psi_\alpha(z)^{1/2}. \end{aligned}$$

Therefore,

$$\|(F(z) \circ z)_+\| + \|(-F(z))_+\| + \|(-z)_+\| \leq \left( \sqrt{\frac{2}{\alpha}} + 2\sqrt{2} \right) \Psi_\alpha(z)^{1/2}.$$

Letting  $\theta = \sqrt{\frac{\sqrt{2}(2\sqrt{\alpha}+1)}{\tau\sqrt{\alpha}}}$  shows the second part result.  $\square$

The condition  $\alpha > 0$  is necessary in Proposition 4.2. From the proof, we see that  $\theta \rightarrow \infty$  as  $\alpha \rightarrow 0$ . Thus,  $\alpha \neq 0$  is necessary. In fact, when  $\alpha = 0$ , we have  $\Psi_\alpha = \Psi_{\text{FB}}$ . This explains that for the FB merit function  $\Psi_{\text{FB}}$ , we can not reach our desired result under the same conditions in Proposition 4.2.

## 5 Boundedness of level sets

The boundedness of level sets of a merit function is also important since it is a necessary condition to ensure that the sequence generated by a descent method has at least one accumulation point. In this section, we will show the boundedness of level sets of  $\Psi_{\text{FB}}$  and  $\Psi_\alpha$  under different conditions. The relation between such conditions will be discussed as well.

**Lemma 5.1** *For any  $z^k, w^k \in \mathcal{H}$ , let  $\lambda_2(z^k) \leq \lambda_1(z^k)$  and  $\mu_2(w^k) \leq \mu_1(w^k)$  denote the spectral values of  $z^k$  and  $w^k$ , respectively. Then,*

(a) *if  $\lambda_2(z^k) \rightarrow -\infty$  or  $\mu_2(w^k) \rightarrow -\infty$ , we have  $\Psi_i \rightarrow \infty$  for  $i = 1, 2$ ,*

$$\text{where } \Psi_1(z) := \frac{1}{2}(\|(-z^k)_+\|^2 + \|(-w^k)_+\|^2) \text{ and } \Psi_2(z) := \frac{1}{2}\|\phi_{\text{FB}}(z^k, w^k)_+\|^2.$$

(b) *if  $\{\lambda_2(z^k)\}$  and  $\{\mu_2(w^k)\}$  are bounded below,  $\lambda_1(z^k) \rightarrow \infty$ ,  $\mu_1(w^k) \rightarrow \infty$  and  $\frac{z^k}{\|z^k\|} \circ \frac{w^k}{\|w^k\|} \not\rightarrow 0$  as  $k \rightarrow \infty$ , then  $\Phi_{\text{FB}}(z^k, w^k) \rightarrow \infty$ .*

**Proof.** (a) See [22, Lemma 4.4].

(b) Suppose that  $\{\phi_{\text{FB}}(z^k, w^k)\}$  is bounded. Define  $u^k := ((z^k)^2 + (w^k)^2)^{1/2}$  for each  $k$ . From the definition of  $\phi_{\text{FB}}$ , we have  $z^k + w^k = u^k - \phi_{\text{FB}}(z^k, w^k)$  for each  $k$ . Squaring two sides of the above equality leads to

$$2z^k \circ w^k = -2u^k \circ \phi_{\text{FB}}(z^k, w^k) + \phi_{\text{FB}}^2(z^k, w^k).$$

Since  $\|z^k\| \geq \frac{1}{\sqrt{2}}\lambda_1(z^k)$  and  $\|w^k\| \geq \frac{1}{\sqrt{2}}\mu_1(w^k)$ , by the conditions of this lemma, we have

$$\lim_{k \rightarrow \infty} \frac{u^k}{\|z^k\| \|w^k\|} = \lim_{k \rightarrow \infty} \left[ \frac{(z^k)^2}{\|z^k\|^2 \|w^k\|^2} + \frac{(w^k)^2}{\|z^k\|^2 \|w^k\|^2} \right]^{1/2} = 0.$$

This together with the boundedness of  $\{\phi_{\text{FB}}(z^k, w^k)\}$  implies

$$\lim_{k \rightarrow \infty} \frac{2z^k \circ w^k}{\|z^k\| \|w^k\|} = \lim_{k \rightarrow \infty} \frac{-2u^k \circ \phi_{\text{FB}}(z^k, w^k) + \phi_{\text{FB}}^2(z^k, w^k)}{\|z^k\| \|w^k\|} = 0.$$

Thus,  $\lim_{k \rightarrow \infty} \frac{z^k}{\|z^k\|} \circ \frac{w^k}{\|w^k\|} = 0$ , which contradicts the given assumption. Hence, the conclusion is proved.  $\square$

**Remark:** The condition of Lemma 5.1(b) was discussed in Lemma 4.2 of paper [11] and Lemma 4.1 of paper [3] in finite-dimensional space as well.

**Condition A**[18]. For any sequence  $\{z^k\} \subseteq \mathcal{H}$  satisfying  $\|z^k\| \rightarrow \infty$ , if  $\{\lambda_2(z^k)\}$  and  $\lambda_2(F(z^k))$  are bounded below, and  $\lambda_1(z^k) \rightarrow \infty$ ,  $\lambda_1(F(z^k)) \rightarrow \infty$  as  $k \rightarrow \infty$ . Then

$$\limsup_{k \rightarrow \infty} \left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|} \right\rangle > 0.$$

Using Lemma 5.1 and similar arguments as in [3, Proposition 4.2], we may obtain the following propositions, which are the boundedness of level sets for  $\Psi_{\text{FB}}$  and  $\Psi_\alpha$ , respectively.

**Proposition 5.1** *Let  $\Psi_{\text{FB}}$  be given as in (4). Assume that  $F$  is a strongly monotone function with modulus  $\mu > 0$  and satisfies Condition A. Then, the level set*

$$\mathcal{L}(\gamma) := \{z \in H \mid \Psi_{\text{FB}}(z) \leq \gamma\}$$

*is bounded for all  $\gamma \geq 0$ .*

**Proof.** We prove this result by contradiction. Suppose there exists an unbounded sequence  $\{z^k\} \subseteq \mathcal{L}(\gamma)$  for some  $\gamma \geq 0$ . We claim that the sequence of the smallest spectral values of  $z^k$  and  $F(z^k)$  are bounded below. If not, by Lemma 5.1, we have  $\Psi_i(z^k) \rightarrow \infty$  for  $i = 1, 2$ , which implies  $\Psi_{\text{FB}}(z^k) \rightarrow \infty$ . This contradicts  $\{z^k\} \subseteq \mathcal{L}(\gamma)$ . On the other hand, by the strongly monotone property of  $F$ , there exists a constant  $\mu > 0$  such that

$$\begin{aligned} \mu \|z^k\|^2 &\leq \langle z^k, F(z^k) - F(0) \rangle \\ &\leq \|z^k\| \|F(z^k) - F(0)\| \\ &\leq \|z^k\| (\|F(z^k)\| + \|F(0)\|). \end{aligned}$$

This implies that  $\mu \|z^k\| \leq \|F(z^k)\| + \|F(0)\|$ . It follows from unboundedness of  $\{z^k\}$  and boundedness of  $F$  that  $\{F(z^k)\}$  is unbounded, which says that  $\lambda_1(z^k) \rightarrow \infty$  and  $\lambda_1(F(z^k)) \rightarrow \infty$  as  $k \rightarrow \infty$ . By Condition A, it gives

$$\limsup_{k \rightarrow \infty} \left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|} \right\rangle > 0,$$

which implies

$$\limsup_{k \rightarrow \infty} \lambda_1 \left[ \frac{z^k}{\|z^k\|} \circ \frac{F(z^k)}{\|F(z^k)\|} \right] > 0.$$

From this, we have  $\frac{z^k}{\|z^k\|} \circ \frac{F(z^k)}{\|F(z^k)\|} \not\rightarrow 0$ . Together with Lemma 5.1(b), we obtain  $\Psi_{\text{FB}}(z^k) = \Phi_{\text{FB}}(z^k, F(z^k)) \rightarrow \infty$ . Hence, this is a contradiction to  $z^k \subseteq \mathcal{L}(\gamma)$ . The proof is complete.  $\square$

In fact, the condition A and the strong monotonicity of  $F$  in Proposition 5.1 can be replaced by the Lipschitz continuity of  $F$  and  $R_{01}$ -function. This can be verified as below. Since the sequence of the smaller spectral values of  $z^k$  and  $F(z^k)$  are bounded below, we have  $(-z^k)_+$  and  $(-F(z^k))_+$  are bounded above. For any sequence  $z^k$  satisfying  $\|z^k\| \rightarrow \infty$ , by the definition of  $R_{01}$ -function, we have

$$\liminf_{k \rightarrow \infty} \frac{\langle z^k, F(z^k) \rangle}{\|z^k\|^2} > 0. \quad (14)$$

This implies that

$$\begin{aligned} \limsup_{k \rightarrow \infty} \left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|} \right\rangle &\geq \liminf_{k \rightarrow \infty} \left\langle \frac{z^k}{\|z^k\|}, \frac{F(z^k)}{\|F(z^k)\|} \right\rangle \\ &= \liminf_{k \rightarrow \infty} \frac{\langle z^k, F(z^k) \rangle}{\|z^k\|^2} \frac{\|z^k\|}{\|F(z^k)\|} \\ &> 0, \end{aligned}$$

where the last inequality is due to (14) and the Lipschitz continuity of  $F$ .

Next, we show the bounded level sets for  $\Psi_\alpha$ . As will be seen, it requires  $F$  being  $R_{02}$ -function to guarantee this property. In view of the above remark and the fact that every  $R_{01}$ -function is an  $R_{02}$ -function, we see this condition is weaker than in Proposition 5.1 though their proofs are similar.

**Proposition 5.2** *Let  $\Psi_\alpha$  be given as in (5). Suppose that  $F$  is an  $R_{02}$ -function. Then, the level sets*

$$\mathcal{L}(\gamma) := \{z \in \mathcal{H} \mid \Psi_\alpha(z) \leq \gamma\}.$$

*is bounded for all  $\gamma \geq 0$ .*

**Proof.** We prove this result by contradiction again. Suppose there exists an unbounded sequence  $\{z^k\} \subseteq \mathcal{L}(\gamma)$  for some  $\gamma \geq 0$ . We claim that the sequence of the smaller spectral values of  $z^k$  and  $F(z^k)$  are bounded below. In fact, if not, by Lemma 5.1, we have  $\Psi_i(z^k) \rightarrow \infty$  for  $i = 1, 2$ , which says  $\Psi_\alpha(z^k) \rightarrow \infty$ . This contradicts  $\{z^k\} \subseteq \mathcal{L}(\gamma)$ . Therefore  $\{(-z^k)_+\}$  and  $\{(-F(z^k))_+\}$  are bounded above. Thus, for any sequence  $\{z^k\}$  satisfying  $\|z^k\| \rightarrow \infty$ , we have

$$\frac{(-z^k)_+}{\|z^k\|} \rightarrow 0 \quad \text{and} \quad \frac{(-F(z^k))_+}{\|z^k\|} \rightarrow 0.$$

By the definition of  $R_{02}$ -function, we have

$$\liminf_{k \rightarrow \infty} \frac{\lambda_1(z^k \circ F(z^k))}{\|z^k\|^2} > 0.$$

This implies that  $\lambda_1(z^k \circ F(z^k)) \rightarrow \infty$ . Hence,  $\|(z^k \circ F(z^k))_+\| \rightarrow \infty$ . This together with definition of  $\Psi_\alpha$ , which leads to  $\Psi_\alpha(z^k) \rightarrow \infty$ . This contradicts  $\{z^k\} \subseteq \mathcal{L}(\gamma)$ . Therefore, the desired result is proved.  $\square$

## 6 Concluding Remarks

In this paper, we have studied the Lorentz cone complementarity problems in real Hilbert space for which we prove the same growth of FB and NR merit functions, provide a global error bound for the solutions via two kinds of merit functions, and discuss the property of the bounded level sets of these two kinds of merit functions under different conditions. Such results will be helpful and useful for further designing solution methods for solving problem (1).

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