Journal of Computational and Applied Mathematics 232 (2009) 455–471

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

An R-linearly convergent derivative-free algorithm for nonlinear complementarity problems based on the generalized Fischer–Burmeister merit function

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ARTICLE INFO

Article history:
Received 12 July 2008

Keywords:
Nonlinear complementarity problem
NCP-function
Merit function
Global error bound
Convergence rate

ABSTRACT

In the paper [J.-S. Chen, S. Pan, A family of NCP-functions and a descent method for the nonlinear complementarity problem, Computational Optimization and Applications, 40 (2008) 389–404], the authors proposed a derivative-free descent algorithm for nonlinear complementarity problems (NCPs) by the generalized Fischer–Burmeister merit function: $\psi_p(a,b) = \frac{1}{2}[\|(a,b)\|_p - (a+b)]^2$, and observed that the choice of the parameter p has a great influence on the numerical performance of the algorithm. In this paper, we analyze the phenomenon theoretically for a derivative-free descent algorithm which is based on a penalized form of ψ_p and uses a different direction from that of Chen and Pan. More specifically, we show that the algorithm proposed is globally convergent and has a locally R-linear convergence rate, and furthermore, its convergence rate will become worse when the parameter p decreases. Numerical results are also reported for the test problems from MCPLIB, which further verify the theoretical results obtained.

1. Introduction

The nonlinear complementarity problem (NCP) is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0,\; F(x) \geq 0, \quad \langle x, F(x)\rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \ldots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. We assume that F is continuously differentiable throughout this paper. The NCP has attracted much attention because of its wide applications in the fields of economics, engineering, and operations research [1,2], to name a few.

Many methods have been proposed to solve the NCP; see [3,2,4] and the references therein. One of the most powerful and popular methods is to reformulate the NCP as a system of nonlinear equations [5–7], or an unconstrained minimization problem [8–15]. The objective function that can constitute an equivalent unconstrained minimization problem is called a merit function, whose global minima are coincident with the solutions of the original NCP. To construct a merit function, a class of functions, called NCP-functions and defined below, plays a significant role.

Definition 1.1. A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if it satisfies

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\; b \geq 0,\; ab = 0. \tag{2}$$

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2009.06.022

The Fischer–Burmeister (FB) function is a well-known NCP-function defined as

$$\phi_{\rm FB}(a,b)=\sqrt{a^2+b^2}-(a+b), \tag{3}$$

by which the NCP can be reformulated as a system of nonsmooth equations:

$$\Phi_{\rm FB}(x)=\begin{pmatrix}\phi_{\rm FB}(x_1\,,F_1(x))\\ \cdot\\ \cdot\\ \cdot\\ \phi_{\rm FB}(x_n\,,F_n(x))\end{pmatrix}=0. \tag{4}$$

Thus, the function $\Psi_{\rm FB}:\mathbb{R}^n\to\mathbb{R}_+$ defined as below is a merit function for the NCP:

$$\Psi_{\rm FB}(x):=\frac{1}{2}\|\Phi_{\rm FB}(x)\|^2=\sum_{i=1}^{n}\psi_{\rm FB}(x_i,F_i(x)), \tag{5}$$

where $\psi_{\rm FB}:\mathbb{R}^2\to\mathbb{R}_+$ is the square of $\phi_{\rm FB}$, i.e.,

$$\psi_{\rm FB}(a,b)=\frac{1}{2}\left|\sqrt{a^2+b^2}-(a+b)\right|^2. \tag{6}$$

Consequently, the NCP is equivalent to an unconstrained minimization problem:

$$\min_{x\in\mathbb{R}^n}\Psi_{\rm FB}(x). \tag{7}$$

Recently, an extension of the FB-function was considered in [16–18] by the authors. More specifically, we define the generalized FB-function $\phi_p:\mathbb{R}^2\to\mathbb{R}$ by

$$\phi_p(a,b):=\|(a,b)\|_p-(a+b), \tag{8}$$

where $p>1$ is an arbitrary fixed real number and $\|(a,b)\|_p$ denotes the p-norm of (a,b), i.e., $\|(a,b)\|_p=\sqrt[p]{|a|^p+|b|^p}$. In other words, in the function ϕ_p, we replace the 2-norm of (a,b) in the FB-function by a more general p-norm. The function ϕ_p is still an NCP-function, which naturally induces another NCP-function $\psi_p:\mathbb{R}^2\to\mathbb{R}_+$ given by

$$\psi_p(a,b):=\frac{1}{2}|\phi_p(a,b)|^2. \tag{9}$$

For any given $p>1$, the function ψ_p is shown to possess all favorable properties of the FB-function $\psi_{\rm FB}$; see [16–18]. For example, ψ_p is also continuously differentiable everywhere on $\mathbb{R}^2$. Like $\phi_{\rm FB}$, the operator $\Phi_p:\mathbb{R}^n\to\mathbb{R}^n$ defined as

$$\Phi_p(x)=\begin{pmatrix}\phi_p(x_1\,,F_1(x))\\ \cdot\\ \cdot\\ \cdot\\ \phi_p(x_n\,,F_n(x))\end{pmatrix} \tag{10}$$

yields a family of merit functions $\Psi_p:\mathbb{R}^n\to\mathbb{R}_+$ for the NCP

$$\Psi_p(x):=\frac{1}{2}\|\Phi_p(x)\|^2=\sum_{i=1}^{n}\psi_p(x_i,F_i(x)). \tag{11}$$

In this paper, we study the following merit function $\Psi_{\alpha,p}:\mathbb{R}^n\to\mathbb{R}$ for the NCP:

$$\Psi_{\alpha,p}(x):=\sum_{i=1}^{n}\psi_{\alpha,p}(x_i\,,F_i(x)), \tag{12}$$

where $\psi_{\alpha,p}:\mathbb{R}^2\to\mathbb{R}_+$ is an NCP-function defined by

$$\psi_{\alpha,p}(a,b):=\frac{\alpha}{2}(\max\{0,ab\})^2+\psi_p(a,b)=\frac{\alpha}{2}(ab)_+^2+\frac{1}{2}(\|(a,b)\|_p-(a+b))^2, \tag{13}$$

with $\alpha\geq 0$ being a real parameter. When $\alpha=0$, the function $\psi_{\alpha,p}$ reduces to ψ_p. Hence, $\psi_{\alpha,p}$ is an extension of ψ_p. Besides, $\psi_{\alpha,p}$ also extends the function ψ_α studied in [19] by Yamada, Yamashita, and Fukushima which corresponds to $p=2$. Indeed, $\psi_{\alpha,p}$ has been studied in [17] by one of the authors (see ψ_4 therein), but there was no investigation on the property of the error bound. In this paper, we present more favorable properties of $\psi_{\alpha,p}$, and particularly, the conditions under which $\Psi_{\alpha,p}$ provides a global error bound for the NCP. With these results, we propose a derivative-free descent algorithm based on

$\phi_{\alpha,p}$ and establish its global convergence and local R-linear convergence rate. Moreover, we also analyze the influence of p on the convergence rate of the proposed algorithm theoretically and obtain the conclusion that the convergence rate of the algorithm will become worse when the value of p decreases. Thus, this paper can be viewed as a follow-up of [17,18].

This paper is organized as follows. In Section 2, we review some definitions and preliminary results to be used in the subsequent analysis. In Section 3, we show some important properties of the proposed merit function. In Section 4, we propose a derivative-free algorithm associated with $\Psi_{\alpha,p}$, prove its global convergence and the R-linear convergence rate, and analyze the influence of p on the convergence rate. Some numerical experiments are reported in Section 5, and we make concluding remarks in Section 6.

Throughout this paper, $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors and $^{\mathrm{T}}$ denotes transpose. For every differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of f at x. For every differentiable mapping $F = (F_1, \dots, F_n)^{\mathrm{T}} : \mathbb{R}^n \to \mathbb{R}^n$, $\nabla F(x) = (\nabla F_1(x) \ \dots \ \nabla F_n(x))$ denotes the transpose Jacobian of F at x. We denote by $\|x\|_p$ the p-norm of x and by $\|x\|$ the Euclidean norm of x. The level set of a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\mathcal{L}(\Psi, c) := \{x \in \mathbb{R}^n \mid \Psi(x) \le c\}$. In addition, we also use the natural residual merit function $\Psi_{\mathrm{NR}} : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$\Psi_{\mathrm{NR}}(x) := \frac{1}{2} \sum_{i=1}^{n} \phi_{\mathrm{NR}}^2(x_i, F_i(x)), \tag{14}$$

where $\phi_{\mathrm{NR}} : \mathbb{R}^2 \to \mathbb{R}$ denotes the minimum NCP-function $\min\{a, b\}$. Unless otherwise stated, in what follows, we always suppose that p is a fixed real number in $(1, \infty)$.

2. Preliminaries

This section mainly recalls some concepts about the mapping F that will be used later.

Definition 2.1. Let $F = (F_1, \dots, F_n)^{\mathrm{T}}$ with $F_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \dots, n$. We say that

(a) F is monotone if $\langle x - y, F(x) - F(y)\rangle \ge 0$ for all $x, y \in \mathbb{R}^n$.
(b) F is strongly monotone with modulus $\mu > 0$ if $\langle x - y, F(x) - F(y)\rangle \ge \mu\|x - y\|^2$ for all $x, y \in \mathbb{R}^n$.
(c) F is a P_0-function if $\max_{\substack{1\le i\le n \\ x_i \ne y_i}} (x_i - y_i)(F_i(x) - F_i(y)) \ge 0$ for all $x, y \in \mathbb{R}^n$ and $x \ne y$.
(d) F is a uniform P-function with modulus $\mu > 0$ if $\max_{1\le i\le n}(x_i - y_i)(F_i(x) - F_i(y)) \ge \mu\|x - y\|^2$ for all $x, y \in \mathbb{R}^n$.
(e) $\nabla F(x)$ is uniformly positive definite with modulus $\mu > 0$ if $d^{\mathrm{T}}\nabla F(x)d \ge \mu\|d\|^2$ for all $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$.
(f) F is Lipschitz continuous if there exists a constant $L > 0$ such that $\|F(x) - F(y)\| \le L\|x - y\|$ for all $x, y \in \mathbb{R}^n$.

From Definition 2.1, it is easy to see that F is a uniform P-function with modulus $\mu > 0$ if F is strongly monotone with modulus $\mu > 0$, and F is a P_0-function if F is monotone. In addition, when F is continuously differentiable, the following results hold:

1. F is monotone if and only if $\nabla F(x)$ is positive semidefinite for all $x \in \mathbb{R}^n$.
2. F is strongly monotone if and only if $\nabla F(x)$ is uniformly positive definite.

3. Properties of the merit function

In this section, we study some favorable properties of the merit function $\psi_{\alpha,p}$ which will be used in the subsequent analysis, and then present some mild conditions under which the merit function $\Psi_{\alpha,p}$ has bounded level sets and provides a global error bound, respectively.

The following lemma states that $\psi_{\alpha,p}$ enjoys many favorable properties as ψ_p holds. Furthermore, when $\alpha > 0$, it has an important property that ψ_p does not have (see Lemma 3.1(f)). Although most results of the lemma were investigated in [17, Prop. 3.3] where only p being integer was considered, we here provide more detailed arguments for the general case where p is any real number greater than one.

Lemma 3.1. *The function $\psi_{\alpha,p}$ defined by (13) has the following favorable properties:*

(a) *$\psi_{\alpha,p}$ is an NCP-function and $\psi_{\alpha,p} \ge 0$ for all $(a, b) \in \mathbb{R}^2$.*
(b) *$\psi_{\alpha,p}$ is continuously differentiable everywhere, and moreover, if $(a, b) \ne (0, 0)$,*

$$\begin{aligned} \nabla_a\psi_{\alpha,p}(a, b) &= \alpha b(ab)_+ + \left(\frac{\operatorname{sgn}(a)\cdot|a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b), \\ \nabla_b\psi_{\alpha,p}(a, b) &= \alpha a(ab)_+ + \left(\frac{\operatorname{sgn}(b)\cdot|b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b); \end{aligned} \tag{15}$$

and otherwise $\nabla_a\psi_{\alpha,p}(0, 0) = \nabla_b\psi_{\alpha,p}(0, 0) = 0$.

(c) *For $p \geq 2$, the gradient of $\psi_{\alpha,p}$ is Lipschitz continuous on any nonempty bounded set S, i.e., there exists $L > 0$ such that, for any (a, b),*
$(c, d) \in S$,

$$\|\nabla\psi_{\alpha,p}(a, b) - \nabla\psi_{\alpha,p}(c, d)\| \leq L\|(a, b) - (c, d)\|.$$

(d) $\nabla_a\psi_{\alpha,p}(a, b) \cdot \nabla_b\psi_{\alpha,p}(a, b) \geq 0$ *for any $(a, b) \in \mathbb{R}^2$, and furthermore, the equality holds if and only if $\psi_{\alpha,p}(a, b) = 0$.*
(e) $\nabla_a\psi_{\alpha,p}(a, b) = 0 \Longleftrightarrow \nabla_b\psi_{\alpha,p}(a, b) = 0 \Longleftrightarrow \psi_{\alpha,p}(a, b) = 0$.
(f) *Suppose that $\alpha > 0$. If $a \to -\infty$ or $b \to -\infty$ or $ab \to \infty$, then $\psi_{\alpha,p}(a, b) \to \infty$.*

Proof. Parts (a), (b) and (f) directly follow from the definition of $\psi_{\alpha,p}$ and Proposition 3.2(a)–(c) and Lemma 3.1 of [18]. It remains to show parts (c)–(e).

(c) Notice that the functions $a(ab)_+$ and $b(ab)_+$ for any $a, b \in \mathbb{R}$ are Lipschitz continuous on any nonempty bounded set S, whereas $\phi_p(a, b)$ is Lipschitz continuous on $\mathbb{R}^2$ by [18, Proposition 3.1 (e)]. Therefore, by the expression of $\nabla\psi_{\alpha,p}(a, b)$ and the boundedness of

$$\left(\frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right) \quad \text{and} \quad \left(\frac{\operatorname{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right),$$

it is not hard to verify that the gradient $\nabla\psi_{\alpha,p}(a, b)$ is Lipschitz continuous on S for $p \geq 2$.

(d) If $(a, b) = (0, 0)$, part (d) clearly holds. Now suppose that $(a, b) \neq (0, 0)$. Then,

$$\begin{aligned}\nabla_a\psi_{\alpha,p}(a, b) \cdot \nabla_b\psi_{\alpha,p}(a, b) &= \left(\frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\left(\frac{\operatorname{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p^2(a, b) \\ &+ \alpha^2 ab(ab)_+^2 + \alpha a(ab)_+\left(\frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b) + \alpha b(ab)_+\left(\frac{\operatorname{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b). \end{aligned} \tag{16}$$

Since

$$ab(ab)_+^2 \geq 0, \quad \frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1 \leq 0, \quad \text{and} \quad \frac{\operatorname{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1 \leq 0, \tag{17}$$

it suffices to show that the last two terms of (16) are nonnegative. We next claim that

$$\alpha a(ab)_+\left(\frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b) \geq 0, \quad \forall\,(a, b) \neq (0, 0). \tag{18}$$

If $a \leq 0$, then $\phi_p(a, b) \geq 0$, which together with the second inequality in (17) implies that (18) holds. If $a > 0$ and $b > 0$, then $\phi_p(a, b) < 0$, which implies (18) by a similar reason. If $a > 0$ and $b \leq 0$, then $(ab)_+ = 0$, and hence (18) holds. Similarly, we have that

$$\alpha b(ab)_+\left(\frac{\operatorname{sgn}(b) \cdot |b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b) \geq 0, \quad \forall\,(a, b) \neq (0, 0).$$

Consequently, $\nabla_a\psi_{\alpha,p}(a, b) \cdot \nabla_b\psi_{\alpha,p}(a, b) \geq 0$. From (16), $\nabla_a\psi_{\alpha,p}(a, b) \cdot \nabla_b\psi_{\alpha,p}(a, b) = 0$ if and only if $\{a = 0$ or $(a \geq 0$ and $b = 0)$ or $\phi_p(a, b) = 0\}$ and $\{b = 0$ or $(b \geq 0$ and $a = 0)$ or $\phi_p(a, b) = 0\}$ and $\{ab = 0\}$. Thus, $\nabla_a\psi_{\alpha}(a, b) \cdot \nabla_b\psi_{\alpha,p}(a, b) = 0$ if and only if $\psi_{\alpha,p}(a, b) = 0$.

(e) If $\psi_{\alpha,p}(a, b) = 0$, then $ab = 0$ and $\phi_p(a, b) = 0$ by part (a), which in turn implies that $\nabla_a\psi_{\alpha,p}(a, b) = 0$ and $\nabla_b\psi_{\alpha,p}(a, b) = 0$. Next, we claim that $\nabla_a\psi_{\alpha,p}(a, b) = 0$ implies $\psi_{\alpha,p}(a, b) = 0$. Suppose that $\nabla_a\psi_{\alpha,p}(a, b) = 0$. Then,

$$\alpha b(ab)_+ = -\left(\frac{\operatorname{sgn}(a) \cdot |a|^{p-1}}{\|(a, b)\|_p^{p-1}} - 1\right)\phi_p(a, b). \tag{19}$$

We can verify that the equality (19) implies $b = 0$, $a \geq 0$ or $b > 0$, $a = 0$. Under the two cases, we both have $\psi_{\alpha,p}(a, b) = 0$. Similarly, $\nabla_b\psi_{\alpha,p}(a, b) = 0$ also implies $\psi_{\alpha,p}(a, b) = 0$. □

Notice that $ab \to \infty$ does not necessarily imply $\psi_p(a, b) \to \infty$, which means ψ_p does not share Lemma 3.1(f). In fact, for $\alpha = 0$, the lemma needs to be modified as "if $(a \to \infty)$ or $(b \to \infty)$ or $(a \to \infty$ and $b \to \infty)$, then $\psi_{\alpha,p}(a, b) \to \infty$". As we will see later, Lemma 3.1(f) is useful for proving that the level sets of $\Psi_{\alpha,p}$ are bounded. Besides, by Lemma 3.1(a), we immediately have the following theorem.

Theorem 3.1. *Let $\Psi_{\alpha,p}$ be defined as in (12). Then $\Psi_{\alpha,p}(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\Psi_{\alpha,p}(x) = 0$ if and only if x solves the NCP. Moreover, if the NCP has at least one solution, then x is a global minimizer of $\Psi_{\alpha,p}$ if and only if x solves the NCP.*

Theorem 3.1 indicates that the NCP can be recast as the unconstrained minimization:

$$\min_{x\in\mathbb{R}^n} \Psi_{\alpha,p}(x). \tag{20}$$

In general, it is hard to find a global minimum of $\Psi_{\alpha,p}$. Therefore, it is important to know under what conditions a stationary point of $\Psi_{\alpha,p}$ is a global minimum. Using Lemma 3.1(d) and the same proof techniques as in [11, Theorem 3.5], we can establish that each stationary point of $\Psi_{\alpha,p}$ is a global minimum only if F is a P_0-function.

Theorem 3.2. *Let F be a P_0-function. Then $x^* \in \mathbb{R}^n$ is a global minimum of the unconstrained optimization problem (20) if and only if x^* is a stationary point of $\Psi_{\alpha,p}$.*

From the following theorem, we see that the unconstrained minimization problem (20) has a stationary point under rather weak conditions of the mapping F. Since similar results and analogous analysis can be found in [17, Proposition 4.1], [11, Theorem 3.8] and [20, Theorem 4.1], we omit the proof here.

Theorem 3.3. *The function $\Psi_{\alpha,p}$ has bounded level sets $\mathcal{L}(\Psi_{\alpha,p}, c)$ for all $c \in \mathbb{R}$, if F is monotone and the NCP is strictly feasible (i.e., there exists $\hat{x} > 0$ such that $F(\hat{x}) > 0$) when $\alpha > 0$, or F is a uniform P-function when $\alpha \geq 0$.*

In what follows, we will show that the merit functions Ψ_p, $\Psi_{\rm NR}$ and $\Psi_{\alpha,p}$ have the same order on every bounded set. For this purpose, we need the following crucial technical lemma, which generalizes the important property of $\phi_{\rm FB}$ proved by Tseng in [21].

Lemma 3.2. *Let $\phi_p : \mathbb{R}^2 \to \mathbb{R}$ be defined as in (8). Then for any $p > 1$ we have*

$$(2-2^{\frac{1}{p}})|\min\{a,b\}| \leq |\phi_p(a,b)| \leq (2+2^{\frac{1}{p}})|\min\{a,b\}|. \tag{21}$$

Proof. Without loss of generality, suppose $a \geq b$. We will prove the desired results by considering the following two cases: (1) $a+b \leq 0$ and (2) $a+b > 0$.

Case (1): $a+b \leq 0$. In this case, we have

$$|\phi_p(a,b)| \geq \|(a,b)\|_p \geq |b| = |\min\{a,b\}| \geq (2-2^{\frac{1}{p}})|\min\{a,b\}|. \tag{22}$$

On the other hand, since $a \geq b$ and $a+b \leq 0$, we have $|b| \geq |a|$. Then

$$|\phi_p(a,b)| \leq \|(a,b)\|_p - 2b = (2+2^{\frac{1}{p}})|b| = (2+2^{\frac{1}{p}})|\min\{a,b\}|. \tag{23}$$

Case (2): $a+b > 0$. If $ab = 0$, then (21) clearly holds. Thus, we discuss by two subcases:

(i) $ab < 0$. In this subcase, we have $a > 0$, $b < 0$, and $|a| > |b|$. Consequently,

$$\phi_p(a,b) \leq |a| + |b| - (a+b) = -2b = 2|\min\{a,b\}| \leq (2+2^{\frac{1}{p}})|\min\{a,b\}|, \tag{24}$$

and

$$\phi_p(a,b) \geq \|(a,b)\|_\infty - (a+b) = -b = |\min\{a,b\}| \geq (2-2^{\frac{1}{p}})|\min\{a,b\}|. \tag{25}$$

(ii) $ab > 0$. Now we have $a \geq b > 0$. Since for any $p > 1$ it holds that

$$0 \geq \phi_p(a,b) \geq \|(a,b)\|_\infty - (a+b) = a - (a+b) = -b = -\min\{a,b\},$$

we immediately obtain that

$$|\phi_p(a,b)| \leq |\min\{a,b\}| \leq (2+2^{\frac{1}{p}})|\min\{a,b\}|. \tag{26}$$

On the other hand, since $\phi_p(a,b) \leq 0$, it follows that

$$|\phi_p(a,b)| = a + b - \|(a,b)\|_p = b\left[\left(\frac{a}{b}+1\right) - \left(\left(\frac{a}{b}\right)^p + 1\right)^{1/p}\right].$$

Let $f(t) = t + 1 - (t^p+1)^{1/p}$ for $t \geq 1$. Then

$$f'(t) = 1 - \left(\frac{t^p}{t^p+1}\right)^{\frac{p-1}{p}}.$$

Notice that $f'(t) > 0$ for $t \geq 1$, and $f(1) = 2 - 2^{\frac{1}{p}}$, and hence we obtain that

$$|\phi_p(a,b)| \geq (2-2^{\frac{1}{p}})b = (2-2^{\frac{1}{p}})|\min\{a,b\}| \quad \text{for any } p > 1. \tag{27}$$

All the aforementioned inequalities (22)–(27) imply that (21) holds. □

Proposition 3.1. *Let Ψ_p, $\Psi_{\rm NR}$ and $\Psi_{\alpha,p}$ be defined as in (11), (12) and (14), respectively. Let S be an arbitrary bounded set. Then, for any $p > 1$, we have*

$$\left(2 - 2^{\frac{1}{p}}\right)^2 \Psi_{\rm NR}(x) \le \Psi_p(x) \le \left(2 + 2^{\frac{1}{p}}\right)^2 \Psi_{\rm NR}(x) \quad \text{for all } x \in \mathbb{R}^n \tag{28}$$

and

$$\left(2 - 2^{\frac{1}{p}}\right)^2 \Psi_{\rm NR}(x) \le \Psi_{\alpha,p}(x) \le \left(\alpha B^2 + (2 + 2^{\frac{1}{p}})^2\right) \Psi_{\rm NR}(x) \quad \text{for all } x \in S, \tag{29}$$

where B is a constant defined by $B = \max_{1\le i\le n}\left\{\sup_{x\in S}\{\max\{|x_i|, |F_i(x)|\}\}\right\} < \infty$.

Proof. The inequality in (28) is direct by Lemma 3.2 and the definitions of Ψ_p and $\Psi_{\rm NR}$. In addition, from Lemma 3.2 and the definition of $\Psi_{\alpha,p}$, it follows that

$$\Psi_{\alpha,p}(x) \ge \left(2 - 2^{\frac{1}{p}}\right)^2 \Psi_{\rm NR}(x) \quad \text{for all } x \in \mathbb{R}^n.$$

We next prove the inequality on the right-hand side of (29). We claim that, for each i,

$$(x_iF_i(x))_+ \le B|\min\{x_i, F_i(x)\}| \quad \text{for all } x \in S. \tag{30}$$

Without loss of generality, suppose $F_i(x) \ge x_i$. If $F_i(x) \ge x_i \ge 0$, it follows that

$$(x_iF_i(x))_+ = x_iF_i(x) = F_i(x)|\min\{x_i, F_i(x)\}| \le B|\min\{x_i, F_i(x)\}|.$$

If $F_i(x) \ge 0 \ge x_i$, then $(x_iF_i(x))_+ = 0$. If $0 \ge F_i(x) \ge x_i$, it follows that

$$(x_iF_i(x))_+ = |x_iF_i(x)| \le |x_i|^2 \le B|\min\{x_i, F_i(x)\}|.$$

Thus, (30) holds for all $x \in S$. By Lemma 3.2 and (30), for all $i = 1, \ldots, n$ and $x \in S$,

$$\psi_{\alpha,p}(x_i, F_i(x)) \le \left\{\alpha B^2 + (2 + 2^{\frac{1}{p}})^2\right\} \min\{x_i, F_i(x)\}^2$$

holds for any $p > 1$. The proof is then complete by the definition of $\Psi_{\alpha,p}$ and $\Psi_{\rm NR}$. □

From Proposition 3.1, we immediately obtain the following result.

Corollary 3.1. *Let Ψ_p and $\Psi_{\alpha,p}$ be defined by (12) and (11), respectively, and S be any bounded set. Then, for any $p > 1$ and all $x \in S$, we have the following inequalities:*

$$\frac{(2 - 2^{\frac{1}{p}})^2}{\left(\alpha B^2 + (2 + 2^{1p})^2\right)} \Psi_{\alpha,p}(x) \le \Psi_p(x) \le \frac{(2 + 2^{\frac{1}{p}})^2}{(2 - 2^{1p})^2} \Psi_{\alpha,p}(x)$$

where B is the constant defined as in Proposition 3.1.

Since Ψ_p, $\Psi_{\rm NR}$ and $\Psi_{\alpha,p}$ have the same order on a bounded set, one will provide a global error bound for the NCP as long as the other one does. Below, we show that $\Psi_{\alpha,p}$ provides a global error bound without the Lipschitz continuity of F when $\alpha > 0$.

Theorem 3.4. *Let $\Psi_{\alpha,p}$ be defined as in (12). Suppose that F is a uniform P-function with modulus $\mu > 0$. If $\alpha > 0$, then there exists a constant $\kappa_1 > 0$ such that*

$$\|x - x^*\| \le \kappa_1 \Psi_{\alpha,p}(x)^{\frac{1}{4}} \quad \text{for all } x \in \mathbb{R}^n;$$

if $\alpha = 0$ and S is any bounded set, there exists a constant $\kappa_2 > 0$ such that

$$\|x - x^*\| \le \kappa_2 \left(\max\left\{\Psi_{\alpha,p}(x), \sqrt{\Psi_{\alpha,p}(x)}\right\}\right)^{\frac{1}{2}} \quad \text{for all } x \in S;$$

where $x^* = (x_1^*, \ldots, x_n^*)$ is the unique solution for the NCP.

Proof. Since F is a uniform P-function, the NCP has a unique solution, and moreover,

$$\begin{aligned}
\mu\|x - x^*\|^2 &\le \max_{1\le i\le n}(x - x^*)(F_i(x) - F_i(x^*)) \\
&= \max_{1\le i\le n}\{x_iF_i(x) - x_i^*F_i(x) - x_iF_i(x^*) + x_i^*F_i(x^*)\} \\
&= \max_{1\le i\le n}\{x_iF_i(x) - x_i^*F_i(x) - x_iF_i(x^*)\} \\
&\le \max_{1\le i\le n} \tau_i\{(x_iF_i(x))_+ + (-F_i(x))_+ + (-x_i)_+\},
\end{aligned} \tag{31}$$

where $\tau_i := \max\{1, x_i^*, F_i(x^*)\}$. We next prove that, for all $(a, b) \in \mathbb{R}^2$,

$$(-a)_+^2 + (-b)_+^2 \le \left[\|(a, b)\|_p - (a + b)\right]^2. \tag{32}$$

Without loss of generality, suppose $a \ge b$. If $a \ge b \ge 0$, then (32) holds obviously. If $a \ge 0 \ge b$, then $\|(a, b)\|_p - (a + b) \ge -b \ge 0$, which in turn implies that

$$(-a)_+^2 + (-b)_+^2 = b^2 \le \left[\|(a, b)\|_p - (a + b)\right]^2.$$

If $0 \ge a \ge b$, then $(-a)_+^2 + (-b)_+^2 = a^2 + b^2 \le \left[\|(a, b)\|_p - (a + b)\right]^2$. Hence, (32) follows.

Suppose that $\alpha > 0$. Using the inequality (32), we then obtain that

$$\begin{aligned}
[(ab)_+ + (-a)_+ + (-b)_+]^2 &= (ab)_+^2 + (-b)_+^2 + (-a)_+^2 + 2(ab)_+(-a)_+ + 2(-a)_+(-b)_+ + 2(ab)_+(-b)_+ \\
&\le (ab)_+^2 + (-b)_+^2 + (-a)_+^2 + (ab)_+^2 + (-a)_+^2 + (-a)_+^2 + (-b)_+^2 + (ab)_+^2 + (-b)_+^2 \\
&\le 3\left[(ab)_+^2 + \left(\|(a, b)\|_p - (a + b)\right)^2\right] \\
&\le \tau\left[\frac{\alpha}{2}(ab)_+^2 + \frac{1}{2}\left(\|(a, b)\|_p - (a + b)\right)^2\right] \\
&= \tau\psi_{\alpha,p}(a, b) \quad \text{for all } (a, b) \in \mathbb{R}^2,
\end{aligned} \tag{33}$$

where $\tau := \max\left\{\frac{6}{\alpha}, 6\right\} > 0$. Combining (33) with (31) and letting $\hat{\tau} = \max_{1\le i\le n} \tau_i$, we get

$$\begin{aligned}
\mu\|x - x^*\|^2 &\le \max_{1\le i\le n} \tau_i \left\{\tau\psi_{\alpha,p}(x_i, F_i(x))\right\}^{1/2} \\
&\le \hat{\tau}\tau^{1/2} \max_{1\le i\le n} \psi_{\alpha,p}(x_i, F(x))^{1/2} \\
&\le \hat{\tau}\tau^{1/2} \left\{\sum_{i=1}^{n}\{\psi_{\alpha,p}(x_i, F_i(x))\}\right\}^{1/2} \\
&= \hat{\tau}\tau^{1/2}\Psi_{\alpha,p}(x, F(x))^{1/2}.
\end{aligned}$$

From this, the first desired result follows immediately by setting $\kappa_1 := \left[\hat{\tau}\tau^{1/2}/\mu\right]^{1/2}$.

Suppose that $\alpha = 0$. From the proof of Proposition 3.1, the inequality (30) holds. Combining with Eqs. (31)–(32), it then follows that, for all $x \in S$,

$$\begin{aligned}
\mu\|x - x^*\|^2 &\le \max_{1\le i\le n} \tau_i \left[B|\min\{x_i, F_i(x)\}| + (\psi_p(x_i, F_i(x)))^{1/2}\right] \\
&\le \hat{\tau} \max_{1\le i\le n} \left[\sqrt{2}\hat{B}\psi_p(x_i, F_i(x)) + (\psi_p(x_i, F_i(x)))^{1/2}\right] \\
&\le \sqrt{2}\hat{\tau}\hat{B}\left(\Psi_p(x) + \sqrt{\Psi_p(x)}\right) \\
&\le 4\hat{\tau}\hat{B}\max\left\{\Psi_p(x), \sqrt{\Psi_p(x)}\right\} \\
&= 4\hat{\tau}\hat{B}\max\left\{\Psi_{\alpha,p}(x), \sqrt{\Psi_{\alpha,p}(x)}\right\}
\end{aligned}$$

where $\hat{B} = B/(2 - 2^{\frac{1}{p}})$ and the second inequality is from Lemma 3.2. Letting $\kappa_2 := 2\left[\hat{\tau}\hat{B}/\mu\right]^{1/2}$, we obtain the desired result from the above inequality. □

The following lemma is needed for the proof of Proposition 3.2, which plays a crucial role in showing the convergence rate of the algorithm described in Section 4.

Lemma 3.3. *For all $(a, b) \neq (0, 0)$ and $p > 1$, we have the following inequality:*

$$\left(\frac{\operatorname{sgn}(a)\cdot|a|^{p-1} + \operatorname{sgn}(b)\cdot|b|^{p-1}}{\|(a, b)\|_p^{p-1}} - 2\right)^2 \ge \left(2 - 2^{\frac{1}{p}}\right)^2.$$

Proof. If $a = 0$ or $b = 0$, the inequality holds obviously. Then we complete the proof by considering three cases: (i) $a > 0$ and $b > 0$, (ii) $a < 0$ and $b < 0$, and (iii) $ab < 0$.

Case (i): Without loss of generality, we suppose $a \geq b > 0$. Then

$$\frac{|a|^{p-1} + |b|^{p-1}}{\|(a,b)\|_p^{p-1}} = \frac{\left(\left|\frac{a}{b}\right|\right)^{p-1} + 1}{\left((|ab|)^p + 1\right)^{1-1p}}. \tag{34}$$

Let $f(t) := \frac{t^{p-1}+1}{(t^p+1)^{1-1p}}$ for any $t > 0$. By computation, we have that

$$f'(t) = \frac{t^{p-2}(p-1)(1-t)}{(t^p+1)^2} \quad \forall t > 0.$$

Since $f'(t) < 0$ for $t \geq 1$ and $f(1) = 2^{\frac{1}{p}}$, it follows that $f(t) \leq 2^{\frac{1}{p}}$ for $t \geq 1$. Therefore,

$$\frac{|a|^{p-1} + |b|^{p-1}}{\|(a,b)\|_p^{p-1}} \leq 2^{\frac{1}{p}} \quad \text{for } p > 1,$$

which in turn implies that $2 - \frac{|a|^{p-1}+|b|^{p-1}}{\|(a,b)\|_p^{p-1}} \geq 2 - 2^{\frac{1}{p}}$ for $p > 1$. Squaring both sides then leads to the desired inequality.

Case (ii): By similar arguments as in Case (i), we obtain

$$2 - 2^{\frac{1}{p}} \leq 2 - \frac{|a|^{p-1} + |b|^{p-1}}{\|(a,b)\|_p^{p-1}} \leq 2 + \frac{|a|^{p-1} + |b|^{p-1}}{\|(a,b)\|_p^{p-1}} \quad \text{for } p > 1,$$

from which the result follows immediately.

Case (iii): Again, we suppose $|a| \geq |b|$, and therefore have

$$2^{\frac{1}{p}} \geq \frac{|a|^{p-1} + |b|^{p-1}}{\|(a,b)\|_p^{p-1}} \geq \frac{|a|^{p-1} - |b|^{p-1}}{\|(a,b)\|_p^{p-1}} \quad \text{for } p > 1.$$

Thus $2 - 2^{\frac{1}{p}} \leq 2 - \frac{|a|^{p-1}-|b|^{p-1}}{\|(a,b)\|_p^{p-1}}$ for $p > 1$, and the desired result is also satisfied. □

Proposition 3.2. *Let $\psi_{\alpha,p}$ be given as in (13). Then, for all $x \in \mathbb{R}^n$ and $p > 1$,*

$$\|\nabla_a \psi_{\alpha,p}(x, F(x)) + \nabla_b \psi_{\alpha,p}(x, F(x))\|^2 \geq 2\left(2 - 2^{\frac{1}{p}}\right)^2 \Psi_p(x),$$

and particularly, for all x belonging to any bounded set S and $p > 1$,

$$\|\nabla_a \psi_{\alpha,p}(x, F(x)) + \nabla_b \psi_{\alpha,p}(x, F(x))\|^2 \geq \frac{2(2 - 2^{\frac{1}{p}})^4}{\left(\alpha B^2 + (2 + 2^{1p})^2\right)} \Psi_{\alpha,p}(x)$$

where B is defined as in Proposition 3.1 and

$$\begin{aligned} \nabla_a \psi_{\alpha,p}(x, F(x)) &:= \left(\nabla_a \psi_{\alpha,p}(x_1, F_1(x)), \ldots, \nabla_a \psi_{\alpha,p}(x_n, F_n(x))\right)^{\mathrm{T}}, \\ \nabla_b \psi_{\alpha,p}(x, F(x)) &:= \left(\nabla_b \psi_{\alpha,p}(x_1, F_1(x)), \ldots, \nabla_b \psi_{\alpha,p}(x_n, F_n(x))\right)^{\mathrm{T}}. \end{aligned} \tag{35}$$

Proof. The second part of the conclusions is direct by Corollary 3.1 and the first part. From the definition of $\nabla_a \psi_{\alpha,p}(x, F(x))$, $\nabla_b \psi_{\alpha,p}(x, F(x))$ and $\Psi_p(x)$, the first part of the conclusions is equivalent to proving that the following inequality,

$$\left(\nabla_a \psi_{\alpha,p}(a,b) + \nabla_b \psi_{\alpha,p}(a,b)\right)^2 \geq 2\left(2 - 2^{\frac{1}{p}}\right)^2 \psi_p(a,b), \tag{36}$$

holds for all $(a, b) \in \mathbb{R}^2$. When $(a, b) = (0, 0)$, the inequality (36) clearly holds. Suppose $(a, b) \neq (0, 0)$. Then, it follows from Eq. (15) that

$$\begin{aligned}\left(\nabla_a\psi_{\alpha,p}(a,b)+\nabla_b\psi_{\alpha,p}(a,b)\right)^2 &= \left\{\alpha(a+b)(ab)_+ + (\phi_p(a,b))\left(\frac{\mathrm{sgn}(a)\cdot|a|^{p-1}+\mathrm{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-2\right)\right\}^2\\ &= \alpha^2(a+b)^2(ab)_+^2 + (\phi_p(a,b))^2\left(\frac{\mathrm{sgn}(a)\cdot|a|^{p-1}+\mathrm{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-2\right)^2\\ &\quad + 2\alpha(a+b)(ab)_+(\phi_p(a,b))\left(\frac{\mathrm{sgn}(a)\cdot|a|^{p-1}+\mathrm{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-2\right). \end{aligned} \tag{37}$$

Now, we claim that, for all $(a,b)\neq(0,0)\in\mathbb{R}^2$,

$$2\alpha(a+b)(ab)_+(\phi_p(a,b))\left(\frac{\mathrm{sgn}(a)\cdot|a|^{p-1}+\mathrm{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-2\right)\geq 0. \tag{38}$$

If $ab\leq 0$, then $(ab)_+=0$ and the inequality (36) is clear. If $a,b>0$, then by noting that

$$\left(\frac{\mathrm{sgn}(a)\cdot|a|^{p-1}+\mathrm{sgn}(b)\cdot|b|^{p-1}}{\|(a,b)\|_p^{p-1}}-2\right)\leq 0,\quad \forall(a,b)\neq(0,0)\in\mathbb{R}^2 \tag{39}$$

and $\phi_p(a,b)\leq 0$, the inequality (38) also holds. If $a,b<0$, then $\phi_p(a,b)\geq 0$, which together with (39) then yields the inequality (38). Thus, we prove that the inequality (38) holds for all $(a,b)\neq(0,0)$. Using Lemma 3.3 and Eqs. (38)–(39), we readily obtain that the inequality (36) holds for all $(a,b)\neq(0,0)$. The proof is thus complete. □

4. A descent algorithm and convergence results

In this section, we propose a derivative-free descent algorithm based on the function $\Psi_{\alpha,p}$. By Lemma 3.1(d), it is easy to verify that $\bar{d}:=-\nabla_b\psi_{\alpha,p}(x,F(x))$ is a descent direction for monotone nonlinear complementarity problems, i.e., the following result holds.

Lemma 4.1. *Let $\Psi_{\alpha,p}$ be defined as in (12). If the mapping F is monotone, then $\bar{d}:=-\nabla_b\psi_{\alpha,p}(x,F(x))$ is a descent direction of $\Psi_{\alpha,p}$ at any $x\in\mathbb{R}^n$, i.e., $\nabla\Psi_{\alpha,p}(x)^{\mathrm{T}}\bar{d}<0$.*

However, we observe that $\bar{d}$ does not involve any information of $\nabla_a\psi_{\alpha,p}(x,F(x))$ and is lacking a certain symmetry, for which we cannot find a constant $c>0$ such that

$$\|\bar{d}\|\geq c\,\psi_{\alpha,p}(x,F(x)).$$

This sets a big obstacle to establishing the convergence rate of the derivative-free algorithm based on $\bar{d}$. In view of this, we follow a similar line as [19] to adopt a search direction of the following form:

$$d^k(\rho):=-\nabla_b\psi_{\alpha,p}(x^k,F(x^k))-\rho\nabla_a\psi_{\alpha,p}(x^k,F(x^k)), \tag{40}$$

where ρ is a parameter such that $\rho\in(0,1)$ and $\nabla_a\psi_{\alpha,p}(x,F(x))$, $\nabla_b\psi_{\alpha,p}(x,F(x))$ are defined as in (35). Although $d^k(\rho)$ for any $\rho\in(0,1)$ is not necessarily a descent direction of $\Psi_{\alpha,p}$ at the iterate x^k, Lemma 4.1 implies that it is a descent one if $\rho\in(0,\bar{\rho}_k)$, where

$$\bar{\rho}_k:=1\quad \text{if } \nabla_a\psi_{\alpha,p}(x^k,F(x^k))^{\mathrm{T}}\nabla\Psi_{\alpha,p}(x^k)\geq 0,$$

and otherwise

$$\bar{\rho}_k:=\min\left\{1,-\frac{\nabla_b\psi_{\alpha,p}(x^k,F(x^k))^{\mathrm{T}}\nabla\Psi_{\alpha,p}(x^k)}{\nabla_a\psi_{\alpha,p}(x^k,F(x^k))^{\mathrm{T}}\nabla\Psi_{\alpha,p}(x^k)}\right\}.$$

Clearly, $\bar{\rho}_k\in(0,1)$ except that x^k is a solution of the NCP. Thus, d^k is a descent direction of $\Psi_{\alpha,p}$ at x^k for monotone NCPs only if ρ is chosen sufficiently small. Similarly to [19], we also determine an appropriate ρ_k by the backtracking search of Armijo type instead of the value of $\bar{\rho}_k$, in our algorithm described below.

Algorithm 4.1. (Step 0) Given real numbers $p>1$ and $\alpha\geq 0$ and a starting point $x^0\in\mathbb{R}^n$. Choose the parameters $\sigma\in(0,1)$, $\beta\in(0,1)$, $\gamma\in(0,1)$ and $\varepsilon\geq 0$. Set $k:=0$.

(Step 1) If $\Psi_{\alpha,p}(x^k)\leq\varepsilon$, then stop.

(Step 2) Let m_k be the smallest nonnegative integer m satisfying

$$\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) \le (1 - \sigma\beta^{2m})\Psi_{\alpha,p}(x^k), \tag{41}$$

where

$$d^k(\gamma^m) := -\nabla_b \psi_{\alpha,p}(x^k, F(x^k)) - \gamma^m \nabla_a \psi_{\alpha,p}(x^k, F(x^k)).$$

(Step 3) Set $x^{k+1} := x^k + \beta^{m_k} d^k(\gamma^{m_k})$, $k := k + 1$ and go to Step 1.

We see that Algorithm 4.1 does not involve the computation of $\nabla\Psi_{\alpha,p}$ and ∇F, and hence it is a derivative-free algorithm. In what follows, we establish the convergence results for Algorithm 4.1, and particularly, analyze its convergence rate under the strongly monotone assumption of F. To this end, we assume that the parameter ε in Algorithm 4.1 is equal to zero and Algorithm 4.1 generates an infinite sequence $\{x^k\}$.

Proposition 4.1. *Suppose that F is monotone. Then Algorithm 4.1 is well-defined for any starting point x^0. Furthermore, if x^* is an accumulation point of the sequence $\{x^k\}$ generated by Algorithm 4.1, then x^* is a solution of the NCP.*

Proof. We first prove that Algorithm 4.1 is well-defined. From the construction of the algorithm, it suffices to show that Step 2 is well-defined. Assume to the contrary that there is no nonnegative integer m satisfying (41). Then, for any integer $m \ge 0$,

$$\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) - \Psi_{\alpha,p}(x^k) > -\sigma\beta^{2m}\Psi_{\alpha,p}(x^k).$$

Dividing the above inequality by β^m and passing to the limit $m \to +\infty$, we obtain that

$$\lim_{m\to+\infty} \frac{\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) - \Psi_{\alpha,p}(x^k)}{\beta^m} \ge 0. \tag{42}$$

Since $\Psi_{\alpha,p}$ is continuously differentiable, we have that $\Psi_{\alpha,p}$ is locally Lipschitz continuous at x^k, which in turn implies that there exists $L > 0$ such that

$$\|\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) - \Psi_{\alpha,p}(x^k + \beta^m d^k(0))\| \le L\beta^m \|d^k(\gamma^m) - d^k(0)\|$$

for all sufficiently large m. Consequently,

$$\begin{aligned}\lim_{m\to+\infty} \frac{\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) - \Psi_{\alpha,p}(x^k)}{\beta^m} &= \lim_{m\to+\infty} \frac{\Psi_{\alpha,p}(x^k + \beta^m d^k(0)) - \Psi_{\alpha,p}(x^k)}{\beta^m} \\ &\quad + \lim_{m\to+\infty} \frac{\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) - \Psi_{\alpha,p}(x^k + \beta^m d^k(0))}{\beta^m} \\ &\le \nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}} d^k(0).\end{aligned}$$

This together with (42) yields that $\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}} d^k(0) \ge 0$. However, by Lemma 4.1, $\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}} d^k(0) < 0$, which leads to a contradiction. Hence, Algorithm 4.1 is well-defined.

Next we prove that any accumulation point x^* of $\{x^k\}$ is a solution of the NCP. Let $\{x^k\}_{k\in K}$ be a subsequence converging to x^*. Notice that $\Psi_{\alpha,p}$ is continuously differentiable everywhere, and hence using the boundedness of $\{x^k\}_{k\in K}$ yields that $\{d^k\}_{k\in K}$ is bounded. We assume that, subsequencing if necessary, $d^k \to d^*$ as $k(\in K) \to +\infty$. Since $\{\Psi_{\alpha,p}(x^k)\}$ is a nonnegative and decreasing sequence, the sequence $\{\Psi_{\alpha,p}(x^k)\}$ is convergent, which together with (41) implies that

$$\lim_{k\to+\infty} \beta^{2m_k}\Psi_{\alpha,p}(x^k) = 0.$$

If $\{m_k\}_{k\in K}$ is bounded, then $\{\beta^{2m_k}\}_{k\in K}$ does not approach to 0, and consequently,

$$\lim_{k\to+\infty,\ k\in K} \Psi_{\alpha,p}(x^k) = \Psi_{\alpha,p}(x^*) = 0.$$

This shows that x^* is a solution of the NCP. Next we suppose that $\{m_k\}_{k\in K}$ is unbounded, which implies $\{\beta^{2m_k}\}_{k\in K} \to 0$. From Step 2 of Algorithm 4.1, it follows that, for all $k \in K$

$$\Psi_{\alpha,p}(x^k + \beta^{m_k-1} d^k(\gamma^{m_k-1})) - \Psi_{\alpha,p}(x^k) > -\sigma\beta^{2(m_k-1)}\Psi_{\alpha,p}(x^k).$$

Dividing the inequality by β^{m_k-1} and passing to the limit $k \to +\infty$ then yields that

$$\lim_{k\to+\infty,\ k\in K} \frac{\Psi_{\alpha,p}(x^k + \beta^{m_k-1} d^k(\gamma^{m_k-1})) - \Psi_{\alpha,p}(x^k)}{\beta^{m_k-1}} \ge 0.$$

In addition, by the Mean-Value theorem, there exists some $t \in (0, 1)$ such that

$$\Psi_{\alpha,p}(x^k + \beta^{m_k-1} d^k(\gamma^{m_k-1})) - \Psi_{\alpha,p}(x^k) = \beta^{m_k-1}\nabla\Psi_{\alpha,p}(x^k + t\beta^{m_k-1} d^k(\gamma^{m_k-1}))^{\mathrm{T}} d^k(\gamma^{m_k-1}).$$

Combining the last two equations and using the continuity of $\nabla\Psi_{\alpha,p}$ and the fact that $\beta^{m_k-1}, \gamma^{m_k-1} \to 0$ as $k(\in K) \to +\infty$, we obtain

$$\nabla\Psi_{\alpha,p}(x^*)^{\mathrm{T}}d^* \geq 0.$$

On the other hand, clearly, $\nabla\Psi_{\alpha,p}(x^*)^{\mathrm{T}}d^* \leq 0$ since $d^* = -\nabla_b\psi_{\alpha,p}(x^*, F(x^*))$. Thus, we prove that $\nabla\Psi_{\alpha,p}(x^*)^{\mathrm{T}}d^* = 0$, i.e.,

$$\nabla_a\psi_{\alpha,p}(x^*, F(x^*))^{\mathrm{T}}\nabla_b\psi_{\alpha,p}(x^*, F(x^*)) + \nabla_b\psi_{\alpha,p}(x^*, F(x^*))^{\mathrm{T}}\nabla F(x^*)\nabla_b\psi_{\alpha,p}(x^*, F(x^*)) = 0.$$

Using the monotonicity of F then yields that

$$\nabla_a\psi_{\alpha,p}(x^*, F(x^*))^{\mathrm{T}}\nabla_b\psi_{\alpha,p}(x^*, F(x^*)) = 0,$$

which, by Lemma 3.1(e), implies that $\Psi_{\alpha,p}(x^*) = 0$. That is, x^* is a solution of the NCP. $\square$

Combining Proposition 4.1 with Theorem 3.3, we get the following convergence result.

Theorem 4.1. *Suppose that either of the following conditions holds:*

(a) *F is monotone and the NCP is strictly feasible when $\alpha > 0$;*
(b) *F is a uniform P-function when $\alpha \geq 0$.*

Then the sequence $\{x^k\}$ generated by Algorithm 4.1 has at least one accumulation point and every accumulation point is a solution of the NCP.

Next we investigate the rate of convergence of the sequence $\{x^k\}$ generated by Algorithm 4.1, for which the following technical lemma will be used.

Lemma 4.2. *Let $\{x^k\}$ be the sequence generated by Algorithm 4.1. Suppose that F is strongly monotone with modulus $\mu > 0$. Then there exists an integer $\hat{m} := \lceil \log_\gamma \frac{2\mu}{\nu^2+2\nu+2} \rceil$ such that for each k and all $m \geq \hat{m}$, the search direction $d^k(\gamma^m)$ satisfies*

$$\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}}d^k(\gamma^m) \leq -\frac{\gamma^m}{2}\left(\|\nabla_a\psi_{\alpha,p}(x^k, F(x^k))\| + \|\nabla_b\psi_{\alpha,p}(x^k, F(x^k))\|\right)^2.$$

Proof. By Theorem 3.3, the level set $\mathcal{L}(\Psi_{\alpha,p}, \Psi_{\alpha,p}(x^0))$ is bounded. Notice that $\{x^k\} \subseteq \mathcal{L}(\Psi_{\alpha,p}, \Psi_{\alpha,p}(x^0))$, and consequently using the continuity of ∇F yields that there exists a constant $\nu > 0$ such that $\|\nabla F(x^k)\| \leq \nu$ for all k. Thus, using Lemma 3.1 (d) and the same arguments as [19, Lemma 5.1], we can prove that the conclusion holds. $\square$

Theorem 4.2. *Let $\{x^k\}$ be the sequence generated by Algorithm 4.1 and $\mathcal{L}(x^0)$ denote the level set $\mathcal{L}(\Psi_{\alpha,p}, \Psi_{\alpha,p}(x^0))$. Suppose that F is strongly monotone and ∇F is Lipschitz continuous in $\mathcal{L}(x^0)$. Then,*

(a) *the sequence $\{\Psi_{\alpha,p}(x^k)\}$ converges Q-linearly to zero;*
(b) *and the sequence $\{x^k\}$ converges R-linearly to the solution of the NCP.*

Proof. (a) By Theorem 3.3, the level set $\mathcal{L}(x^0)$ is bounded. Since F is continuously differentiable and strongly monotone, F is Lipschitz continuous. Then, using Lemma 3.1 and the Lipschitz continuity of ∇F on $\mathcal{L}(x^0)$, it is easy to verify that $\nabla\Psi_{\alpha,p}$ is Lipschitz continuous on $\mathcal{L}(x^0)$, i.e., there exists a constant $L_1 > 0$ such that

$$\|\nabla\Psi_{\alpha,p}(x) - \nabla\Psi_{\alpha,p}(x')\| \leq L_1\|x - x'\| \quad \forall x, x' \in \mathcal{L}(x^0). \tag{43}$$

Notice that $\{x^k\} \subseteq \mathcal{L}(x^0)$ since the sequence $\{\Psi_{\alpha,p}(x^k)\}$ is nonincreasing. Therefore, for any $t \in [0, 1]$, we have that x^k, $x^k + td^k \in \mathcal{L}(x^0)$ and

$$\begin{aligned}
\Psi_{\alpha,p}(x^k + td^k) - \Psi_{\alpha,p}(x^k) &= \int_0^t \nabla\Psi_{\alpha,p}(x^k + sd^k)^{\mathrm{T}}\mathrm{d}^k\mathrm{d}s \\
&= \int_0^t [\nabla\Psi_{\alpha,p}(x^k + sd^k) - \nabla\Psi_{\alpha,p}(x^k)]^{\mathrm{T}}\mathrm{d}^k\mathrm{d}s + t\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}}\mathrm{d}^k \\
&\leq t\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}}d^k + L_1\int_0^t s\|d^k\|^2\mathrm{d}s \\
&= t\left(\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}}d^k + \frac{L_1 t}{2}\|d^k\|^2\right),
\end{aligned}$$

where the inequality is by (43) and the Cauchy–Schwarz inequality. From Lemma 4.2,

$$\nabla\Psi_{\alpha,p}(x^k)^{\mathrm{T}}d^k \leq -\frac{\gamma^m}{2}\left(\|\nabla_a\psi_{\alpha,p}(x^k, F(x^k))\| + \|\nabla_b\psi_{\alpha,p}(x^k, F(x^k))\|\right)^2$$

for all $m \geq \hat{m}$ with $\hat{m}$ given by Lemma 4.2. In addition, from the definition of d^k,

$$\begin{aligned}\|d^k\|^2 &= \|\nabla_b \Psi_{\alpha,p}(x^k, F(x^k)) + \gamma^m \nabla_a \Psi_{\alpha,p}(x^k, F(x^k))\|^2 \\ &\leq \|\nabla_b \Psi_{\alpha,p}(x^k, F(x^k))\|^2 + \gamma^{2m}\|\nabla_a \Psi_{\alpha,p}(x^k, F(x^k))\|^2 + 2\gamma^m \langle \nabla_a \Psi_{\alpha,p}(x^k, F(x^k)), \nabla_b \Psi_{\alpha,p}(x^k, F(x^k))\rangle \\ &\leq \|\nabla_b \Psi_{\alpha,p}(x^k, F(x^k))\|^2 + \|\nabla_a \Psi_{\alpha,p}(x^k, F(x^k))\|^2 + 2\langle \nabla_a \Psi_{\alpha,p}(x^k, F(x^k)), \nabla_b \Psi_{\alpha,p}(x^k, F(x^k))\rangle \\ &= \left(\|\nabla_a \psi_{\alpha,p}(x^k, F(x^k))\| + \|\nabla_b \psi_{\alpha,p}(x^k, F(x^k))\|\right)^2\end{aligned}$$

for all nonnegative integer m, where the second inequality is due to Lemma 3.1(d) and the fact that $\gamma^m \leq 1$. Using the last three inequalities, we thus obtain that

$$\Psi_{\alpha,p}(x^k + \beta^m d^k) - \Psi_{\alpha,p}(x^k) \leq -\frac{\beta^m}{2}(\gamma^m - L_1\beta^m)\left(\|\nabla_a \psi_{\alpha,p}(x^k, F(x^k))\| + \|\nabla_b \psi_{\alpha,p}(x^k, F(x^k))\|\right)^2 \tag{44}$$

for all nonnegative integer $m \geq \hat{m}$. Noting that $\{x^k\} \subseteq \mathcal{L}(x^0)$, and using the inequality (44) and Proposition 3.2, we have that condition (41) is satisfied whenever $m \geq \hat{m}$ and

$$\beta^m(\gamma^m - L_1\beta^m)C(B, \alpha, p) \geq \sigma\beta^{2m}, \tag{45}$$

where

$$C(B, \alpha, p) = \frac{\left(2 - 2^{\frac{1}{p}}\right)^4}{\alpha B^2 + \left(2 + 2^{1p}\right)^2} \tag{46}$$

with $B = \max_{1\leq i\leq n}\left\{\sup_{x\in\mathcal{L}(x^0)}\{\max\{|x_i|, |F_i(x)|\}\}\right\}$. Notice that the inequality (45) is equivalent to requiring that

$$m \geq \log_{\frac{\gamma}{\beta}}\left(L_1 + \frac{\sigma}{C(B, \alpha, p)}\right).$$

Consequently, condition (41) is satisfied for all $m \geq \bar{m}$, where

$$\bar{m} := \max\left\{\hat{m}, \left\lceil \log_{\frac{\gamma}{\beta}}\left(L_1 + \frac{\sigma}{C(B, \alpha, p)}\right)\right\rceil\right\}.$$

Since m_k is the smallest nonnegative integer m satisfying (41), we have $m_k \leq \bar{m}$ for all k, which together with (41) implies that

$$\Psi_{\alpha,p}(x^{k+1}) - \Psi_{\alpha,p}(x^k) \leq -\sigma\beta^{2m_k}\Psi_{\alpha,p}(x^k) \leq -\sigma\beta^{2\bar{m}}\Psi_{\alpha,p}(x^k).$$

Therefore,

$$\Psi_{\alpha,p}(x^{k+1}) \leq (1 - \sigma\beta^{2\bar{m}})\Psi_{\alpha,p}(x^k).$$

This means that $\{\Psi_{\alpha,p}(x^k)\}$ converges Q-linearly to zero since $0 < 1 - \sigma\beta^{2\bar{m}} < 1$.

(b) Since F is strongly monotone, the NCP has a unique solution, denoted by x^*. From Theorem 3.4, there exists a positive constant κ_1 such that

$$\|x^k - x^*\| \leq \kappa_1 \Psi_{\alpha,p}(x^k)^{\frac{1}{4}} \quad \text{when } \alpha > 0$$

and there exists a positive constant κ_2 such that

$$\|x^k - x^*\| \leq \kappa_2\left(\max\left\{\Psi_{\alpha,p}(x^k), \sqrt{\Psi_{\alpha,p}(x^k)}\right\}\right)^{\frac{1}{2}} \quad \text{when } \alpha = 0.$$

Since the sequence $\{\Psi_{\alpha,p}(x^k)\}$ converges Q-linearly to zero, the sequence $\{x^k\}$ converges R-linearly to the solution of the NCP. □

From the proof of Theorem 4.2, we see that the convergence rate of Algorithm 4.1 has a close relation with the constant $\left\lceil \log_{\frac{\gamma}{\beta}}\left(L_1 + \frac{\sigma}{C(B,\alpha,p)}\right)\right\rceil$.

Remark 4.1. (a) If $\gamma < \beta$, the value of $C(B, \alpha, p)$ has an influence on the convergence rate only when $L_1 + \frac{\sigma}{C(B,\alpha,p)} < 1$. For this case, when p decreases, $C(B, \alpha, p)$ also decreases, which in turn implies that $\left\lceil \log_{\frac{\gamma}{\beta}}\left(L_1 + \frac{\sigma}{C(B,\alpha,p)}\right)\right\rceil$ and $1 - \sigma\beta^{2\bar{m}}$ increases. This shows that the convergence rate of Algorithm 4.1 becomes worse when p decreases. Similarly, if $\gamma > \beta$ and $L_1 + \frac{\sigma}{C(B,\alpha,p)} > 1$, the convergence rate of Algorithm 4.1 also becomes worse as p decreases. Therefore, when the value of p decreases, the convergence rate of Algorithm 4.1 becomes worse and worse.

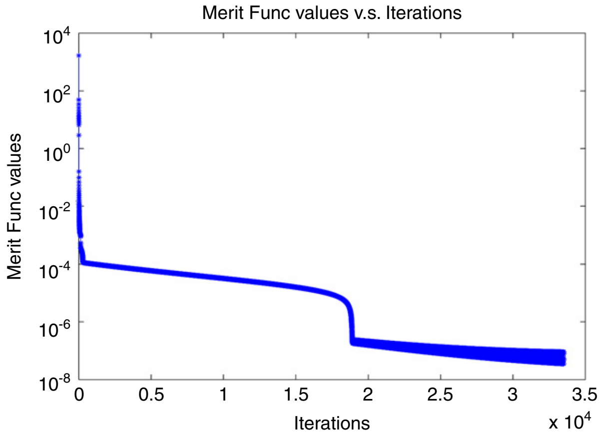

Fig. 1. Convergent behavior of "gafni(1)" with $p = 1.1$.

(b) Assume that p is fixed. Then the value of $\left\lceil \log_{\frac{\gamma}{\beta}} \left(L_1 + \frac{\sigma}{C(B,\alpha,p)} \right) \right\rceil$ is nondecreasing as α increases, which in turn implies that the convergence rate of Algorithm 4.1 becomes worse and worse when α increases. Thus, when $\alpha = 0$, Algorithm 4.1 has the best convergence rate, but it has a worse global convergence by Theorem 3.3.

Of course, we should point out that the property of the mapping F itself has a great influence on the convergence rate of Algorithm 4.1.

5. Numerical experiments

In this section, we test how the numerical performance of Algorithm 4.1 varies with the value of p. We implemented Algorithm 4.1 with our code in MATLAB 6.5 for the test problems with all available starting points in MCPLIB [22]. All numerical experiments were done on a PC with CPU of 2.8 GHz and RAM of 512 MB. During the tests, we replaced the standard (monotone) Armijo rule by a nonmonotone line search as described in [23], i.e., we computed the smallest nonnegative integer m such that

$$\Psi_{\alpha,p}(x^k + \beta^m d^k(\gamma^m)) \le \mathcal{W}_k - \sigma \beta^{2m} \Psi_{\alpha,p}(x^k),$$

where $\mathcal{W}_k$ is given by

$$\mathcal{W}_k = \max_{j=k-m_k,\dots,k} \Psi_{\alpha,p}(x^j)$$

and where, for given nonnegative integers $\tilde{m}$ and s, we set

$$m_k = \begin{cases} 0 & \text{if } k \le s \\ \min\left\{m_{k-1} + 1, \tilde{m}\right\} & \text{otherwise.} \end{cases}$$

Throughout the experiments, we adopted $\tilde{m} = 5$ and $s = 5$. The algorithm was terminated whenever one of the following conditions was satisfied:

(1) $\Psi_{\alpha,p}(x^k) \le 1.0\text{e}{-6}$ and $|(x^k)^{\mathrm{T}} F(x^k)| \le 1.0\text{e}{-3}$;
(2) the steplength β^{m_k} is less than 1.0 e−10;
(3) the number of iteration is more than 500 000.

We first took "gafni(1)" for example to observe the convergence of Algorithm 4.1 with different p. The parameters in Algorithm 4.1 were chosen as follows:

$$\alpha = 1.0\text{e}{-2}, \qquad \gamma = 0.1, \qquad \beta = 0.2, \qquad \sigma = 1.0\text{e}{-10}.$$

Figs. 1 and 2 depict the detailed iteration process of Algorithm 4.1 with $p = 1.1$ and $p = 1000$, respectively. From the two figures, we see that, when $p = 1.1$, the merit function $\Psi_{\alpha,p}$ has a faster decrease than the case where $p = 1000$ within the first several thousands of iterations, but it has a much slower convergence speed once the value of $\Psi_{\alpha,p}$ is less than 1.0e−6. This exactly coincides with the analysis in Remark 4.1(a).

Then, we took "bertsekas(2)" for example to observe the convergence of Algorithm 4.1 with different α. The parameters in Algorithm 4.1 were chosen as follows:

$$p = 4, \qquad \gamma = 0.1, \qquad \beta = 0.2, \qquad \sigma = 1.0\text{e}{-10}.$$

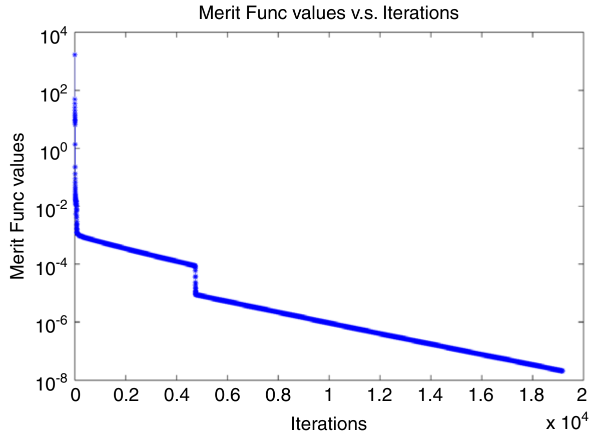

Fig. 2. Convergent behavior of "gafni(1)" with $p = 1000$.

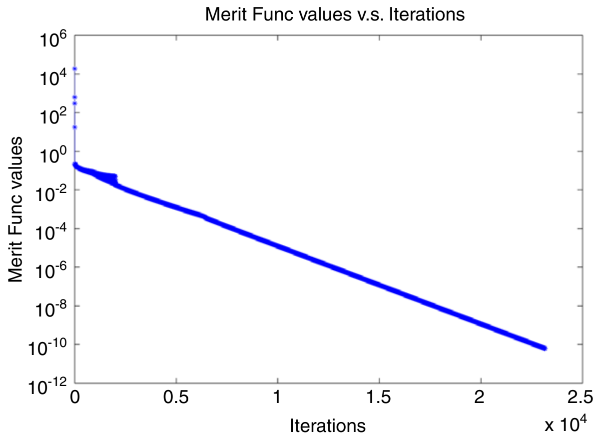

Fig. 3. Convergent behavior of "bertsekas(2)" with $\alpha = 1$.

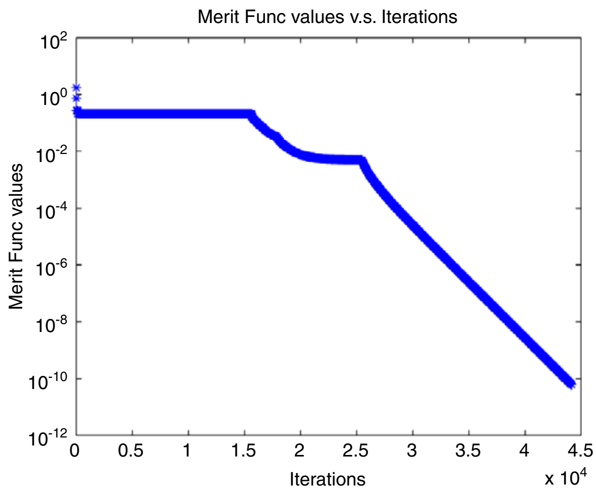

Fig. 4. Convergent behavior of "bertsekas(2)" with $\alpha = 0$.

Figs. 3 and 4 depict the detailed iteration process of Algorithm 4.1 with $\alpha = 1$ and $\alpha = 0$, respectively. From the two figures, we see that, when $\alpha = 0$, the value of $\Psi_{\alpha,p}$ has a faster decrease than the case where $\alpha = 1$ once $\Psi_{\alpha,p}$ is less than 1.0e−4, but it has a much slower speed to decrease the value of $\Psi_{\alpha,p}$ within the first 10 000 iterations. This shows that Algorithm 4.1

Table 1
Numerical results for MCPLIB problems based on $\Psi_{1.5}(x)$, $\Psi_2(x)$ and $\Psi_3(x)$.

Problem	$\Psi_{1.5}(x)$			$\Psi_2(x)$			$\Psi_3(x)$		
	Gap	NF	Time	Gap	NF	Time	Gap	NF	Time
bertsekas(1)	3.50e−7	90 679	9.25e−4	6.18e−11	92 986	9.98e−4	6.17e−11	98 657	9.99e−4
bertsekas(2)	3.50e−7	83 681	9.25e−4	6.19e−11	96 880	9.99e−4	6.16e−11	86 855	9.99e−4
bertsekas(3)	–	–	–	–	–	–	–	–	–
colvdual(1)	3.37e−9	60 327	9.99e−4	3.33e−9	61 502	9.99e−3	3.34e−9	61 699	9.99e−4
colvdual(2)	3.20e−9[a]	60 203	9.99e−4	3.33e−9	61 098	9.99e−4	3.34e−9	61 772	9.99e−4
colvnlp(1)	3.33e−9	14 409	9.98e−4	8.03e−9	8 238	9.98e−4	7.88e−9	7 180	9.97e−4
colvnlp(2)	3.34e−9	15 190	1.00e−3	3.34e−9	15 455	9.99e−4	3.31e−9	15 493	9.98e−4
cycle	1.19e−9	9	3.44e−4	7.30e−17	6	8.54e−8	1.18e−19	5	4.87e−9
explcp	2.50e−7	46	7.27e−4	7.24e−10	61	3.88e−5	2.87e−20	73	4.04e−10
gafni(1)	2.05e−8	92 784	9.99e−4	2.69e−8	68 063	1.00e−3	2.08e−8	68 230	1.00e−3
gafni(2)	3.75e−8	91 011	9.98e−4	3.58e−8	77 193	1.00e−3	3.56e−8	78 908	1.00e−3
gafni(3)	3.63e−8	108 748	9.99e−4	3.52e−8	72 459	1.00e−3	3.51e−8	71 950	9.98e−4
hanskoop(1)	5.43e−7[a]	46	2.20e−4	7.22e−10	1 849	1.19e−6	4.98e−14	432	9.99e−9
hanskoop(2)	6.42e−7	1 022	2.41e−4	5.38e−10	1 387	1.18e−6	1.73e−12	973	1.60e−7
hanskoop(3)	1.53e−7	23	1.09e−5	1.30e−7	12	4.20e−6	4.25e−7	5	5.71e−5
hanskoop(4)	1.86e−7	32	1.87e−4	1.30e−10	36	1.48e−6	2.08e−9[a]	25	1.85e−6
josephy(1)	3.98e−8	720	9.93e−4	3.34e−8	652	9.84e−4	1.86e−7	57	7.98e−4
josephy(2)	3.45e−8	841	9.85e−4	2.88e−8	745	9.97e−4	2.77e−8	731	9.97e−4
josephy(3)	2.22e−7	1 071	8.68e−4	1.80e−8	580	9.82e−4	1.84e−7	70	8.05e−4
josephy(4)	2.22e−7	773	8.68e−4	2.51e−8	652	9.93e−4	2.48e−8	644	9.86e−4
josephy(5)	2.22e−7	748	8.68e−4	1.84e−7	51	7.80e−4	2.05e−7	50	8.47e−4
josephy(6)	3.48e−8	842	9.98e−4	2.85e−8	748	9.97e−4	2.08e−7	107	7.19e−4

The – in Table 1 means that Algorithm 4.1 fails for the problem.
[a] Means that the numerical results are obtained with a different α.

Table 2
Numerical results for MCPLIB problems based on $\Psi_{1.5}(x)$, $\Psi_2(x)$ and $\Psi_3(x)$.

Problem	$\Psi_{1.5}(x)$			$\Psi_2(x)$			$\Psi_3(x)$		
	Gap	NF	Time	Gap	NF	Time	Gap	NF	Time
kojshin(1)	2.70e−7	702	9.95e−4	3.13e−7	600	9.93e−4	3.09e−7	586	9.85e−4
kojshin(2)	1.58e−8[a]	921	9.87e−4	3.15e−7	643	9.96e−4	3.14e−7	621	9.93e−4
kojshin(3)	2.64e−7	910	9.84e−4	3.11e−7	588	9.90e−4	3.09e−7	588	9.85e−4
kojshin(4)	5.24e−8	470	8.84e−4	6.11e−8	112	9.65e−4	4.17e−8	121	7.98e−4
kojshin(5)	2.66e−7	646	9.86e−4	3.11e−7	590	9.89e−4	3.06e−7	586	9.80e−4
kojshin(6)	2.21e−7	81	9.83e−5	2.35e−8	102	5.98e−4	3.19e−8	102	6.98e−4
mathinum(1)	9.55e−7	292	2.68e−4	2.66e−7	168	6.02e−4	2.77e−7	134	8.76e−4
mathinum(2)	8.13e−7	129	4.71e−4	2.54e−7	134	2.97e−4	3.80e−7	68	2.04e−4
mathinum(3)	8.72e−7[a]	610	7.02e−4	2.29e−7	277	6.96e−4	3.71e−7	150	8.26e−4
mathinum(4)	5.06e−7[a]	541	8.79e−4	1.04e−7	247	9.20e−4	5.89e−7	164	7.47e−4
mathisum(1)	1.00e−6	217 019	88.18	1.00e−6	216 101	80.34	1.00e−6	216 007	87.25
mathisum(2)	1.00e−6	219 150	90.90	1.00e−6	218 369	83.31	1.00e−6	218 293	88.04
mathisum(3)	1.00e−6	216 990	89.11	1.00e−6	216 096	80.89	1.00e−6	216 000	87.59
mathisum(4)	1.00e−6	217 038	90.57	1.00e−6	216 123	80.06	1.00e−6	216 038	92.26
nash(1)	1.39e−10[a]	26	5.03e−5	7.68e−10	28	2.67e−4	1.15e−12	29	7.31e−6
nash(2)	1.06e−9[a]	28	5.30e−4	9.68e−8	23	2.56e−4	2.38e−15	23	9.58e−8
sppe(1)	7.33e−9	26 751	9.74e−4	5.10e−9	38 859	9.93e−4	1.16e−8	41 981	9.99e−4
sppe(2)	7.65e−9	28 254	1.00e−3	1.74e−8	35 500	1.00e−3	2.46e−8	56 524	9.88e−4
tobin(1)	9.79e−7	670	8.42e−4	4.48e−8	900	9.95e−4	2.75e−10	318	5.73e−4
tobin(2)	9.95e−7	786	5.32e−4	3.98e−7	776	9.97e−4	1.47e−7	322	3.32e−4

[a] Means that the numerical results are obtained with a different α.

corresponding to a smaller α will have a faster convergence rate but a worse global convergence, which is coincident with Remark 4.1(b).

We finally computed the test problems from MCPLIB [22] with all available starting points by Algorithm 4.1. From the analysis in Remark 4.1, we know that the ratio between γ and β may have an influence on the numerical performance of Algorithm 4.1. Hence, we considered two cases: $\gamma/\beta < 1$ and $\gamma/\beta > 1$, respectively, with $p = 1.5$, $p = 2$ and $p = 3$. For the former case, we chose $\gamma = 0.1$ and $\beta = 0.2$, whereas $\gamma = 0.2$ and $\beta = 0.1$ for the latter case. The other parameters of Algorithm 4.1 were chosen as follows:

$$\alpha = 1.0\text{e}{-2}, \qquad \sigma = 1.0\text{e}{-10}, \qquad \varepsilon_1 = 1.0\text{e}{-6}, \qquad \varepsilon_2 = 1.0\text{e}{-3}.$$

Our computational results are summarized in Tables 1–4. Among others, Tables 1 and 2 list the numerical results of the test problems for the case where $\gamma/\beta < 1$, whereas Tables 3 and 4 list the numerical results of the test problems for the case where $\gamma/\beta > 1$. In these tables, the first column presents the name of the problems and the starting point number in

Table 3
Numerical results for MCPLIB problems based on $\Psi_{1.5}(x)$, $\Psi_2(x)$ and $\Psi_3(x)$.

Problem	$p = 1.5$			$p = 2$			$p = 3$		
	$\Psi_{\alpha,p}(x^k)$	NF	Gap	$\Psi_{\alpha,p}(x^k)$	NF	Gap	$\Psi_{\alpha,p}(x^k)$	NF	Gap
bertsekas(1)	6.15e−11	82 830	9.99e−4	6.15e−11	83 937	1.0e−3	–	–	–
bertsekas(2)	6.15e−11	82 470	1.00e−3	6.15e−11	83 438	1.00e−3	–	–	–
bertsekas(3)	–	–	–	–	–	–	–	–	–
colvdual(1)	3.38e−9	115 012	9.97e−4	3.34e−9	141 067	1.00e−3	3.36e−9	128 222	9.99e−4
colvdual(2)	3.20e−9	88 135	1.00e−3	3.21e−9	88 083	1.00e−3	3.20e−9	88 411	1.00e−3
colvnlp(1)	9.50e−7	90 830	3.37e−4	5.84e−9	23 763	1.00e−3	4.98e−9	23 974	1.00e−3
colvnlp(2)	9.50e−7	92 022	3.37e−4	5.17e−9	24 697	9.99e−4	5.45e−9	24 763	9.99e−4
cycle	1.19e−9	9	3.44e−4	7.30e−17	6	8.55e−8	1.18e−19	5	4.87e−9
explcp	3.46e−7	13	8.45e−4	1.99e−7	43	8.42e−4	1.37e−11	55	5.23e−6
gafni(1)	2.03e−8	31 661	1.00e−3	2.05e−8	30 961	1.00e−3	2.05e−8	30 345	1.00e−3
gafni(2)	3.84e−8	32 952	1.00e−3	3.66e−8	31 181	9.99e−4	3.64e−8	32 504	1.00e−3
gafni(3)	3.68e−8	35 341	1.00e−3	3.55e−8	31 780	1.00e−3	3.52e−8	34 411	1.00e−3
hanskoop(1)	–	–	–	8.20e−7	65 606	3.98e−5	4.31e−14	2 335	2.86e−8
hanskoop(2)	–	–	–	5.13e−10	56 359	1.22e−6	3.79e−8	28 226	1.15e−5
hanskoop(3)	1.50e−7	25	1.06e−5	1.30e−7	12	4.20e−6	4.25e−7	5	5.71e−5
hanskoop(4)	5.10e−8	26	9.54e−5	7.15e−7	157	4.69e−5	1.51e−10	45	3.80e−7
josephy(1)	4.94e−8	819	9.99e−4	2.89e−7	1 071	9.99e−4	5.70e−8	745	9.97e−4
josephy(2)	3.87e−8	877	9.90e−4	2.99e−8	763	9.85e−4	2.82e−8	743	9.92e−4
josephy(3)	2.13e−7	1 254	8.70e−4	2.89e−7	1 081	9.99e−4	5.69e−8	747	9.97e−4
josephy(4)	2.60e−8	727	9.92e−4	2.61e−8	661	9.89e−4	2.60e−8	651	9.88e−4
josephy(5)	1.01e−8	114	1.87e−4	2.89e−7	1 029	9.99e−4	2.87e−7	3 079	9.95e−4
josephy(6)	3.74e−8	879	9.85e−4	2.90e−8	765	9.81e−4	2.74e−8	745	9.84e−4

The – in the table means that Algorithm 4.1 fails for the problem.

Table 4
Numerical results for MCPLIB problems based on $\Psi_{1.5}(x)$, $\Psi_2(x)$and $\Psi_3(x)$.

Problem	$p = 1.5$			$p = 2$			$p = 3$		
	$\Psi_{\alpha,p}(x^k)$	NF	Gap	$\Psi_{\alpha,p}(x^k)$	NF	Gap	$\Psi_{\alpha,p}(x^k)$	NF	Gap
kojshin(1)	4.97e−8[a]	58 810	1.00e−3	5.03e−8[a]	52 291	1.00e−3	6.30e−8	45 272	1.00e−3
kojshin(2)	9.78e−7	949	1.37e−4	9.66e−7	749	2.81e−4	9.89e−8	705	2.21e−4
kojshin(3)	6.88e−8	49 731	1.00e−3	6.30e−8	45 386	1.00e−3	6.30e−8	45 274	1.00e−3
kojshin(4)	5.26e−8	238	9.02e−4	4.68e−8	255	8.48e−4	4.18e−8	237	8.02e−4
kojshin(5)	6.15e−8	108 267	1.00e−3	6.30e−8	45 279	1.00e−3	6.30e−8	44 989	1.00e−3
kojshin(6)	4.15e−7	107	1.08e−4	5.82e−8	244	9.46e−4	4.13e−8	236	7.97e−4
mathinum(1)	6.97e−7	308	4.68e−4	1.67e−7	200	9.21e−4	6.85e−8	169	6.39e−4
mathinum(2)	4.56e−7	211	6.57e−5	2.70e−7	194	8.46e−4	3.34e−7	181	5.83e−4
mathinum(3)	8.31e−7	257	4.45e−4	2.08e−7	158	3.06e−4	6.96e−7	126	3.63e−4
mathinum(4)	6.66e−7[a]	386	6.46e−4	6.79e−7	306	9.05e−4	4.14e−7	188	4.31e−4
mathisum(1)	–	–	–	–	–	–	–	–	–
mathisum(2)	–	–	–	–	–	–	–	–	–
mathisum(3)	–	–	–	–	–	–	–	–	–
mathisum(4)	–	–	–	–	–	–	–	–	–
nash(1)	8.29e−8	451	8.81e−4	1.53e−7	352	9.97e−4	2.35e−7	301	3.68e−4
nash(2)	2.04e−9[a]	184	9.74e−4	1.26e−9	106	3.52e−5	2.11e−17	56	9.29e−9
sppe(1)	3.24e−9	12 364	9.96e−4	1.20e−9	13 947	9.94e−4	5.58e−9	8 952	9.88e−4
sppe(2)	2.29e−9	13 821	9.72e−4	1.34e−9	10 939	9.97e−4	3.22e−9	11 647	9.99e−4
tobin(1)	4.78e−10	441	8.10e−4	3.83e−10	288	7.83e−4	5.34e−7	219	2.32e−4
tobin(2)	3.74e−10	414	9.96e−4	7.73e−10	946	9.48e−4	3.77e−10	318	8.39e−4

[a] Means that the numerical results are obtained with a different α.

MCPLIB, $\Psi_{\alpha,p}(x^k)$ denotes the value of $\Psi_{\alpha,p}(x)$ at the final iteration, **NF** indicates the number of function evaluations of $\Psi_{\alpha,p}$ for solving each problem, and **Gap** denotes the value of $|x^TF(x)|$ at the final iteration.

The results in Tables 1–4 show that the derivative-free descent algorithm based on $\Psi_{\alpha,p}$ with $p = 1.5, p = 2$ and $p = 3$ can solve most complementarity problems in MCPLIB with favorable accuracy. More precisely, there are eight failures (**billups, pgvon105, pgvon106, powell, scarfanum, scarfasum, scarfbnum, scarfbsum**) for Algorithm 4.1 due to a too small steplength. These problems are also regarded as difficult ones for Newton-type algorithms [24]. From these tables, we see that, for most of the test problems, Algorithm 4.1 with $p = 3$ requires fewer function evaluations than the cases where $p = 1.5$ and $p = 2$, whether $\gamma/\beta > 1$ or $\gamma/\beta < 1$ not. Moreover, when $p = 1.5$, Algorithm 4.1 has worse stability. This indicates that Algorithm 4.1 with $p = 3$ or $p = 2$ has better numerical performance than the case where $p = 1.5$. In addition, comparing the numerical results of Tables 1 and 2 with those in Tables 3 and 4, we may find that the value of γ/β has a great influence on the number of function evaluations and it seems that Algorithm 4.1 with $\gamma/\beta < 1$ has better numerical performance for most of the test problems.

6. Concluding remarks

In this paper, we proposed a new merit function $\Psi_{\alpha,p}$ for the NCPs based on the generalized Fischer–Burmeister function ϕ_p, and extended all of the results in [19] to a more general situation. With these results, we presented a derivative-free descent algorithm and established the global convergence and locally R-linear convergence rate of the algorithm. Also, we analyzed the influence of the parameter p on the convergence rate of the algorithm, and verified these theoretical results via numerical experiments by solving the test problems from MCPLIB. Compared with [18], it seems that this paper yields an "opposite" numerical conclusion. However, they are actually coincident. In [18], the termination conditions of the algorithm only required the dual gap $(x^k)^T F(x^k) <= 5.0e-3$, which means that the numerical performance of the algorithm is dominated by the global convergence, while the termination conditions used in this paper tend to reflecting the convergence rate of the algorithm. In particular, combining the numerical results of the two papers, we may obtain the conclusion that the merit function method based on ϕ_p has a better a global convergence and a worse convergence rate when p decreases.

Acknowledgement

The first author is a member of Mathematics Division, National Center for Theoretical Sciences, Taipei Office. The author's work is partially supported by the National Science Council of Taiwan.

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