Comput Optim Appl (2016) 63:495–522
DOI 10.1007/s10589-015-9781-1

Constructions of complementarity functions and merit functions for circular cone complementarity problem

Xin-He Miao[1] · **Shengjuan Guo**[1] · **Nuo Qi**[1] · **Jein-Shan Chen**[2]

Received: 29 March 2015 / Published online: 20 August 2015

Abstract In this paper, we consider complementarity problem associated with circular cone, which is a type of nonsymmetric cone complementarity problem. The main purpose of this paper is to show the readers how to construct complementarity functions for such nonsymmetric cone complementarity problem, and propose a few merit functions for solving such a complementarity problem. In addition, we study the conditions under which the level sets of the corresponding merit functions are bounded, and we also show that these merit functions provide an error bound for the circular cone complementarity problem. These results ensure that the sequence generated by descent methods has at least one accumulation point, and build up a theoretical basis for designing the merit function method for solving circular cone complementarity problem.

Keywords Circular cone complementarity problem · Complementarity function · Merit function · The level sets · Strong coerciveness

✉ Jein-Shan Chen
jschen@math.ntnu.edu.tw

Xin-He Miao
xinhemiao@tju.edu.cn

Shengjuan Guo
gshengjuan@163.com

Nuo Qi
qinuo@163.com

1 Department of Mathematics, School of Science, Tianjin University, Tianjin 300072, People's Republic of China

2 Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

1 Motivation and introduction

The general conic complementarity problem is to find an element $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \quad F(x) \in \mathcal{K}^* \quad \text{and} \quad \langle x, F(x) \rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable mapping, $\mathcal{K}$ represents a closed convex cone, and $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$ given by

$$\mathcal{K}^* := \{v \in \mathbb{R}^n \mid \langle v, x \rangle \geq 0, \ \forall x \in \mathcal{K}\}.$$

When $\mathcal{K}$ is a symmetric cone, the problem (1) is called the symmetric cone complementarity problem [12,14,18,20]. In particular, when $\mathcal{K}$ is the so-called second-order cone which is defined as

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\},$$

the problem (1) reduces to the second-order cone complementarity problem [1,3–5,10,11]. In contrast to symmetric cone programming and symmetric cone complementarity problem, we are not familiar with their nonsymmetric counterparts. Referring the reader to [16,19] and the bibliographies therein, we observe that there is no any unified way to handle nonsymmetric cone constraints, and the study on each item for such problems usually uses certain specific features of the nonsymmetric cones under consideration.

In this paper, we pay attention to a special nonsymmetric cone $\mathcal{K}$ for problem (1). In particular, we focus on the case of $\mathcal{K}$ being the circular cone defined as below, which enables the problem (1) reduce to the circular cone complementarity problem (CCCP for short). Indeed in $\mathbb{R}^n$, the circular cone [7,23] is a pointed closed convex cone having hyper-spherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its half-aperture angle be θ with $\theta \in (0, \frac{\pi}{2})$. Then, the circular cone denoted by $\mathcal{L}_\theta$ can be expressed as

$$\begin{aligned} \mathcal{L}_\theta &:= \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x\| \cos\theta \leq x_1 \right\} \\ &= \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1 \tan\theta \right\}. \end{aligned} \tag{2}$$

When $\theta = \frac{\pi}{4}$, the circular cone is exactly the second-order cone, which means the circular cone complementarity problem is actually the second-order cone complementarity problem. Thus, the circular cone complementarity problem (CCCP) can be viewed as the generalization of the second-order cone complementarity problem. Moreover, the CCCP includes the KKT system of the circular programming problem [13] as a special case. For real world applications of optimization problems involving circular cones, please refer to [6]. Note that in [23], Zhou and Chen characterize the relation between circular cone $\mathcal{L}_\theta$ and second-order cone as follows:

$$\mathcal{L}_\theta = A^{-1}\mathcal{K}^n \quad \text{and} \quad \mathcal{K}^n = A\mathcal{L}_\theta \quad \text{with} \quad A = \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}.$$

In other words, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, there have

$$x \in \mathcal{L}_\theta \Longleftrightarrow Ax \in \mathcal{K}^n, \qquad y \in \mathcal{L}_\theta^* \Longleftrightarrow A^{-1}y \in \mathcal{K}^n. \tag{3}$$

Relation (3) indicates that after scaling the circular cone complementarity problem and the second-order cone complementarity problem are equivalent. However, when dealing with the circular cone complementarity problem, this approach may not be acceptable from both theoretical and numerical viewpoints. Indeed, if the appropriate scaling is not found or checked, some scaling step can cause undesirable numerical performance due to round-off errors in computers, which has been confirmed by experiments. Moreover, it usually need to exploits its associated merit functions or complementarity functions, which plays an important role in tackling complementarity problem. To this end, we are devoted to seeking a way to construct complementarity functions and merit functions for the circular cone complementarity problem directly. Thus, we pay our attention to the circular cone complementarity problem and the structure of $\mathcal{L}_\theta$ mainly. There is another relationship between the circular cone and the (nonsymmetric) matrix cone introduced in [8,9], where the authors study the epigraph of six different matrix norms, such as the Frobeninus norm, the l_∞ norm, l_1 norm, the spectral or the operator norm, the nuclear norm, the Ky Fan k-norm. If we regard a matrix as a high-dimensional vector, then the circular cone is equivalent to the matrix cone with Frobeninus norm, see [24] for more details.

While there have been much attention to the symmetric cone complementarity problem and the second-order cone complementarity problem, the study about nonsymmetric cone complementarity problem is very limited. The main difficulty is that the idea for constructing complementarity functions (C-functions for short) and merit functions is not clear. Hence, The main goal of this paper is showing the readers how to construct C-functions and merit functions for such complementarity problem, and studying the properties of these merit functions. To our best knowledge, the idea is new and we believe that it will help in analyzing other types of nonsymmetric cone complementarity problems.

Recall that for solving the problem (1), a popular approach is to reformulate it as an unconstrained smooth minimization problem or a system of nonsmooth equations. In this category of methods, it is important to adapt a merit function. Officially, a *merit function* for the circular cone complementarity problem is a function $h : \mathbb{R}^n \to [0, +\infty)$, provided that

$$h(x) = 0 \quad \Longleftrightarrow \quad x \ \text{ solves the CCCP (1).}$$

Hence, solving the problem (1) is equivalent to handling the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} h(x)$$

with the optimal value zero. For constructing the merit functions in finite dimensional vector space, please refer to [17]. Until now, for solving symmetric cone complementarity problem, a number of merit functions have been proposed. Among them, one of the most popular merit functions is the natural residual (NR) merit function $\Psi_{NR} : \mathbb{R}^n \to \mathbb{R}$, which is defined as

$$\Psi_{NR}(x) := \frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2 = \frac{1}{2}\|x - (x - F(x))_+\|^2,$$

where $(\cdot)_+$ denotes the projection onto the symmetric cone $\mathcal{K}$. It is well known that $\Psi_{NR}(x) = 0$ if and only if x is a solution to the symmetric cone complementarity problem. In this paper, we present two classes of complementarity functions and four types of merit functions for the circular cone complementarity problem. Moreover, we investigate the properties of these proposed merit functions, and study conditions under which these merit functions provide bounded level sets. Note that such properties will guarantee that the sequence generated by descent methods has at least one accumulation point, and build up a theoretical basis for designing the merit function method for solving circular cone complementarity problem.

2 Preliminaries

In this section, we briefly review some basic concepts and background materials about the circular cone and second-order cone, which will be extensively used in subsequent analysis.

As defined in (2), the circular cone $\mathcal{L}_\theta$ is a pointed closed convex cone and has a revolution axis which is the ray generated by the canonical vector $e_1 := (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. Its dual cone denoted by $\mathcal{L}_\theta^*$ is given as

$$\mathcal{L}_\theta^* := \{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|y\| \sin\theta \le y_1\}.$$

Note that the circular cone $\mathcal{L}_\theta$ is not a self-dual cone when $\theta \neq \frac{\pi}{4}$, that is, $\mathcal{L}_\theta^* \neq \mathcal{L}_\theta$, whenever $\theta \neq 45^\circ$. Hence, $\mathcal{L}_\theta$ is not a symmetric cone for $\theta \in \left(0, \frac{\pi}{2}\right) \setminus \{\frac{\pi}{4}\}$. It is also known from [23] that the dual cone of $\mathcal{L}_\theta$ can be expressed as

$$\mathcal{L}_\theta^* = \{y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|y_2\| \le y_1 \cot\theta\} = \mathcal{L}_{\frac{\pi}{2}-\theta}.$$

Now, we talk about the projection onto $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. To this end, we let x_+ denote the projection of x onto the circular cone $\mathcal{L}_\theta$, and x_- be the projection of $-x$ onto the dual cone $\mathcal{L}_\theta^*$. With these notations, for any $x \in \mathbb{R}^n$, it can be verified that $x = x_+ - x_-$. Moreover, due to the special structure of the circular cone $\mathcal{L}_\theta$, the explicit formula of projection of $x \in \mathbb{R}^n$ onto $\mathcal{L}_\theta$ is obtained in [23] as below:

$$x_+ = \begin{cases} x \text{ if } x \in \mathcal{L}_\theta, \\ 0 \text{ if } x \in -\mathcal{L}_\theta^*, \\ u \text{ otherwise}, \end{cases} \tag{4}$$

where

$$u = \begin{bmatrix} \dfrac{x_1 + \|x_2\| \tan\theta}{1 + \tan^2\theta} \\ \left(\dfrac{x_1 + \|x_2\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \dfrac{x_2}{\|x_2\|} \end{bmatrix}.$$

Similarly, we can obtain the expression of x_- as below:

$$x_- = \begin{cases} 0 & \text{if } x \in \mathcal{L}_\theta, \\ -x & \text{if } x \in -\mathcal{L}_\theta^*, \\ w & \text{otherwise}, \end{cases} \tag{5}$$

where

$$w = \begin{bmatrix} -\dfrac{x_1 - \|x_2\| \cot\theta}{1 + \cot^2\theta} \\ \left(\dfrac{x_1 - \|x_2\| \cot\theta}{1 + \cot^2\theta} \cot\theta \right) \dfrac{x_2}{\|x_2\|} \end{bmatrix}.$$

From the expressions (4)–(5) for x_+ and x_-, it is easy to verity that $\langle x_+, x_- \rangle = 0$ for any $x \in \mathbb{R}^n$.

Next, we introduce the Jordan product associated with second-order cone. As mentioned earlier, the SOC in $\mathbb{R}^n$ (also called Lorentz cone or ice-cream cone) is defined by

$$\mathcal{K}^n := \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1\}.$$

It is well known that the dual cone of $\mathcal{K}^n$ is itself, and the second-order cone $\mathcal{K}^n$ belongs to a class of symmetric cones. In addition, $\mathcal{K}^n$ is a special case of $\mathcal{L}_\theta$ corresponding to $\theta = \frac{\pi}{4}$. In fact, there is a relationship between $\mathcal{L}_\theta$ and $\mathcal{K}^n$, which is described in (3). In the SOC setting, there is so-called Jordan algebra associated with SOC. More specifically, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, in the setting of the SOC, the **Jordan product** of x and y is defined as

$$x \circ y := \begin{bmatrix} \langle x, y \rangle \\ y_1 x_2 + x_1 y_2 \end{bmatrix}.$$

The Jordan product "$\circ$", unlike scalar or matrix multiplication, is not associative. The identity element under Jordan product is $e = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$. In this paper, we write x^2 to mean $x \circ x$. It is known that $x^2 \in \mathcal{K}^n$ for any $x \in \mathbb{R}^n$, and if $x \in \mathcal{K}^n$, there exists a unique vector denoted by $x^{\frac{1}{2}}$ in $\mathcal{K}^n$ such that $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}} \circ x^{\frac{1}{2}} = x$. For any $x \in \mathbb{R}^n$, we denote $|x| := \sqrt{x^2}$ and x_+^{soc} means the orthogonal projection of x onto the second-order cone $\mathcal{K}^n$. Then, it follows that $x_+^{\mathrm{soc}} = \dfrac{x + |x|}{2}$. For further details regarding the SOC and Jordan product, please refer to [1,3,5,10].

Lemma 2.1 ([10, Proposition 2.1]) *For any $x, y \in \mathbb{R}^n$, the following holds:*

$$x \in \mathcal{K}^n,\ y \in \mathcal{K}^n,\ \text{and } \langle x, y\rangle = 0 \iff x \in \mathcal{K}^n,\ y \in \mathcal{K}^n,\ \text{and } x \circ y = 0.$$

With the help of (3) and Lemma 2.1, we obtain the following theorem which explains the relationship between SOCCP and CCCP.

Theorem 2.1 *Let $A = \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}$. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the following are equivalent:*

(a) $x \in \mathcal{L}_\theta,\ y \in \mathcal{L}_\theta^*$ *and* $\langle x, y\rangle = 0$.
(b) $Ax \in \mathcal{K}^n,\ A^{-1}y \in \mathcal{K}^n$ *and* $\langle Ax, A^{-1}y\rangle = 0$.
(c) $Ax \in \mathcal{K}^n,\ A^{-1}y \in \mathcal{K}^n$ *and* $Ax \circ A^{-1}y = 0$.
(d) $x \in \mathcal{L}_\theta,\ y \in \mathcal{L}_\theta^*$ *and* $Ax \circ A^{-1}y = 0$.

In each case, elements x and y satisfy the condition that either y_2 is a multiple of x_2 or x_2 is a multiple of y_2.

Proof From the relation between $\mathcal{K}^n$ and $\mathcal{L}_\theta$ given as in (3), we know that

$$x \in \mathcal{L}_\theta \iff Ax \in \mathcal{K}^n \quad \text{and} \quad y \in \mathcal{L}_\theta^* \iff A^{-1}y \in \mathcal{K}^n.$$

Moreover, under condition (a), there holds

$$\langle Ax, A^{-1}y\rangle = \langle A^{-1}Ax, y\rangle = \langle x, y\rangle = 0.$$

Hence, it follows that (a) and (b) are equivalent. The equivalence of (b) and (c) has been shown in Lemma 2.1. In addition, based on the relation between $\mathcal{K}^n$ and $\mathcal{L}_\theta$ again, the equivalence of (c) and (d) is obvious.
Now, under condition (a), we prove that either y_2 is a multiple of x_2 or x_2 is a multiple of y_2. To see this, note that $x \in \mathcal{L}_\theta$ and $y \in \mathcal{L}_\theta^*$ which gives

$$\|x_2\| \le x_1 \tan\theta \quad \text{and} \quad \|y_2\| \le y_1 \cot\theta.$$

This together with $\langle x, y\rangle = 0$ yields

$$\begin{aligned} 0 &= \langle x, y\rangle \\ &= x_1 y_1 + \langle x_2, y_2\rangle \\ &\ge x_1 y_1 - \|x_2\|\|y_2\| \\ &\ge x_1 y_1 - x_1 y_1 \\ &= 0 \end{aligned}$$

which implies $\langle x_2, y_2\rangle = \|x_2\|\|y_2\|$. This says that either y_2 is a multiple of x_2 or x_2 is a multiple of y_2. Thus, the proof is complete. □

3 C-functions for CCCP

In this section, we define C-functions for CCCP and the product of elements in the setting of the circular cone. Moreover, based on the product of elements, we construct some C-functions which play an important role in solving the circular cone complementarity problems by merit function methods.

Definition 3.1 Given a mapping $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, we call ϕ an C-function for CCCP if, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, it satisfies

$$\phi(x, y) = 0 \Longleftrightarrow x \in \mathcal{L}_\theta,\ \ y \in \mathcal{L}_\theta^*,\ \ \langle x, y \rangle = 0.$$

When $\theta = \frac{\pi}{4}$, an C-function for CCCP reduces to an C-function for SOCCP, i.e.,

$$\phi(x, y) = 0 \Longleftrightarrow x \in \mathcal{K}^n,\ \ y \in \mathcal{K}^n,\ \ \langle x, y \rangle = 0.$$

Two popular and well-known C-functions for SOCCP are Fischer-Burmeister (FB) function and natural residual (NR) function:

$$\begin{aligned} \phi_{\rm FB}(x, y) &= \left(x^2 + y^2\right)^{1/2} - (x + y), \\ \phi_{\rm NR}(x, y) &= x - (x - y)_+^{\rm soc}. \end{aligned}$$

We may ask whether we can modify the above two C-functions for SOCCP to form C-functions for CCCP. The answer is affirmative. In fact, we consider

$$\begin{aligned} \widetilde{\phi_{\rm FB}}(x, y) &:= \left[(Ax)^2 + (A^{-1}y)^2\right]^{\frac{1}{2}} - (Ax + A^{-1}y), \\ \widetilde{\phi_{\rm NR}}(x, y) &:= Ax - [Ax - A^{-1}y]_+^{\rm soc}. \end{aligned}$$

Then, these two functions are C-functions for CCCP.

Proposition 3.1 *Let $\widetilde{\phi_{\rm FB}}$ and $\widetilde{\phi_{\rm NR}}$ be defined as above where $(Ax)^2$ equals $(Ax) \circ (Ax)$ under Jordan product. Then, $\widetilde{\phi_{\rm FB}}$ and $\widetilde{\phi_{\rm NR}}$ are both C-functions for CCCP.*

Proof In view of Theorem 2.1 and Definition 3.1, it is not hard to verify that

$$\begin{aligned} \widetilde{\phi_{\rm FB}}(x, y) = 0 &\Longleftrightarrow x \in \mathcal{L}_\theta,\ \ y \in \mathcal{L}_\theta^*,\ \ \langle x, y \rangle = 0, \\ \widetilde{\phi_{\rm NR}}(x, y) = 0 &\Longleftrightarrow x \in \mathcal{L}_\theta,\ \ y \in \mathcal{L}_\theta^*,\ \ \langle x, y \rangle = 0, \end{aligned}$$

which says that these two functions are C-functions for CCCP. □

We point out that if we consider directly the FB function $\phi_{\rm FB}(x, y)$ for CCCP, unfortunately, it cannot be C-function for CCCP because x^2 is not well-defined associated with the circular cone $\mathcal{L}_\theta$ for any $x \in \mathbb{R}^n$. More specifically, because x^2 is defined under the Jordan product in the setting of SOC, i.e.,

$$x^2 := x \circ x = \begin{bmatrix} \langle x, y \rangle \\ x_1 y_2 + y_1 x_2 \end{bmatrix},$$

it follows that $x^2 \in \mathcal{K}^n$, which implies x^2 may not belong to $\mathcal{L}_\theta$ or $\mathcal{L}_\theta^*$. Furthermore, when $\phi_{\rm FB}(x, y) = 0$, we have $x + y = \left(x^2 + y^2\right)^{\frac{1}{2}} \in \mathcal{K}^n$, which yields that $x, y \in \mathcal{K}^n$. This says that either $x \notin \mathcal{L}_\theta$ or $y \notin \mathcal{L}_\theta^*$. All the above explains that the FB function $\phi_{\rm FB}$ cannot be an C-function for CCCP. Nonetheless, the NR function $\phi_{\rm NR} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\phi_{\rm NR}(x, y) := x - (x - y)_+ \tag{6}$$

is always an C-function for CCCP. Moreover, it is also an C-function for general cone complementarity problem, see [11, Proposition 1.5.8].

Are there any other types of C-functions for CCCP and how to construct an C-function for CCCP? As mentioned earlier, The FB function $\phi_{\rm FB}$ cannot serve as C-functions for CCCP because "x^2" is not well-defined in the setting of circular cone. This inspires us to define a special product associated with circular cone, and find other C-functions for CCCP.

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define one type of product of x and y as follows:

$$x \bullet y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \bullet \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \langle x, y \rangle \\ \max\{\tan^2\theta, 1\}\, x_1 y_2 + \max\{\cot^2\theta, 1\}\, y_1 x_2 \end{bmatrix}. \tag{7}$$

From the above product and direct calculation, it is easy to verify that

$$\langle x \bullet y, z \rangle = \langle x, z \bullet y \rangle, \quad \forall z \in \mathbb{R}^n \text{ with } \theta \in \left(0, \frac{\pi}{4}\right] \tag{8}$$

and

$$\langle x \bullet y, z \rangle = \langle y, x \bullet z \rangle, \quad \forall z \in \mathbb{R}^n \text{ with } \theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right). \tag{9}$$

Moreover, we also obtain the following inequalities which are crucial to establishing our main results.

Lemma 3.1 *For any $x, y \in \mathbb{R}^n$,*

(a) *if $\theta \in (0, \frac{\pi}{4}]$, we have $\langle x_-, x_+ \bullet (-y)_- \rangle \le 0$;*
(b) *if $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, we have $\langle (-y)_+, x_+ \bullet (-y)_- \rangle \le 0$.*

Proof (a) When $\theta \in (0, \frac{\pi}{4}]$, let $x_+ := (s, u) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $x_- := (t, v) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $(-y)_- := (k, w) \in \mathbb{R} \times \mathbb{R}^{n-1}$. For the elements x_+, x_- and $(-y)_-$, if there exist at least one in them is zero, it is easy to obtain

$$\langle x_-, x_+ \bullet (-y)_- \rangle = 0.$$

If all the three elements are not equal to zero, from the definition of x_+, x_-, and $(-y)_-$, we have $k\cot\theta \geq \|w\|$, $s\tan\theta = \|u\|$, $t\cot\theta = \|v\|$ and

$$u = \alpha v \text{ or } v = \alpha u \quad \text{with} \quad \alpha < 0.$$

Without loss of generality, we consider the case $u = \alpha v$ with $\alpha < 0$ for the following analysis. In fact, using this, we know that

$$\begin{aligned}
&\langle x_-, x_+ \bullet (-y)_- \rangle \\
&= stk + t\langle u, w\rangle + s\langle v, w\rangle + k\langle u, v\rangle \cot^2\theta \\
&= \|u\|\|v\|k - k\|u\|\|v\|\cot^2\theta - \|u\|\langle v, w\rangle \tan\theta + \|u\|\langle v, w\rangle \cot\theta \\
&= (1-\cot^2\theta)k\|u\|\|v\| - (1-\cot^2\theta)(\|u\|\langle v, w\rangle \tan\theta) \\
&= (1-\cot^2\theta)[k\|u\|\|v\| - \|u\|\langle v, w\rangle \tan\theta] \\
&\leq (1-\cot^2\theta)[k\|u\|\|v\| - \|u\|\|v\|\|w\|\tan\theta] \\
&= (1-\cot^2\theta)\|u\|\|v\|[k - \|w\|\tan\theta] \\
&\leq 0.
\end{aligned}$$

Here the second equality is true due to $\alpha t = \alpha\|v\|\tan\theta = -\|u\|\tan\theta$. The last inequality holds due to $k\cot\theta \geq \|w\|$ and $\theta \in (0, \frac{\pi}{4}]$. Hence, the desired result follows.
(b) When $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, with the same skills, we also conclude that

$$\langle (-y)_+, x_+ \bullet (-y)_- \rangle \leq 0.$$

Then, the desired result follows. □

Besides the inequalities in Lemma 3.1, "$\bullet$" defined as in (7) plays the similar role like what "$\circ$" does in the setting of second-order cone. This is shown as below.

Theorem 3.1 *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the following statements are equivalent:*

(a) $x \in \mathcal{L}_\theta$, $y \in \mathcal{L}_\theta^*$ *and* $\langle x, y\rangle = 0$.
(b) $x \in \mathcal{L}_\theta$, $y \in \mathcal{L}_\theta^*$ *and* $x \bullet y = 0$.

In each case, x and y satisfy the condition that either y_2 is a multiple of x_2 or x_2 is a multiple of y_2.

Proof In view of Theorem 2.1, we know that part (a) is equivalent to

$$x \in \mathcal{L}_\theta, \quad y \in \mathcal{L}_\theta^* \quad \text{and} \quad Ax \circ A^{-1}y = 0.$$

To proceed the proof, we discuss the following two cases.
Case 1 For $\theta \in (0, \frac{\pi}{4}]$, from the definition of the product of x and y, we have

$$x \bullet y = \begin{bmatrix} \langle x, y\rangle \\ x_1 y_2 + \cot^2\theta \ y_1 x_2 \end{bmatrix}$$

which implies

$$Ax \circ A^{-1}y = \begin{bmatrix} \langle x, y\rangle \\ x_1 \tan\theta \; y_2 + \cot\theta \; y_1 x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & (\tan\theta) I \end{bmatrix} (x \bullet y).$$

This together with Theorem 2.1 yields the conclusion.

Case 2 For $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, from the definition of the product of x and y again, we have

$$x \bullet y = \begin{bmatrix} \langle x, y\rangle \\ \tan^2\theta \; x_1 y_2 + y_1 x_2 \end{bmatrix}$$

which says

$$Ax \circ A^{-1}y = \begin{bmatrix} 1 & 0 \\ 0 & (\cot\theta) I \end{bmatrix} (x \bullet y).$$

Then, applying Theorem 2.1 again, the desired result follows. □

Based on the product $x \bullet y$ of x and y. we now introduce a class of functions $\phi_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which is called the penalized natural residual function and defined as

$$\phi_p(x, y) = x - (x - y)_+ + p\,(x_+ \bullet (-y)_-)\,, \quad p > 0. \tag{10}$$

Note that when $p = 0$, $\phi_p(x, y)$ reduces to $\phi_{\mathrm{NR}}(x, y)$. In the following, we show that the function ϕ_p is an C-function for CCCP. To achieve the conclusion, a technical lemma is needed.

Lemma 3.2 *Let $\phi_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined as in (10). Then, for any $x, y \in \mathbb{R}^n$, we have*

$$\|\phi_p(x, y)\| \geq \max\{\|x_-\|, \|(-y)_+\|\}.$$

Proof First, we prove that $\|\phi_p(x, y)\| \geq \|x_-\|$. To see this, we observe that

$$\begin{aligned}
&\|\phi_p(x, y)\|^2 \\
&= \langle x - (x-y)_+ + p\,x_+ \bullet (-y)_-,\; x - (x-y)_+ + p\,x_+ \bullet (-y)_-\rangle \\
&= \langle x_+ - x_- - (x-y)_+ + p\,x_+ \bullet (-y)_-,\; x_+ - x_- - (x-y)_+ + p\,x_+ \bullet (-y)_-\rangle \\
&= \|x_-\|^2 + \|x_+ - (x-y)_+ + p\,x_+ \bullet (-y)_-\|^2 - 2\,\langle x_-, x_+ - (x-y)_+ + p\,x_+ \bullet (-y)_-\rangle \\
&\geq \|x_-\|^2 - 2\langle x_-, x_+\rangle + 2\,\langle x_-, (x-y)_+\rangle - 2\,\langle x_-,\; p\,x_+ \bullet (-y)_-\rangle \\
&\geq \|x_-\|^2 - 2p\;\langle x_-,\; x_+ \bullet (-y)_-\rangle\,.
\end{aligned}$$

Here, the last inequality is true due to $x_+, (x-y)_+ \in \mathcal{L}_\theta$, $x_- \in \mathcal{L}_\theta^*$, $\langle x_+, x_-\rangle = 0$ and the relation between $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. When $\theta \in (0, \frac{\pi}{4}]$, by Lemma 3.1(a), we have

$$\langle x_-,\; x_+ \bullet (-y)_-\rangle \leq 0.$$

When $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, from Eq. (9), we have

$$\langle x_-,\ x_+ \bullet (-y)_- \rangle = \langle (-y)_-,\ x_+ \bullet x_- \rangle = 0$$

where the second equality holds due to $x_+ \bullet x_- = 0$. In summary, from all the above, we prove that

$$\|\phi_p(x, y)\|^2 \geq \|x_-\|^2.$$

With similar arguments, we also obtain

$$\begin{aligned}
&\|\phi_p(x, y)\|^2 \\
&\quad = \langle x - (x-y)_+ + p\, x_+ \bullet (-y)_-,\ x - (x-y)_+ + p\, x_+ \bullet (-y)_- \rangle \\
&\quad = \langle y - (x-y)_- + p\, x_+ \bullet (-y)_-,\ y - (x-y)_- + p\, x_+ \bullet (-y)_- \rangle \\
&\quad = \langle (-y)_- - (-y)_+ - (x-y)_- + p\, x_+ \bullet (-y)_-,\ (-y)_- - (-y)_+ - (x-y)_- \\
&\qquad + p x_+ \bullet (-y)_- \rangle \\
&\quad = \|(-y)_+\|^2 + \|(-y)_- - (x-y)_- + p\, x_+ \bullet (-y)_-\|^2 - 2\langle (-y)_+, (-y)_- \\
&\qquad - (x-y)_- + p x_+ \bullet (-y)_- \rangle \\
&\quad \geq \|(-y)_+\|^2 - 2\langle (-y)_+, (-y)_- \rangle + 2\langle (-y)_+, (x-y)_- \rangle - 2\langle (-y)_+, p\, x_+ \bullet (-y)_- \rangle \\
&\quad \geq \|(-y)_+\|^2 - 2p\, \langle (-y)_+,\ x_+ \bullet (-y)_- \rangle \\
&\quad \geq \|(-y)_+\|^2,
\end{aligned}$$

where the second inequality holds due to due to $(-y)_+ \in \mathcal{L}_\theta$, $(-y)_-, (x-y)_- \in \mathcal{L}_\theta^*$, $\langle (-y)_+, (-y)_- \rangle = 0$ and the relation between $\mathcal{L}_\theta$ and $\mathcal{L}_\theta^*$. The last inequality holds due to equation (8) and Lemma 3.1(b). Therefore, we prove that $\|\phi_p(x, y)\| \geq \|(-y)_+\|$. Then, the proof is complete. □

Remark 3.1 From the proof of Lemma 3.2, it also can be seen that

$$\|\phi_{\mathrm{NR}}(x, y)\| \geq \max\{\|x_-\|, \|(-y)_+\|\}.$$

Theorem 3.2 *Let $\phi_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be defined as in* (10). *Then, ϕ_p is an C-function for CCCP, i.e., for any $x, y \in \mathbb{R}^n$,*

$$\phi_p(x, y) = 0 \iff x \in \mathcal{L}_\theta,\ y \in \mathcal{L}_\theta^* \text{ and } \langle x, y \rangle = 0.$$

Proof "$\Longrightarrow$" Suppose that $\phi_p(x, y) = 0$. If either $x \notin \mathcal{L}_\theta$ or $y \notin \mathcal{L}_\theta^*$, applying Lemma 3.2 yields

$$\|\phi_p(x, y)\| \geq \max\{\|x_-\|, \|(-y)_+\|\} > 0.$$

This contradicts with $\phi_p(x, y) = 0$. Hence, there must have $x \in \mathcal{L}_\theta$ and $y \in \mathcal{L}_\theta^*$. Next, we argue that $\langle x, y \rangle = 0$. To see this, we consider the first component of $\phi_p(x, y)$, which is denoted by $\left[\phi_p(x,y)\right]_1$. In other words,

$$\begin{aligned}\left[\phi_p(x,y)\right]_1 &= \left[x-(x-y)_+ + p\, x \bullet y\right]_1\\ &= \begin{cases} y_1 + p\,\langle x,y\rangle & \text{if } x-y\in\mathcal{L}_\theta,\\ x_1 + p\,\langle x,y\rangle & \text{if } x-y\in-\mathcal{L}_\theta^*,\\ w + p\,\langle x,y\rangle & \text{otherwise,}\end{cases}\end{aligned}$$

where

$$w = x_1 - \frac{x_1 - y_1 + \|x_2-y_2\|\tan\theta}{1+\tan^2\theta} = \frac{x_1\tan^2\theta + y_1 - \|x_2-y_2\|\tan\theta}{1+\tan^2\theta}.$$

Since $x\in\mathcal{L}_\theta$ and $y\in\mathcal{L}_\theta^*$, it follows that $x_1, y_1 \geq 0$, $\langle x,y\rangle \geq 0$ and

$$\frac{x_1\tan^2\theta + y_1 - \|x_2-y_2\|\tan\theta}{1+\tan^2\theta} \geq \frac{\tan\theta(x_1\tan\theta - \|x_2\| + y_1\cot\theta - \|y_2\|)}{1+\tan^2\theta} \geq 0.$$

This together with $\phi_p(x,y)=0$ gives $p\langle x,y\rangle = 0$. Thus, we conclude that $\langle x,y\rangle = 0$ because $p>0$.
"$\Longleftarrow$" Suppose that $x\in\mathcal{L}_\theta$, $y\in\mathcal{L}_\theta^*$ and $\langle x,y\rangle = 0$. Since ϕ_{NR} is always an C-function for CCCP, we have $x-(x-y)_+ = 0$. Using Theorem 3.1 again yields $x_+\bullet(-y)_- = x\bullet y = 0$, which says $\phi_p(x,y)=0$. □

Remark 3.2 In fact, for any $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ and $y=(y_1,y_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, we define another type of product of x and y as follows:

$$x\bullet y = \begin{bmatrix}x_1\\x_2\end{bmatrix}\bullet\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}\langle x,y\rangle\\ \min\{\tan^2\theta,1\}\,x_1y_2 + \min\{\cot^2\theta,1\}\,y_1x_2\end{bmatrix}.$$

With the same skills, we may obtain the same results.

Motivated by the construction of ϕ_p given as in (10), we consider another function $\phi_r:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ defined by

$$\phi_r(x,y) = x-(x-y)_+ + r\,(x\bullet y)_+^{\Omega} \quad r>0, \tag{11}$$

where $\Omega := \mathcal{L}_\theta\cap\mathcal{L}_\theta^* = \begin{cases}\mathcal{L}_\theta & \text{if } \theta\in(0,\frac{\pi}{4}],\\ \mathcal{L}_\theta^* & \text{if } \theta\in[\frac{\pi}{4},\frac{\pi}{2}).\end{cases}$ We point out that the function ϕ_r defined as in (11) is not an C-function for CCCP. The reason come from that if $\phi_r(x,y)=0$, we have $\phi_{\text{NR}}(x,y) = x-(x-y)_+ = -r\,(x\bullet y)_+^{\Omega}$. Combining with the expression of ϕ_p, this implies that

$$-r\,(x\bullet y)_+^{\Omega} + p\,(x_+\bullet(-y)_-) \neq 0$$

due to $(x\bullet y)_+^{\Omega}\in\Omega = \mathcal{L}_\theta\cap\mathcal{L}_\theta^*$ and $x_+\bullet(-y)_-\notin\mathcal{K}^n\supseteq\mathcal{L}_\theta$ (or $\mathcal{L}_\theta^*$) when $\theta\in(0,\frac{\pi}{4}]$ (or $\theta\in[\frac{\pi}{4},\frac{\pi}{2})$). This explains that $\phi_p(x,y)\neq 0$, which contradicts $\phi_p(x,y)$ being an C-function for CCCP.

However, there is a merit function related to ϕ_r which possesses property of bounded level sets. We will explore it in next section.

4 Merit functions for circular cone complementarity problem

In this section, based on the product (7) of x and y in $\mathbb{R}^n$, we propose four classes of merit functions for the circular cone complementarity problem and investigate their important properties, respectively.

First, we recall that a function $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *monotone* if, for any $x, y \in \mathbb{R}^n$, there holds

$$\langle x - y, F(x) - F(y)\rangle \geq 0;$$

and *strictly monotone* if, for any $x \neq y$, the above inequality holds strictly; and *strongly monotone* with modulus $\rho > 0$ if, for any $x, y \in \mathbb{R}^n$, the following inequality holds

$$\langle x - y, F(x) - F(y)\rangle \geq \rho\|x - y\|^2.$$

The following technical lemma is crucial for achieving the property of bounded level sets.

Lemma 4.1 *Suppose that CCCP has a strictly feasible point $\bar{x}$, i.e., $\bar{x} \in \mathrm{int}(\mathcal{L}_\theta)$ and $F(\bar{x}) \in \mathrm{int}(\mathcal{L}_\theta^*)$ and that F is a monotone function. Then, for any sequence $\{x^k\}$ satisfying*

$$\left\|x^k\right\| \to \infty, \quad \limsup_{k\to\infty} \left\|x_-^k\right\| < \infty \quad \textit{and} \quad \limsup_{k\to\infty} \left\|\left(-F\left(x^k\right)\right)_+\right\| < \infty,$$

we have

$$\left\langle x^k, F\left(x^k\right)\right\rangle \to \infty \quad \textit{and} \quad \left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle \to \infty.$$

Proof Since F is monotone, for all $x^k \in \mathbb{R}^n$, we know

$$\left\langle x^k - \bar{x}, F\left(x^k\right) - F(\bar{x})\right\rangle \geq 0,$$

which says

$$\left\langle x^k, F\left(x^k\right)\right\rangle + \langle \bar{x}, F(\bar{x})\rangle \geq \left\langle x^k, F(\bar{x})\right\rangle + \left\langle \bar{x}, F\left(x^k\right)\right\rangle. \tag{12}$$

Using $x^k = x_+^k - x_-^k$ and $F\left(x^k\right) = \left(-F\left(x^k\right)\right)_- - \left(-F\left(x^k\right)\right)_+$, it follows from (12) that

$$\begin{aligned} &\left\langle x^k, F\left(x^k\right)\right\rangle + \langle \bar{x}, F(\bar{x})\rangle \\ &\quad \geq \left\langle x_+^k, F(\bar{x})\right\rangle - \left\langle x_-^k, F(\bar{x})\right\rangle + \left\langle \bar{x}, \left(-F\left(x^k\right)\right)_-\right\rangle - \left\langle \bar{x}, \left(-F\left(x^k\right)\right)_+\right\rangle. \quad (13) \end{aligned}$$

We look into the first term in the right-hand side of (13).

$$\begin{aligned} \left\langle x_+^k, F(\bar{x})\right\rangle &= \left[x_+^k\right]_1 [f(\bar{x})]_1 + \left\langle \left[x_+^k\right]_2, [f(\bar{x})]_2\right\rangle \\ &\geq \left[x_+^k\right]_1 [f(\bar{x})]_1 - \left\| \left[x_+^k\right]_2 \right\| \cdot \left\| [f(\bar{x})]_2 \right\| \\ &\geq \left[x_+^k\right]_1 [f(\bar{x})]_1 - \left[x_+^k\right]_1 \tan\theta \left\| [f(\bar{x})]_2 \right\| \\ &= \left[x_+^k\right]_1 \left\{ [f(\bar{x})]_1 - \tan\theta \left\| [f(\bar{x})]_2 \right\| \right\}. \end{aligned} \quad (14)$$

Note that $x^k = x_+^k - x_-^k$, it gives $\|x_+^k\| \geq \|x^k\| - \|x_-^k\|$. From the assumptions on $\{x^k\}$, i.e., $\|x^k\| \to \infty$, and $\limsup_{k\to\infty} \|x_-^k\| < \infty$, we see that $\|x_+^k\| \to \infty$, and hence $[x_+^k]_1 \to \infty$. Because CCCP has a strictly feasible point $\bar{x}$, we have $[f(\bar{x})]_1 - \tan\theta \|[f(\bar{x})]_2\| > 0$, which together with (14) implies that

$$\left\langle x_+^k, F(\bar{x})\right\rangle \to \infty \qquad (k \to \infty). \quad (15)$$

On the other hand, we observe that

$$\begin{aligned} \limsup_{k\to\infty} \langle x_-^k, F(\bar{x})\rangle &\leq \limsup_{k\to\infty} \|x_-^k\| \|F(\bar{x})\| < \infty \\ \limsup_{k\to\infty} \langle \bar{x}, \left(-F\left(x^k\right)\right)_+ \rangle &\leq \limsup_{k\to\infty} \|\bar{x}\| \| \left(-F\left(x^k\right)\right)_+ \| < \infty \end{aligned}$$

and $\langle \bar{x}, \left(-F\left(x^k\right)\right)_- \rangle \geq 0$. All of these together with (13) and (15) yield

$$\left\langle x^k, F\left(x^k\right)\right\rangle \to \infty,$$

which is the first part of the desired result.
Next, we prove that $\left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle \to \infty$. Suppose not, that is, $\lim_{k\to\infty} \left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle < \infty$. Then, we obtain

$$\frac{\left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle}{\|x_+^k\|} = \left\langle \frac{x_+^k}{\|x_+^k\|}, \left(-F\left(x^k\right)\right)_- \right\rangle \to 0.$$

This means that there exists $\bar{x} \in \mathbb{R}^n$ such that $\frac{x_+^k}{\|x_+^k\|} \to \frac{\bar{x}_+}{\|\bar{x}_+\|}$ and

$$\left\langle \frac{\bar{x}_+}{\|\bar{x}_+\|}, (-F(\bar{x}))_- \right\rangle = 0. \tag{16}$$

Denote $z := \frac{\bar{x}_+}{\|\bar{x}_+\|}$ and apply Theorem 3.1, there exists $\alpha \in \mathbb{R}$ such that

$$\left[(-F(\bar{x}))_-\right]_2 = \alpha z_2 \quad \text{or} \quad \alpha z_2 = \left[(-F(\bar{x}))_-\right]_2.$$

It is obvious that $z \in \mathcal{L}_\theta$ and $(-F(\bar{x}))_- \in \mathcal{L}_\theta^*$. Hence, Eq. (16) implies that $\alpha < 0$, which says that z_2 and $\left[(-F(\bar{x}))_-\right]_2$ are in opposite direction to each other. From the expression of $(-F(\bar{x}))_+$ and $(-F(\bar{x}))_-$ again, it follows that $\left[(-F(\bar{x}))_+\right]_2$ and $\left[(-F(\bar{x}))_-\right]_2$ are in the opposite direction, to each other. These conclude that z_2 and $\left[(-F(\bar{x}))_+\right]_2$ are in the same direction, which means $[\bar{x}_+]_2$ and $\left[(-F(\bar{x}))_+\right]_2$ are also in the same direction. Now, combining with the fact that $\bar{x}_+, (-F(\bar{x}))_+ \in \mathcal{L}_\theta$, we have

$$\langle \bar{x}_+, (-F(\bar{x}))_+ \rangle \geq 0.$$

Similarly, by the the relation between $\bar{x}_+$ and $\bar{x}_-$, we know $[\bar{x}_-]_2$ and $[(-F(\bar{x}))_-]_2$ are in the same direction. Then, combining with $\bar{x}_-, (-F(\bar{x}))_- \in \mathcal{L}_\theta^*$, it leads to

$$\langle \bar{x}_-, (-F(\bar{x}))_- \rangle \geq 0.$$

Moreover, writing out the expression for $\langle \bar{x}, F(\bar{x}) \rangle$, we see that

$$\langle \bar{x}, F(\bar{x}) \rangle = \langle \bar{x}_+, (-F(\bar{x}))_- \rangle - \langle \bar{x}_+, (-F(\bar{x}))_+ \rangle - \langle \bar{x}_-, (-F(\bar{x}))_- \rangle + \langle \bar{x}_-, (-F(\bar{x}))_+ \rangle.$$

Note that the second and third terms of the right-hand side are nonpositive and the fourth is bounded from above. Hence, from the assumptions $\lim_{k\to\infty} \left\langle x_+^k, \left(-F\left(x^k\right)\right)_- \right\rangle < \infty$, we conclude that $\langle \bar{x}, F(\bar{x}) \rangle < \infty$, which contradict

$$\langle \bar{x}, F(\bar{x}) \rangle = \lim_{k\to\infty} \left\langle x^k, F\left(x^k\right) \right\rangle = \infty.$$

Thus, we prove that $\left\langle x_+^k, \left(-F\left(x^k\right)\right)_- \right\rangle \to \infty$. □

4.1 The first class of merit functions

For any $x \in \mathbb{R}^n$, from the analysis of the Sect. 3, we know that the function ϕ_p and $\phi_{\rm NR}$ are complementarity function for CCCP. In this subsection, we focus on the property of bounded level sets of merit functions based on $\phi_{\rm NR}$ and ϕ_p with the product of elements, which is a property to guarantee that the existence of accumulation points of sequence generated by some descent algorithms.

Theorem 4.1 *Let ϕ_p be defined as in* (10). *Suppose that CCCP has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathcal{L}_p(\alpha) = \{x \in \mathbb{R}^n \mid \left\|\phi_p(x, F(x))\right\| \leq \alpha\}$$

is bounded for all $\alpha \geq 0$.

Proof We prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subset \mathcal{L}_p(\alpha)$ for some $\alpha \geq 0$. If $\|x_-^k\| \to \infty$ or $\|\left(-F\left(x^k\right)\right)_+\| \to \infty$, by Lemma 3.2, we have $\|\phi_p(x^k, F\left(x^k\right))\| \to \infty$, which contradicts $\|\phi_p(x^k, F\left(x^k\right))\| \leq \alpha$. On the other hand, if

$$\limsup_{k\to\infty} \|x_-^k\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \left\|\left(-F\left(x^k\right)\right)_+\right\| < \infty,$$

it follows from Lemma 4.1 that $\left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle \to \infty$. From the proof of Lemma 4.1, there exists a constant κ_0 such that

$$\left[\phi_{\rm NR}\left(x^k, f\left(x^k\right)\right)\right]_1 \geq \begin{cases} [x_+^k]_1 - \kappa_0 & \text{if } x^k - F\left(x^k\right) \in -\mathcal{L}_\theta^*, \\ \left[\left(-F\left(x^k\right)\right)_-\right]_1 - \kappa_0 & \text{if } x^k - F\left(x^k\right) \in \mathcal{L}_\theta, \\ \frac{[x_+^k]_1 \tan^2\theta + \left[\left(-F\left(x^k\right)\right)_-\right]_1 - \|[x_+^k]_2\|\tan\theta - \|[\left(-F\left(x^k\right)\right)_-]_2\|\tan\theta}{1+\tan^2\theta} \\ -\frac{2\kappa_0(1+\tan\theta)}{1+\tan^2\theta}, & \text{if } x^k - F\left(x^k\right) \notin \mathcal{L}_\theta \cup -\mathcal{L}_\theta^*, \end{cases}$$

which means $\liminf \left[\phi_{\rm NR}(x^k, f\left(x^k\right))\right]_1 > -\infty$. Hence, it follows that

$$\begin{aligned} \left[\phi_p\left(x^k, f\left(x^k\right)\right)\right]_1 &= \left[\phi_{\rm NR}\left(x^k, f\left(x^k\right)\right)\right]_1 + \left[\left(x_+^k \bullet \left(-F\left(x^k\right)\right)_-\right]_1 \\ &= \left[\phi_{\rm NR}\left(x^k, f\left(x^k\right)\right)\right]_1 + \left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle \\ &\to \infty, \end{aligned}$$

where the limit comes from

$$\left\langle x_+^k, \left(-F\left(x^k\right)\right)_-\right\rangle \to \infty \quad \text{and} \quad \liminf \left[\phi_{\rm NR}\left(x^k, f\left(x^k\right)\right)\right]_1 > -\infty.$$

Thus, we obtain that $\|\phi_p(x^k, F\left(x^k\right))\| \to \infty$ which contradicts $\|\phi_p(x^k, F\left(x^k\right))\| \leq \alpha$. Then, the proof is complete. □

Note that, under the conditions of Lemma 4.1 or Theorem 4.1, we cannot guarantee the boundedness of the level set on the NR function $\phi_{\rm NR}$. For example, let $F(x) = 1-\frac{1}{x}$ and $x > 0$, it is easy to verify that the level set

$$\mathcal{L}_{\rm NR}(2) = \left\{x \in \mathbb{R}^n \mid \|\phi_{\rm NR}(x, F(x))\| \leq 2\right\}$$

is unbounded. In fact, In order to establish the boundedness of the level set on the natural residual function $\phi_{\rm NR}$, we need the following concept.

Definition 4.1 A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be strongly coercive if

$$\lim_{\|x\|\to\infty} \frac{\langle F(x), x-y\rangle}{\|x-y\|} = \infty.$$

holds for all $y \in \mathbb{R}^n$.

Theorem 4.2 *Suppose that F is strongly coercive. Then, the level set*

$$\mathcal{L}_{\rm NR}(\alpha) = \left\{x \in \mathbb{R}^n \mid \left\|\phi_{\rm NR}(x, F(x))\right\| \leq \alpha\right\}$$

is bounded for all $\alpha \geq 0$.

Proof Again, we prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subset \mathcal{L}_{\rm NR}(\alpha)$ for some $\alpha \geq 0$, i.e., $\left\|x^k\right\| \to \infty$. Note that the sequence $\left\{\phi_{\rm NR}(x^k, F\left(x^k\right)) = x^k - \left(x^k - F\left(x^k\right)\right)_+\right\}$ is bounded. It follows from the unboundedness of the sequence $\{x^k\}$ that the sequence $\{\left(x^k - F\left(x^k\right)\right)_+\}$ is also unbounded. Then, for any $y \in \mathcal{L}_\theta$, there exist $N \in \mathbb{N}$ and $\beta > 0$ such that

$$\left\|\left(x^k - F\left(x^k\right)\right)_+ - y\right\| > \beta, \quad \forall k > N.$$

From the property of projection mapping, we have

$$\left\langle x^k - F\left(x^k\right) - \left(x^k - F\left(x^k\right)\right)_+,\ y - \left(x^k - F\left(x^k\right)\right)_+\right\rangle \leq 0 \tag{17}$$

for each $k > N$. On the other hand,

$$\begin{aligned}
&\left\langle x^k - F\left(x^k\right) - \left(x^k - F\left(x^k\right)\right)_+,\ y - \left(x^k - F\left(x^k\right)\right)_+\right\rangle \\
&= \left\langle x^k - \left(x^k - F\left(x^k\right)\right)_+, y - \left(x^k - F\left(x^k\right)\right)_+\right\rangle + \left\langle F\left(x^k\right), \left(x^k - F\left(x^k\right)\right)_+ - y\right\rangle \\
&\geq -\left\|x^k - \left(x^k - F\left(x^k\right)\right)_+\right\| \cdot \left\|y - \left(x^k - F\left(x^k\right)\right)_+\right\| + \left\langle F\left(x^k\right), \left(x^k - F\left(x^k\right)\right)_+ - y\right\rangle \\
&\geq \left\|y - \left(x^k - F\left(x^k\right)\right)_+\right\| \left(\frac{\langle F\left(x^k\right), \left(x^k - F\left(x^k\right)\right)_+ - y\rangle}{\|y - (x^k - F(x^k))_+\|} - \alpha\right).
\end{aligned}$$

Plugging in $y^k := x^k - \left(x^k - F\left(x^k\right)\right)_+ - y$, we obtain

$$\lim_{k\to\infty} \frac{\left\langle F\left(x^k\right), \left(x^k - F\left(x^k\right)\right)_+ - y\right\rangle}{\|y - \left(x^k - F\left(x^k\right)\right)_+\|} = \lim_{k\to\infty} \frac{\left\langle F\left(x^k\right), x^k - y^k\right\rangle}{\|x^k - y^k\|} = \infty,$$

where the last equality holds due to the strong coercivity of F and [22, Theorem 2.1]. This implies that

$$\lim_{k\to\infty} \left\langle x^k - F\left(x^k\right) - \left(x^k - F\left(x^k\right)\right)_+, y - \left(x^k - F\left(x^k\right)\right)_+\right\rangle = \infty,$$

which contradicts (17). Therefore, the level set

$$\mathcal{L}_{\mathrm{NR}}(\alpha) = \left\{x \in \mathbb{R}^n \mid \left\|\phi_{\mathrm{NR}}(x, F(x))\right\| \le \alpha\right\}$$

is bounded for all $\alpha \ge 0$. □

4.2 The second class of merit functions

For any $x \in \mathbb{R}^n$, LT (standing for Luo-Tseng) merit function for the circular cone complementarity problem is given as follows:

$$f_{LT}(x) := \varphi(\langle x, F(x)\rangle) + \frac{1}{2}\left[\|(x)_-\|^2 + \|(-F(x))_+\|^2\right], \tag{18}$$

where $\varphi : \mathbb{R} \to \mathbb{R}_+$ is an arbitrary smooth function satisfying

$$\varphi(t) = 0,\ \forall t \le 0 \ \text{ and } \ \varphi'(t) > 0,\ \forall t > 0.$$

Notice that we have $\varphi(t) \ge 0$ for all $t \in \mathbb{R}$ from the above condition. Indeed, this class of functions has been considered for the SDCP case (positive semidefinite complementarity problem) by Tseng in [21], for the SOCCP case (second-order cone complementarity problem) by Chen in [2] and for the general SCCP case by Pan and Chen in [18], respectively. For the case of generally closed convex cone complementarity problems, the LT merit function has been studied by Lu and Huang in [15]. In view of the results in [15], it is easy to obtain the following results directly for the circular cone complementarity problem.

Proposition 4.1 *Let $f_{LT} : \mathbb{R}^n \to \mathbb{R}$ be given as in (18). Then, the following results hold.*

(a) *For all $x \in \mathbb{R}^n$, we have $f_{LT}(x) \ge 0$; and $f_{LT}(x) = 0$ if and only if x solves the circular cone complementarity problem.*
(b) *If $F(\cdot)$ is differentiable, then so is $f_{LT}(\cdot)$. Moreover,*

$$\nabla f_{LT}(x) = \nabla\varphi(\langle x, F(x)\rangle)[F(x) + x\nabla F(x)] - x_- - \nabla F(x)(-F(x))_+$$

for all $x \in \mathbb{R}^n$.

Proof See Lemma 3.1 and Theorem 3.4 in [15]. □

Proposition 4.2 *Let* f_{LT} *be given as in* (18)*. Suppose that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a strongly monotone mapping and that the circular cone complementarity problem has a solution* x^**. Then, there exists a constant* $\tau > 0$ *such that*

$$\tau \|x - x^*\|^2 \leq \max\{0, \langle x, F(x)\rangle\} + \|x_-\| + \|(-F(x))_+\|, \quad \forall x \in \mathbb{R}^n.$$

Moreover,

$$\tau \|x - x^*\|^2 \leq \varphi^{-1}(f_{LT}(x)) + 2[f_{LT}(x)]^{\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n.$$

Proof See Theorem 3.6 in [15]. □

In the following theorem, we present the condition which ensures the boundedness of the level sets for LT merit function f_{LT} to solve the circular cone complementarity problem.

Theorem 4.3 *Suppose that the circular cone complementarity problem has a strictly feasible point and that* F *is monotone. Then, the level set*

$$\mathcal{L}_{f_{LT}}(\alpha) := \{x \in \mathbb{R}^n \mid f_{LT}(x) \leq \alpha\}$$

is bounded for all $\alpha \geq 0$.

Proof We prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subseteq \mathcal{L}_{f_{LT}}(\alpha)$ for some $\alpha \geq 0$. We may assert that the sequences $\{x^k_-\}$ and $\{\left(-F\left(x^k\right)\right)_+\}$ are bounded. If not, from the expression (18) of LT merit function f_{LT} and the property $\varphi(t) \geq 0$ for all $t \in \mathbb{R}$, it follows that

$$f_{LT}\left(x^k\right) \geq \frac{1}{2}\left[\|x^k_-\|^2 + \|\left(-F\left(x^k\right)\right)_+\|^2\right] \to \infty,$$

which contradicts $\{x^k\} \subseteq \mathcal{L}_{f_{LT}}(\alpha)$, i.e., $f_{LT}\left(x^k\right) \leq \alpha$. Therefore, we have

$$\limsup_{k\to\infty} \|x^k_-\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \|\left(-F\left(x^k\right)\right)_+\| < \infty.$$

Then, by Lemma 4.1, we get that

$$\left\langle x^k, F\left(x^k\right)\right\rangle \to \infty.$$

By the properties of the function φ again, we obtain that $\varphi(\langle x^k, F\left(x^k\right)\rangle) \to \infty$, which implies $f_{LT}\left(x^k\right) \to \infty$. This contradicts $\{x^k\} \subseteq \mathcal{L}_{f_{LT}}(\alpha)$. Hence, the level set $\mathcal{L}_{f_{LT}}(\alpha)$ is bounded for all $\alpha \geq 0$. □

4.3 The third class of merit functions

To achieve the third class of merit functions, we make a slight modification of LT merit function f_{LT} for the circular cone complementarity problem. More specifically, we consider the set Ω as follows:

$$\Omega := \mathcal{L}_\theta \cap \mathcal{L}_\theta^* = \begin{cases} \mathcal{L}_\theta \text{ for } 0 < \theta \leq \frac{\pi}{4}, \\ \mathcal{L}_\theta^* \text{ for } \frac{\pi}{4} < \theta < \frac{\pi}{2}. \end{cases}$$

Indeed, Ω is also a closed convex cone. In light of this Ω, another function is considered:

$$\widehat{f_{LT}}(x) := \frac{1}{2} \left\| (x \bullet F(x))_+^{\Omega} \right\|^2 + \frac{1}{2} \left[\|x_-\|^2 + \|(-F(x))_+\|^2 \right], \tag{19}$$

where $(x \bullet y)_+^{\Omega}$ denotes the projection of $x \bullet y$ onto Ω. Then, together with the expressions (7) of $x \bullet y$, we can verify that the function $\widehat{f_{LT}}$ is also a type of merit function for the circular cone complementarity problem, which will be shown in following theorem.

Theorem 4.4 *Let the function $\widehat{f_{LT}}$ be given by (19). Then, for all $x \in \mathbb{R}^n$, we have*

$$\widehat{f_{LT}}(x) = 0 \iff x \in \mathcal{L}_\theta, \quad F(x) \in \mathcal{L}_\theta^* \text{ and } \langle x, F(x) \rangle = 0,$$

where $\mathcal{L}_\theta^*$ denotes the dual cone of $\mathcal{L}_\theta$, i.e., $\mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta}$.

Proof By the definition of the function $\widehat{f_{LT}}$ given by (19), we have

$$\begin{aligned} \widehat{f_{LT}}(x) = 0 &\Leftrightarrow \left\| (x \bullet F(x))_+^{\Omega} \right\| = 0, \quad \|x_-\| = 0 \text{ and } \|(-F(x))_+\| = 0, \\ &\Leftrightarrow (x \bullet F(x))_+^{\Omega} = 0, \quad x_- = 0 \text{ and } (-F(x))_+ = 0, \\ &\Leftrightarrow x \bullet F(x) \in -\mathcal{L}_\theta \text{ or } x \bullet F(x) \in -\mathcal{L}_\theta^*, \quad x \in \mathcal{L}_\theta, \quad \text{and} \quad F(x) \in \mathcal{L}_\theta^*, \\ &\Leftrightarrow x \in \mathcal{L}_\theta, \quad F(x) \in \mathcal{L}_\theta^* \text{ and } \langle x, F(x) \rangle = 0, \end{aligned}$$

where the last equivalence holds due to the properties of the cone $-\mathcal{L}_\theta$ or $-\mathcal{L}_\theta^*$. Thus, the proof is complete. □

From Theorem 4.4, we know that the function $\widehat{f_{LT}}$ is a merit function for the circular cone complementarity problem. As below, according to the type of dot product (7), we establish the differentiability of $\widehat{f_{LT}}$.

Theorem 4.5 *Let $\widehat{f_{LT}} : \mathbb{R}^n \to \mathbb{R}$ be given by (19). Suppose that the type of dot product (7) is employed. If $F(\cdot)$ is differentiable, then so is $\widehat{f_{LT}}(\cdot)$. Moreover, for all $x \in \mathbb{R}^n$, we have*

$$\nabla \widehat{f_{LT}}(x) = (L_y + \nabla F(x) L_x) \cdot (x \bullet F(x))_+^{\Omega} - x_- - \nabla F(x)(-F(x))_+,$$

where

$$L_x = \begin{bmatrix} y_1 & y_2^T \\ \max\{\tan^2\theta, 1\}y_2 & \max\{\cot^2\theta, 1\}y_1 I \end{bmatrix}$$

and

$$L_y = \begin{bmatrix} x_1 & x_2^T \\ \max\{\tan^2\theta, 1\}x_2 & \max\{\cot^2\theta, 1\}x_1 I \end{bmatrix}$$

with I being the identity matrix.

Proof From the proof of Lemma 3.1(b) in [15], we have

$$\nabla \left(\frac{1}{2} \|(z)_+^\Omega\|^2 \right) = (z)_+^\Omega, \quad \forall z \in \mathbb{R}^n.$$

Then, by the chain rule again, it follows that

$$\begin{aligned} \nabla \left(\frac{1}{2} \|(x \bullet F(x))_+^\Omega\|^2 \right) &= \nabla_x (x \bullet F(x)) \cdot (x \bullet F(x))_+^\Omega \\ &= \left[L_y + \nabla F(x) L_x \right] \cdot (x \bullet F(x))_+^\Omega, \end{aligned}$$

where

$$L_x = \begin{bmatrix} y_1 & y_2^T \\ \max\{\tan^2\theta, 1\}y_2 & \max\{\cot^2\theta, 1\}y_1 I \end{bmatrix}$$

and

$$L_y = \begin{bmatrix} x_1 & x_2^T \\ \max\{\tan^2\theta, 1\}x_2 & \max\{\cot^2\theta, 1\}x_1 I \end{bmatrix}$$

with I being the identity matrix. Thus, we obtain that

$$\nabla \widehat{f_{LT}}(x) = (L_y + \nabla F(x) L_x) \cdot (x \bullet F(x))_+^\Omega - (x)_- - \nabla F(x)(-F(x))_+$$

for all $x \in \mathbb{R}^n$. □

In order to establish error bound property of the merit function $\widehat{f_{LT}}$ for the circular cone complementarity problem, we need a technical lemma as below.

Lemma 4.2 *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, we have*

$$\langle x, y \rangle \le \max \left\{ \frac{1 + \tan^2\theta}{\sqrt{2}}, \frac{1 + \cot^2\theta}{\sqrt{2}} \right\} \left\| (x \bullet y)_+^\Omega \right\|$$

where $\bullet$ is defined as in (7).

Proof Given any two vectors $x = (x_1, x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$. For the type of dot product (7), we know that

$$x \bullet y = \begin{bmatrix} \langle x, y\rangle \\ \max\{\tan^2\theta, 1\}x_1 y_2 + \max\{\cot^2\theta, 1\}y_1 x_2 \end{bmatrix} := \begin{bmatrix} \langle x, y\rangle \\ w \end{bmatrix}.$$

To proceed the arguments, we consider the following three cases:

Case 1 When $x \bullet y \in \Omega$, we have $(x \bullet y)_+^\Omega = x \bullet y$. Then, it is easy to verify that

$$\|(x \bullet y)_+^\Omega\| \geq \langle x, y\rangle.$$

Case 2 When $x \bullet y \in -\Omega^*$, where Ω^* denotes the dual cone of Ω, we have $(x \bullet y)_+^\Omega = 0$ and $\langle x, y\rangle \leq 0$. This implies that

$$\|(x \bullet y)_+^\Omega\| \geq \langle x, y\rangle.$$

Case 3 When $x \bullet y \notin \Omega \cup (-\Omega^*)$, we consider the expression of $(x \bullet y)_+^\Omega$. If $\langle x, y\rangle \leq 0$, then the result is obvious. Thus, we only need to look into the case of $\langle x, y\rangle > 0$. If $\Omega = \mathcal{L}_\theta$, by the expression of projection on Ω, we have

$$(x \bullet y)_+^\Omega = \begin{bmatrix} \dfrac{\langle x, y\rangle + \|w\|\tan\theta}{1+\tan^2\theta} \\ \dfrac{\langle x, y\rangle + \|w\|\tan\theta}{1+\tan^2\theta}\dfrac{w}{\|w\|} \end{bmatrix}.$$

This implies that

$$\begin{aligned} \|(x \bullet y)_+^\Omega\|^2 &= 2\left(\frac{\langle x, y\rangle + \|w\|\tan\theta}{1+\tan^2\theta}\right)^2 \\ &= \frac{2}{(1+\tan^2\theta)^2}\left[(\langle x, y\rangle)^2 + 2\langle x, y\rangle\|w\|\tan\theta + \|w\|^2\tan^2\theta\right] \\ &\geq \frac{2}{(1+\tan^2\theta)^2}(\langle x, y\rangle)^2. \end{aligned}$$

Hence, we see that

$$\langle x, y\rangle \leq \frac{1+\tan^2\theta}{\sqrt{2}}\|(x \bullet y)_+^\Omega\|.$$

With similar arguments, for $\Omega = \mathcal{L}_\theta^* = L_{\frac{\pi}{2}-\theta}$, it follows that

$$\langle x, y\rangle \leq \frac{1+\cot^2\theta}{\sqrt{2}}\|(x \bullet y)_+^\Omega\|.$$

From all the above analysis for three cases, we obtain that

$$\langle x, y\rangle \le \max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}\right\} \left\|(x \bullet y)_+^{\Omega}\right\|.$$

Thus, the proof is complete. □

Theorem 4.6 *Let the function* $\widehat{f_{LT}}$ *be given by* (19). *Suppose that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is strongly monotone mapping and that* x^* *is a solution to the circular cone complementarity problem. Then, there exists a scalar* $\tau > 0$ *such that*

$$\tau\|x - x^*\|^2 \le (2+\sqrt{2})\left[\widehat{f_{LT}}(x)\right]^{\frac{1}{2}}.$$

Proof Since the function F is strongly monotone and x^* is a solution to the circular cone complementarity problem, there exists a scalar $\rho > 0$ such that, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\rho\|x - x^*\|^2 &\le \langle F(x) - F(x^*), x - x^*\rangle \\
&= \langle F(x), x\rangle + \langle F(x^*), -x\rangle + \langle -F(x), x^*\rangle \\
&= \langle F(x), x\rangle + \langle F(x^*), x_- - x_+\rangle + \langle (-F(x))_+ - (-F(x))_-, x^*\rangle \\
&\le \langle F(x), x\rangle + \langle F(x^*), x_-\rangle + \langle (-F(x))_+, x^*\rangle \\
&\le \max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}\right\} \|(x \bullet F(x))_+^{\Omega}\| + \|x_-\|\|F(x^*)\| \\
&\quad + \|x^*\|\|(-F(x))_+\| \\
&\le \max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}, \|F(x^*)\|, \|x^*\|\right\} (\|(x \bullet F(x))_+^{\Omega}\| + \|x_-\| \\
&\quad + \|(-F(x))_+\|),
\end{aligned}$$

where the second inequality holds due to the properties of the cone $\mathcal{L}_\theta$ and its dual cone $\mathcal{L}_\theta^*$, and the third inequality follows from Lemma 4.2 and properties of inner product. Then, setting $\tau := \dfrac{\rho}{\max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}, \|F(x^*)\|, \|x^*\|\right\}}$ yields

$$\tau\|x - x^*\|^2 \le \|(x \bullet F(x))_+^{\Omega}\| + \|x_-\| + \|(-F(x))_+\|.$$

Moreover,

$$\|(x \bullet F(x))_+^{\Omega}\| = \sqrt{2}\left(\frac{1}{2}\|(x \bullet F(x))_+^{\Omega}\|^2\right)^{\frac{1}{2}} \le \sqrt{2}\left[\widehat{f_{LT}}(x)\right]^{\frac{1}{2}},$$

and

$$\|x_-\| + \|(-F(x))_+\| \le \sqrt{2}\left(\|x_-\|^2 + \|(-F(x))_+\|^2\right)^{\frac{1}{2}} \le 2\left[\widehat{f_{LT}}(x)\right]^{\frac{1}{2}}.$$

Hence, we have

$$\tau \|x - x^*\|^2 \le (2 + \sqrt{2}) \left[\widehat{f_{LT}}(x)\right]^{\frac{1}{2}},$$

which is the desired result. □

Next, we focus on the boundedness of level sets of merit function $\widehat{f_{LT}}$.

Theorem 4.7 *Let the merit function $\widehat{f_{LT}}$ be given by (19). Suppose that the circular cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level set*

$$\mathcal{L}_{\widehat{f_{LT}}}(\alpha) = \left\{ x \in \mathbb{R}^n \,\middle|\, \widehat{f_{LT}}(x) \le \alpha \right\}$$

is bounded for all $\alpha \ge 0$.

Proof Similar to the proof of Theorem 4.3, we prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subseteq \mathcal{L}_{\widehat{f_{LT}}}(\alpha)$ for some $\alpha \ge 0$. We assert that the sequences $\{x_-^k\}$ and $\{\left(-F\left(x^k\right)\right)_+\}$ are bounded. In fact, if not, it follows from the expression (19) of $\widehat{f_{LT}}$ that

$$\widehat{f_{LT}}\left(x^k\right) \ge \frac{1}{2}\left[\|x_-^k\|^2 + \|\left(-F\left(x^k\right)\right)_+\|^2\right] \to \infty,$$

which contradicts $\{x^k\} \subseteq \mathcal{L}_{\widehat{f_{LT}}}(\alpha)$, i.e., $\widehat{f_{LT}}\left(x^k\right) \le \alpha$. Therefore, we have

$$\limsup_{k\to\infty} \|x_-^k\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \|\left(-F\left(x^k\right)\right)_+\| < \infty.$$

Then, by Lemma 4.1 again, we get

$$\left\langle x^k, F\left(x^k\right)\right\rangle \to \infty.$$

Then, together with Lemma 4.2, it is easy to verify that

$$\max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}\right\} \left\|\left(x^k \bullet F\left(x^k\right)\right)_+^{\Omega}\right\| \ge \langle x^k, F\left(x^k\right)\rangle \to \infty.$$

This leads to $\widehat{f_{LT}}\left(x^k\right) \to \infty$, which contradicts $\{x^k\} \subseteq \mathcal{L}_{\widehat{f_{LT}}}(\alpha)$. Therefore, the level set $\mathcal{L}_{\widehat{f_{LT}}}(\alpha)$ is bounded. □

4.4 The fourth class of merit function

For any $x \in \mathbb{R}^n$, combining NR merit function and the merit function $\widehat{f_{LT}}$, we consider another merit function as below:

$$f_r(x) := \frac{1}{2}\|\phi_{\mathrm{NR}}(x, F(x))\|^2 + \frac{1}{2}\left\|(x \bullet F(x))_+^{\Omega}\right\|^2. \tag{20}$$

Based on the dot product (7) for $x \bullet y$, we show that $f_r(x)$ is also a merit function for the circular cone complementarity problem.

Theorem 4.8 *Let the function f_r be given by (20). Then, for all $x \in \mathbb{R}^n$, we have*

$$f_r(x) = 0 \iff x \in \mathcal{L}_\theta, \quad F(x) \in \mathcal{L}_\theta^* \text{ and } \langle x, F(x)\rangle = 0,$$

where $\mathcal{L}_\theta^*$ denotes the dual cone of $\mathcal{L}_\theta$, i.e., $\mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta}$.

Proof In light of the definition of f_r given by (20), we have

$$\begin{aligned} f_r(x) = 0 &\iff \left\|(x \bullet F(x))_+^{\Omega}\right\| = 0 \text{ and } \|\phi_{\mathrm{NR}}(x, F(x))\| = 0, \\ &\iff x \in \mathcal{L}_\theta, \ F(x) \in \mathcal{L}_\theta^* \text{ and } \langle x, F(x)\rangle = 0, \end{aligned}$$

where the second equivalence holds because the function $\phi_{\mathrm{NR}}(x, y)$ is a complementarity function for the circular cone complementarity problem. Thus, the proof is complete. □

From Theorem 4.8, we see that the function f_r is a merit function for the circular cone complementarity problem. In fact, if the squared exponent in the expression of f_r is deleted. In other words, we consider

$$\widetilde{f_r}(x) := \|\phi_{\mathrm{NR}}(x, F(x))\| + \left\|(x \bullet F(x))_+^{\Omega}\right\|, \tag{21}$$

then $\widetilde{f_r}$ is also a merit function for the CCCP. For these two merit functions f_r and $\widetilde{f_r}$, there has no big differences between them in addition to the nature of f_r is better than $\widetilde{f_r}$. As below, we establish the error bound properties for f_r and $\widetilde{f_r}$.

Theorem 4.9 *Let f_r and $\widetilde{f_r}$ be given by (20) and (21), respectively. Suppose that $F : \mathbb{R}^n \to \mathbb{R}^n$ is strongly monotone mapping and that x^* is a solution to the circular cone complementarity problem. Then, there exists a scalar $\tau > 0$ such that*

$$\tau\|x - x^*\|^2 \le 3\sqrt{2}\,[f_r(x)]^{\frac{1}{2}} \quad \textit{and} \quad \tau\|x - x^*\|^2 \le 2\widetilde{f_r}(x).$$

Proof From Remark 3.1, we have

$$\|\phi_{\mathrm{NR}}(x, F(x))\| \ge \max\{\|x_-\|, \|(-F(x))_+\|\}.$$

This together with the proof of Theorem 4.6 imply that

$$\begin{aligned}
\tau\|x-x^*\|^2 &\leq \|(x \bullet F(x))_+^{\Omega}\| + \|x_-\| + \|(-F(x))_+\| \\
&\leq \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\|\phi_{\rm NR}(x, F(x))\| \\
&= \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\sqrt{2}\left(\frac{1}{2}\|\phi_{\rm NR}(x, F(x))\|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}(f_r(x))^{\frac{1}{2}} + 2\sqrt{2}(f_r(x))^{\frac{1}{2}} \\
&= 3\sqrt{2}\,[f_r(x)]^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\tau\|x-x^*\|^2 &\leq \|(x \bullet F(x))_+^{\Omega}\| + \|x_-\| + \|(-F(x))_+\| \\
&\leq \|(x \bullet F(x))_+^{\Omega}\| + 2\|\phi_{\rm NR}(x, F(x))\| \\
&\leq 2\widetilde{f}_r(x),
\end{aligned}$$

where $\tau := \dfrac{\rho}{\max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}, \|F(x^*)\|, \|x^*\|\right\}}$. Thus, the proof is complete. □

The following theorem will show that the boundedness of the level sets of the function $\widetilde{f}_r$ and f_r.

Theorem 4.10 *Let f_r and $\widetilde{f}_r$ be given by* (20) *and* (21)*, respectively. Suppose that that the circular cone complementarity problem has a strictly feasible point and that F is monotone. Then, the level sets*

$$\mathcal{L}_{f_r}(\alpha) = \left\{x \in \mathbb{R}^n \mid f_r(x) \leq \alpha\right\}$$

and

$$\mathcal{L}_{\widetilde{f}_r}(\alpha) = \left\{x \in \mathbb{R}^n \mid \widetilde{f}_r(x) \leq \alpha\right\}$$

are both bounded for all $\alpha \geq 0$.

Proof Here we only prove that the level sets of the function $\widetilde{f}_r$ are bounded for all $\alpha \geq 0$. With the same arguments, we also easily obtain the boundedness of the level sets of the function f_r.

We prove this result by contradiction. Suppose there exists an unbounded sequence $\{x^k\} \subset \mathcal{L}_{\widetilde{f}_r}(\alpha)$ for some $\alpha \geq 0$. If $\|x_-^k\| \to \infty$ or $\left\|\left(-F\left(x^k\right)\right)_+\right\| \to \infty$, by Remark 3.1, we have

$$\widetilde{f}_r\left(x^k\right) \geq \left\|\phi_{\rm NR}\left(x^k, F\left(x^k\right)\right)\right\| \to \infty,$$

which contradicts $x^k \in \mathcal{L}_{\widetilde{f}_r}(\alpha)$, i.e., $\widetilde{f}_r\left(x^k\right) \leq \alpha$. On the other hand, if

$$\limsup_{k\to\infty} \left\|x_-^k\right\| < \infty \quad \text{and} \quad \limsup_{k\to\infty} \left\|\left(-F\left(x^k\right)\right)_+\right\| < \infty,$$

it follows from Lemma 4.1 that $\langle x^k, F\left(x^k\right)\rangle \to \infty$. Then, applying Lemma 4.2 gives

$$\langle x^k, F\left(x^k\right)\rangle \leq \max\left\{\frac{1+\tan^2\theta}{\sqrt{2}}, \frac{1+\cot^2\theta}{\sqrt{2}}\right\} \left\|\left(x^k \bullet F\left(x^k\right)\right)_+^{\mathcal{K}}\right\| \to \infty.$$

This implies that

$$\left\|\left(x^k \bullet F\left(x^k\right)\right)_+^{\mathcal{K}}\right\| \to \infty,$$

which says $\widetilde{f}_r\left(x^k\right) \to \infty$. This is a contradiction because $\widetilde{f}_r\left(x^k\right) \leq \alpha$. Thus, the proof is complete. □

5 Conclusion and future direction

In this paper, we have shown that how to construct complementarity functions for the circular cone complementarity problem, and have proposed four classes of merit functions for the circular cone complementarity problem, which belongs to nonsymmetric cone complementarity problems. In addition, we have also shown conditions under which these merit functions have properties of error bounds and bounded level sets. These results not only build up a theoretical basis for designing the merit function method for solving circular cone complementarity problem, but also open a way to tackle nonsymmetric cone complementarity. In particular, with these properties, it is possible to construct a descent algorithm for the circular cone complementarity problem, even for general nonsymmetric cone complementarity problem. Hence, the future study will be about the descent methods including numerical examples for solving the unconstrained minimization via these merit functions.

We also want to point out that our approach to constructing complementarity functions for the circular cone complementarity problem is based on careful observation of the structure of circular cone. Alternatively, as mentioned by one reviewer, we think that it may be possible to achieve some results of this paper directly by exploiting the relation between the circular cone and the second-order cone. For example, Proposition 3.1, Proposition 4.2, Theorem 4.2, and Theorem 4.3 may be reached by this way. More precisely, if we replace the product $x \bullet y$ for the elements x and y as $Ax \circ A^{-1}y$, and change the formula $(x \bullet F(x))_+^{\Omega}$ as in (19), the analysis may be do-able. However, we are not sure about how far this approach can go by now. We will keep an eye on this approach in the future.

Acknowledgments Xin-He Miao work is supported by National Young Natural Science Foundation (No. 11101302 and No. 61002027) and National Natural Science Foundation of China (No. 11471241). Jein-Shan Chen work is supported by Ministry of Science and Technology, Taiwan.

References

1. Chen, J.-S.: A new merit function and its related properties for the second-order cone complementarity problem. Pac. J. Optim. **2**, 167–179 (2006)
2. Chen, J.-S.: Two classes of merit functions for the second-order cone complementarity problem. Math. Methods Oper. Res. **64**, 495–519 (2006)
3. Chen, J.-S.: Conditions for error bounds and bounded level sets of some merit functions for the second-order cone complementarity problem. J. Optim. Theory Appl. **135**, 459–473 (2007)
4. Chen, X.D., Sun, D., Sun, J.: Complementarity functions and numerical experiments on some smoothing Newton methods for second-order cone complementarity problems. Comput. Optim. Appl. **25**, 39–56 (2003)
5. Chen, J.-S., Tseng, P.: An unconstrained smooth minimization reformulation of second-order cone complementarity problem. Math. Program. **104**, 293–327 (2005)
6. Chang, Y.-L., Yang, C.-Y., Chen, J.-S.: Smooth and nonsmooth analysis of vector-valued functions associated with circular cones. Nonlinear Anal.: Theory Methods Appl. **85**, 160–173 (2013)
7. Dattorro, J.: Convex Optimization and Euclidean Distance Geometry. Meboo Publishing, California (2005)
8. Ding, C., Sun, D.F., Toh, K.-C.: An introduction to a class of matrix cone programming. Math. Program. **144**, 141–179 (2014)
9. Ding, C.: An introduction to a class of matrix optimization problems, Ph.D. Thesis, National University of Singapore (2012)
10. Fukushima, M., Luo, Z.-Q., Tseng, P.: Smoothing functions for second- order-cone complimentarity problems. SIAM J. Optim. **12**, 436–460 (2002)
11. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer, New York (2003)
12. Han, D.R.: On the coerciveness of some merit functions for complimentarity problems over symmetric cones. J. Math. Anal. Appl. **336**, 727–737 (2007)
13. Ko, C.-H., Chen, J.-S.: Optimal grasping manipulation for multifingered robots using semismooth Newton method. Math. Prob. Eng. **2013**, Article ID 681710, 9 p. (2013)
14. Kong, L., Sun, J., Xiu, N.: A regularized smoothing Newton method for symmetric cone complementarity problems. SIAM J. Optim. **19**, 1028–1047 (2008)
15. Lu, N., Huang, Z.-H.: Three classes of merit functions for the complementarity problem over a closed convex cone. Optimization **62**, 545–560 (2013)
16. Matsukawa, Y., Yoshise, A.: A primal barrier function Phase I algorithm for nonsymmetric conic optimization problems. Jpn. J. Ind. Appl. Math. **29**, 499–517 (2012)
17. Pang, J.S.: Complementarity problems. In: Horst, R., Pardalos, P. (eds.) Handbook in Global Optimization. Kluwer Academic Publishers, Boston (1994)
18. Pan, S.-H., Chen, J.-S.: A one-parametric class of merit functions for the symmetric cone complementarity problem. J. Math. Anal. Appl. **355**, 195–215 (2009)
19. Skajaa, A., Jorgensen, J.B., Hansen, P.C.: On implementing a homogeneous interior-point algorithm for nonsymmetric conic optimization, IMM-Technical Report, No. 2011–02 (2011)
20. Sun, D., Sun, J.: Löwner's operator and spectral functions in Euclidean Jordan algebras. Math. Oper. Res. **33**, 421–445 (2008)
21. Tseng, P.: Merit functions for semi-definite complementarity problems. Math. Program. **83**, 159–185 (1998)
22. Yamashita, N., Fukushima, M.: On the level-boundedness of the natural residual function for variational inequality problems. Pac. J. Optim. **1**, 625–630 (2005)
23. Zhou, J.-C., Chen, J.-S.: Properties of circular cone and spectral factorization associated with circular cone. J. Nonlinear Convex Anal. **14**(4), 807–816 (2013)
24. Zhou, J.-C., Chen, J.-S.: On the vector-valued functions associated with circular cones. Abstract Appl. Anal. **2014**, Article ID 603542, 21 p. (2014)