Comp. Appl. Math.
https://doi.org/10.1007/s40314-018-0660-0

Discovery of new complementarity functions for NCP and SOCCP

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Received: 25 February 2017 / Revised: 20 May 2018 / Accepted: 30 May 2018

Abstract It is well known that complementarity functions play an important role in dealing with complementarity problems. In this paper, we propose a few new classes of complementarity functions for nonlinear complementarity problems and second-order cone complementarity problems. The constructions of such new complementarity functions are based on discrete generalization which is a novel idea in contrast to the continuous generalization of Fischer–Burmeister function. Surprisingly, these new families of complementarity functions possess continuous differentiability even though they are discrete-oriented extensions. This feature enables that some methods like derivative-free algorithm can be employed directly for solving nonlinear complementarity problems and second-order cone complementarity problems. This is a new discovery to the literature and we believe that such new complementarity functions can also be used in many other contexts.

Communicated by Jinyun Yuan.

Peng-Fei Ma This research was supported by a grant from the National Natural Science Foundation of China(No.11626212).
Jein-Shan Chen The author's work is supported by Ministry of Science and Technology, Taiwan.

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Published online: 06 June 2018

Keywords NCP · SOCCP · Natural residual · Complementarity function

Mathematics Subject Classification 26B05 · 26B35 · 90C33 · 65K05

1 Introduction

In general, the complementarity problem comes from the Karush–Kuhn–Tucker (KKT) conditions of linear and nonlinear programming problems. For different types of optimization problems, there arise various complementarity problems, for example, linear complementarity problem, nonlinear complementarity problem (NCP), semidefinite complementarity problem, second-order cone complementarity problem (SOCCP), and symmetric cone complementarity problem. To deal with complementarity problems, the so-called complementarity functions play an important role therein. In this paper, we focus on two classes of complementarity functions, which are used for the NCP and SOCCP, respectively.

The first class is the NCP that has attracted much attention since 1970s because of its wide applications in the fields of economics, engineering, and operations research, see (Cottle et al. 1992; Facchinei and Pang 2003; Harker and Pang 1990) and references therein. In mathematical format, the NCP is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \ldots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. For solving NCP, the so-called NCP function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as below

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a, b \geq 0, \ ab = 0$$

plays a crucial role. Generally speaking, with such NCP functions, the NCP can be reformulated as nonsmooth equations (Mangasarian 1976; Pang 1990; Yamashita and Fukushima 1997) or unconstrained minimization (Facchinei and Soares 1997; Fischer 1992; Geiger and Kanzow 1996; Jiang 1996; Kanzow 1996; Pang and Chan 1982; Yamashita and Fukushima 1995). Then, different kinds of approaches and algorithms are designed based on the aforementioned reformulations and various NCP functions. During the past four decades, around thirty NCP functions are proposed, see (Galántai 2012) for a survey.

The second class is the SOCCP, which can be viewed as a natural extension of NCP and is to seek a $\zeta \in \mathbb{R}^n$ such that

$$\zeta \in \mathcal{K}, \quad F(\zeta) \in \mathcal{K}, \quad \langle \zeta, F(\zeta) \rangle = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map and $\mathcal{K}$ is the Cartesian product of second-order cones (SOC), also called Lorentz cones (Faraut and Korányi 1994). In other words, $\mathcal{K}$ is expressed as:

$$\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_m},$$

where $m, n_1, \ldots, n_m \geq 1$, $n_1 + \cdots + n_m = n$, and

$$\mathcal{K}^{n_i} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid \|x_2\| \leq x_1\},$$

with $\|\cdot\|$ denoting the Euclidean norm. The SOCCP has important applications in engineering problems (Kanno et al. 2006) and robust Nash equilibria (Hayashi et al. 2005). Another important special case of SOCCP corresponds to the KKT optimality conditions for the second-order cone program (see Chen and Tseng 2005 for details):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \in \mathcal{K}, \end{array}$$

where $A \in \mathbb{R}^{m\times n}$ has full row rank, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Many solution methods have been proposed for solving SOCCP, see (Chen and Pan 2012) for a survey. For example, merit function approach based on reformulating the SOCCP as an unconstrained smooth minimization problem is studied in Chen and Tseng (2005), Chen (2006b), Pan et al. (2014). In such approach, it is to find a smooth function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\psi(x, y) = 0 \quad \Longleftrightarrow \quad \langle x, y\rangle = 0, \quad x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n. \tag{1}$$

Then, the SOCCP can be expressed as an unconstrained smooth (global) minimization problem:

$$\min_{\zeta\in\mathbb{R}^n} \psi(\zeta, F(\zeta)). \tag{2}$$

In fact, a function ψ satisfying the condition in (1) (not necessarily smooth) is called a complementarity function for SOCCP (or complementarity function associated with $\mathcal{K}^n$). Various gradient methods such as conjugate gradient methods and quasi-Newton methods (Bertsekas 1999; Fletcher 1987) can be applied for solving (2). In general, for this approach to be effective, the choice of complementarity function ψ is also crucial.

Back to the complementarity functions for NCP, two popular choices of NCP functions are the well-known Fischer–Burmeister function (FB function, in short) $\phi_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ defined by see (Fischer 1992, 1997)

$$\phi_{\text{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b),$$

and the squared norm of Fischer–Burmeister function given by

$$\psi_{\text{FB}}(a, b) = \frac{1}{2}\left|\phi_{\text{FB}}(a, b)\right|^2.$$

In addition, the generalized Fischer–Burmeister function $\phi_p : \mathbb{R}^2 \to \mathbb{R}$, which includes the Fischer–Burmeister as a special case, is considered in Chen (2006a, 2007), Chen et al. (2009), Chen and Pan (2008), Hu et al. (2009), Tsai and Chen (2014). In particular, the function ϕ_p is a natural "continuous extension" of ϕ_{FB}, in which the 2-norm in $\phi_{\text{FB}}(a, b)$ is replaced by general p-norm. In other words, $\phi_p : \mathbb{R}^2 \to \mathbb{R}$ is defined as:

$$\phi_p(a, b) = \|(a, b)\|_p - (a + b), \quad p > 1 \tag{3}$$

and its geometric view is depicted in Tsai and Chen (2014). The effect of perturbing p for different kinds of algorithms is investigated in Chen et al. (2010, 2011), and Chen and Pan (2008) . We point it out that the generalized Fischer–Burmeister ϕ_p given as in (3) is not differentiable, whereas the squared norm of generalized Fischer–Burmeister function is smooth so that it is usually adapted as a differentiable NCP function Pan et al. (2014). Moreover, all the aforementioned functions including Fischer–Burmeister function, generalized Fischer–Burmeister function and their squared norm can be extended to the setting of SOCCP via Jordan algebra.

A different type of popular NCP function is the natural residual function $\phi_{\text{NR}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi_{\text{NR}}(a, b) = a - (a - b)_+ = \min\{a, b\}.$$

Recently, Chen et al. propose a family of generalized natural residual functions ϕ^p_{NR} defined by

$$\phi^p_{\text{NR}}(a, b) = a^p - (a - b)^p_+,$$

where $p > 1$ is a positive odd integer, $(a-b)_+^p = [(a-b)_+]^p$, and $(a-b)_+ = \max\{a-b, 0\}$. When $p = 1$, ϕ_{NR}^p reduces to the natural residual function ϕ_{NR}, i.e.,

$$\phi_{\mathrm{NR}}^1(a,b) = a - (a-b)_+ = \min\{a,b\} = \phi_{\mathrm{NR}}(a,b).$$

As remarked in Chen et al. (2016), this extension is "discrete generalization", not "continuous generalization". Nonetheless, it possesses twice differentiability surprisingly so that the squared norm of ϕ_{NR}^p is not needed. Based on this discrete generalization, two families of NCP functions are further proposed in Chang et al. (2015) which have the feature of symmetric surfaces. To the contrast, it is very natural to ask whether there is a similar "discrete extension" for Fischer–Burmeister function. We answer this question affirmatively.

In this paper, we apply the idea of "discrete generalization" to the Fischer–Burmeister function which gives the following function (denoted by $\phi_{\mathrm{D-FB}}^p$):

$$\phi_{\mathrm{D-FB}}^p(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p, \tag{4}$$

where $p > 1$ is a positive odd integer and $(a,b) \in \mathbb{R}^2$. Notice that when $p = 1$, $\phi_{\mathrm{D-FB}}^p$ reduces to the Fischer–Burmeister function. In Sect. 3, we will see that $\phi_{\mathrm{D-FB}}^p$ is an NCP function and is twice differentiable directly without taking its squared norm. Note that if p is even, it is no longer an NCP function. Even though we have the feature of differentiability, we point out that the Newton method may not be applied directly because the Jacobian at a degenerate solution to NCP is singular see (Kanzow 1996; Kanzow and Kleinmichel 1995). Nonetheless, this feature may enable that many methods like derivative-free algorithm can be employed directly for solving NCP. In addition, we investigate the differentiable properties of $\phi_{\mathrm{D-FB}}^p$, the computable formulas for their gradients and Jacobians. In order to have more insight for this new family of NCP function, we also depict the surfaces of $\phi_{\mathrm{D-FB}}^p(a,b)$ with various values of p.

In Sect. 4, we show that the new function $\phi_{\mathrm{D-FB}}^p$ can be further employed to the SOCCP setting as complementarity functions and merit functions. In other words, in the terms of Jordan algebra, we define $\phi_{\mathrm{D-FB}}^p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi_{\mathrm{D-FB}}^p(x,y) = \left(\sqrt{x^2+y^2}\right)^p - (x+y)^p, \tag{5}$$

where $p > 1$ is a positive odd integer, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x^2 = x \circ x$ is the Jordan product of x with itself and $\sqrt{x}$ with $x \in \mathcal{K}^n$ being the unique vector such that $\sqrt{x} \circ \sqrt{x} = x$. We prove that each $\phi_{\mathrm{D-FB}}^p(x,y)$ is a complementarity function associated with $\mathcal{K}^n$ and establish formulas for its gradient and Jacobian. These properties and formulas can be used to design and analyze non-interior continuation methods for solving second-order cone programs and complementarity problems. In addition, several variants of $\phi_{\mathrm{D-FB}}^p$ are also shown to be complementarity functions for SOCCP.

Throughout the paper, we assume $\mathcal{K} = \mathcal{K}^n$ for simplicity and all the analysis can be carried over to the case where $\mathcal{K}$ is a product of SOC without difficulty. The following notations will be used. The identity matrix is denoted by I and $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors. For any given $x \in \mathbb{R}^n$ with $n > 1$, we write $x = (x_1, x_2)$ where x_1 is the first entry of x and x_2 is the subvector that consists of the remaining entries. For every differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of f at x. For every differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla F(x)$ is an $n \times m$ matrix which denotes the transposed Jacobian of F at x. For nonnegative scalar functions α and β, we write $\alpha = o(\beta)$ to mean $\lim_{\beta \to 0} \dfrac{\alpha}{\beta} = 0$.

2 Preliminaries

In this section, we review some background materials about the Jordan algebra in Faraut and Korányi (1994), Fukushima et al. (2002). Then, we present some technical lemmas which are needed in subsequent analysis.

For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define the *Jordan product* associated with $\mathcal{K}^n$ as:

$$x \circ y \ := \ (\langle x, y\rangle, \ y_1 x_2 + x_1 y_2).$$

The identity element under this product is $e := (1, 0, \dots, 0)^T \in \mathbb{R}^n$. For any given $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define symmetric matrix

$$L_x := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix}$$

which can be viewed as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is easy to verify that

$$L_x y = x \circ y, \quad \forall x \in \mathbb{R}^n.$$

Moreover, we have L_x is invertible for $x \succ_{\mathcal{K}^n} 0$ and

$$L_x^{-1} = \frac{1}{\det(x)} \begin{bmatrix} x_1 & -x_2^T \\ -x_2 & \dfrac{\det(x)}{x_1} I + \dfrac{1}{x_1} x_2 x_2^T \end{bmatrix},$$

where $\det(x) = x_1^2 - \|x_2\|^2$. We next recall from Chen and Pan (2012); Fukushima et al. (2002) that each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ admits a spectral factorization, associated with $\mathcal{K}^n$, of the form

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{6}$$

where λ_1, λ_2 and $u^{(1)}, u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|,$$
$$u^{(i)} = \begin{cases} \frac{1}{2}\left(1, \ (-1)^i \dfrac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0; \\ \frac{1}{2}\left(1, \ (-1)^i w_2\right) & \text{if } x_2 = 0, \end{cases}$$

for $i = 1, 2$, with w_2 being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w_2\| = 1$. If $x_2 \neq 0$, the factorization is unique.

Given a real-valued function $g : \mathbb{R} \to \mathbb{R}$, we can define a vector-valued SOC function $g^{\mathrm{soc}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g^{\mathrm{soc}}(x) := g(\lambda_1) u^{(1)} + g(\lambda_2) u^{(2)}.$$

If g is defined on a subset of $\mathbb{R}$, then g^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition of g^{soc} is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. In this paper, we will often use the vector-valued functions corresponding to t^p ($t \in \mathbb{R}$) and $\sqrt{t}$ ($t \geq 0$), respectively, which are expressed as:

$$x^p \ := (\lambda_1(x))^p u^{(1)} + (\lambda_2(x))^p u^{(2)}, \qquad \forall x \in \mathbb{R}^n$$
$$\sqrt{x} := \sqrt{\lambda_1(x)} u^{(1)} + \sqrt{\lambda_2(x)} u^{(2)}, \qquad \forall x \in \mathcal{K}^n.$$

We will see that the above two vector-valued functions play a role, showing that $\phi^p_{\text{D-FB}}$ given as in (5) is well defined in the SOC setting for any $x, y \in \mathbb{R}^n$. Note that the other way to define x^p and $\sqrt{x}$ is through Jordan product. In other words, x^p represents $x \circ x \circ \cdots \circ x$ for p-times and $\sqrt{x} \in \mathcal{K}^n$ satisfies $\sqrt{x} \circ \sqrt{x} = x$.

Lemma 2.1 *Suppose that $p = 2k + 1$ where $k = 1, 2, 3, \cdots$. Then, for any $u, v \in \mathbb{R}$, we have $u^p = v^p$ if and only if $u = v$.*

Proof The proof is straightforward and can be found in (Baggett et al. 2012, Theorem 1.12). Here, we provide an alternative proof.
"$\Leftarrow$" It is trivial.
"$\Rightarrow$" For $v = 0$, since $u^p = v^p$, we have $u = v = 0$. For $v \neq 0$, from $f(t) = t^p - 1$ being a strictly monotone increasing function for any $t \in \mathbb{R}$, we have $\left(\frac{u}{v}\right)^p - 1 = 0$ if and only if $\frac{u}{v} = 1$, which implies $u = v$. Thus, the proof is complete. □

Lemma 2.2 *For $p = 2m + 1$ with $m = 1, 2, 3, \cdots$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, suppose that x^p and y^p represent $x \circ x \circ \cdots \circ x$ and $y \circ y \circ \cdots \circ y$ for p-times, respectively. Then, $x^p = y^p$ if and only if $x = y$.*

Proof "$\Leftarrow$" This direction is trivial.
"$\Rightarrow$" Suppose that $x^p = y^p$. By the spectral decomposition (6), we write

$$\begin{aligned} x &= \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \\ y &= \lambda_1(y) u_y^{(1)} + \lambda_2(y) u_y^{(2)}. \end{aligned}$$

Then, $x^p = (\lambda_1(x))^p u_x^{(1)} + (\lambda_2(x))^p u_x^{(2)}$ and $y^p = (\lambda_1(y))^p u_y^{(1)} + (\lambda_2(y))^p u_y^{(2)}$. Since $x^p = y^p$ and eigenvalues are unique, we obtain $(\lambda_1(x))^p = (\lambda_1(y))^p$ and $(\lambda_2(x))^p = (\lambda_2(y))^p$. By Lemma 2.1, this implies $\lambda_1(x) = \lambda_1(y)$ and $\lambda_2(x) = \lambda_2(y)$. Moreover, $\{u_x^{(1)}, u_x^{(2)}\}$ and $\{u_y^{(1)}, u_y^{(2)}\}$ are Jordan frames, we have $u_x^{(1)} + u_x^{(2)} = u_y^{(1)} + u_y^{(2)} = e$, where e is the identity element. From $x^p = y^p$ and $u_x^{(1)} + u_x^{(2)} = u_y^{(1)} + u_y^{(2)}$, we get

$$\left[(\lambda_1(x))^p - (\lambda_2(x))^p\right] (u_x^{(1)} - u_y^{(1)}) = 0.$$

If $(\lambda_1(x))^p = (\lambda_2(x))^p$, we have $\lambda_1(x) = \lambda_2(x)$ and $\lambda_1(y) = \lambda_2(y)$, that is, $x = \lambda_1(x) e = y$. Otherwise, if $(\lambda_1(x))^p \neq (\lambda_2(x))^p$, we must have $u_x^{(1)} = u_y^{(1)}$, which implies $u_x^{(2)} = u_y^{(2)}$. □

3 New generalized Fischer–Burmeister function for NCP

In this section, we show that the function $\phi^p_{\text{D-FB}}$ defined as in (4) is an NCP function and present its twice differentiability. At the same time, we also depict the surfaces of $\phi^p_{\text{D-FB}}$ with various values of p to have more insight for this new family of NCP functions.

Proposition 3.1 *Let $\phi^p_{\text{D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, $\phi^p_{\text{D-FB}}$ is an NCP function.*

Proof Suppose $\phi^p_{\text{D-FB}}(a, b) = 0$, which says $\left(\sqrt{a^2 + b^2}\right)^p = (a + b)^p$. Using p being a positive odd integer and applying Lemma 2.1, we have

$$\left(\sqrt{a^2 + b^2}\right)^p = (a + b)^p \iff \sqrt{a^2 + b^2} = a + b.$$

It is well known that $\sqrt{a^2+b^2}=a+b$ is equivalent to $a, b \geq 0, ab=0$ because ϕ_{FB} is an NCP function. This shows that $\phi^p_{\mathrm{D-FB}}(a,b)=0$ implies $a, b \geq 0, ab=0$. The converse direction is trivial. Thus, we prove that $\phi^p_{\mathrm{D-FB}}$ is an NCP function. □

Remark 3.1 We elaborate more about the new NCP function $\phi^p_{\mathrm{D-FB}}$.

(a) For p being an even integer, $\phi^p_{\mathrm{D-FB}}$ is not an NCP function. A counterexample is given as below.

$$\phi^p_{\mathrm{D-FB}}(-5,0)=(-5)^2-(-5)^2=0.$$

(b) The surface of $\phi^p_{\mathrm{D-FB}}$ is symmetric, i.e., $\phi^p_{\mathrm{D-FB}}(a,b)=\phi^p_{\mathrm{D-FB}}(b,a)$.
(c) The function $\phi^p_{\mathrm{D-FB}}(a,b)$ is positive homogenous of degree p, i.e., $\phi^p_{\mathrm{D-FB}}(\alpha(a,b)) = \alpha^p\phi^p_{\mathrm{D-FB}}(a,b)$ for $\alpha \geq 0$.
(d) The function $\phi^p_{\mathrm{D-FB}}$ is neither convex nor concave function. To see this, taking $p=3$ and using the following argument verify the assertion.

$$\begin{aligned}5^3-7^3=\phi^3_{\mathrm{D-FB}}(3,4) &> \frac{1}{2}\phi^3_{\mathrm{D-FB}}(0,0)+\frac{1}{2}\phi^3_{\mathrm{D-FB}}(6,8)\\ &= \frac{1}{2}\times 0+\frac{1}{2}\left(10^3-14^3\right)=4\left(5^3-7^3\right)\end{aligned}$$

and

$$0=\phi^3_{\mathrm{D-FB}}(0,0)<\frac{1}{2}\phi^3_{\mathrm{D-FB}}(-2,0)+\frac{1}{2}\phi^3_{\mathrm{D-FB}}(2,0)=\frac{1}{2}\times 16+\frac{1}{2}\times 0=8.$$

Proposition 3.2 *Let $\phi^p_{\mathrm{D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, the following hold.*

(a) *For $p>1$, $\phi^p_{\mathrm{D-FB}}$ is continuously differentiable with*

$$\nabla\phi^p_{\mathrm{D-FB}}(a,b)=p\begin{bmatrix} a(\sqrt{a^2+b^2})^{p-2}-(a+b)^{p-1}\\ b(\sqrt{a^2+b^2})^{p-2}-(a+b)^{p-1}\end{bmatrix}.$$

(b) *For $p>3$, $\phi^p_{\mathrm{D-FB}}$ is twice continuously differentiable with*

$$\nabla^2\phi^p_{\mathrm{D-FB}}(a,b)=\begin{bmatrix}\frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial a^2} & \frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial a\partial b}\\ \frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial b\partial a} & \frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial b^2}\end{bmatrix},$$

where

$$\begin{aligned}\frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial a^2} &= p\left\{\left[(p-1)a^2+b^2\right](\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}\right\},\\ \frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial a\partial b} &= p\left[(p-2)ab(\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}\right]=\frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial b\partial a},\\ \frac{\partial^2\phi^p_{\mathrm{D-FB}}}{\partial b^2} &= p\left\{\left[a^2+(p-1)b^2\right](\sqrt{a^2+b^2})^{p-4}-(p-1)(a+b)^{p-2}\right\}.\end{aligned}$$

Proof The verifications of differentiability and computations of first and second derivatives are straightforward, we omit them. □

Next, we present some variants of $\phi^p_{\text{D-FB}}$. Indeed, analogous to those functions in Sun and Qi (1999), the variants of $\phi^p_{\text{D-FB}}$ as below can be verified being NCP functions.

$$\begin{aligned}
\phi_1(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha(a)_+(b)_+, \ \alpha > 0.\\
\phi_2(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha\left((a)_+(b)_+\right)^2, \ \alpha > 0.\\
\phi_3(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^4, \ \alpha > 0.\\
\phi_4(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^2, \ \alpha > 0.
\end{aligned}$$

In the above expressions, for any $t \in \mathbb{R}$, we define t_+ as $\max\{0, t\}$.

Lemma 3.1 *Let $\phi^p_{\text{D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, the value of $\phi^p_{\text{D-FB}}(a,b)$ is negative only in the first quadrant, i.e., $\phi^p_{\text{D-FB}}(a,b) < 0$ if and only if $a > 0$, $b > 0$.*

Proof We know that $f(t) = t^p$ is a strictly increasing function when p is odd. Using this fact yields

$$\begin{aligned}
& a > 0, \ b > 0\\
&\iff a+b > 0 \quad \text{and} \quad ab > 0\\
&\iff \sqrt{a^2+b^2} < a+b\\
&\iff \left(\sqrt{a^2+b^2}\right)^p < (a+b)^p\\
&\iff \phi^p_{\text{D-FB}}(a,b) < 0,
\end{aligned}$$

which proves the desired result. □

Proposition 3.3 *All the above functions ϕ_i for $i \in \{1,2,3,4\}$ are NCP functions.*

Proof Applying Lemma 3.1, the arguments are similar to those in [Chen et al. (2016), Proposition 2.4], which are omitted here. □

In fact, in light of Lemma 2.1, we can construct more variants of $\phi^p_{\text{D-FB}}$, which are also new NCP function. More specifically, consider that k and m are positive integers, $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ with $g(a,b) \neq 0$ for all $a, b \in \mathbb{R}$, the following functions are new variants of $\phi^p_{\text{D-FB}}$.

$$\begin{aligned}
\phi_5(a,b) &= \left[g(a,b)\big(\sqrt{a^2+b^2} + f(a,b)\big)\right]^{\frac{2k+1}{2m+1}} - \left[g(a,b)\big(a+b+f(a,b)\big)\right]^{\frac{2k+1}{2m+1}}.\\
\phi_6(a,b) &= \left[g(a,b)(\sqrt{a^2+b^2} - a - b)\right]^{\frac{k}{m}}.\\
\phi_7(a,b) &= \left[g(a,b)(\sqrt{a^2+b^2} - a + f(a,b))\right]^{\frac{2k+1}{2m+1}} - [g(a,b)(b+f(a,b))]^{\frac{2k+1}{2m+1}}.\\
\phi_8(a,b) &= \left[g(a,b)(\sqrt{a^2+b^2} - a + f(a,b))\right]^{\frac{2k+1}{2m+1}} - [g(a,b)(b+f(a,b))]^{\frac{2k+1}{2m+1}}.\\
\phi_9(a,b) &= e^{\phi_i(a,b)} - 1 \text{ where } i = 5,6,7,8.\\
\phi_{10}(a,b) &= \ln(|\phi_i(a,b)| + 1) \text{ where } i = 5,6,7,8.
\end{aligned}$$

Proposition 3.4 *All the above functions ϕ_i for $i \in \{5, 6, 7, 8, 9, 10\}$ are NCP functions.*

Proof This is an immediate consequence of Propositions 3.1, 3.1, 3.3. By Lemma 2.1 and $g(a, b) \neq 0$ for $a, b \in \mathbb{R}$, we have

$$
\begin{aligned}
&\phi_5(a, b) = 0 \\
&\quad\Longleftrightarrow \Big[g(a, b)\big(\sqrt{a^2 + b^2} + f(a, b)\big)\Big]^{\frac{2k+1}{2m+1}} = \big[g(a, b)\big(a + b + f(a, b)\big)\big]^{\frac{2k+1}{2m+1}} \\
&\quad\Longleftrightarrow \Big\{\Big[g(a, b)\big(\sqrt{a^2 + b^2} + f(a, b)\big)\Big]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} \\
&\quad= \Big\{\big[g(a, b)\big(a + b + f(a, b)\big)\big]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} \\
&\quad\Longleftrightarrow \Big[g(a, b)\big(\sqrt{a^2 + b^2} + f(a, b)\big)\Big]^{2k+1} = \big[g(a, b)\big(a + b + f(a, b)\big)\big]^{2k+1} \\
&\quad\Longleftrightarrow g(a, b)\big(\sqrt{a^2 + b^2} + f(a, b)\big) = g(a, b)\big(a + b + f(a, b)\big) \\
&\quad\Longleftrightarrow \big(\sqrt{a^2 + b^2} + f(a, b)\big) = \big(a + b + f(a, b)\big) \\
&\quad\Longleftrightarrow \sqrt{a^2 + b^2} = a + b.
\end{aligned}
$$

The other functions ϕ_i for $i \in \{6, 7, 8, 9, 10\}$ are similar to ϕ_5. □

According to the above results, we immediately obtain the following theorem.

Theorem 3.1 *Suppose that $\phi(a, b) = \varphi_1(a, b) - \varphi_2(a, b)$ is an NCP function on $\mathbb{R} \times \mathbb{R}$ and k and m are positive integers. Then, $\big[\phi(a, b)\big]^{\frac{k}{m}}$ and $\big[\varphi_1(a, b)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(a, b)]^{\frac{2k+1}{2m+1}}$ are NCP functions.*

Proof Using k and m being positive integers and applying Lemma 2.1, we have

$$
\begin{aligned}
&\big[\phi(a, b)\big]^{\frac{k}{m}} = 0 \\
&\quad\Longleftrightarrow \Big\{\big[\phi(a, b)\big]^{\frac{k}{m}}\Big\}^m = 0 \\
&\quad\Longleftrightarrow \big[\phi(a, b)\big]^k = 0 \\
&\quad\Longleftrightarrow \phi(a, b) = 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\big[\varphi_1(a, b)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(a, b)]^{\frac{2k+1}{2m+1}} = 0 \\
&\quad\Longleftrightarrow \big[\varphi_1(a, b)\big]^{\frac{2k+1}{2m+1}} = [\varphi_2(a, b)]^{\frac{2k+1}{2m+1}} \\
&\quad\Longleftrightarrow \Big\{\big[\varphi_1(a, b)\big]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} = \Big\{[\varphi_2(a, b)]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} \\
&\quad\Longleftrightarrow \big[\varphi_1(a, b)\big]^{2k+1} = \big[\varphi_2(a, b)\big]^{2k+1} \\
&\quad\Longleftrightarrow \varphi_1(a, b) = \varphi_2(a, b) \\
&\quad\Longleftrightarrow \phi(a, b) = 0.
\end{aligned}
$$

The above arguments together with the assumption of $\phi(a, b)$ being an NCP function yield the desired result. □

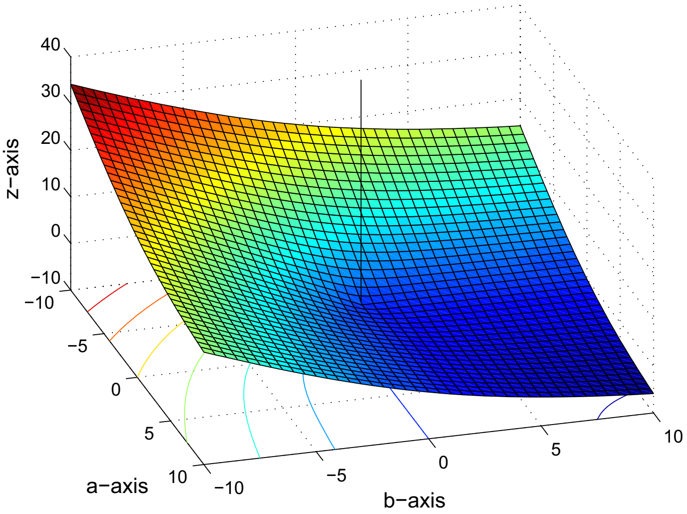


Fig. 1 The surface of $z = \phi_{\text{D-FB}}(a, b)$ and $(a, b) \in [-10, 10] \times [-10, 10]$

Remark 3.2 We elaborate more about Theorem 3.1.

(a) Based on the existing well-known NCP functions, we can construct new NCP functions in light of Theorem 3.1. This is a novel way to construct new NCP functions.

(b) When k is a positive integer, $\left[\phi(a, b)\right]^k$ is an NCP function. This means that perturbing the parameter k gives new NCP functions. In addition, if $\phi(a, b)$ is an NCP function, for any positive integer m, $\left[\phi(a, b)\right]^{\frac{k}{m}}$ is also an NCP function. Thus, we can determine suitable and nice NCP functions among these functions according to their numerical performance.

To close this section, we depict the surfaces of $\phi^p_{\text{D-FB}}$ with different values of p so that we may have deeper insight for this new family of NCP functions. Figure 1 shows the surface if $\phi_{\text{D-FB}}(a, b)$ from which we see that it is convex. Figure 2 presents the surface of $\phi^3_{\text{D-FB}}(a, b)$ in which we see that it is neither convex nor concave as mentioned in Remark 3.1(c). In addition, the value of $\phi^p_{\text{D-FB}}(a, b)$ is negative only when $a > 0$ and $b > 0$ as mentioned in Lemma 3.1. The surfaces of $\phi^p_{\text{D-FB}}$ with various values of p are shown in Fig. 3.

4 Extending $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} to SOCCP

In this section, we extend the new function $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} to SOC setting. More specifically, we show that the function $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} are complementarity functions associated with $\mathcal{K}^n$. In addition, we present the computing formulas for its Jacobian.

Proposition 4.1 *Let $\phi^p_{\text{D-FB}}$ be defined by (5). Then, $\phi^p_{\text{D-FB}}$ is a complementarity function associated with $\mathcal{K}^n$, i.e., it satisfies*

$$\phi^p_{\text{D-FB}}(x, y) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{K}^n, \; y \in \mathcal{K}^n, \; \langle x, y \rangle = 0.$$

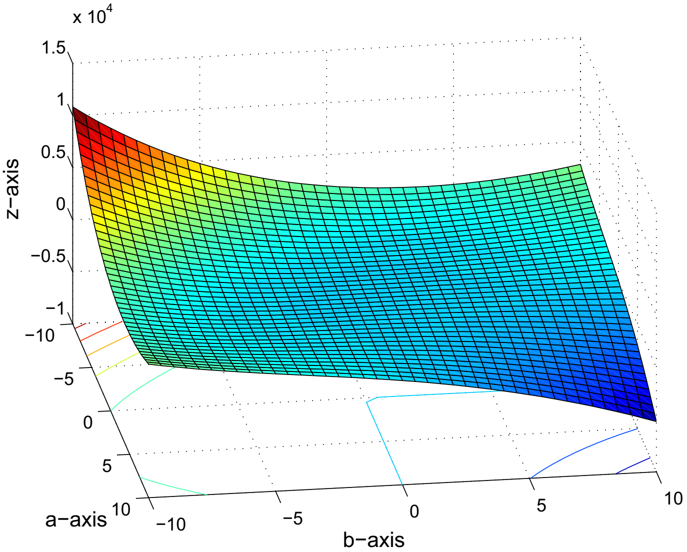
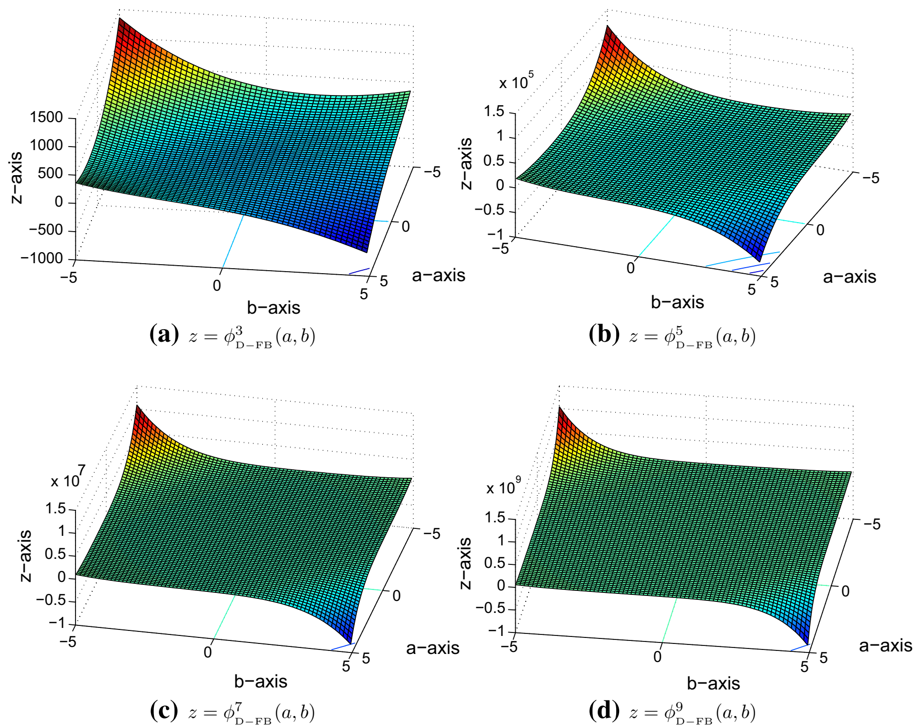


Fig. 2 The surface of $z = \phi^3_{\mathrm{D-FB}}(a, b)$ and $(a, b) \in [-10, 10] \times [-10, 10]$



(a) $z = \phi^3_{\mathrm{D-FB}}(a, b)$ **(b)** $z = \phi^5_{\mathrm{D-FB}}(a, b)$

(c) $z = \phi^7_{\mathrm{D-FB}}(a, b)$ **(d)** $z = \phi^9_{\mathrm{D-FB}}(a, b)$

Fig. 3 The surface of $z = \phi^p_{\mathrm{D-FB}}(a, b)$ with different values of p.

Proof Since $\phi^p_{\text{D-FB}}(x, y) = 0$, we have $\left(\sqrt{x^2 + y^2}\right)^p = (x + y)^p$. Using p being a positive odd integer and applying Lemma 2.2 yield

$$\left(\sqrt{x^2 + y^2}\right)^p = (x + y)^p \quad \Longleftrightarrow \quad \sqrt{x^2 + y^2} = x + y.$$

It is known that $\phi_{\text{FB}}(x, y) := \sqrt{x^2 + y^2} - (x + y)$ is a complementarity function associated with $\mathcal{K}^n$. This indicates that $\phi^p_{\text{D-FB}}$ is a complementarity function associated with $\mathcal{K}^n$. $\square$

With similar technique, we can prove that ϕ^p_{NR} can be extended as a complementarity function for SOCCP.

Proposition 4.2 *The function* $\phi^p_{\text{NR}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ *defined by*

$$\phi^p_{\text{NR}}(x, y) = x^p - [(x - y)_+]^p \tag{7}$$

is a complementarity function associated with $\mathcal{K}^n$*, where* $p > 1$ *is a positive odd integer and* $(\cdot)_+$ *means the projection onto* $\mathcal{K}^n$*.*

Proof From Lemma 2.2, we see that $\phi^p_{\text{NR}}(x, y) = 0$ if and only if $x = (x - y)_+$. On the other hand, it is known that $\phi_{\text{NR}}(x, y) = x - (x - y)_+$ is a complementarity function for SOCCP, which implies $x - (x - y)_+ = 0$ if and only if $x \in \mathcal{K}^n$, $y \in \mathcal{K}^n$, and $\langle x, y\rangle = 0$. Hence, ϕ^p_{NR} is a complementarity function associated with $\mathcal{K}^n$. $\square$

To compute the Jacobian of $\phi^p_{\text{D-FB}}$, we need to introduce some notations for convenience. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define

$$w(x, y) := x^2 + y^2 = (w_1(x, y), w_2(x, y)) \in \mathbb{R} \times \mathbb{R}^{n-1} \;\text{ and }\; v(x, y) := x + y.$$

Then, it is clear that $w(x, y) \in \mathcal{K}^n$ and $\lambda_i(w) \geq 0$, $i = 1, 2$.

Proposition 4.3 *Let* $\phi^p_{\text{D-FB}}$ *be defined as in* (5) *and* $g^{\text{soc}}(x) = (\sqrt{|x|})^p$, $h^{\text{soc}}(x) = x^p$ *are the vector-valued functions corresponding to* $g(t) = |t|^{\frac{p}{2}}$ *and* $h(t) = t^p$ *for* $t \in \mathbb{R}$*, respectively. Then,* $\phi^p_{\text{D-FB}}$ *is continuously differentiable at any* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$*. Moreover, we have*

$$\begin{aligned}\nabla_x \phi^p_{\text{D-FB}}(x, y) &= 2L_x \nabla g^{\text{soc}}(w) - \nabla h^{\text{soc}}(v),\\ \nabla_y \phi^p_{\text{D-FB}}(x, y) &= 2L_y \nabla g^{\text{soc}}(w) - \nabla h^{\text{soc}}(v),\end{aligned}$$

where $w := w(x, y) = x^2 + y^2$, $v := v(x, y) = x + y$, $t \mapsto \operatorname{sign}(t)$ *is the sign function, and*

$$\begin{aligned}
\nabla g^{\text{soc}}(w) &= \begin{cases} \dfrac{p}{2}|w_1|^{\frac{p}{2}-1} \cdot \operatorname{sign}(w_1) I & \text{if } \;w_2 = 0;\\ \begin{bmatrix} b_1(w) & c_1(w)\bar{w}_2^T \\ c_1(w)\bar{w}_2 & a_1(w)I + (b_1(w) - a_1(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if } \;w_2 \neq 0;\end{cases}\\
\bar{w}_2 &= \frac{w_2}{\|w_2\|},\\
a_1(w) &= \frac{|\lambda_2(w)|^{\frac{p}{2}} - |\lambda_1(w)|^{\frac{p}{2}}}{\lambda_2(w) - \lambda_1(w)},\\
b_1(w) &= \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} + |\lambda_1(w)|^{\frac{p}{2}-1}\right],\\
c_1(w) &= \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} - |\lambda_1(w)|^{\frac{p}{2}-1}\right],
\end{aligned}$$

and

$$\nabla h^{\rm soc}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_2(v) & c_2(v)\bar{v}_2^T \\ c_2(v)\bar{v}_2 & a_2(v)I + (b_2(v) - a_2(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases} \tag{8}$$

$$\bar{v}_2 = \frac{v_2}{\|v_2\|}, \tag{9}$$

$$a_2(v) = \frac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, \tag{10}$$

$$b_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], \tag{11}$$

$$c_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right], \tag{12}$$

Proof From the definition of $\phi^p_{\rm D-FB}$, it is clear to see that for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \phi^p_{\rm D-FB}(x, y) &= \left(\sqrt{x^2 + y^2}\right)^p - (x + y)^p \\ &= \left(\sqrt{|x^2 + y^2|}\right)^p - (x + y)^p \\ &= \left[|\lambda_1(w)|^{\frac{p}{2}} u^{(1)}(w) + |\lambda_2(w)|^{\frac{p}{2}} u^{(2)}(w)\right] \\ &\quad - \left[(\lambda_1(v))^p u^{(1)}(v) + (\lambda_2(v))^p u^{(2)}(v)\right] \\ &= g^{\rm soc}(w) - h^{\rm soc}(v). \end{aligned} \tag{13}$$

For $p \geq 3$, since both $|t|^{\frac{p}{2}}$ and t^p are continuously differentiable on $\mathbb{R}$, by [Chen et al. (2004), Proposition 5] and [Fukushima et al. (2002), Proposition 5.2], we know that the function $g^{\rm soc}$ and $h^{\rm soc}$ are continuously differentiable on $\mathbb{R}^n$. Moreover, it is clear that $w(x, y) = x^2 + y^2$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$, then we conclude that $\phi^p_{\rm D-FB}$ is continuously differentiable. Moreover, from the formula in [Chen et al. (2004), Proposition 4] and [Fukushima et al. (2002), Proposition 5.2], we have

$$\nabla g^{\rm soc}(w) = \begin{cases} \frac{p}{2}|w_1|^{\frac{p}{2}-1} \cdot \operatorname{sign}(w_1) I & \text{if} \quad w_2 = 0; \\ \begin{bmatrix} b_1(w) & c_1(w)\bar{w}_2^T \\ c_1(w)\bar{w}_2 & a_1(w)I + (b_1(w) - a_1(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if} \quad w_2 \neq 0; \end{cases}$$

$$\nabla h^{\rm soc}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_2(v) & c_2(v)\bar{v}_2^T \\ c_2(v)\bar{v}_2 & a_2(v)I + (b_2(v) - a_2(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases}$$

where

$$\begin{aligned} \bar{w}_2 &= \tfrac{w_2}{\|w_2\|}, & \bar{v}_2 &= \tfrac{v_2}{\|v_2\|} \\ a_1(w) &= \tfrac{|\lambda_2(w)|^{\frac{p}{2}} - |\lambda_1(w)|^{\frac{p}{2}}}{\lambda_2(w) - \lambda_1(w)}, & a_2(v) &= \tfrac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, \\ b_1(w) &= \tfrac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} + |\lambda_1(w)|^{\frac{p}{2}-1}\right], & b_2(v) &= \tfrac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], \\ c_1(w) &= \tfrac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} - |\lambda_1(w)|^{\frac{p}{2}-1}\right], & c_2(v) &= \tfrac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right]. \end{aligned}$$

By taking differentiation on both sides about x and y for (13), respectively, and applying the chain rule for differentiation, it follows that

$$\nabla_x \phi^p_{\mathrm{D-FB}}(x, y) = 2L_x \nabla g^{\mathrm{soc}}(w) - \nabla h^{\mathrm{soc}}(v),$$
$$\nabla_y \phi^p_{\mathrm{D-FB}}(x, y) = 2L_y \nabla g^{\mathrm{soc}}(w) - \nabla h^{\mathrm{soc}}(v).$$

Hence, we complete the proof. □

With Lemma 2.2 and Proposition 4.1, we can construct more complementarity functions for SOCCP which are variants of $\phi^p_{\mathrm{D-FB}}(x, y)$. More specifically, consider that k and m are positive integers and $f^{\mathrm{soc}}(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector-valued function corresponding to a given real-valued function f, the following functions are new variants of $\phi^p_{\mathrm{D-FB}}(x, y)$.

$$\widetilde{\phi_1}(x, y) = \left[\sqrt{x^2 + y^2} + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}} - \left[x + y + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}}.$$

$$\widetilde{\phi_2}(x, y) = \left[\sqrt{x^2 + y^2} - x - y\right]^{\frac{k}{m}}.$$

$$\widetilde{\phi_3}(x, y) = \left[\sqrt{x^2 + y^2} - x + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}} - \left[y + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}}.$$

$$\widetilde{\phi_4}(x, y) = \left[\sqrt{x^2 + y^2} - y + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}} - \left[x + f^{\mathrm{soc}}(x, y)\right]^{\frac{2k+1}{2m+1}}.$$

Proposition 4.4 *All the above functions $\widetilde{\phi_i}$ for $i \in \{1, 2, 3, 4\}$ are complementarity functions associated with $\mathcal{K}^n$.*

Proof The results follow from applying Lemma 2.2 and Proposition 4.1. □

In general, for complementarity functions associated with $\mathcal{K}^n$, we have the following parallel result to Theorem 3.1.

Theorem 4.1 *Suppose that $\phi(x, y) = \varphi_1(x, y) - \varphi_2(x, y)$ is a complementarity function associated with $\mathcal{K}^n$ on $\mathbb{R}^n \times \mathbb{R}^n$, and k, m are positive integers. Then, $\left[\phi(x, y)\right]^{\frac{k}{m}}$ and $\left[\varphi_1(x, y)\right]^{\frac{2k+1}{2m+1}} - [\varphi_2(x, y)]^{\frac{2k+1}{2m+1}}$ are complementarity functions associated with $\mathcal{K}^n$.*

Proof According to k and m are positive integers and using Lemma 2.2, we have

$$\begin{aligned}
&\left[\phi(x, y)\right]^{\frac{k}{m}} = 0 \\
&\Longleftrightarrow \left\{\left[\phi(x, y)\right]^{\frac{k}{m}}\right\}^m = 0 \\
&\Longleftrightarrow \left[\phi(x, y)\right]^k = 0 \\
&\Longleftrightarrow \phi(x, y) = 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\left[\varphi_1(x, y)\right]^{\frac{2k+1}{2m+1}} - [\varphi_2(x, y)]^{\frac{2k+1}{2m+1}} = 0 \\
&\Longleftrightarrow \left[\varphi_1(x, y)\right]^{\frac{2k+1}{2m+1}} = [\varphi_2(x, y)]^{\frac{2k+1}{2m+1}}
\end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow \left\{\left[\varphi_1(x,y)\right]^{\frac{2k+1}{2m+1}}\right\}^{2m+1} = \left\{[\varphi_2(x,y)]^{\frac{2k+1}{2m+1}}\right\}^{2m+1} \\
&\Longleftrightarrow \left[\varphi_1(x,y)\right]^{2k+1} = \left[\varphi_2(x,y)\right]^{2k+1} \\
&\Longleftrightarrow \varphi_1(x,y) = \varphi_2(x,y) \\
&\Longleftrightarrow \phi(x,y) = 0.
\end{aligned}
$$

From the above arguments and the assumption, the proof is complete. □

Remark 4.1 We elaborate more about Theorem 4.1.

(a) Based on the existing complementarity functions, we can construct new complementarity functions associated with $\mathcal{K}^n$ in light of Theorem 4.1.
(b) When k is a positive odd integer, $\phi(x,y)^k$ is a complementarity function associated with $\mathcal{K}^n$. This means that perturbing the odd integer parameter k, we obtain the new complementarity functions associated with $\mathcal{K}^n$. In addition, if $\phi(x,y)$ is a complementarity function, then for any positive integer m, $\left[\phi(x,y)\right]^{\frac{k}{m}}$ is also a complementarity function. We can determine nice complementarity functions associated with $\mathcal{K}^n$ among these functions by their numerical performance.

Finally, we establish formula for Jacobian of ϕ^p_{NR} and the smoothness of ϕ^p_{NR}. To this aim, we need the following technical lemma.

Lemma 4.1 *Let $p > 1$. Then, the real-valued function $f(t) = (t_+)^p$ is continuously differentiable with $f'(t) = p(t_+)^{p-1}$ where $t_+ = \max\{0, t\}$.*

Proof By the definition of t_+, we have

$$
f(t) = (t_+)^p = \begin{cases} t^p & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0, \end{cases}
$$

which implies

$$
f'(t) = \begin{cases} pt^{p-1} & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0. \end{cases}
$$

Then, it is easy to see that $f'(t) = p(t_+)^{p-1}$ is continuous for $p > 1$. □

Proposition 4.5 *Let ϕ^p_{NR} be defined as in (7) and $h^{\mathrm{soc}}(x) = x^p$, $l^{\mathrm{soc}}(x) = (x_+)^p$ be the vector-valued functions corresponding to the real-valued functions $h(t) = t^p$ and $l(t) = (t_+)^p$, respectively. Then, ϕ^p_{NR} is continuously differentiable at any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, and its Jacobian is given by*

$$
\begin{aligned}
\nabla_x \phi^p_{\mathrm{NR}}(x,y) &= \nabla h^{\mathrm{soc}}(x) - \nabla l^{\mathrm{soc}}(x-y), \\
\nabla_y \phi^p_{\mathrm{NR}}(x,y) &= \nabla l^{\mathrm{soc}}(x-y),
\end{aligned}
$$

where ∇h^{soc} satisfies (8)–(12) and

$$
\begin{aligned}
\nabla l^{\mathrm{soc}}(u) &= \begin{cases} p((u_1)_+)^{p-1} I & \text{if} \quad u_2 = 0; \\ \begin{bmatrix} b_3(u) & c_3(u)\bar{u}_2^T \\ c_3(u)\bar{u}_2 & a_3(u)I + (b_3(u) - a_3(u))\,\bar{u}_2\bar{u}_2^T \end{bmatrix} & \text{if} \quad u_2 \neq 0; \end{cases} \\
\bar{u}_2 &= \frac{u_2}{\|u_2\|},
\end{aligned}
$$

$$
\begin{aligned}
a_3(u) &= \frac{(\lambda_2(u)_+)^p - (\lambda_1(u)_+)^p}{\lambda_2(u) - \lambda_1(u)}, \\
b_3(u) &= \frac{p}{2}\left[(\lambda_2(u)_+)^{p-1} + (\lambda_1(u)_+)^{p-1}\right], \\
c_3(u) &= \frac{p}{2}\left[(\lambda_2(u)_+)^{p-1} - (\lambda_1(u)_+)^{p-1}\right],
\end{aligned}
$$

Proof In light of [Chen et al. 2004, Proposition 5] and [Fukushima et al. 2002, Proposition 5.2], the results follow from applying Lemma 4.1 and using the chain rule for differentiation. □

5 Numerical experiments

As mentioned, the Newton method may not be appropriate for numerical implementation, due to possible singularity of Jacobian at a degenerate solution. In view of this, in this section, we employ the derivative-free descent method studied in Pan and Chen (2010) to test the numerical performance based on various value of p. The target of the derivative-free descent method studied in Pan and Chen (2010) is mainly on SOCCP. Hence, we consider the following SOCCP:

$$
z \in \mathcal{K}, \quad Mz + b \in \mathcal{K}, \quad z^T(Mz + b) = 0, \\
\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_r.
$$

According to our results, the above SOCCP can be recast as an unconstrained minimization problem:

$$
\min_{\zeta \in \mathbb{R}^n} \Psi_p(\zeta) = \frac{1}{2}\|\phi^p_{\text{D-FB}}(\zeta, F(\zeta))\|^2,
$$

where $F(\zeta) = M\zeta + b$.

All tests are done on a PC using Inter core i7-5600U with 2.6 GHz and 8 GB RAM, and the codes are written in Matlab 2010b. The test instances are generated randomly. In particular, we first generate random sparse square matrices $N_i (i = 1, 2 \ldots r)$ with density 0.01, in which non-zero elements are chosen randomly from a normal distribution with mean -1 and variance 4. Then, we create the positive semidefinite matrix M_i for $(i = 1, 2 \ldots r)$ by setting $M_i := N_i N_i^T$ and let $M := \text{diag}(M_1, \ldots, M_r)$. In addition, we take vector $b := -Mw$ with $w = (w_1, \ldots, w_r)$ and $w_i \in \mathcal{K}_i$. With these M and b, it is not hard to verify that the corresponding SOCCP has at least a feasible solution. To construct SOCs of various types, we set $n_1 = n_2 = \cdots = n_r$.

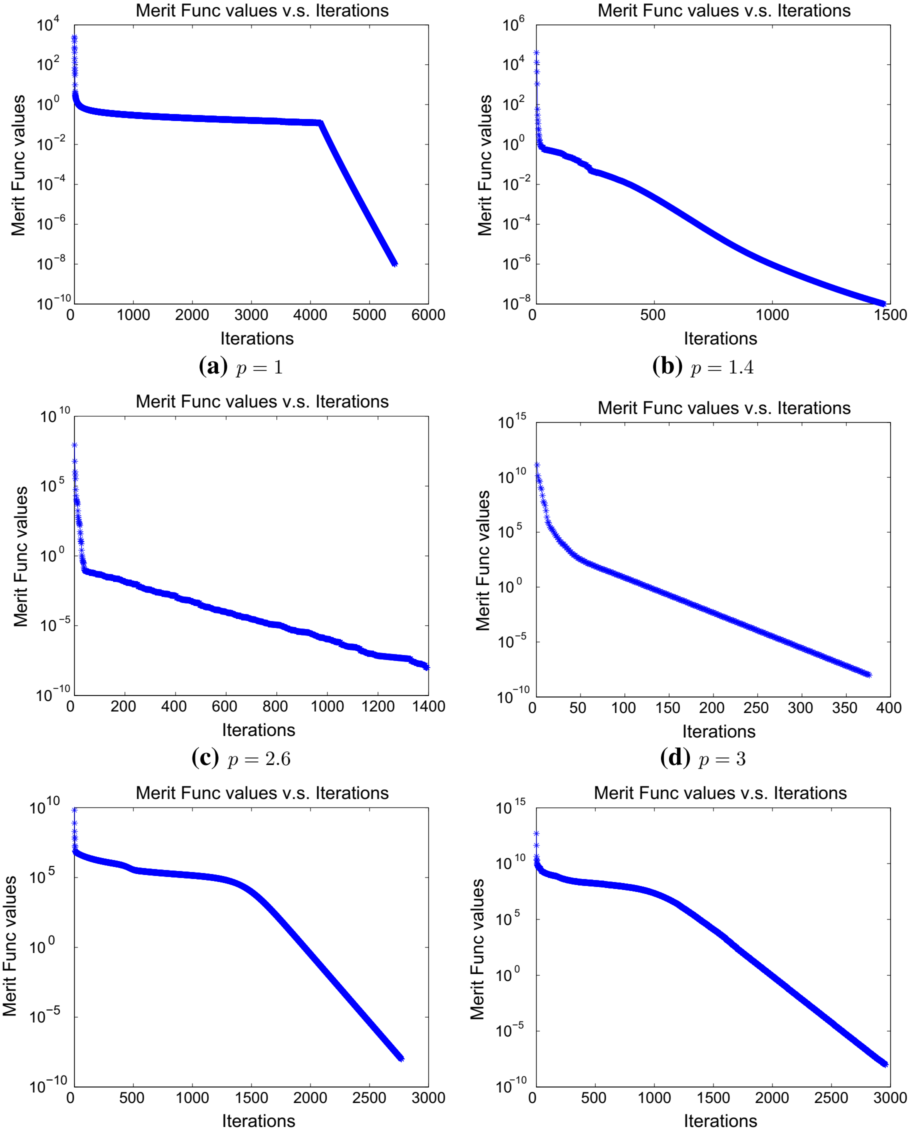
We implement a test problem generated as above with $n = 1000$ and $r = 100$. The parameters in the algorithm are set as:

$$
\beta = 0.9, \quad \gamma = 0.8, \quad \sigma = 10^{-4}, \text{ and } \epsilon = 10^{-8}.
$$

We start with the initial point

$$
\zeta_0 = (\zeta_{n_1}, \cdots, \zeta_{n_r}) \text{ where } \zeta_{n_i} = \left(10, \frac{w_i}{\|w_i\|}\right)
$$

with $w_i \in \mathbb{R}^{n_i - 1}$ being generated randomly. The stopping criteria, i.e., $\Psi_p(\zeta^k) \leq \epsilon$, are either the number of iteration is over 10^5 or a step-length is less than 10^{-12}. Figure 4 depicts the detailed iteration process of the algorithm corresponding to different value of p.



(a) $p = 1$ **(b)** $p = 1.4$

(c) $p = 2.6$ **(d)** $p = 3$

(e) $p = 3.4$ **(f)** $p = 5$

Fig. 4 Convergence behavior of $\Phi_p(\zeta^k)$ with different value of p

The algorithm fails for the problem when $p \geq 5$. The main reason is that the step-length is too small eventually. We also suspect that larger p leads to tedious computation of the complementarity function in Jordan algebra. Anyway, this phenomenon indicates that the discrete-type of complementarity functions only work well for small value of p. The convergence in Fig. 4 shows the method with a bigger p has a faster reduction of Ψ_p at the beginning, and the method with a smaller p has a faster reduction of Ψ_p eventually. Moreover, the bigger p applies, the total number of iterations of the algorithm is less.

In order to check numerical performance of the algorithm corresponding to different value of p, we solve the test problems with different dimension. The numerical results are summarized in Tables 1, 2. "$\Psi_p(\zeta^*)$" and "Gap" denote the merit function value and the value of $\left|\zeta^T F(\zeta)\right|$ at the final iteration, respectively. "NF", "Iter", and "Time" indicate the number of function evaluations of Ψ_p, the number of iteration required in order to satisfy the termination condition, and the CPU time in second for solving each problem, respectively.

We also use the performance profiles introduced by Dolan and Morè (2002) to compare the performance of algorithm with different p. The performance profiles are generated by executing solvers $\mathcal{S}$ on the test set $\mathcal{P}$. Let $n_{p,s}$ be the number of iteration (or the computing time) required to solve problem $p \in \mathcal{P}$ by solver $s \in \mathcal{S}$, and define the performance ratio as:

$$r_{p,s} = \frac{n_{p,s}}{\min\{n_{p,s} : 1 \leq s \leq n_s\}},$$

where n_s is the number of solvers. Whenever the solver s does not solve problem p successfully, set $r_{p,s} = r_M$. Here, r_M is a very large preset positive constant. Then, performance profile for each solver s is defined by

$$\rho_s(\chi) = \frac{1}{n_p}\text{size}\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}.$$

where size$\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}$ is the number of elements in the set $\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}$. $\rho_s(\chi)$ represents the probability that the performance ratio $r_{p,s}$ is within the factor 2^χ. It is easy to see that $\rho_s(0)$ is the probability that the solver s wins over the rest of solvers. See Dolan and Morè (2002) for more details about the performance profile.

From Fig. 5a, it shows that the algorithm with $p = 1$ and $p = 1.4$ performs better than $p = 2.6$ and $p = 3$ on function evaluations. Similarly, from Fig. 5b, c, we observe that the algorithm with $p = 3$ performs best on the number of iterations, while the algorithm with $p = 1.4$ is the best one on CPU time. This provides evidence that the discrete type of complementarity function may be better than the well-known function ϕ_{FB} in some cases.

6 Conclusion

In this paper, we propose a few families of new NCP functions and investigate their differentiability. Then, these new families of NCP functions have also shown that they can serve as complementarity functions associated with second-order cone in light of Jordan algebra. We also construct several variants of such complementarity functions for NCP and SOCCP. The behind idea for constructing all such new complementarity functions is based on "discrete generalization" which is a novel thinking. In contrast to the traditional "continuous generalization", this opens a new direction for future research.

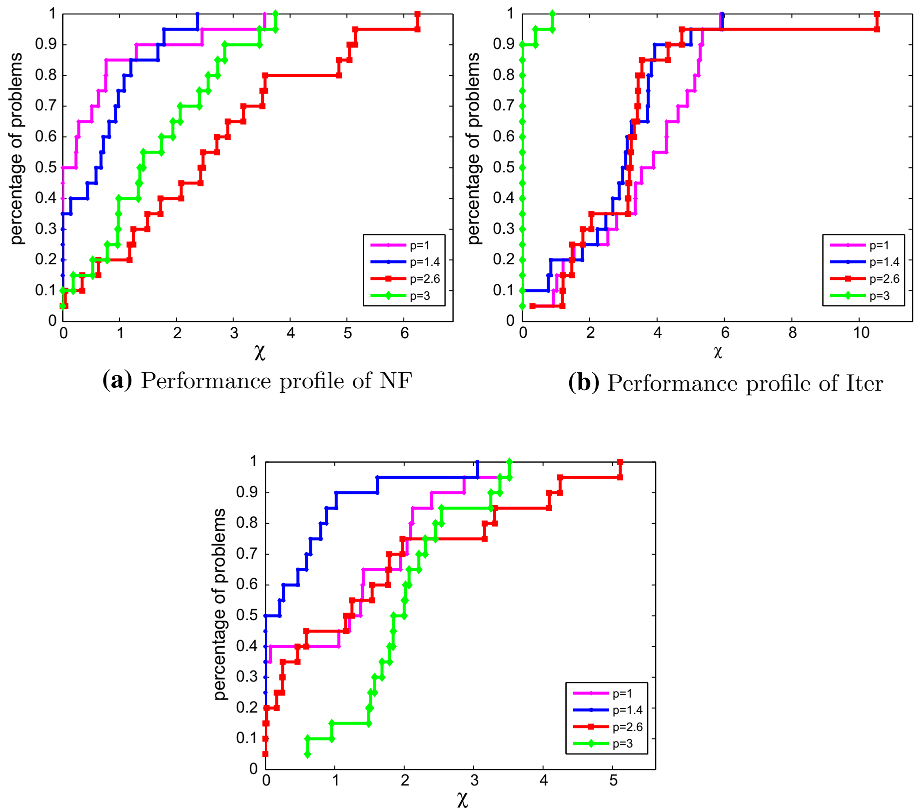
As below, we explain why we adopt "discrete-type" for our new NCP functions. First, for the generalized Natural-Residual function $\phi^p_{\text{NR}}(a, b) = a^p - (a - b)^p_+$, as remarked in Chen et al. (2016), the parameter p must be odd integer to ensure that the generalization is also an NCP function. This means that the main idea to create the new functions relies on "discrete generalization", it is totally different from the concept of generalization of Fischer–Burmeister function $\phi^p_{\text{FB}}(a, b) = \sqrt[p]{|a|^p + |b|^p} - (a + b)$, as remarked in Chen (2007), the parameter p may be any real number which is great or equal to one. That is why we call our generalization "discrete-type".

Table 1 Numerical results with different value of p

Problem	$p=1$					$p=1.4$				
(n,r)	$\Phi_p(\zeta^*)$	NF	$Iter$	Gap	$time$	$\Phi_p(\zeta^*)$	NF	$Iter$	Gap	$time$
(100,10)	9.8e-9	5350	4952	2.75e-4	9.3	1.0e-8	4401	1474	5.92e-5	3.5
(200,20)	9.4e-9	5064	4914	3.74e-5	16.5	1.0e-8	16,179	5649	3.84e-5	25.9
(300,30)	1.0e-8	7445	5273	2.26e-4	30.3	9.9e-9	7000	1266	2.40e-5	11.5
(400,40)	9.8e-9	5342	5016	1.62e-4	50.0	9.9e-9	3747	857	4.31e-5	9.5
(500,50)	1.0e-8	23,533	13,749	6.81e-4	126.4	9.6e-9	29,454	6257	3.39e-4	93.9
(600,60)	1.0e-8	18,260	11,119	16.1e-4	65.1	1.0e-8	24,685	8320	8.69e-5	119.7
(700,70)	1.0e-8	8320	5690	6.16e-4	38.3	1.0e-8	13,458	4493	1.79e-4	77.7
(800,80)	1.0e-8	29,415	10,149	4.43e-5	199.2	9.3e-9	2507	1838	1.54e-4	27.4
(900,90)	1.0e-8	14,648	10,888	1.46e-3	159.8	9.9e-9	5970	1621	8.77e-5	44.9
(1000,100)	1.0e-8	14,590	9672	2.78e-4	238.3	1.0e-8	12, 337	2570	7.58e-5	92.0
(1100,110)	9.9e-9	5994	5406	4.64e-6	109.6	1.0e-8	13,767	2948	3.51e-4	126.5
(1200,120)	9.8e-9	6100	5528	6.12e-5	121.7	9.9e-9	20,990	5650	1.51e-5	211.4
(1300,130)	9.8e-9	4253	3612	2.42e-4	115.5	9.7e-9	777	316	5.78e-5	10.1
(1400,140)	1.0e-8	9827	7136	1.46e-4	307.5	1.0e-8	6357	2736	2.20e-4	70.6
(1500,150)	9.9e-9	4701	4211	3.04e-4	156.9	9.9e-9	7060	1823	6.56e-6	67.8
(1600,160)	9.9e-9	5744	3843	4.61e-4	172.8	1.0e-8	9434	2583	1.39e-4	82.9
(1700,170)	1.0e-8	11,163	5581	2.74e-4	195.1	1.0e-8	12,307	2740	9.87e-5	185.7
(1800,180)	1.0e-8	7449	5985	3.77e-4	204.5	1.0e-8	38,524	9469	2.43e-4	439.8
(1900,190)	1.0e-8	4205	2102	7.19e-5	83.2	1.0e-8	7413	1636	3.40e-4	125.4
(2000,200)	9.9e-9	5189	4953	2.12e-4	212.9	9.15e-9	10,230	480	2.32e-5	294.9

Table 2 Numerical results with different value of p

Problem	$p = 2.6$					$p = 3$				
(n, r)	$\Phi_p(\zeta^*)$	NF	$Iter$	Gap	$time$	$\Phi_p(\zeta^*)$	NF	$Iter$	Gap	$time$
(100,10)	9.9e-9	28,878	1866	2.40e-6	11.9	9.2e-9	11,281	201	3.80e-7	14.7
(200,20)	1.0e-8	57,844	3743	1.64e-6	47.9	9.5e-9	21,221	422	1.15e-6	52.9
(300,30)	9.9e-9	14,452	963	3.14e-6	17.3	9.2e-9	4383	89	5.97e-7	17.5
(400,40)	9.8e-9	20,747	1417	2.31e-6	32.7	9.9e-9	7419	133	8.34e-7	34.0
(500,50)	9.8e-9	13,929	1084	1.53e-6	30.7	8.4e-9	27,229	474	1.04e-6	87.8
(600,60)	9.9e-9	28,224	2032	2.48e-7	77.1	9.9e-9	48,809	878	4.19e-7	193.8
(700,70)	9.9e-9	16,739	1230	1.93e-5	52.8	7.9e-9	7069	140	6.16e-4	58.4
(800,80)	9.9e-9	72,745	5342	7.69e-7	270.5	9.8e-9	27,620	534	5.95e-7	260.1
(900,90)	9.5e-9	7574	522	6.09e-7	37.5	8.0e-9	10, 276	187	1.35e-7	129.6
(1000,100)	1.0e-8	14,5414	8664	4.92e-7	821.6	9.6e-9	17,790	325	2.26e-7	258.2
(1100,110)	9.7e-9	16,834	1465	3.76e-7	111.0	9.5e-9	31,750	528	6.41e-7	507.2
(1200,120)	9.9e-9	45,621	3346	1.82e-6	271.5	9.8e-9	20,326	370	4.82e-7	437.4
(1300,130)	1.0e-8	25,661	1739	3.21e-6	171.8	8.9e-9	10,399	185	7.16e-7	115.5
(1400,140)	9.8e-9	57,526	4116	2.09e-5	277.6	8.9e-9	12,529	205	1.09e-6	348.4
(1500,150)	1.0e-8	355,478	321,117	1.50e-5	2343.0	4.7e-3	11,824	217	1.54e-5	393.5
(1600,160)	9.3e-9	12,995	5961	1.70e-6	98.5	9.9e-9	33,843	550	5.43e-7	862.2
(1700,170)	1.0e-8	47,367	3380	8.64e-7	441.0	1.0e-8	80,519	5084	1.73e-7	742.8
(1800,180)	9.8e-9	7697	536	1.67e-6	53.0	7.4e-9	8472	154	4.15e-8	289.6
(1900,190)	1.0e-8	149,019	10,644	2.59e-6	1577.9	1.0e-8	16,128	909	5.84e-7	161.5
(2000,200)	1.0e-8	27,876	1991	2.64e-6	238.5	1.0e-8	34,310	630	1.37e-7	862.2



(a) Performance profile of NF

(b) Performance profile of Iter

(c) Performance profile of CPU time

Fig. 5 Performance profiles with different value of p

In fact, there is another way to achieve $\phi^p_{\mathrm{D-FB}}$ and ϕ^p_{NR} which was proposed in Galántai (2012). More specifically, it is a construction based on monotone transformations to create new NCP functions from the existing ones. The construction is stated as below.

Remark 6.1 (Galántai 2012, Lemma 15) Assume that ϕ is continuous and $\phi(a, b) = f_1(a, b) - f_2(a, b)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a strictly monotone increasing and continuous function. Then, ϕ is an NCP function if and only if $\psi_\theta(a, b) = \theta(f_1(a, b)) - \theta(f_2(a, b))$ is an NCP function.

In light of this, we let the function $\theta = \theta_p$ be $\theta_p(t) = \mathrm{sign}(t)|t|^p$, where "sign$(t)$" is the sign function and $p \geq 1$. For Fischer–Burmeister function, we choose $f_1(a, b) = \sqrt{a^2 + b^2}$, $f_2(a, b) = a + b$, and for Natural-Residual function, we choose $f_1(a, b) = a$, $f_2(a, b) = (a - b)_+$, then it can be verified that both $\phi^p_{\mathrm{D-FB}}$ and ϕ^p_{NR} (only with odd integer p) can be obtained from the function ψ_{θ_p}. In other words, the function ψ_{θ_p} includes both of them as special cases, from which we may view it as a "continuous generalization". Yes, the Galantai's method Galántai (2012) is more general than ours. Nonetheless, we emphasize that the NCP functions generated by our approach are shown to be complementarity functions in the SOCCP setting. This can be used to generate new SOCCP functions, which is one of the main contributions of this paper. It will be a future direction to check whether Galantai's NCP functions can be extended to SOCCP setting as well and describe the relation therein.

In general, the Newton method may not be applicable even though we have the differentiability for some new complementarity functions because the Jacobian at a degenerate solution is singular see (Kanzow 1996; Kanzow and Kleinmichel 1995). Nonetheless, some derivative-free algorithm may be employed due to the differentiability. On the other hand, we can reformulate NCP and SOCCP as nonsmooth equations or unconstrained minimization, for which merit function approach, nonsmooth function approach, smoothing function approach, and regularization approach can be studied. All the new complementarity functions can be employed in these approaches. How these new families of complementarity functions perform in contrast to the existing ones? This is the first question that we are eager to know. Some other questions, like are there any benefits for "discrete generalization" compared to "continuous generalization", can these proposed complementarity functions be employed for other types of problems including semi-definite complementarity problems and symmetric cone complementarity problems, etc.? We leave them as future research topics.

References

Baggett L (2012) Analysis of Functions of a Single Variable, University of Colorado, http://cnx.org/content/col11249/1.1/. Accessed 10 Sept 2017

Bertsekas DP (1999) Nonlinear programming, 2nd edn. Athena Scientific, Belmont

Chang Y-L, Chen J-S, Yang C-Y (2015) Symmetrization of generalized natural residual function for NCP. Oper Res Lett 43:354–358

Chen J-S, Tseng P (2005) An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. Mathe Program 104:293–327

Chen J-S (2006) The semismooth-related properties of a merit function and a descent method for the nonlinear complementarity problem. J Glob Optim 36:565–580

Chen J-S (2006) Two classes of merit functions for the second-order cone complementarity problem. Mathe Methods Oper Res 64:495–519

Chen J-S (2007) On some NCP-functions based on the generalized Fischer-Burmeister function. Asia Pac J Oper Res 24:401–420

Chen J-S, Gao H-T, Pan S (2009) A R-linearly convergent derivative-free algorithm for the NCPs based on the generalized Fischer-Burmeister merit function. J Comput Appl Math 232:455–471

Chen J-S, Huang Z-H, She C-Y (2011) A new class of penalized NCP-functions and its properties. Comput Optim Appl 50:49–73

Chen J-S, Ko C-H, Pan S-H (2010) A neural network based on generalized Fischer-Burmeister function for nonlinear complementarity problems. Inf Sci 180:697–711

Chen J-S, Pan S-H (2008) A family of NCP-functions and a descent method for the nonlinear complementarity problem. Comput Optim Appl 40:389–404

Chen J-S, Pan S-H (2012) A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs. Pac J Optim 8:33–74

Chen J-S, Chen X, Tseng P (2004) Analysis of nonsmooth vector-valued functions associated with second-order cones. Math Program 101:95–117

Chen J-S, Pan S-H, Lin T-C (2010) A smoothing Newton method based on the generalized Fischer-Burmeister function for MCPs. Nonlinear Anal 72:3739–3758

Chen J-S, Pan S-H, Yang C-Y (2010) Numerical comparison of two effective methods for mixed complementarity problems. J Comput Appl Math 234:667–683

Chen J-S, Ko C-H, Wu X-R (2016) What is the generalization of natural residual function for NCP. Pac J Optim 12:19–27

Cottle RW, Pang J-S, Stone RE (1992) The linear complementarity problem. Academic Press, New York

Dolan ED, Morè JJ (2002) Benchmarking optimization software with performance profiles. Math Program 91:201–213

Faraut J, Korányi A (1994) Analysis on symmetric cones. Oxford University Press, Oxford Mathematical Monographs, New York

Fletcher R (1987) Practical methods of optimization, 2nd edn. Wiley-Interscience, Chichester

Facchinei F, Pang J-S (2003) Finite-dimensional variational inequalities and complementarity problems. Springer Verlag, New York

Facchinei F, Soares J (1997) A new merit function for nonlinear complementarity problems and a related algorithm. SIAM J Optim 7:225–247

Fischer A (1992) A special Newton-type optimization methods. Optimization 24:269–284

Fischer A (1997) Solution of the monotone complementarity problem with locally Lipschitzian functions. Math Program 76:513–532

Fukushima M, Luo Z-Q, Tseng P (2002) Tseng smoothing functions for second-order cone complementarity problems. SIAM J Optim 12:436–460

Galántai A (2012) Properties and construction of NCP functions. Comput Optim Appl 52:805–824

Geiger C, Kanzow C (1996) On the resolution of monotone complementarity problems. Comput Optim Appl 5:155–173

Hayashi S, Yamashita N, Fukushima M (2005) Robust Nash equilibria and second-order cone complementarity problems. J Nonlinear Convex Anal 6:283–296

Harker PT, Pang J-S (1990) Finite dimensional variational inequality and nonlinear complementarity problem: a survey of theory, algorithms and applications. Math Program 48:161–220

Hu S-L, Huang Z-H, Chen J-S (2009) Properties of a family of generalized NCP-functions and a derevative free algotithm for complementarity problems. J Comput Appl Math 230:69–82

Jiang H (1996) Unconstrained minimization approaches to nonlinear complementarity problems. J Glob Optim 9:169–181

Kanzow C (1996) Nonlinear complementarity as unconstrained optimization. J Optim Theory Appl 88:139–155

Kanzow C, Kleinmichel H (1995) A class of Netwton-type methods for equality and ineqality constrained optimization. Optim Methods Softw 5:173–198

Koshy T (2007) Elementary number theory with applications, 2nd edn. Academic Press, Cambridge, London

Kanno Y, Martins AC, da Costa A pinto (2006) Three-dimensional quasi-static frictional contact by using second-order cone linear complementarity problem. Int J Numer Methods Eng 65:62–83

Mangasarian OL (1976) Equivalence of the complementarity problem to a system of nonlinear equations. SIAM J Appl Math 31:89–92

Pan S-H, Chen J-S (2010) A linearly convergent derivative-free descent method for the second-order cone complementarity problem. Optimization 59:1173–1197

Pan S-H, Kum S, Lim Y, Chen J-S (2014) On the generalized Fischer-Burmeister merit function for the second-order cone complementarity problem. Math Comput 83:1143–1171

Pang J-S (1990) Newton's method for B-differentiable equations. Math Oper Res 15:311–341

Pang J-S, Chan D (1982) Iterative methods for variational and complemantarity problems. Math Program 27:284–313

Sun D, Qi L (1999) On NCP-functions. Comput Optim Appl 13:201–220

Tsai H-Y, Chen J-S (2014) Geometric views of the generalized Fischer-Burmeister function and its induced merit function. Appl Math Comput 237:31–59

Yamashita N, Fukushima M (1995) On stationary points of the implict Lagrangian for nonlinear complementarity problems. J Optim Theory Appl 84:653–663

Yamashita N, Fukushima M (1997) Modified Newton methods for solving a semismooth reformulation of monotone complementarity problems. Math Program 76:469–491