Computational and Applied Mathematics, vol. 37, no. 5, pp. 5727-5749, 2018

Discovery of new complementarity functions for NCP and SOCCP

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February 26, 2017
(1st revised on November 18, 2017)
(2nd revised on March 9, 2018)
(3rd revised on May 20, 2018)

[1]E-mail: mathpengfeima@126.com. This research was supported by a grant from the National Natural Science Foundation of China(No.11626212).

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Abstract. It is well known that complementarity functions play an important role in dealing with complementarity problems. In this paper, we propose a few new classes of complementarity functions for nonlinear complementarity problems and second-order cone complementarity problems. The constructions of such new complementarity functions are based on discrete generalization which is a novel idea in contrast to the continuous generalization of Fischer-Burmeister function. Surprisingly, these new families of complementarity functions possess continuous differentiability even though they are discrete-oriented extensions. This feature enables that some methods like derivative-free algorithm can be employed directly for solving nonlinear complementarity problems and second-order cone complementarity problems. This is a new discovery to the literature and we believe that such new complementarity functions can also be used in many other contexts.

Keywords. NCP, SOCCP, natural residual, complementarity function.

1 Introduction

In general, the complementarity problem comes from the KKT conditions of linear and nonlinear programming problems. For different types of optimization problems, there arise various complementarity problems, for example, linear complementarity problem, nonlinear complementarity problem, semidefinite complementarity problem, second-order cone complementarity problem, and symmetric cone complementarity problem. To deal with complementarity problems, the so-called complementarity functions play an important role therein. In this paper, we focus on two classes of complementarity functions, which are used for the nonlinear complementarity problem (NCP) and the second-order cone complementarity problem (SOCCP), respectively.

The first class is the nonlinear complementarity problem (NCP) that has attracted much attention since 1970s because of its wide applications in the fields of economics, engineering, and operations research, see [17, 21, 29] and references therein. In mathematical format, the NCP is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \ldots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. For solving NCP, the so-called NCP-function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as below

$$\phi(a, b) = 0 \quad \Longleftrightarrow \quad a, b \geq 0, \ ab = 0$$

plays a crucial role. Generally speaking, with such NCP-functions, the NCP can be reformulated as nonsmooth equations [36, 39, 44] or unconstrained minimization [22, 23, 27, 31, 32, 40, 43]. Then, different kinds of approaches and algorithms are designed based on the aforementioned reformulations and various NCP-functions. During the past four decades, around thirty NCP-functions are proposed, see [26] for a survey.

The second class is the second-order cone complementarity problem (SOCCP), which can be viewed as a natural extension of NCP and is to seek a $\zeta \in \mathbb{R}^n$ such that

$$\zeta \in \mathcal{K}, \quad F(\zeta) \in \mathcal{K}, \quad \langle \zeta, F(\zeta) \rangle = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a map and $\mathcal{K}$ is the Cartesian product of second-order cones (SOC), also called Lorentz cones [19]. In other words, $\mathcal{K}$ is expressed as

$$\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_m},$$

where $m, n_1, \ldots, n_m \geq 1$, $n_1 + \cdots + n_m = n$, and

$$\mathcal{K}^{n_i} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid \|x_2\| \leq x_1\},$$

with $\|\cdot\|$ denoting the Euclidean norm. The SOCCP has important applications in engineering problems [35] and robust Nash equilibria [28]. Another important special case of SOCCP corresponds to the Karush-Kuhn-Tucker (KKT) optimality conditions for the second-order cone program (SOCP) (see [4] for details):

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \in \mathcal{K}, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ has full row rank, $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. Many solution methods have been proposed for solving SOCCP, see [12] for a survey. For example, merit function approach based on reformulating the SOCCP as an unconstrained smooth minimization problem is studied in [4, 6, 38]. In such approach, it is to find a smooth function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\psi(x, y) = 0 \quad \Longleftrightarrow \quad \langle x, y \rangle = 0, \quad x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n. \tag{1}$$

Then, the SOCCP can be expressed as an unconstrained smooth (global) minimization problem:

$$\min_{\zeta \in \mathbb{R}^n} \ \psi(\zeta, F(\zeta)). \tag{2}$$

In fact, a function ψ satisfying the condition in (1) (not necessarily smooth) is called a complementarity function for SOCCP (or complementarity function associated with $\mathcal{K}^n$). Various gradient methods such as conjugate gradient methods and quasi-Newton methods [2, 20] can be applied for solving (2). In general, for this approach to be effective, the choice of complementarity function ψ is also crucial.

Back to the complementarity functions for NCP, two popular choices of NCP-functions are the well-known Fischer-Burmeister function (FB function, in short) $\phi_{\text{FB}} : \mathbb{R}^2 \to \mathbb{R}$ defined by (see [23, 24])

$$\phi_{\text{FB}}(a, b) = \sqrt{a^2 + b^2} - (a + b),$$

and the squared norm of Fischer-Burmeister function given by

$$\psi_{\text{FB}}(a, b) = \frac{1}{2}\left|\phi_{\text{FB}}(a, b)\right|^2.$$

In addition, the generalized Fischer-Burmeister function $\phi_p : \mathbb{R}^2 \to \mathbb{R}$, which includes the Fischer-Burmeister as a special case, is considered in [5, 7, 8, 11, 30, 42]. In particular, the function ϕ_p is a natural "continuous extension" of ϕ_{FB}, in which the 2-norm in $\phi_{\text{FB}}(a, b)$ is replaced by general p-norm. In other words, $\phi_p : \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$\phi_p(a, b) = \|(a, b)\|_p - (a + b), \quad p > 1 \tag{3}$$

and its geometric view is depicted in [42]. The effect of perturbing p for different kinds of algorithms is investigated in [9–11, 14, 15]. We point it out that the generalized Fischer-Burmeister ϕ_p given as in (3) is not differentiable, whereas the squared norm of generalized Fischer-Burmeister function is smooth so that it is usually adapted as a differentiable NCP-function [38]. Moreover, all the aforementioned functions including Fischer-Burmeister function, generalized Fischer-Burmeister function and their squared norm can be extended to the setting of SOCCP via Jordan algebra.

A different type of popular NCP-function is the natural residual function $\phi_{\text{NR}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\phi_{\text{NR}}(a, b) = a - (a - b)_+ = \min\{a, b\}.$$

Recently, Chen et al. propose a family of generalized natural residual functions ϕ^p_{NR} defined by

$$\phi^p_{\text{NR}}(a, b) = a^p - (a - b)^p_+,$$

where $p > 1$ is a positive odd integer, $(a-b)^p_+ = [(a-b)_+]^p$, and $(a-b)_+ = \max\{a-b, 0\}$. When $p = 1$, ϕ^p_{NR} reduces to the natural residual function ϕ_{NR}, i.e.,

$$\phi^1_{\text{NR}}(a, b) = a - (a - b)_+ = \min\{a, b\} = \phi_{\text{NR}}(a, b).$$

As remarked in [16], this extension is "discrete generalization", not "continuous generalization". Nonetheless, it possesses twice differentiability surprisingly so that the squared norm of ϕ^p_{NR} is not needed. Based on this discrete generalization, two families of NCP-functions are further proposed in [3] which have the feature of symmetric surfaces. To the contrast, it is very natural to ask whether there is a similar "discrete extension" for Fischer-Burmeister function. We answer this question affirmatively.

In this paper, we apply the idea of "discrete generalization" to the Fischer-Burmeister function which gives the following function (denoted by $\phi^p_{\text{D-FB}}$):

$$\phi^p_{\text{D-FB}}(a,b) = \left(\sqrt{a^2+b^2}\right)^p - (a+b)^p, \tag{4}$$

where $p > 1$ is a positive odd integer and $(a,b) \in \mathbb{R}^2$. Notice that when $p = 1$, $\phi^p_{\text{D-FB}}$ reduces to the Fischer-Burmeister function. In Section 3, we will see that $\phi^p_{\text{D-FB}}$ is an NCP-function and is twice differentiable directly without taking its squared norm. Note that if p is even, it is no longer an NCP-function. Even though we have the feature of differentiability, we point out that the Newton method may not applied directly because the Jacobian at a degenerate solution to NCP is singular (see [32, 33]). Nonetheless, this feature may enable that many methods like derivative-free algorithm can be employed directly for solving NCP. In addition, we investigate the differentiable properties of $\phi^p_{\text{D-FB}}$, the computable formulas for their gradients and Jacobians. In order to have more insight for this new family of NCP-function, we also depict the surfaces of $\phi^p_{\text{D-FB}}(a,b)$ with various values of p.

In Section 4, we show that the new function $\phi^p_{\text{D-FB}}$ can be further employed to the SOCCP setting as complementarity functions and merit functions. In other words, in the terms of Jordan algebra, we define $\phi^p_{\text{D-FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi^p_{\text{D-FB}}(x,y) = \left(\sqrt{x^2+y^2}\right)^p - (x+y)^p, \tag{5}$$

where $p > 1$ is a positive odd integer, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $x^2 = x \circ x$ is the Jordan product of x with itself and $\sqrt{x}$ with $x \in \mathcal{K}^n$ being the unique vector such that $\sqrt{x} \circ \sqrt{x} = x$. We prove that each $\phi^p_{\text{D-FB}}(x,y)$ is a complementarity function associated with $\mathcal{K}^n$ and establish formulas for its gradient and Jacobian. These properties and formulas can be used to design and analyze non-interior continuation methods for solving second-order cone programs and complementarity problems. In addition, several variants of $\phi^p_{\text{D-FB}}$ are also shown to be complementarity functions for SOCCP.

Throughout the paper, we assume $\mathcal{K} = \mathcal{K}^n$ for simplicity and all the analysis can be carried over to the case where $\mathcal{K}$ is a product of second-order cones without difficulty. The following notations will be used. The identity matrix is denoted by I and $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors. For any given $x \in \mathbb{R}^n$ with $n > 1$, we write $x = (x_1, x_2)$ where x_1 is the first entry of x and x_2 is the subvector that consists of the remaining entries. For every differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of f at x. For every differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla F(x)$ is an $n \times m$ matrix which denotes the transposed Jacobian of F at x. For nonnegative scalar functions α and β, we write $\alpha = o(\beta)$ to mean $\lim\limits_{\beta \to 0} \dfrac{\alpha}{\beta} = 0$.

2 Preliminaries

In this section, we review some background materials about the Jordan algebra in [19, 25]. Then, we present some technical lemmas which are needed in subsequent analysis.

For any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define the *Jordan product* associated with $\mathcal{K}^n$ as

$$x \circ y \ := \ (\langle x, y \rangle, \ y_1 x_2 + x_1 y_2).$$

The identity element under this product is $e := (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. For any given $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define symmetric matrix

$$L_x := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix}$$

which can be viewed as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. It is easy to verify that

$$L_x y = x \circ y, \quad \forall x \in \mathbb{R}^n.$$

Moreover, we have L_x is invertible for $x \succ_{\mathcal{K}^n} 0$ and

$$L_x^{-1} = \frac{1}{\det(x)} \begin{bmatrix} x_1 & -x_2^T \\ -x_2 & \dfrac{\det(x)}{x_1} I + \dfrac{1}{x_1} x_2 x_2^T \end{bmatrix},$$

where $\det(x) = x_1^2 - \|x_2\|^2$. We next recall from [12, 25] that each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ admits a spectral factorization, associated with $\mathcal{K}^n$, of the form

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{6}$$

where λ_1, λ_2 and $u^{(1)}, u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\begin{aligned} \lambda_i &= x_1 + (-1)^i \|x_2\|, \\ u^{(i)} &= \begin{cases} \frac{1}{2}\left(1, \ (-1)^i \dfrac{x_2}{\|x_2\|}\right) & \text{if } x_2 \neq 0; \\ \frac{1}{2}\left(1, \ (-1)^i w_2\right) & \text{if } x_2 = 0, \end{cases} \end{aligned}$$

for $i = 1, 2$, with w_2 being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w_2\| = 1$. If $x_2 \neq 0$, the factorization is unique.

Given a real-valued function $g : \mathbb{R} \to \mathbb{R}$, we can define a vector-valued SOC-function $g^{\rm soc} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g^{\rm soc}(x) := g(\lambda_1) u^{(1)} + g(\lambda_2) u^{(2)}.$$

If g is defined on a subset of $\mathbb{R}$, then $g^{\rm soc}$ is defined on the corresponding subset of $\mathbb{R}^n$. The definition of $g^{\rm soc}$ is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. In this paper, we will often use the vector-valued functions corresponding to t^p ($t \in \mathbb{R}$) and $\sqrt{t}$ ($t \geq 0$), respectively, which are expressed as

$$\begin{array}{rcl} x^p & := & (\lambda_1(x))^p u^{(1)} + (\lambda_2(x))^p u^{(2)}, \quad \forall x \in \mathbb{R}^n \\ \sqrt{x} & := & \sqrt{\lambda_1(x)}u^{(1)} + \sqrt{\lambda_2(x)}u^{(2)}, \quad \forall x \in \mathcal{K}^n. \end{array}$$

We will see that the above two vector-valued functions play a role in showing that $\phi^p_{\rm D-FB}$ given as in (5) is well-defined in the SOC setting for any $x, y \in \mathbb{R}^n$. Note that the other way to define x^p and $\sqrt{x}$ is through Jordan product. In other words, x^p represents $x \circ x \circ \cdots \circ x$ for p-times and $\sqrt{x} \in \mathcal{K}^n$ satisfies $\sqrt{x} \circ \sqrt{x} = x$.

Lemma 2.1. *Suppose that $p = 2k+1$ where $k = 1, 2, 3, \cdots$. Then, for any $u, v \in \mathbb{R}$, we have $u^p = v^p$ if and only if $u = v$.*

Proof. The proof is straightforward and can be found in [1, Theorem 1.12]. Here, we provide an alternative proof.
"$\Leftarrow$" It is trivial.
"$\Rightarrow$" For $v = 0$, since $u^p = v^p$, we have $u = v = 0$. For $v \neq 0$, from $f(t) = t^p - 1$ being a strictly monotone increasing function for any $t \in \mathbb{R}$, we have $\left(\frac{u}{v}\right)^p - 1 = 0$ if and only if $\frac{u}{v} = 1$, which implies $u = v$. Thus, the proof is complete. $\Box$

Lemma 2.2. *For $p = 2m+1$ with $m = 1, 2, 3, \cdots$ and $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, suppose that x^p and y^p represent $x \circ x \circ \cdots \circ x$ and $y \circ y \circ \cdots \circ y$ for p-times, respectively. Then, $x^p = y^p$ if and only if $x = y$.*

Proof. "$\Leftarrow$" This direction is trivial.

"$\Rightarrow$" Suppose that $x^p = y^p$. By the spectral decomposition (6), we write

$$\begin{array}{rcl} x & = & \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \\ y & = & \lambda_1(y)u_y^{(1)} + \lambda_2(y)u_y^{(2)}. \end{array}$$

Then, $x^p = (\lambda_1(x))^p u_x^{(1)} + (\lambda_2(x))^p u_x^{(2)}$ and $y^p = (\lambda_1(y))^p u_y^{(1)} + (\lambda_2(y))^p u_y^{(2)}$. Since $x^p = y^p$ and eigenvalues are unique, we obtain $(\lambda_1(x))^p = (\lambda_1(y))^p$ and $(\lambda_2(x))^p = (\lambda_2(y))^p$. By Lemma 2.1, this implies $\lambda_1(x) = \lambda_1(y)$ and $\lambda_2(x) = \lambda_2(y)$. Moreover, $\{u_x^{(1)}, u_x^{(2)}\}$ and $\{u_y^{(1)}, u_y^{(2)}\}$ are Jordan frames, we have $u_x^{(1)} + u_x^{(2)} = u_y^{(1)} + u_y^{(2)} = e$, where e is the identity element. From $x^p = y^p$ and $u_x^{(1)} + u_x^{(2)} = u_y^{(1)} + u_y^{(2)}$, we get

$$[(\lambda_1(x))^p - (\lambda_2(x))^p]\,(u_x^{(1)} - u_y^{(1)}) = 0.$$

If $(\lambda_1(x))^p = (\lambda_2(x))^p$, we have $\lambda_1(x) = \lambda_2(x)$ and $\lambda_1(y) = \lambda_2(y)$, that is, $x = \lambda_1(x)e = y$. Otherwise, if $(\lambda_1(x))^p \neq (\lambda_2(x))^p$, we must have $u_x^{(1)} = u_y^{(1)}$, which implies $u_x^{(2)} = u_y^{(2)}$. $\Box$

3 New generalized Fischer-Burmeister function for NCP

In this section, we show that the function $\phi^p_{_{\rm D-FB}}$ defined as in (4) is an NCP-function and present its twice differentiability. At the same time, we also depict the surfaces of $\phi^p_{_{\rm D-FB}}$ with various values of p to have more insight for this new family of NCP-functions.

Proposition 3.1. *Let $\phi^p_{_{\rm D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, $\phi^p_{_{\rm D-FB}}$ is an NCP-function.*

Proof. Suppose $\phi^p_{_{\rm D-FB}}(a,b) = 0$, which says $\left(\sqrt{a^2+b^2}\right)^p = (a+b)^p$. Using p being a positive odd integer and applying Lemma 2.1, we have

$$\left(\sqrt{a^2+b^2}\right)^p = (a+b)^p \quad \Longleftrightarrow \quad \sqrt{a^2+b^2} = a+b.$$

It is well known that $\sqrt{a^2+b^2} = a+b$ is equivalent to $a, b \geq 0, ab = 0$ because $\phi_{_{\rm FB}}$ is an NCP-function. This shows that $\phi^p_{_{\rm D-FB}}(a,b) = 0$ implies $a, b \geq 0, ab = 0$. The converse direction is trivial. Thus, we prove that $\phi^p_{_{\rm D-FB}}$ is an NCP-function. $\Box$

Remark 3.1: We elaborate more about the new NCP-function $\phi^p_{_{\rm D-FB}}$.

(a) For p being an even integer, $\phi^p_{_{\rm D-FB}}$ is not a NCP-function. A counterexample is given as below.

$$\phi^p_{_{\rm D-FB}}(-5,0) = (-5)^2 - (-5)^2 = 0.$$

(b) The surface of $\phi^p_{_{\rm D-FB}}$ is symmetric, i.e., $\phi^p_{_{\rm D-FB}}(a,b) = \phi^p_{_{\rm D-FB}}(b,a)$.

(c) The function $\phi^p_{_{\rm D-FB}}(a,b)$ is positive homogenous of degree p, i.e., $\phi^p_{_{\rm D-FB}}(\alpha(a,b)) = \alpha^p \phi^p_{_{\rm D-FB}}(a,b)$ for $\alpha \geq 0$.

(d) The function $\phi^p_{_{\rm D-FB}}$ is neither convex nor concave function. To see this, taking $p=3$ and using the following argument verify the assertion.

$$\begin{aligned} 5^3 - 7^3 = \phi^3_{_{\rm D-FB}}(3,4) &> \frac{1}{2}\phi^3_{_{\rm D-FB}}(0,0) + \frac{1}{2}\phi^3_{_{\rm D-FB}}(6,8) \\ &= \frac{1}{2} \times 0 + \frac{1}{2}\left(10^3 - 14^3\right) = 4\left(5^3 - 7^3\right) \end{aligned}$$

and

$$0 = \phi^3_{_{\rm D-FB}}(0,0) < \frac{1}{2}\phi^3_{_{\rm D-FB}}(-2,0) + \frac{1}{2}\phi^3_{_{\rm D-FB}}(2,0) = \frac{1}{2} \times 16 + \frac{1}{2} \times 0 = 8.$$

Proposition 3.2. *Let $\phi^p_{\text{D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, the following hold.*

(a) *For $p > 1$, $\phi^p_{\text{D-FB}}$ is continuously differentiable with*

$$\nabla\phi^p_{\text{D-FB}}(a,b) = p\begin{bmatrix} a(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \\ b(\sqrt{a^2+b^2})^{p-2} - (a+b)^{p-1} \end{bmatrix}.$$

(b) *For $p > 3$, $\phi^p_{\text{D-FB}}$ is twice continuously differentiable with*

$$\nabla^2\phi^p_{\text{D-FB}}(a,b) = \begin{bmatrix} \dfrac{\partial^2\phi^p_{\text{D-FB}}}{\partial a^2} & \dfrac{\partial^2\phi^p_{\text{D-FB}}}{\partial a\partial b} \\ \dfrac{\partial^2\phi^p_{\text{D-FB}}}{\partial b\partial a} & \dfrac{\partial^2\phi^p_{\text{D-FB}}}{\partial b^2} \end{bmatrix},$$

where

$$\begin{aligned}
\frac{\partial^2\phi^p_{\text{D-FB}}}{\partial a^2} &= p\left\{[(p-1)a^2+b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}, \\
\frac{\partial^2\phi^p_{\text{D-FB}}}{\partial a\partial b} &= p[(p-2)ab(\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}] = \frac{\partial^2\phi^p_{\text{D-FB}}}{\partial b\partial a}, \\
\frac{\partial^2\phi^p_{\text{D-FB}}}{\partial b^2} &= p\left\{[a^2+(p-1)b^2](\sqrt{a^2+b^2})^{p-4} - (p-1)(a+b)^{p-2}\right\}.
\end{aligned}$$

Proof. The verifications of differentiability and computations of first and second derivatives are straightforward, we omit them. □

Next, we present some variants of $\phi^p_{\text{D-FB}}$. Indeed, analogous to those functions in [41], the variants of $\phi^p_{\text{D-FB}}$ as below can be verified being NCP-functions.

$$\begin{aligned}
\phi_1(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha(a)_+(b)_+,\ \alpha > 0. \\
\phi_2(a,b) &= \phi^p_{\text{D-FB}}(a,b) - \alpha\left((a)_+(b)_+\right)^2,\ \alpha > 0. \\
\phi_3(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^4,\ \alpha > 0. \\
\phi_4(a,b) &= [\phi^p_{\text{D-FB}}(a,b)]^2 + \alpha\left((ab)_+\right)^2,\ \alpha > 0.
\end{aligned}$$

In the above expressions, for any $t \in \mathbb{R}$, we define t_+ as $\max\{0,t\}$.

Lemma 3.1. *Let $\phi^p_{\text{D-FB}}$ be defined as in (4) where p is a positive odd integer. Then, the value of $\phi^p_{\text{D-FB}}(a,b)$ is negative only in the first quadrant, i.e., $\phi^p_{\text{D-FB}}(a,b) < 0$ if and only if $a > 0$, $b > 0$.*

Proof. We know that $f(t) = t^p$ is a strictly increasing function when p is odd. Using this fact yields

$$\begin{aligned} & a > 0,\ b > 0 \\ \Longleftrightarrow\ & a + b > 0 \quad \text{and} \quad ab > 0 \\ \Longleftrightarrow\ & \sqrt{a^2 + b^2} < a + b \\ \Longleftrightarrow\ & \left(\sqrt{a^2 + b^2}\right)^p < (a + b)^p \\ \Longleftrightarrow\ & \phi^p_{_{\rm D-FB}}(a, b) < 0, \end{aligned}$$

which proves the desired result. $\square$

Proposition 3.3. *All the above functions ϕ_i for $i \in \{1, 2, 3, 4\}$ are NCP-functions.*

Proof. Applying Lemma 3.1, the arguments are similar to those in [16, Proposition 2.4], which are omitted here. $\square$

In fact, in light of Lemma 2.1, we can construct more variants of $\phi^p_{_{\rm D-FB}}$, which are also new NCP-function. More specifically, consider that k and m are positive integers, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $g(a, b) \neq 0$ for all $a, b \in \mathbb{R}$, the following functions are new variants of $\phi^p_{_{\rm D-FB}}$.

$$\begin{aligned} \phi_5(a, b) &= \left[g(a, b)\big(\sqrt{a^2 + b^2} + f(a, b)\big)\right]^{\frac{2k+1}{2m+1}} - \big[g(a, b)\big(a + b + f(a, b)\big)\big]^{\frac{2k+1}{2m+1}}. \\ \phi_6(a, b) &= \left[g(a, b)(\sqrt{a^2 + b^2} - a - b)\right]^{\frac{k}{m}}. \\ \phi_7(a, b) &= \left[g(a, b)(\sqrt{a^2 + b^2} - a + f(a, b))\right]^{\frac{2k+1}{2m+1}} - [g(a, b)(b + f(a, b))]^{\frac{2k+1}{2m+1}}. \\ \phi_8(a, b) &= \left[g(a, b)(\sqrt{a^2 + b^2} - a + f(a, b))\right]^{\frac{2k+1}{2m+1}} - [g(a, b)(b + f(a, b))]^{\frac{2k+1}{2m+1}}. \\ \phi_9(a, b) &= e^{\phi_i(a,b)} - 1 \text{ where } i = 5, 6, 7, 8. \\ \phi_{10}(a, b) &= \ln(|\phi_i(a, b)| + 1) \text{ where } i = 5, 6, 7, 8. \end{aligned}$$

Proposition 3.4. *All the above functions ϕ_i for $i \in \{5, 6, 7, 8, 9, 10\}$ are NCP-functions.*

Proof. This is an immediate consequence of Propositions 3.1-3.3. By Lemma 2.1 and

$g(a,b) \neq 0$ for $a, b \in \mathbb{R}$, we have

$$
\begin{aligned}
& \phi_5(a,b) = 0 \\
\Longleftrightarrow \quad & \Big[g(a,b)\big(\sqrt{a^2+b^2} + f(a,b)\big) \Big]^{\frac{2k+1}{2m+1}} = \big[g(a,b)\big(a+b+f(a,b)\big) \big]^{\frac{2k+1}{2m+1}} \\
\Longleftrightarrow \quad & \left\{ \Big[g(a,b)\big(\sqrt{a^2+b^2} + f(a,b)\big) \Big]^{\frac{2k+1}{2m+1}} \right\}^{2m+1} = \left\{ \big[g(a,b)\big(a+b+f(a,b)\big) \big]^{\frac{2k+1}{2m+1}} \right\}^{2m+1} \\
\Longleftrightarrow \quad & \Big[g(a,b)\big(\sqrt{a^2+b^2} + f(a,b)\big) \Big]^{2k+1} = \big[g(a,b)\big(a+b+f(a,b)\big) \big]^{2k+1} \\
\Longleftrightarrow \quad & g(a,b)\big(\sqrt{a^2+b^2} + f(a,b)\big) = g(a,b)\big(a+b+f(a,b)\big) \\
\Longleftrightarrow \quad & \big(\sqrt{a^2+b^2} + f(a,b)\big) = \big(a+b+f(a,b)\big) \\
\Longleftrightarrow \quad & \sqrt{a^2+b^2} = a+b.
\end{aligned}
$$

The other functions ϕ_i for $i \in \{6, 7, 8, 9, 10\}$ are similar to ϕ_5. $\Box$

According to the above results, we immediately obtain the following theorem.

Theorem 3.1. *Suppose that $\phi(a,b) = \varphi_1(a,b) - \varphi_2(a,b)$ is an NCP-function on $\mathbb{R} \times \mathbb{R}$ and k and m are positive integers. Then, $\big[\phi(a,b)\big]^{\frac{k}{m}}$ and $\big[\varphi_1(a,b)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(a,b)]^{\frac{2k+1}{2m+1}}$ are NCP-functions.*

Proof. Using k and m being positive integers and applying Lemma 2.1, we have

$$
\begin{aligned}
& \big[\phi(a,b)\big]^{\frac{k}{m}} = 0 \\
\Longleftrightarrow \quad & \left\{ \big[\phi(a,b)\big]^{\frac{k}{m}} \right\}^{m} = 0 \\
\Longleftrightarrow \quad & \big[\phi(a,b)\big]^{k} = 0 \\
\Longleftrightarrow \quad & \phi(a,b) = 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \big[\varphi_1(a,b)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(a,b)]^{\frac{2k+1}{2m+1}} = 0 \\
\Longleftrightarrow \quad & \big[\varphi_1(a,b)\big]^{\frac{2k+1}{2m+1}} = [\varphi_2(a,b)]^{\frac{2k+1}{2m+1}} \\
\Longleftrightarrow \quad & \left\{ \big[\varphi_1(a,b)\big]^{\frac{2k+1}{2m+1}} \right\}^{2m+1} = \left\{ [\varphi_2(a,b)]^{\frac{2k+1}{2m+1}} \right\}^{2m+1} \\
\Longleftrightarrow \quad & \big[\varphi_1(a,b)\big]^{2k+1} = \big[\varphi_2(a,b)\big]^{2k+1} \\
\Longleftrightarrow \quad & \varphi_1(a,b) = \varphi_2(a,b) \\
\Longleftrightarrow \quad & \phi(a,b) = 0.
\end{aligned}
$$

The above arguments together with the assumption of $\phi(a,b)$ being an NCP-function yield the desired result. $\Box$

Remark 3.2: We elaborate more about Theorem 3.1.

(a) Based on the existing well-known NCP-functions, we can construct new NCP-functions in light of Theorem 3.1. This is a novel way to construct new NCP-functions.

(b) When k is a positive integer, $\big[\phi(a,b)\big]^k$ is an NCP-function. This means that perturbing the parameter k gives new NCP-functions. In addition, if $\phi(a,b)$ is an NCP-function, for any positive integer m, $\big[\phi(a,b)\big]^{\frac{k}{m}}$ is also an NCP-function. Thus, we can determine suitable and nice NCP-functions among these functions according to their numerical performance.

To close this section, we depict the surfaces of $\phi^p_{\text{D-FB}}$ with different values of p so that we may have deeper insight for this new family of NCP-functions. Figure 1 is the surface if $\phi_{\text{D-FB}}(a,b)$ from which we see that it is convex. Figure 2 presents the surface of $\phi^3_{\text{D-FB}}(a,b)$ in which we see that it is neither convex nor concave as mentioned in Remark 3.1(c). In addition, the value of $\phi^p_{\text{D-FB}}(a,b)$ is negative only when $a>0$ and $b>0$ as mentioned in Lemma 3.1. The surfaces of $\phi^p_{\text{D-FB}}$ with various values of p are shown in Figure 3.

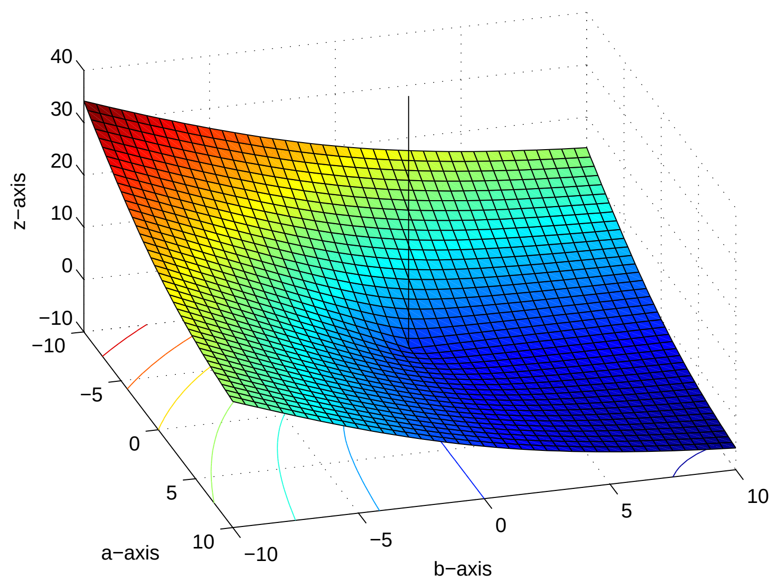


Figure 1: The surface of $z=\phi_{\text{D-FB}}(a,b)$ and $(a,b)\in[-10,10]\times[-10,10]$

4 Extending $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} to SOCCP

In this section, we extend the new function $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} to SOC setting. More specifically, we show that the function $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} are complementarity functions associated

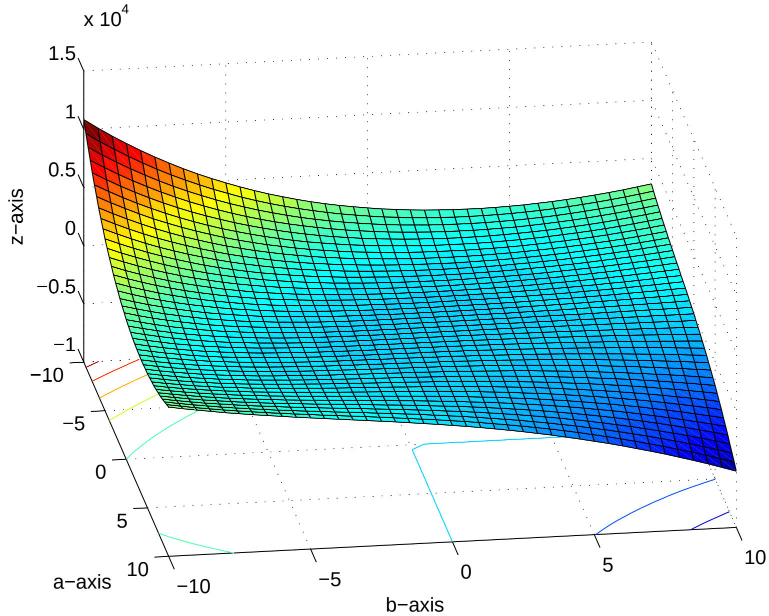


Figure 2: The surface of $z = \phi^3_{\text{D-FB}}(a,b)$ and $(a,b) \in [-10,10] \times [-10,10]$

with $\mathcal{K}^n$. In addition, we present the computing formulas for its Jacobian.

Proposition 4.1. *Let $\phi^p_{\text{D-FB}}$ be defined by (5). Then, $\phi^p_{\text{D-FB}}$ is a complementarity function associated with $\mathcal{K}^n$, i.e., it satisfies*

$$\phi^p_{\text{D-FB}}(x,y) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{K}^n,\ y \in \mathcal{K}^n,\ \langle x, y \rangle = 0.$$

Proof. Since $\phi^p_{\text{D-FB}}(x,y) = 0$, we have $\left(\sqrt{x^2+y^2}\right)^p = (x+y)^p$. Using p being a positive odd integer and applying Lemma 2.2 yield

$$\left(\sqrt{x^2+y^2}\right)^p = (x+y)^p \quad \Longleftrightarrow \quad \sqrt{x^2+y^2} = x+y.$$

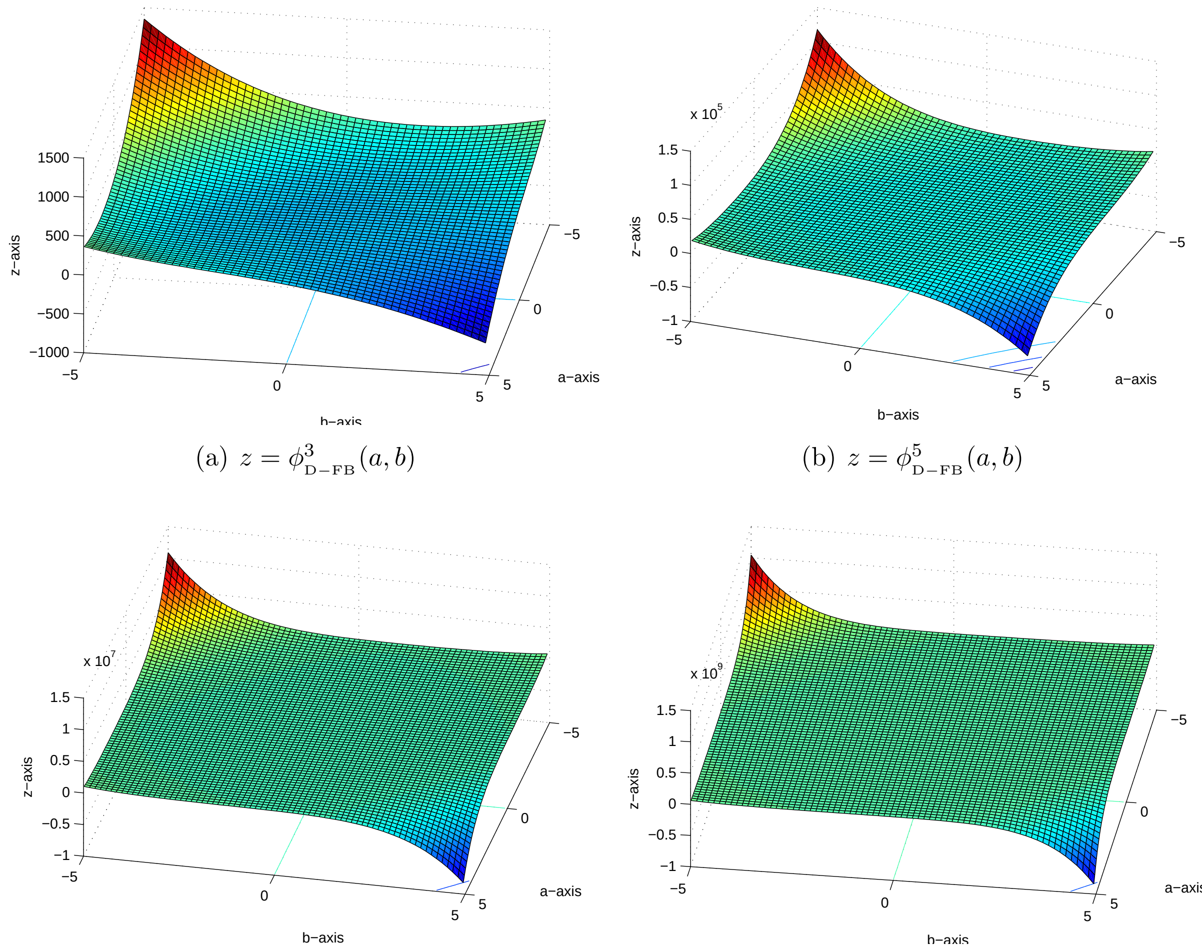
It is known that $\phi_{\text{FB}}(x,y) := \sqrt{x^2+y^2} - (x+y)$ is a complementarity function associated with $\mathcal{K}^n$. This indicates that $\phi^p_{\text{D-FB}}$ is a complementarity function associate with $\mathcal{K}^n$. $\square$

With similar technique, we can prove that ϕ^p_{NR} can be extended as a complementarity function for SOCCP.

Proposition 4.2. *The function $\phi^p_{\text{NR}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by*

$$\phi^p_{\text{NR}}(x,y) = x^p - [(x-y)_+]^p \tag{7}$$

is a complementarity function associated with $\mathcal{K}^n$, where $p > 1$ is a positive odd integer and $(\cdot)_+$ means the projection onto $\mathcal{K}^n$.



(a) $z = \phi^3_{\mathrm{D-FB}}(a,b)$ (b) $z = \phi^5_{\mathrm{D-FB}}(a,b)$

(c) $z = \phi^7_{\mathrm{D-FB}}(a,b)$ (d) $z = \phi^9_{\mathrm{D-FB}}(a,b)$

Figure 3: The surface of $z = \phi^p_{\mathrm{D-FB}}(a,b)$ with different values of p

Proof. From Lemma 2.2, we see that $\phi^p_{\mathrm{NR}}(x,y) = 0$ if and only if $x = (x-y)_+$. On the other hand, it is known that $\phi_{\mathrm{NR}}(x,y) = x - (x-y)_+$ is a complementarity function for SOCCP, which implies $x - (x-y)_+ = 0$ if and only if $x \in \mathcal{K}^n$, $y \in \mathcal{K}^n$, and $\langle x, y\rangle = 0$. Hence, ϕ^p_{NR} is a complementarity function associated with $\mathcal{K}^n$. $\square$

In order to compute the Jacobian of $\phi^p_{\mathrm{D-FB}}$, we need to introduce some notations for convenience. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define

$$w(x,y) := x^2 + y^2 = (w_1(x,y), w_2(x,y)) \in \mathbb{R} \times \mathbb{R}^{n-1} \quad \text{and} \quad v(x,y) := x + y.$$

Then, it is clear that $w(x,y) \in \mathcal{K}^n$ and $\lambda_i(w) \geq 0$, $i = 1, 2$.

Proposition 4.3. *Let $\phi^p_{\mathrm{D-FB}}$ be defined as in (5) and $g^{\mathrm{soc}}(x) = (\sqrt{|x|})^p$, $h^{\mathrm{soc}}(x) = x^p$ are the vector-valued functions corresponding to $g(t) = |t|^{\frac{p}{2}}$ and $h(t) = t^p$ for $t \in$*

$\mathbb{R}$, *respectively. Then,* $\phi^p_{\text{D-FB}}$ *is continuously differentiable at any* $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$. *Moreover, we have*

$$\nabla_x \phi^p_{\text{D-FB}}(x,y) = 2L_x \nabla g^{\text{soc}}(w) - \nabla h^{\text{soc}}(v),$$
$$\nabla_y \phi^p_{\text{D-FB}}(x,y) = 2L_y \nabla g^{\text{soc}}(w) - \nabla h^{\text{soc}}(v),$$

where $w := w(x,y) = x^2 + y^2$, $v := v(x,y) = x + y$, $t \mapsto \text{sign}(t)$ *is the sign function, and*

$$\nabla g^{\text{soc}}(w) = \begin{cases} \frac{p}{2}|w_1|^{\frac{p}{2}-1} \cdot \text{sign}(w_1) I & \text{if} \quad w_2 = 0; \\ \begin{bmatrix} b_1(w) & c_1(w)\bar{w}_2^T \\ c_1(w)\bar{w}_2 & a_1(w)I + (b_1(w) - a_1(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if} \quad w_2 \neq 0; \end{cases}$$

$$\begin{aligned} \bar{w}_2 &= \frac{w_2}{\|w_2\|}, \\ a_1(w) &= \frac{|\lambda_2(w)|^{\frac{p}{2}} - |\lambda_1(w)|^{\frac{p}{2}}}{\lambda_2(w) - \lambda_1(w)}, \\ b_1(w) &= \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} + |\lambda_1(w)|^{\frac{p}{2}-1}\right], \\ c_1(w) &= \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} - |\lambda_1(w)|^{\frac{p}{2}-1}\right], \end{aligned}$$

and

$$\nabla h^{\text{soc}}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_2(v) & c_2(v)\bar{v}_2^T \\ c_2(v)\bar{v}_2 & a_2(v)I + (b_2(v) - a_2(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases} \tag{8}$$

$$\bar{v}_2 = \frac{v_2}{\|v_2\|}, \tag{9}$$
$$a_2(v) = \frac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, \tag{10}$$
$$b_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], \tag{11}$$
$$c_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right], \tag{12}$$

Proof. From the definition of $\phi^p_{\text{D-FB}}$, it is clear to see that for any $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} \phi^p_{\text{D-FB}}(x,y) &= \left(\sqrt{x^2+y^2}\right)^p - (x+y)^p \\ &= \left(\sqrt{|x^2+y^2|}\right)^p - (x+y)^p \\ &= \left[|\lambda_1(w)|^{\frac{p}{2}} u^{(1)}(w) + |\lambda_2(w)|^{\frac{p}{2}} u^{(2)}(w)\right] - \left[(\lambda_1(v))^p u^{(1)}(v) + (\lambda_2(v))^p u^{(2)}(v)\right] \\ &= g^{\text{soc}}(w) - h^{\text{soc}}(v). \end{aligned} \tag{13}$$

For $p \geq 3$, since both $|t|^{\frac{p}{2}}$ and t^p are continuously differentiable on $\mathbb{R}$, by [13, Proposition 5] and [25, Proposition 5.2], we know that the function g^{soc} and h^{soc} are continuously differentiable on $\mathbb{R}^n$. Moreover, it is clear that $w(x,y) = x^2 + y^2$ is continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^n$, then we conclude that $\phi^p_{\mathrm{D-FB}}$ is continuously differentiable. Moreover, from the formula in [13, Proposition 4] and [25, Proposition 5.2], we have

$$\nabla g^{\mathrm{soc}}(w) = \begin{cases} \dfrac{p}{2}|w_1|^{\frac{p}{2}-1} \cdot \mathrm{sign}(w_1) I & \text{if} \quad w_2 = 0; \\ \begin{bmatrix} b_1(w) & c_1(w)\bar{w}_2^T \\ c_1(w)\bar{w}_2 & a_1(w)I + (b_1(w) - a_1(w))\,\bar{w}_2\bar{w}_2^T \end{bmatrix} & \text{if} \quad w_2 \neq 0; \end{cases}$$

$$\nabla h^{\mathrm{soc}}(v) = \begin{cases} p v_1^{p-1} I & \text{if} \quad v_2 = 0; \\ \begin{bmatrix} b_2(v) & c_2(v)\bar{v}_2^T \\ c_2(v)\bar{v}_2 & a_2(v)I + (b_2(v) - a_2(v))\,\bar{v}_2\bar{v}_2^T \end{bmatrix} & \text{if} \quad v_2 \neq 0; \end{cases}$$

where

$$\begin{array}{ll} \bar{w}_2 = \frac{w_2}{\|w_2\|}, & \bar{v}_2 = \frac{v_2}{\|v_2\|} \\ a_1(w) = \frac{|\lambda_2(w)|^{\frac{p}{2}} - |\lambda_1(w)|^{\frac{p}{2}}}{\lambda_2(w) - \lambda_1(w)}, & a_2(v) = \frac{(\lambda_2(v))^p - (\lambda_1(v))^p}{\lambda_2(v) - \lambda_1(v)}, \\ b_1(w) = \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} + |\lambda_1(w)|^{\frac{p}{2}-1}\right], & b_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} + (\lambda_1(v))^{p-1}\right], \\ c_1(w) = \frac{p}{4}\left[|\lambda_2(w)|^{\frac{p}{2}-1} - |\lambda_1(w)|^{\frac{p}{2}-1}\right], & c_2(v) = \frac{p}{2}\left[(\lambda_2(v))^{p-1} - (\lambda_1(v))^{p-1}\right]. \end{array}$$

By taking differentiation on both sides about x and y for (13), respectively, and applying the chain rule for differentiation, it follows that

$$\begin{aligned} \nabla_x \phi^p_{\mathrm{D-FB}}(x,y) &= 2L_x \nabla g^{\mathrm{soc}}(w) - \nabla h^{\mathrm{soc}}(v), \\ \nabla_y \phi^p_{\mathrm{D-FB}}(x,y) &= 2L_y \nabla g^{\mathrm{soc}}(w) - \nabla h^{\mathrm{soc}}(v). \end{aligned}$$

Hence, we complete the proof. $\Box$

With Lemma 2.2 and Proposition 4.1, we can construct more complementarity functions for SOCCP which are variants of $\phi^p_{\mathrm{D-FB}}(x,y)$. More specifically, consider that k and m are positive integers and $f^{\mathrm{soc}}(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the vector-valued function corresponding to a given real-valued function f, the following functions are new variants of $\phi^p_{\mathrm{D-FB}}(x,y)$.

$$\begin{aligned} \widetilde{\phi}_1(x,y) &= \left[\sqrt{x^2+y^2} + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}} - \left[x + y + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}}. \\ \widetilde{\phi}_2(x,y) &= \left[\sqrt{x^2+y^2} - x - y\right]^{\frac{k}{m}}. \\ \widetilde{\phi}_3(x,y) &= \left[\sqrt{x^2+y^2} - x + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}} - \left[y + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}}. \\ \widetilde{\phi}_4(x,y) &= \left[\sqrt{x^2+y^2} - y + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}} - \left[x + f^{\mathrm{soc}}(x,y)\right]^{\frac{2k+1}{2m+1}}. \end{aligned}$$

Proposition 4.4. *All the above functions $\widetilde{\phi_i}$ for $i \in \{1,2,3,4\}$ are complementarity functions associated with $\mathcal{K}^n$.*

Proof. The results follow from applying Lemma 2.2 and Proposition 4.1. $\square$

In general, for complementarity functions associated with $\mathcal{K}^n$, we have the following parallel result to Theorem 3.1.

Theorem 4.1. *Suppose that $\phi(x,y) = \varphi_1(x,y) - \varphi_2(x,y)$ is a complementarity function associated with $\mathcal{K}^n$ on $\mathbb{R}^n \times \mathbb{R}^n$, and k,m are positive integers. Then $\big[\phi(x,y)\big]^{\frac{k}{m}}$ and $\big[\varphi_1(x,y)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(x,y)]^{\frac{2k+1}{2m+1}}$ are complementarity functions associated with $\mathcal{K}^n$.*

Proof. According to k and m are positive integers and by using Lemma 2.2, we have

$$
\begin{aligned}
& \big[\phi(x,y)\big]^{\frac{k}{m}} = 0 \\
\Longleftrightarrow\ & \Big\{\big[\phi(x,y)\big]^{\frac{k}{m}}\Big\}^{m} = 0 \\
\Longleftrightarrow\ & \big[\phi(x,y)\big]^{k} = 0 \\
\Longleftrightarrow\ & \phi(x,y) = 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
& \big[\varphi_1(x,y)\big]^{\frac{2k+1}{2m+1}} - [\varphi_2(x,y)]^{\frac{2k+1}{2m+1}} = 0 \\
\Longleftrightarrow\ & \big[\varphi_1(x,y)\big]^{\frac{2k+1}{2m+1}} = [\varphi_2(x,y)]^{\frac{2k+1}{2m+1}} \\
\Longleftrightarrow\ & \Big\{\big[\varphi_1(x,y)\big]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} = \Big\{[\varphi_2(x,y)]^{\frac{2k+1}{2m+1}}\Big\}^{2m+1} \\
\Longleftrightarrow\ & [\varphi_1(x,y)]^{2k+1} = \big[\varphi_2(x,y)]^{2k+1} \\
\Longleftrightarrow\ & \varphi_1(x,y) = \varphi_2(x,y) \\
\Longleftrightarrow\ & \phi(x,y) = 0.
\end{aligned}
$$

From the above arguments and the assumption, the proof is complete. $\square$

Remark 4.1: We elaborate more about Theorem 4.1.

(a) Based existing complementarity functions, we can construct new complementarity functions associated with $\mathcal{K}^n$ in light of Theorem 4.1.

(b) When k is a positive odd integer, $\phi(x,y)^k$ is a complementarity function associated with $\mathcal{K}^n$. This means that perturbing the odd integer parameter k, we obtain the new complementarity functions associated with $\mathcal{K}^n$. In addition, if $\phi(x,y)$ is a complementarity function, then for any positive integer m, $\big[\phi(x,y)\big]^{\frac{k}{m}}$ is also a complementarity function. We can determine nice complementarity functions associated with $\mathcal{K}^n$ among these functions by their numerical performance.

Finally, we establish formula for Jacobian of ϕ^p_{NR} and the smoothness of ϕ^p_{NR}. To this aim, we need the following technical lemma.

Lemma 4.1. *Let $p > 1$. Then, the real-valued function $f(t) = (t_+)^p$ is continuously differentiable with $f'(t) = p(t_+)^{p-1}$ where $t_+ = \max\{0, t\}$.*

Proof. By the definition of t_+, we have

$$f(t) = (t_+)^p = \begin{cases} t^p & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0, \end{cases}$$

which implies

$$f'(t) = \begin{cases} pt^{p-1} & \text{if} \quad t \geq 0, \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Then, it is easy to see that $f'(t) = p(t_+)^{p-1}$ is continuous for $p > 1$. $\square$

Proposition 4.5. *Let ϕ^p_{NR} be defined as in (7) and $h^{\mathrm{soc}}(x) = x^p$, $l^{\mathrm{soc}}(x) = (x_+)^p$ be the vector-valued functions corresponding to the real-valued functions $h(t) = t^p$ and $l(t) = (t_+)^p$, respectively. Then, ϕ^p_{NR} is continuously differentiable at any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, and its Jacobian is given by*

$$\begin{aligned} \nabla_x \phi^p_{\mathrm{NR}}(x, y) &= \nabla h^{\mathrm{soc}}(x) - \nabla l^{\mathrm{soc}}(x - y), \\ \nabla_y \phi^p_{\mathrm{NR}}(x, y) &= \nabla l^{\mathrm{soc}}(x - y), \end{aligned}$$

where ∇h^{soc} satisfies (8)-(12) and

$$\nabla l^{\mathrm{soc}}(u) = \begin{cases} p((u_1)_+)^{p-1} I & \text{if} \quad u_2 = 0; \\ \begin{bmatrix} b_3(u) & c_3(u)\bar{u}_2^T \\ c_3(u)\bar{u}_2 & a_3(u)I + (b_3(u) - a_3(u))\,\bar{u}_2\bar{u}_2^T \end{bmatrix} & \text{if} \quad u_2 \neq 0; \end{cases}$$

$$\begin{aligned} \bar{u}_2 &= \frac{u_2}{\|u_2\|}, \\ a_3(u) &= \frac{(\lambda_2(u)_+)^p - (\lambda_1(u)_+)^p}{\lambda_2(u) - \lambda_1(u)}, \\ b_3(u) &= \frac{p}{2}\left[(\lambda_2(u)_+)^{p-1} + (\lambda_1(u)_+)^{p-1}\right], \\ c_3(u) &= \frac{p}{2}\left[(\lambda_2(u)_+)^{p-1} - (\lambda_1(u)_+)^{p-1}\right], \end{aligned}$$

Proof. In light of [13, Proposition 5] and [25, Proposition 5.2], the results follow from applying Lemma 4.1 and using the chain rule for differentiation. $\square$

5 Numerical experiments

As mentioned, the Newton method may not be appropriate for numerical implementation, due to possible singularity of Jacobian at a degenerate solution. In view of this, in this section, we employ the derivative-free descent method studied in [37] to test the numerical performance based on various value of p. The target of the derivative-free descent method studied in [37] is mainly on SOCCP (second-order cone complementarity problem). Hence, we consider the following SOCCP:

$$\begin{gathered} z \in \mathcal{K}, \quad Mz + b \in \mathcal{K}, \quad z^T(Mz+b) = 0, \\ \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_r. \end{gathered}$$

According to our results, the above SOCCP can be recast as an unconstrained minimization problem:

$$\min_{\zeta \in \mathbb{R}^n} \Psi_p(\zeta) = \frac{1}{2} \|\phi^p_{\text{D-FB}}(\zeta, F(\zeta))\|^2,$$

where $F(\zeta) = M\zeta + b$.

All tests are done on a PC using Inter core i7-5600U with 2.6GHz and 8GB RAM, and the codes are written in Matlab 2010b. The test instances are generated randomly. In particular, we first generate random sparse square matrices $N_i (i = 1, 2 \ldots r)$ with density 0.01, in which non-zero elements are chosen randomly from a normal distribution with mean -1 and variance 4. Then, we create the positive semidefinite matrix M_i for $(i = 1, 2 \ldots r)$ by setting $M_i := N_i N_i^T$ and let $M := \text{diag}(M_1, \ldots, M_r)$. In addition, we take vector $b := -Mw$ with $w = (w_1, \ldots, w_r)$ and $w_i \in \mathcal{K}_i$. With these M and b, it is not hard to verify that the corresponding SOCCP has at least a feasible solution. To construct SOCs of various types, we set $n_1 = n_2 = \cdots = n_r$.

We implement a test problem generated as above with $n = 1000$ and $r = 100$. The parameters in the algorithm are set as

$$\beta = 0.9, \quad \gamma = 0.8, \quad \sigma = 10^{-4}, \text{ and } \epsilon = 10^{-8}.$$

We start with the initial point

$$\zeta_0 = (\zeta_{n_1}, \cdots, \zeta_{n_r}) \text{ where } \zeta_{n_i} = \left(10, \frac{w_i}{\|w_i\|}\right)$$

with $w_i \in \mathbb{R}^{n_i - 1}$ being generated randomly. The stopping criteria, i.e., $\Psi_p(\zeta^k) \le \epsilon$, is either the number of iteration is over 10^5 or a step-length is less than 10^{-12}. The Figure 4 depicts detailed iteration process of the algorithm corresponding to different value of p.

The algorithm fails for the problem when $p \geq 5$. The main reason is that the step-length is too small eventually. We also suspect that larger p leads to tedious computation of the complementarity function in Jordan algebra. Anyway, this phenomenon indicates that the discrete-type of complementarity functions only work well for small value of p. The convergence in Figure 4 shows the method with a bigger p has a faster reduction of Ψ_p at the beginning, and the method with a smaller p has a faster reduction of Ψ_p eventually. Moreover, the bigger p applies, the total number of iterations of the algorithm is less.

In order to check numerical performance of the algorithm corresponding to different value of p, we solve the test problems with different dimension. The numerical results are summarized in Tables 1. "$\Psi_p(\zeta^*)$" and "Gap" denote the merit function value and the value of $\left|\zeta^T F(\zeta)\right|$ at the final iteration, respectively. "NF", "Iter", and "Time" indicate the number of function evaluations of Ψ_p, the number of iteration required in order to satisfy the termination condition, and the CPU time in second for solving each problem, respectively.

Table 1: Numerical results with different value of p

Problem	$p=1$					$p=1.4$				
(n,r)	$\Phi_p(\zeta^*)$	NF	Iter	Gap	time	$\Phi_p(\zeta^*)$	NF	Iter	Gap	time
(100,10)	9.8e-9	5350	4952	2.75e-4	9.3	1.0e-8	4401	1474	5.92e-5	3.5
(200,20)	9.4e-9	5064	4914	3.74e-5	16.5	1.0e-8	16179	5649	3.84e-5	25.9
(300,30)	1.0e-8	7445	5273	2.26e-4	30.3	9.9e-9	7000	1266	2.40e-5	11.5
(400,40)	9.8e-9	5342	5016	1.62e-4	50.0	9.9e-9	3747	857	4.31e-5	9.5
(500,50)	1.0e-8	23533	13749	6.81e-4	126.4	9.6e-9	29454	6257	3.39e-4	93.9
(600,60)	1.0e-8	18260	11119	16.1e-4	65.1	1.0e-8	24685	8320	8.69e-5	119.7
(700,70)	1.0e-8	8320	5690	6.16e-4	38.3	1.0e-8	13458	4493	1.79e-4	77.7
(800,80)	1.0e-8	29415	10149	4.43e-5	199.2	9.3e-9	2507	1838	1.54e-4	27.4
(900,90)	1.0e-8	14648	10888	1.46e-3	159.8	9.9e-9	5970	1621	8.77e-5	44.9
(1000,100)	1.0e-8	14590	9672	2.78e-4	238.3	1.0e-8	12337	2570	7.58e-5	92.0
(1100,110)	9.9e-9	5994	5406	4.64e-6	109.6	1.0e-8	13767	2948	3.51e-4	126.5
(1200,120)	9.8e-9	6100	5528	6.12e-5	121.7	9.9e-9	20990	5650	1.51e-5	211.4
(1300,130)	9.8e-9	4253	3612	2.42e-4	115.5	9.7e-9	777	316	5.78e-5	10.1
(1400,140)	1.0e-8	9827	7136	1.46e-4	307.5	1.0e-8	6357	2736	2.20e-4	70.6
(1500,150)	9.9e-9	4701	4211	3.04e-4	156.9	9.9e-9	7060	1823	6.56e-6	67.8
(1600,160)	9.9e-9	5744	3843	4.61e-4	172.8	1.0e-8	9434	2583	1.39e-4	82.9
(1700,170)	1.0e-8	11163	5581	2.74e-4	195.1	1.0e-8	12307	2740	9.87e-5	185.7
(1800,180)	1.0e-8	7449	5985	3.77e-4	204.5	1.0e-8	38524	9469	2.43e-4	439.8
(1900,190)	1.0e-8	4205	2102	7.19e-5	83.2	1.0e-8	7413	1636	3.40e-4	125.4
(2000,200)	9.9e-9	5189	4953	2.12e-4	212.9	9.15e-9	10230	480	2.32e-5	294.9

We also use the performance profiles introduced by Dolan and Morè [18] to compare the performance of algorithm with different p. The performance profiles are generated by executing solvers $\mathcal{S}$ on the test set $\mathcal{P}$. Let $n_{p,s}$ be the number of iteration (or the

Table 2: Numerical results with different value of p

Problem	$p=2.6$					$p=3$				
(n,r)	$\Phi_p(\zeta^*)$	NF	Iter	Gap	time	$\Phi_p(\zeta^*)$	NF	Iter	Gap	time
(100,10)	9.9e-9	28878	1866	2.40e-6	11.9	9.2e-9	11281	201	3.80e-7	14.7
(200,20)	1.0e-8	57844	3743	1.64e-6	47.9	9.5e-9	21221	422	1.15e-6	52.9
(300,30)	9.9e-9	14452	963	3.14e-6	17.3	9.2e-9	4383	89	5.97e-7	17.5
(400,40)	9.8e-9	20747	1417	2.31e-6	32.7	9.9e-9	7419	133	8.34e-7	34.0
(500,50)	9.8e-9	13929	1084	1.53e-6	30.7	8.4e-9	27229	474	1.04e-6	87.8
(600,60)	9.9e-9	28224	2032	2.48e-7	77.1	9.9e-9	48809	878	4.19e-7	193.8
(700,70)	9.9e-9	16739	1230	1.93e-5	52.8	7.9e-9	7069	140	6.16e-4	58.4
(800,80)	9.9e-9	72745	5342	7.69e-7	270.5	9.8e-9	27620	534	5.95e-7	260.1
(900,90)	9.5e-9	7574	522	6.09e-7	37.5	8.0e-9	10276	187	1.35e-7	129.6
(1000,100)	1.0e-8	145414	8664	4.92e-7	821.6	9.6e-9	17790	325	2.26e-7	258.2
(1100,110)	9.7e-9	16834	1465	3.76e-7	111.0	9.5e-9	31750	528	6.41e-7	507.2
(1200,120)	9.9e-9	45621	3346	1.82e-6	271.5	9.8e-9	20326	370	4.82e-7	437.4
(1300,130)	1.0e-8	25661	1739	3.21e-6	171.8	8.9e-9	10399	185	7.16e-7	115.5
(1400,140)	9.8e-9	57526	4116	2.09e-5	277.6	8.9e-9	12529	205	1.09e-6	348.4
(1500,150)	1.0e-8	355478	321117	1.50e-5	2343.0	4.7e-3	11824	217	1.54e-5	393.5
(1600,160)	9.3e-9	12995	5961	1.70e-6	98.5	9.9e-9	33843	550	5.43e-7	862.2
(1700,170)	1.0e-8	47367	3380	8.64e-7	441.0	1.0e-8	80519	5084	1.73e-7	742.8
(1800,180)	9.8e-9	7697	536	1.67e-6	53.0	7.4e-9	8472	154	4.15e-8	289.6
(1900,190)	1.0e-8	149019	10644	2.59e-6	1577.9	1.0e-8	16128	909	5.84e-7	161.5
(2000,200)	1.0e-8	27876	1991	2.64e-6	238.5	1.0e-8	34310	630	1.37e-7	862.2

computing time) required to solve problem $p \in \mathcal{P}$ by solver $s \in \mathcal{S}$, and define the performance ratio as

$$r_{p,s} = \frac{n_{p,s}}{\min\{n_{p,s} : 1 \leq s \leq n_s\}},$$

where n_s is the number of solvers. Whenever the solver s does not solve problem p successfully, set $r_{p,s} = r_M$. Here r_M is a very large preset positive constant. Then, performance profile for each solver s is defined by

$$\rho_s(\chi) = \frac{1}{n_p}\text{size}\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}.$$

where size$\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}$ is the number of elements in the set $\{p \in \mathcal{P} : \log_2(r_{p,s}) \leq \chi\}$. $\rho_s(\chi)$ represents the probability that the performance ratio $r_{p,s}$ is within the factor 2^χ. It is easy to see that $\rho_s(0)$ is the probability that the solver s wins over the rest of solvers. See [18] for more details about the performance profile.

From Figure 5(a), it shows that the algorithm with $p = 1$ and $p = 1.4$ performs better than $p = 2.6$ and $p = 3$ on function evaluations. Similarly, from Figure 5(b) and Figure 5(c), we observe that the algorithm with $p = 3$ performs best on the number of iterations, while the algorithm with $p = 1.4$ is the best one on CPU time. This provides evidence that the discrete type of complementarity function may be better than the well-known function ϕ_{FB} in some cases.

6 Conclusion

In this paper, we propose a few families of new NCP-functions and investigate their differentiability. Then, these new families of NCP-functions have also shown that they can serve as complementarity functions associated with second-order cone in light of Jordan algebra. We also construct several variants of such complementarity functions for NCP and SOCCP. The behind idea for constructing all such new complementarity functions is based on "discrete generalization" which is a novel thinking. In contrast to the traditional "continuous generalization", this opens a new direction for future research.

As below, we explain why we adopt "discrete-type" for our new NCP-functions. First, for the generalized Natural-Residual function $\phi^p_{\text{NR}}(a,b) = a^p - (a-b)^p_+$, as remarked in [16], the parameter p must be odd integer to ensure that the generalization is also an NCP-function. This means that the main idea to create the new functions relies on "discrete generalization", it is totally different from the concept of generalization of Fischer-Burmeister function $\phi^p_{\text{FB}}(a,b) = \sqrt[p]{|a|^p + |b|^p} - (a+b)$, as remarked in [7], the parameter p may be any real number which is great or equal to 1. That is why we call our generalization "discrete-type".

In fact, there is another way to achieve $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} which was proposed in [26]. More specifically, it is a construction based on monotone transformations to create new NCP-functions from the existing ones. The construction is stated as below.

Remark 6.1. ([26, Lemma 15]) *Assume that ϕ is continuous and $\phi(a,b) = f_1(a,b) - f_2(a,b)$. Let $\theta : \mathbb{R} \to \mathbb{R}$ be a strictly monotone increasing and continuous function. Then ϕ is an NCP function if and only if $\psi_\theta(a,b) = \theta(f_1(a,b)) - \theta(f_2(a,b))$ is an NCP-function.*

In light of this, we let the function $\theta = \theta_p$ be $\theta_p(t) = \text{sign}(t)|t|^p$, where "sign$(t)$" is the sign function and $p \geq 1$. For Fischer-Burmeister function, we choose $f_1(a,b) = \sqrt{a^2+b^2}$, $f_2(a,b) = a+b$, and for Natural-Residual function, we choose $f_1(a,b) = a$, $f_2(a,b) = (a-b)_+$, then it can be verified that both $\phi^p_{\text{D-FB}}$ and ϕ^p_{NR} (only with odd integer p) can be obtained from the function ψ_{θ_p}. In other words, the function ψ_{θ_p} includes both them as special cases, from which we may view it as a "continuous generalization". Yes, the Galantai's method [26] is more general than our way. Nonetheless, we emphasize that the NCP-functions generated by our approach are shown to be complementarity functions in the SOCCP setting. This can be used to generate new SOCCP-functions, which is one of the main contributions of this paper. It will be a future direction to check whether Galantai's NCP-functions can be extended to SOCCP setting as well and describe the relation therein.

In general, the Newton method may not be applicable even though we have the differentiability for some new complementarity functions because the Jacobian at a degenerate solution is singular (see [32, 33]). Nonetheless, some derivative-free algorithm may be employed due to the differentiability. On the other hand, we can reformulate NCP and SOCCP as nonsmooth equations or unconstrained minimization, for which merit function approach, nonsmooth function approach, smoothing function approach, and regularization approach can be studied. All the new complementarity functions can be employed in these approaches. How these new families of complementarity functions perform in contrast to the existing ones? This is the first question that we are eager to know. Some other questions, like are there any benefits for "discrete generalization" compared to "continuous generalization", can these proposed complementarity functions be employed for other types of problems including semi-definite complementarity problems and symmetric cone complementarity problems, etc? We leave them as future research topics.

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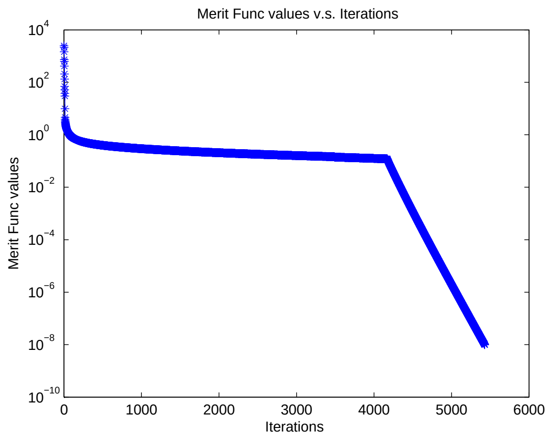
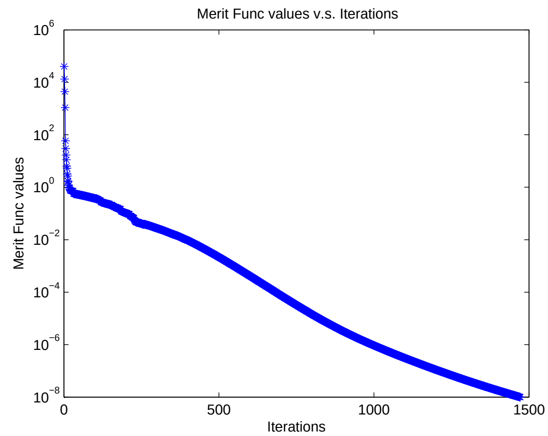
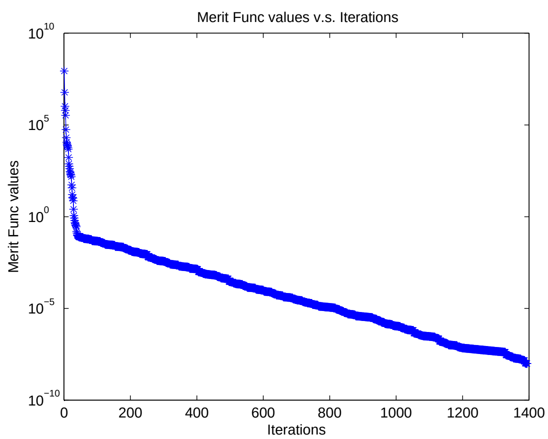


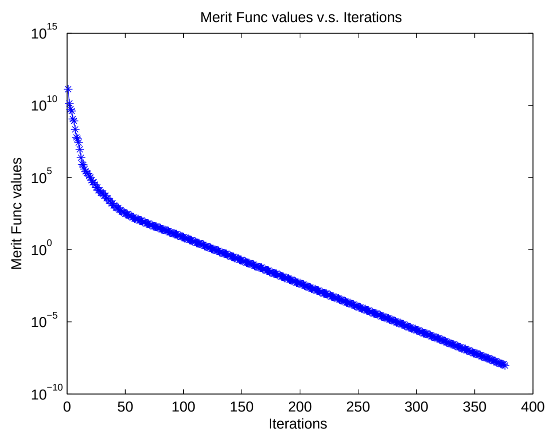
(a) $p = 1$



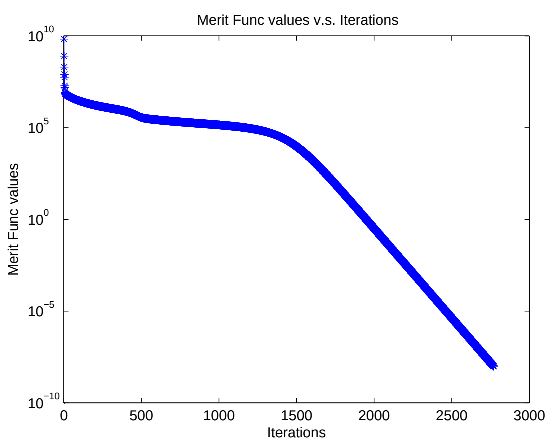
(b) $p = 1.4$



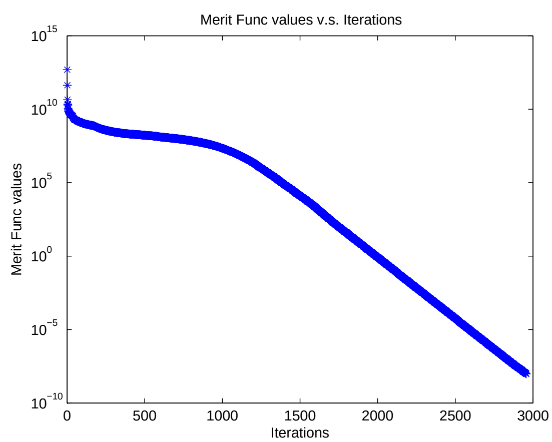
(c) $p = 2.6$



(d) $p = 3$

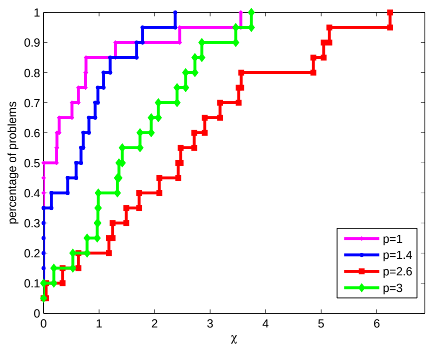


(e) $p = 3.4$

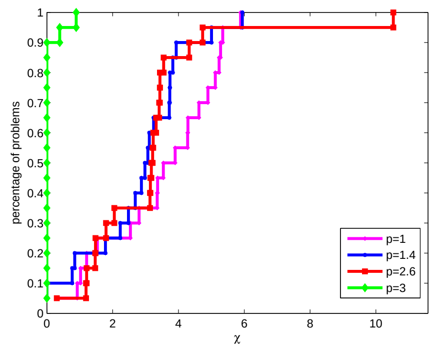


(f) $p = 5$

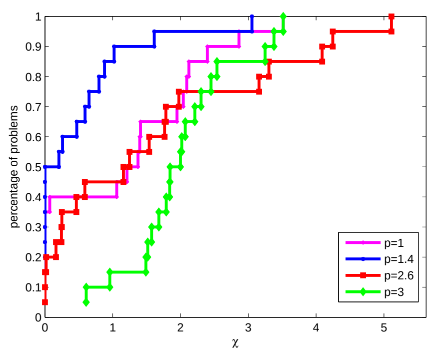
Figure 4: Convergence behaviour of $\Phi_p(\zeta^k)$ with different value of p



(a) Performance profile of NF



(b) Performance profile of Iter



(c) Performance profile of CPU time

Figure 5: Performance profiles with different value of p