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From symmetric cone optimization to nonsymmetric cone optimization: Spectral decomposition, nonsmooth analysis, and projections onto nonsymmetric cones [1]

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Abstract. It is well known that Euclidean Jordan algebra is an unified framework for symmetric cone programs, including positive semidefinite programs and second-order cone programs. Unlike symmetric cone programs, there is no unified analysis technique to deal with nonsymmetric cone programs. Nonetheless, there are several common concepts

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when dealing with general conic optimization. More specifically, we believe that spectral decomposition associated with cones, nonsmooth analysis regarding cone-functions, projections onto cones, and cone-convexity are the bridges between symmetric cone programs and nonsymmetric cone programs. Hence, this paper is devoted to looking into the first three items in the setting of nonsymmetric cones. The importance of cone-convexity is recognized in the literature so that it is not discussed here. All results presented in this paper are very crucial to subsequent study about the optimization problems associated with nonsymmetric cones.

Key words. Spectral decomposition, nonsmooth analysis, projection, symmetric cone, nonsymmetric cone.

Mathematics Subject Classification: 49M27, 90C25

1 Introduction

Symmetric cone optimization, including SDP (positive semidefinite programming) and SOCP (second-order cone programming) as special cases, has been a popular topic during the past two decades. In fact, for many years, there has been much attention on symmetric cone optimization, see [10, 11, 14, 20, 27, 30, 32, 35, 38] and references therein. Recently, some researchers have paid attention to nonsymmetric cones, for example, homogeneous cone [9, 28, 40], matrix norm cone [18], p-order cone [1, 23, 41], hyperbolicity cone [24, 26, 36], circular cone [13, 15, 42] and copositive cone [16], etc.. In general, the structure of symmetric cone is quite different from the one of non-symmetric cone. In particular, unlike the symmetric cone optimization in which the Euclidean Jordan algebra can unify the analysis, so far no unified algebra structure has been found for non-symmetric cone optimization. This motivates us to find the common bridge between them. Based on our earlier experience, we think the following four items are crucial:

- spectral decomposition associated with cones.
- smooth and nonsmooth analysis for cone-functions.
- projection onto cones.
- cone-convexity.

The role of cone-convexity had been recognized in the literature. In this paper, we focus on the other three items that are newly explored recently by the authors. Moreover, we look into several kinds of nonsymmetric cones, that is, the circular cone, the p-order cone, the geometric cone, the exponential cone and the power cone, respectively. The symmetric cone can be unified under Euclidean Jordan algebra, which will be introduced later. Unlike the symmetric cone, there is no unified framework for dealing with non-symmetric cones. This is the main source where the difficulty arises from. Note that the homogeneous cone can be unified under so-called T-algebra [28, 39, 40].

We begin with introducing Euclidean Jordan algebra [29] and symmetric cone [19]. Let $\mathbb{V}$ be an n-dimensional vector space over the real field $\mathbb{R}$, endowed with a bilinear mapping $(x, y) \mapsto x \circ y$ from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}$. The pair $(\mathbb{V}, \circ)$ is called a *Jordan algebra* if

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$.

Note that a Jordan algebra is not necessarily associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$ may not hold for all $x, y, z \in \mathbb{V}$. We call an element $e \in \mathbb{V}$ the *identity* element if $x \circ e = e \circ x = x$ for all $x \in \mathbb{V}$. A Jordan algebra $(\mathbb{V}, \circ)$ with an identity element e is called a *Euclidean Jordan algebra* if there is an inner product, $\langle \cdot, \cdot \rangle_{\mathbb{V}}$, such that

(iii) $\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Given a Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$, we denote the set of squares as

$$\mathcal{K} := \left\{ x^2 \,|\, x \in \mathbb{V} \right\}.$$

By [19, Theorem III.2.1], $\mathcal{K}$ is a symmetric cone. This means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior and for any two elements $x, y \in \text{int}\mathcal{K}$, there exists an invertible linear transformation $\mathcal{T} : \mathbb{V} \to \mathbb{V}$ such that $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(x) = y$.

Below are three well-known examples of Euclidean Jordan algebras.

Example 1.1. Consider $\mathbb{R}^n$ with the (usual) inner product and Jordan product defined respectively as

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad x \circ y = x * y \qquad \forall x, y \in \mathbb{R}^n$$

where x_i denotes the ith component of x, etc., and $x * y$ denotes the componentwise product of vectors x and y. Then, $\mathbb{R}^n$ is a Euclidean Jordan algebra with the nonnegative orthant $\mathbb{R}^n_+$ as its cone of squares.

Example 1.2. Let $\mathbb{S}^n$ be the space of all $n \times n$ real symmetric matrices with the trace inner product and Jordan product, respectively, defined by

$$\langle X, Y \rangle_{\mathrm{T}} := \mathrm{Tr}(XY) \quad \text{and} \quad X \circ Y := \frac{1}{2}(XY + YX) \qquad \forall X, Y \in \mathbb{S}^n.$$

Then, $(\mathbb{S}^n, \circ, \langle \cdot, \cdot \rangle_{\mathrm{T}})$ is a Euclidean Jordan algebra, and we write it as $\mathbb{S}_n$. The cone of squares $\mathbb{S}^n_+$ in $\mathbb{S}_n$ is the set of all positive semidefinite matrices.

Example 1.3. The Jordan spin algebra $\mathbb{L}_n$. Consider $\mathbb{R}^n$ $(n > 1)$ with the inner product $\langle\cdot,\cdot\rangle$ and Jordan product

$$x \circ y := \begin{bmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}$$

for any $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. We denote the Euclidean Jordan algebra $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ by $\mathbb{L}_n$. The cone of squares, called the Lorentz cone (or second-order cone), is given by

$$\mathbb{L}_n^+ := \left\{(x_0; \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_0 \geq \|\bar{x}\|\right\}.$$

For any given $x \in \mathbb{A}$, let $\zeta(x)$ be the degree of the minimal polynomial of x, i.e.,

$$\zeta(x) := \min\left\{k : \{e, x, x^2, \cdots, x^k\} \text{ are linearly dependent}\right\}.$$

Then, the *rank* of $\mathbb{A}$ is defined as $\max\{\zeta(x) : x \in \mathbb{V}\}$. In this paper, we use r to denote the rank of the underlying Euclidean Jordan algebra. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$. Two idempotents c_i and c_j are said to be *orthogonal* if $c_i \circ c_j = 0$. One says that $\{c_1, c_2, \ldots, c_k\}$ is a *complete system of orthogonal idempotents* if

$$c_j^2 = c_j, \quad c_j \circ c_i = 0 \text{ if } j \neq i \text{ for all } j, i = 1, 2, \cdots, k, \quad \text{and } \sum_{j=1}^{k} c_j = e.$$

An idempotent is *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Now we state the second version of the spectral decomposition theorem.

Theorem 1.1. *[19, Theorem III.1.2] Suppose that $\mathbb{A}$ is a Euclidean Jordan algebra with the rank r. Then, for any $x \in \mathbb{V}$, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues and $\mathrm{tr}(x) = \sum_{j=1}^{r} \lambda_j(x)$ the trace of x.

From [19, Prop. III.1.5], a Jordan algebra $(\mathbb{V}, \circ)$ with an identity element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\mathrm{tr}(x \circ y)$ is positive definite. Then, we may define another inner product on $\mathbb{V}$ by $\langle x, y\rangle := \mathrm{tr}(x \circ y)$ for any $x, y \in \mathbb{V}$. The inner product $\langle\cdot,\cdot\rangle$ is associative by [19, Prop. II. 4.3], i.e., $\langle x, y \circ z\rangle = \langle y, x \circ z\rangle$ for any $x, y, z \in \mathbb{V}$. Every Euclidean Jordan algebra can be written as a direct sum of so-called *simple* ones, which are not themselves direct sums in a nontrivial way. In finite dimensions, the simple Euclidean Jordan algebras come from the following five basic structures.

Theorem 1.2. *[19, Chapter V.3.7] Every simple Euclidean Jordan algebra is isomorphic to one of the following.*

(i) *The Jordan spin algebra* $\mathbb{L}^n$.

(ii) *The algebra* $\mathbb{S}^n$ *of* $n \times n$ *real symmetric matrices.*

(iii) *The algebra* $\mathbb{H}^n$ *of all* $n \times n$ *complex Hermitian matrices.*

(iv) *The algebra* $\mathbb{Q}^n$ *of all* $n \times n$ *quaternion Hermitian matrices.*

(v) *The algebra* $\mathbb{O}^3$ *of all* 3×3 *octonion Hermitian matrices.*

Given an n-dimensional Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \langle \cdot, \cdot \rangle, \circ)$ with $\mathcal{K}$ being its corresponding symmetric cone in $\mathbb{V}$. For any scalar function $f : \mathbb{R} \to \mathbb{R}$, we define a vector-valued function $f^{\rm sc}(x)$ (called Löwner function) on $\mathbb{V}$ as

$$f^{\rm sc}(x) = f(\lambda_1(x))c_1 + f(\lambda_2(x))c_2 + \cdots + f(\lambda_r(x))c_r \tag{1}$$

where $x \in \mathbb{V}$ has the spectral decomposition

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

When $\mathbb{V}$ is the space $\mathcal{S}^n$ which means $n \times n$ real symmetric matrices. The spectral decomposition reduces to the following: for any $X \in \mathcal{S}^n$,

$$X = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} P^T,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues of X and P is orthogonal (i.e., $P^T = P^{-1}$). Under this setting, for any function $f : \mathbb{R} \to \mathbb{R}$, we define a corresponding matrix valued function associated with the Euclidean Jordan algebra $\mathcal{S}^n := Sym(n, \mathbb{R})$, denoted by $f^{\rm mat} : \mathcal{S}^n \to \mathcal{S}^n$, as

$$f^{\rm mat}(X) = P \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^T.$$

For this case, Chen, Qi and Tseng in [12] show that the function $f^{\rm mat}$ inherits from f the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. We state them as below.

Theorem 1.3. **(a)** $f^{\rm mat}$ *is continuous if and only if* f *is continuous.*

(b) f^{mat} *is directionally differentiable if and only if* f *is directionally differentiable.*

(c) f^{mat} *is Fréchet-differentiable if and only if* f *is Fréchet-differentiable.*

(d) f^{mat} *is continuously differentiable if and only if* f *is continuously differentiable.*

(e) f^{mat} *is locally Lipschitz continuous if and only if* f *is locally Lipschitz continuous.*

(f) f^{mat} *is globally Lipschitz continuous with constant* κ *if and only if* f *is globally Lipschitz continuous with constant* κ*.*

(g) f^{mat} *is semismooth if and only if* f *is semismooth.*

When $\mathbb{V}$ is the Jordan spin algebra $\mathbb{L}_n$ in which $\mathcal{K}$ corresponds to the second-order cone (SOC), which is defined as

$$\mathcal{K}^n := \{(x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|\bar{x}\| \leq x_1\},$$

the function f^{sc} reduces to so-called SOC-function f^{soc} studied in [4, 6, 7, 8]. More specifically, under such case, the spectral decomposition for any $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ becomes

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{2}$$

where $\lambda_1(x)$, $\lambda_2(x)$, $u_x^{(1)}$ and $u_x^{(2)}$ with respect to $\mathcal{K}^n$ are given by

$$\begin{aligned} \lambda_i(x) &= x_1 + (-1)^i \|\bar{x}\|, \\ u_x^{(i)} &= \begin{cases} \frac{1}{2}\Big(1, (-1)^i \dfrac{\bar{x}}{\|\bar{x}\|}\Big) & \text{if } \ \bar{x} \neq 0, \\ \frac{1}{2}\Big(1, (-1)^i w\Big) & \text{if } \ \bar{x} = 0, \end{cases} \end{aligned}$$

for $i = 1, 2$, with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $\bar{x} \neq 0$, the decomposition (2) is unique. With this spectral decomposition, for any function $f : \mathbb{R} \to \mathbb{R}$, the Löwner function f^{sc} associated with $\mathcal{K}^n$ reduces to f^{soc} as below:

$$f^{\mathrm{soc}}(x) = f(\lambda_1(x)) u_x^{(1)} + f(\lambda_2(x)) u_x^{(2)} \qquad \forall x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{3}$$

The picture of second-order cone $\mathcal{K}^n$ in $\mathbb{R}^3$ is depicted in Figure 1.

For general symmetric cone case, Baes [2] consider the convexity and differentiability properties of spectral functions. For this SOC setting, Chen, Chen and Tseng in [8] show that the function f^{soc} inherits from f the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. In other words, the following hold.

Theorem 1.4. (a) f^{soc} *is continuous if and only if* f *is continuous.*

(b) f^{soc} *is directionally differentiable if and only if* f *is directionally differentiable.*

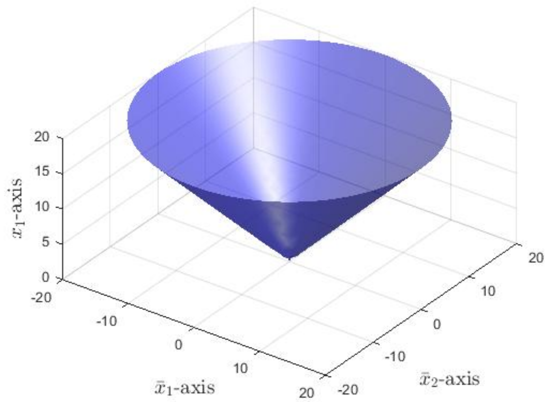

Figure 1: The second-order cone in $\mathbb{R}^3$

(c) f^{soc} *is Fréchet-differentiable if and only if* f *is Fréchet-differentiable.*

(d) f^{soc} *is continuously differentiable if and only if* f *is continuously differentiable.*

(e) f^{soc} *is locally Lipschitz continuous if and only if* f *is locally Lipschitz continuous.*

(f) f^{soc} *is globally Lipschitz continuous with constant* κ *if and only if* f *is globally Lipschitz continuous with constant* κ*.*

(g) f^{soc} *is semismooth if and only if* f *is semismooth.*

As for general symmetric cone case, Sun and Sun [38] uses $\phi_{\mathbb{V}}$ to denote f^{sc} defined as in (1). More specifically, for any function $\phi : \mathbb{R} \to \mathbb{R}$, they define a corresponding function associated with the Euclidean Jordan algebra $\mathbb{V}$ by

$$\phi_{\mathbb{V}}(x) = \phi(\lambda_1(x))c_1 + \phi(\lambda_2(x))c_2 + \cdots + \phi(\lambda_r(x))c_r,$$

where $\lambda_1(x), \lambda_2(x), \cdots, \lambda_r(x)$ and $c_1, c_2, \cdots, c_r$ are the spectral values and spectral vectors of x, respectively. In addition, Sun and Sun [38] extend some of the aforementioned results to more general symmetric cone case regarding f^{sc} (i.e., $\phi_{\mathbb{V}}$).

Theorem 1.5. *Assume that the symmetric cone is simple in the Euclidean Jordan algebra* $\mathbb{V}$*.*

(a) $\phi_{\mathbb{V}}$ *is continuous if and only if* ϕ *is continuous.*

(b) $\phi_{\mathbb{V}}$ *is directionally differentiable if and only if* ϕ *is directionally differentiable.*

(c) $\phi_{\mathbb{V}}$ *is Fréchet-differentiable if and only if* ϕ *is Fréchet-differentiable.*

(d) $\phi_{\mathbb{V}}$ *is continuously differentiable if and only if* ϕ *is continuously differentiable.*

(e) *$\phi_{\mathbb{V}}$ is semismooth if and only if ϕ is semismooth.*

With respect to matrix cones, Ding et al. [17] recently introduce a class of matrix-valued functions, which is called spectral operator of matrices. This class of functions generalizes the well known Löwner operator and has been used in many important applications related to structured low rank matrices and other matrix optimization problems in machine learning and statistics. Similar to Theorem 1.4 and Theorem 1.5, the continuity, directional differentiability and Frechet-differentiability of spectral operator are also obtained. See [17, Theorem 3, 4 and 5] for more details.

For subsequent needs, for a closed convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, we also recall its dual cone, polar cone, and the projection onto itself. For any a given closed convex cone $\mathcal{K} \subseteq \mathbb{R}^n$, its dual cone is defined by

$$\mathcal{K}^* := \{y \in \mathbb{R}^n \,|\, \langle y, x\rangle \geq 0, \quad \forall x \in \mathcal{K}\},$$

and its polar cone is $\mathcal{K}^\circ := -\mathcal{K}^*$. Let $\Pi_{\mathcal{K}}(z)$ denote the Euclidean projection of $z \in \mathbb{R}^n$ onto the closed convex cone $\mathcal{K}$. Then, it follows that $z = \Pi_{\mathcal{K}}(z) - \Pi_{\mathcal{K}^*}(-z)$ and

$$\Pi_{\mathcal{K}}(z) = \mathrm{argmin}_{x\in\mathcal{K}} \frac{1}{2}\|x - z\|^2.$$

2 Circular cone

The definition of the circular cone $\mathcal{L}_\theta$ is defined as [42]:

$$\begin{aligned} \mathcal{L}_\theta &:= \left\{x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|x\| \cos\theta \leq x_1\right\} \\ &= \left\{x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|\bar{x}\| \leq x_1 \tan\theta\right\}. \end{aligned}$$

From the concept of the circular cone $\mathcal{L}_\theta$, we know that when $\theta = \frac{\pi}{4}$, the circular cone is exactly the second-order cone $\mathcal{K}^n$. In addition, we also see that $\mathcal{L}_\theta$ is solid (i.e., $\mathrm{int}\,\mathcal{L}_\theta \neq \emptyset$), pointed (i.e., $\mathcal{L}_\theta \cap -\mathcal{L}_\theta = 0$), closed convex cone, and has a revolution axis which is the ray generated by the canonical vector $e_1 := (1, 0, \cdots, 0)^T \in \mathbb{R}^n$. Moreover, its dual cone is given by

$$\begin{aligned} \mathcal{L}_\theta^* &:= \{y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|y\| \sin\theta \leq y_1\} \\ &= \{y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, \|\bar{y}\| \leq y_1 \cot\theta\} \\ &= \mathcal{L}_{\frac{\pi}{2}-\theta}. \end{aligned}$$

The pictures of circular cone $\mathcal{L}_\theta$ in $\mathbb{R}^3$ are depicted in Figure 2.

In view of the expression of the dual cone $\mathcal{L}_\theta^*$, we see that the dual cone $\mathcal{L}_\theta^*$ is also a solid, pointed, closed convex cone. By the reference [42], the explicit formula of projection onto the circular cone $\mathcal{L}_\theta$ can be expressed by in the following theorem.

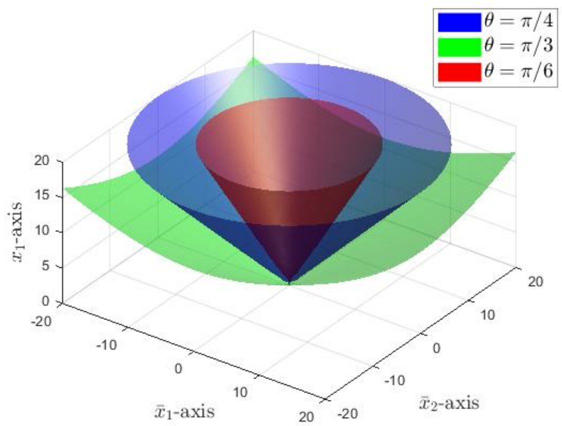

Figure 2: Three different circular cones in $\mathbb{R}^3$.

Theorem 2.1. ([42]) *Let $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and x_+ denote the projection of x onto the circular cone $\mathcal{L}_\theta$. Then x_+ is given below:*

$$x_+ = \begin{cases} x & \text{if } x \in \mathcal{L}_\theta, \\ 0 & \text{if } x \in -\mathcal{L}_\theta^*, \\ u & \text{otherwise,} \end{cases}$$

where

$$u = \begin{bmatrix} \dfrac{x_1 + \|\bar{x}\| \tan\theta}{1 + \tan^2\theta} \\ \left(\dfrac{x_1 + \|\bar{x}\| \tan\theta}{1 + \tan^2\theta} \tan\theta \right) \dfrac{\bar{x}}{\|\bar{x}\|} \end{bmatrix}.$$

Zhou and Chen [42] also present the decomposition of x, which is similar to the one in the setting of second-order cone.

Theorem 2.2. ([42, Theorem 3.1]) *For any $x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, one has*

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where

$$\begin{aligned} \lambda_1(x) &= x_1 - \|\bar{x}\| \cot\theta \\ \lambda_2(x) &= x_1 + \|\bar{x}\| \tan\theta \end{aligned}$$

and

$$\begin{aligned} u_x^{(1)} &= \frac{1}{1 + \cot^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \cot\theta \end{bmatrix} \begin{pmatrix} 1 \\ -w \end{pmatrix} \\ u_x^{(2)} &= \frac{1}{1 + \tan^2\theta} \begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \end{bmatrix} \begin{pmatrix} 1 \\ w \end{pmatrix} \end{aligned}$$

with $w = \frac{\bar{x}}{\|\bar{x}\|}$ if $\bar{x} \neq 0$, and any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$ if $\bar{x} = 0$.

Theorem 2.3. ([42, Theorem 3.2]) *For any $x = (x_1, \bar{x}) \in \mathbb{R}^n \times \mathbb{R}$, we have*

$$x_+ = (\lambda_1(x))_+ u_x^{(1)} + (\lambda_2(x))_+ u_x^{(2)},$$

where $(a)_+ := \max\{0, a\}$, $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are given as in Theorem 2.2.

With this spectral decomposition of x, for any function $f : \mathbb{R} \to \mathbb{R}$, the Löwner function f^{circ} associated with $\mathcal{L}_\theta$ is defined as below:

$$f^{circ}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \quad \forall x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{4}$$

In [15], Chang, Yang and Chen have obtained that many properties of the function f^{circ} are inherited from the function f, which is represented in the following theorem.

Theorem 2.4. ([15]) *For any the function $f : \mathbb{R} \to \mathbb{R}$, the vector-valued function f^{circ} is defined by (4). Then, the following results hold.*

(a) *f^{circ} is continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is continuous at $\lambda_1(x), \lambda_2(x)$.*

(b) *f^{circ} is directionally differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is directionally differentiable at $\lambda_1(x), \lambda_2(x)$.*

(c) *f^{circ} is differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is differentiable at $\lambda_1(x), \lambda_2(x)$.*

(d) *f^{circ} is strictly continuous at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is strictly continuous at $\lambda_1(x), \lambda_2(x)$.*

(e) *f^{circ} is semismooth at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is semismooth at $\lambda_1(x), \lambda_2(x)$.*

(f) *f^{circ} is continuously differentiable at $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x), \lambda_2(x)$ if and only if f is continuously differentiable at $\lambda_1(x), \lambda_2(x)$.*

We point out that there is a close relation between $\mathcal{L}_\theta$ and $\mathcal{K}^n$ (see [34, 42]) as below

$$\mathcal{K}^n = A\mathcal{L}_\theta \quad \text{where} \quad A := \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}.$$

We point out a few points regarding circular cones. First, as mentioned in [43], it is possible to construct a new inner product which ensures the circular cone is self-dual. However, it is not possible to make both $\mathcal{L}_\theta$ and $\mathcal{K}^n$ are self-dual under a certain inner product. Secondly, as shown in [43], the relation $\mathcal{K}^n = A\mathcal{L}_\theta$ does not guarantee that there exists a similar close relation between f^{circ} and f^{soc}. The third point is that the structure of circular cone helps on constructing complementarity functions for the circular cone complementarity problem as indicated in [34].

3 The p-order cone

The p-order cone in $\mathbb{R}^n$, which is a generalization of the second-order cone $\mathcal{K}^n$[14], is defined as

$$\mathcal{K}_p := \left\{ x \in \mathbb{R}^n \,\middle|\, x_1 \geq \left(\sum_{i=2}^{n} |x_i|^p \right)^{\frac{1}{p}} \right\} \quad (p \geq 1). \tag{5}$$

In fact, the p-order cone $\mathcal{K}_p$ can be equivalently expressed as

$$\mathcal{K}_p = \left\{ x = (x_1, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|\bar{x}\|_p \right\}, \quad (p \geq 1),$$

where $\bar{x} := (x_2, \cdots, x_n)^T \in \mathbb{R}^{n-1}$. From (5), it is clear to see that when $p = 2$, $\mathcal{K}_2$ is exactly the second-order cone $\mathcal{K}^n$. That means that the second-order cone is a special case of p-order cone. Moreover, it is known that $\mathcal{K}_p$ is a convex cone and its dual cone is given by

$$\mathcal{K}_p^* = \left\{ y \in \mathbb{R}^n \,\middle|\, y_1 \geq \left(\sum_{i=2}^{n} |y_i|^q \right)^{\frac{1}{q}} \right\}$$

or equivalently

$$\mathcal{K}_p^* = \left\{ y = (y_1, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, y_1 \geq \|\bar{y}\|_q \right\} = \mathcal{K}_q$$

with $\bar{y} := (y_2, \cdots, y_n)^T \in \mathbb{R}^{n-1}$, where $q \geq 1$ and satisfies $\frac{1}{p} + \frac{1}{q} = 1$. From the expression of the dual cone $\mathcal{K}_p^*$, we see that the cone $\mathcal{K}_p^*$ is also a convex cone. For an application of p-order cone programming, we refer the readers to [41], in which a primal-dual potential reduction algorithm for p-order cone constrained optimization problems is studied. Besides, in [41], a special optimization problem called sum of p-norms is transformed into an p-order cone constrained optimization problems. The pictures of three different cones $\mathcal{K}_p$ in $\mathbb{R}^3$ are depicted in Figure 3.

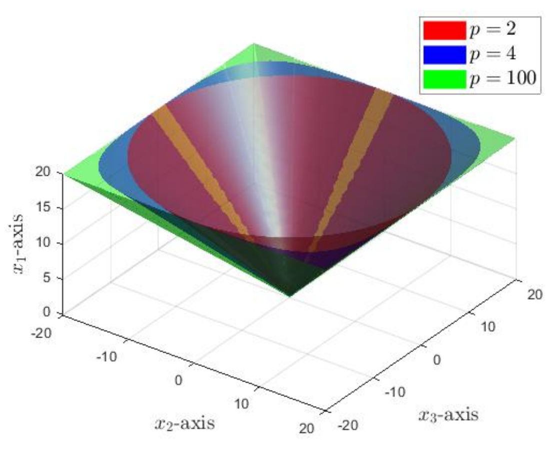

Figure 3: Three different p-order cones in $\mathbb{R}^3$

In [33], Miao, Qi and Chen explore the expression of the projection onto p-order cone and the spectral decomposition associated with p-order cone, which are shown the following theorems.

Theorem 3.1. ([33, Theorem 2.1]) *For any $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{n-1}$, then the projection of z onto $\mathcal{K}_p$ is given by*

$$\Pi_{\mathcal{K}_p}(z) = \begin{cases} z, & z \in \mathcal{K}_p \\ 0, & z \in -\mathcal{K}_p^* = -\mathcal{K}_q \\ u, & \textit{otherwise} \quad (i.e., -\|\bar{z}\|_q < z_1 < \|\bar{z}\|_p) \end{cases}$$

where $u = (u_1, \bar{u})$ with $\bar{u} = (u_2, u_3, \cdots, u_n)^T \in \mathbb{R}^{n-1}$ satisfying

$$u_1 = \|\bar{u}\|_p = (|u_2|^p + |u_3|^p + \cdots + |u_n|^p)^{\frac{1}{p}}$$

and

$$u_i - z_i + \frac{u_1 - z_1}{u_1^{p-1}} |u_i|^{p-2} u_i = 0, \qquad \forall i = 2, \cdots, n.$$

Theorem 3.2. ([33, Theorem 2.2]) *Let $z = (z_1, \bar{z}) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, z can be decomposed as*

$$z = \alpha_1(z) \cdot v^{(1)}(z) + \alpha_2(z) \cdot v^{(2)}(z),$$

where

$$\begin{cases} \alpha_1(z) &= \dfrac{z_1 + \|\bar{z}\|_p}{2} \\ \alpha_2(z) &= \dfrac{z_1 - \|\bar{z}\|_p}{2} \end{cases} \quad \text{and} \quad \begin{cases} v^{(1)}(z) &= \begin{pmatrix} 1 \\ \bar{w} \end{pmatrix} \\ v^{(2)}(z) &= \begin{pmatrix} 1 \\ -\bar{w} \end{pmatrix} \end{cases}$$

with $\bar{w} = \frac{\bar{z}}{\|\bar{z}\|_p}$ if $\bar{z} \neq 0$; while $\bar{w}$ being an arbitrary element satisfying $\|\bar{w}\|_p = 1$ if $\bar{z} = 0$.

For the projection onto p-order cone, we notice that this projection is not an explicit formula because it is hard to solve the equations which in Theorem 3.1. Moreover, the decomposition for z is not an orthogonal decomposition, which is different from the case in the second-order cone and circular cone setting. Because the decomposition for z is not an orthogonal decomposition, the corresponding nonsmooth analysis for its cone-functions is not established.

4 Geometric cone

The geometric cone is defined as bellow [22]:

$$\mathcal{G}^n := \left\{ (x, \theta) \in \mathbb{R}_+^n \times \mathbb{R}_+ \;\middle|\; \sum_{i=1}^{n} e^{-\frac{x_i}{\theta}} \leq 1 \right\}$$

where $x = (x_1, \cdots, x_n)^T \in \mathbb{R}^n_+$ and we also use the convention $e^{-\frac{x_i}{0}} = 0$. From the definition of the geometric cone $\mathcal{G}^n$, we know that $\mathcal{G}^n$ is solid (i.e., $\operatorname{int} \mathcal{G}^n \neq \emptyset$), pointed (i.e., $\mathcal{G}^n \cap -\mathcal{G}^n = 0$), closed convex cone, and its dual cone is given by

$$(\mathcal{G}^n)^* = \left\{ (y, \mu) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,\middle|\, \mu \geq \sum_{y_i > 0} y_i \ln \frac{y_i}{\sum_{i=1}^n y_i} \right\}$$

where $\mu \in \mathbb{R}_+$ and $y = (y_1, \cdots, y_n)^T \in \mathbb{R}^n_+$. In view of the expression of the dual cone $(\mathcal{G}^n)^*$, we see that the dual cone $(\mathcal{G}^n)^*$ is also a solid, pointed, closed convex cone, and $((\mathcal{G}^n)^*)^* = \mathcal{G}^n$. When $n = 1$, we note that the geometric cone $\mathcal{G}^1$ is just nonnegative octant cone $\mathbb{R}^2_+$. In addition, by the expression of the geometric cone $\mathcal{G}^n$ and its dual cone $(\mathcal{G}^n)^*$, it is not hard to verify that the boundary of the geometric cone $\mathcal{G}^n$ and its dual cone $(\mathcal{G}^n)^*$ can be respectively expressed as follows:

$$\operatorname{bd} \mathcal{G}^n = \left\{ (x, \theta) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,\middle|\, \sum_{i=1}^n e^{-\frac{x_i}{\theta}} = 1 \right\}$$

and

$$\operatorname{bd} (\mathcal{G}^n)^* = \left\{ (y, \mu) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \,\middle|\, \mu = \sum_{y_i > 0} y_i \ln \frac{y_i}{\sum_{i=1}^n y_i} \right\}.$$

For an application of geometric cone programming, we refer the readers to [21], in which the author shows how to transform a prime-dual pair of geometric optimization problem into a constrained optimization problem related with $\mathcal{G}^n$ and $(\mathcal{G}^n)^*$. The pictures of $\mathcal{G}^n$ and its dual cone $(\mathcal{G}^n)^*$ in $\mathbb{R}^3$ are depicted in Figure 4.

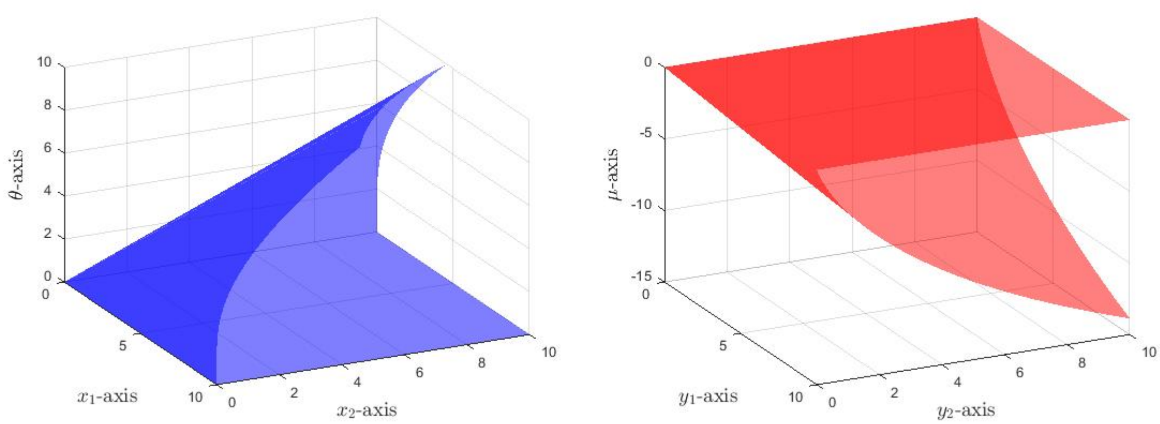

Figure 4: The geometric cone (left) and its dual cone (right) in $\mathbb{R}^3$

Next, we present the projection of $(x, \theta) \in \mathbb{R}^n \times \mathbb{R}$ onto the geometric cone $\mathcal{G}^n$.

Theorem 4.1. *Let $\mathbf{x} = (x, \theta) \in \mathbb{R}^n \times \mathbb{R}$. Then the projection of $\mathbf{x}$ onto the geometric cone $\mathcal{G}^n$ is given by*

$$\Pi_{\mathcal{G}^n}(\mathbf{x}) = \begin{cases} \mathbf{x}, & \text{if } \mathbf{x} \in \mathcal{G}^n, \\ 0, & \text{if } \mathbf{x} \in (\mathcal{G}^n)^\circ, \\ \mathbf{u}, & \text{otherwise,} \end{cases} \tag{6}$$

where $\mathbf{u} = (u, \lambda) \in \mathbb{R}^n_+ \times \mathbb{R}_+$ with $u = (u_1, u_2, \cdots, u_n)^T \in \mathbb{R}^n_+$ satisfying

$$u_i - x_i + \frac{\lambda(\lambda - \theta)}{\sum_{j=1}^n e^{-\frac{u_j}{\lambda}} u_j} e^{-\frac{u_i}{\lambda}} = 0, \quad i = 1, 2, \cdots, n \tag{7}$$

and

$$\sum_{i=1}^n e^{-\frac{u_i}{\lambda}} = 1. \tag{8}$$

Proof. From Projection Theorem [3, Prop. 2.2.1], we know that, for every $\mathbf{x} = (x, \theta) \in \mathbb{R}^n \times \mathbb{R}$, a vector $\mathbf{u} \in \mathcal{G}^n$ is equal to the projection point $\Pi_{\mathcal{G}^n}(\mathbf{x})$ if and only if

$$\mathbf{u} \in \mathcal{G}^n, \quad \mathbf{x} - \mathbf{u} \in (\mathcal{G}^n)^\circ, \quad \text{and} \quad \langle \mathbf{x} - \mathbf{u}, \mathbf{u} \rangle = 0.$$

With this, the first two cases of (6) are obvious. Hence, we only need to consider the third case. Based on (8) and the definition of $\mathcal{G}^n$, it is obvious that $\mathbf{u} \in \mathcal{G}^n$. In addition, from (7), we obtain that $\sum_{i=1}^n u_i(u_i - x_i) + \lambda(\lambda - \theta) = 0$, which explains that $\langle \mathbf{x} - \mathbf{u}, \mathbf{u} \rangle = 0$. Next, we argue that $\mathbf{x} - \mathbf{u} \in (\mathcal{G}^n)^\circ$. To see this, by (7) and (8), we have

$$\sum_{i=1}^n (u_i - x_i) = -\frac{\lambda(\lambda - \theta)}{\sum_{j=1}^n e^{-\frac{u_j}{\lambda}} u_j}.$$

Together with (7) again, it follows that $\frac{u_i - x_i}{\sum_{j=1}^n (u_j - x_j)} = e^{-\frac{u_i}{\lambda}}$, which leads to $\ln \frac{u_i - x_i}{\sum_{j=1}^n (u_j - x_j)} = -\frac{u_i}{\lambda}$. Hence, we have

$$\begin{aligned} & \sum_{u_i - x_i > 0} (u_i - x_i) \ln \frac{u_i - x_i}{\sum_{j=1}^n (u_j - x_j)} \\ = & -\sum_{u_i - x_i > 0} (u_i - x_i) \frac{u_i}{\lambda} \\ = & -\frac{1}{\lambda} \sum_{u_i - x_i > 0} (u_i - x_i) u_i \\ \leq & \frac{1}{\lambda} \cdot \lambda(\lambda - \theta) = \lambda - \theta, \end{aligned}$$

where the inequality holds since $\sum_{i=1}^n u_i(u_i - x_i) + \lambda(\lambda - \theta) = 0$. This explains that $\mathbf{u} - \mathbf{x} \in (\mathcal{G}^n)^*$, i.e, $\mathbf{x} - \mathbf{u} \in (\mathcal{G}^n)^\circ$. Then, the proof is complete. $\square$

For the projection onto geometric cone $\mathcal{G}^n$, we notice again that this projection is not an explicit formula since the equations (7) and 8 cannot be easily solved. Moreover, the decomposition associated with the geometric cone $\mathcal{G}^n$ and the corresponding nonsmooth analysis for its cone-functions are not established.

5 The exponential cone

The exponential cone is defined as bellow [5, 37]:

$$\mathcal{K}_e := \mathrm{cl}\left\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \leq x_3, \quad x_2 > 0\right\}.$$

In fact, the exponential cone can be expressed as the union of two sets, i.e.,

$$\mathcal{K}_e := \left\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \leq x_3, \quad x_2 > 0\right\} \cup \left\{(x_1, 0, x_3)^T \,\middle|\, x_1 \leq 0, \ x_3 \geq 0\right\}.$$

As shown in [5], the dual cone $\mathcal{K}_e^*$ of the exponential cone $\mathcal{K}_e$ is given by

$$\mathcal{K}_e^* = \mathrm{cl}\left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3, \quad y_1 < 0\right\}.$$

In addition, the dual cone $\mathcal{K}_e^*$ is expressed as the union of the two following sets:

$$\mathcal{K}_e^* = \left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} \leq e y_3, \quad y_1 < 0\right\} \cup \left\{(0, y_2, y_3)^T \,\middle|\, y_2 \geq 0, \ y_3 \geq 0\right\}.$$

From the expression of the exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$, it is known that the exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$ are closed convex cone in $\mathbb{R}^3$. Moreover, based on the expression of $\mathcal{K}_e$ and $\mathcal{K}_e^*$, it is easy to verify that their boundary can be respectively expressed as follows:

$$\mathrm{bd}\,\mathcal{K}_e := \left\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \,\middle|\, x_2 e^{\frac{x_1}{x_2}} = x_3, \quad x_2 > 0\right\} \cup \left\{(x_1, 0, x_3)^T \,\middle|\, x_1 \leq 0, \ x_3 \geq 0\right\}.$$

and

$$\mathrm{bd}\,\mathcal{K}_e^* := \left\{(y_1, y_2, y_3)^T \in \mathbb{R}^3 \,\middle|\, -y_1 e^{\frac{y_2}{y_1}} = e y_3, \quad y_1 < 0\right\} \cup \left\{(0, y_2, y_3)^T \,\middle|\, y_2 \geq 0, \ y_3 \geq 0\right\}.$$

For an application of exponential cone programming, we refer the readers to [5], in which interior-point algorithms for structured convex optimization involving exponential have been investigated. The pictures of the exponential cone $\mathcal{K}_e$ and its dual cone $\mathcal{K}_e^*$ in $\mathbb{R}^3$ are depicted in Figure 5.

For the geometric cone $\mathcal{G}^n$ and the exponential cone $\mathcal{K}_e$, there exists the relationship between these two types of cones, which is described in the following proposition.

Proposition 5.1. *Under the suitable conditions, there is a corresponding relationship between the geometric cone $\mathcal{G}^n$ and exponential cone $\mathcal{K}_e$.*

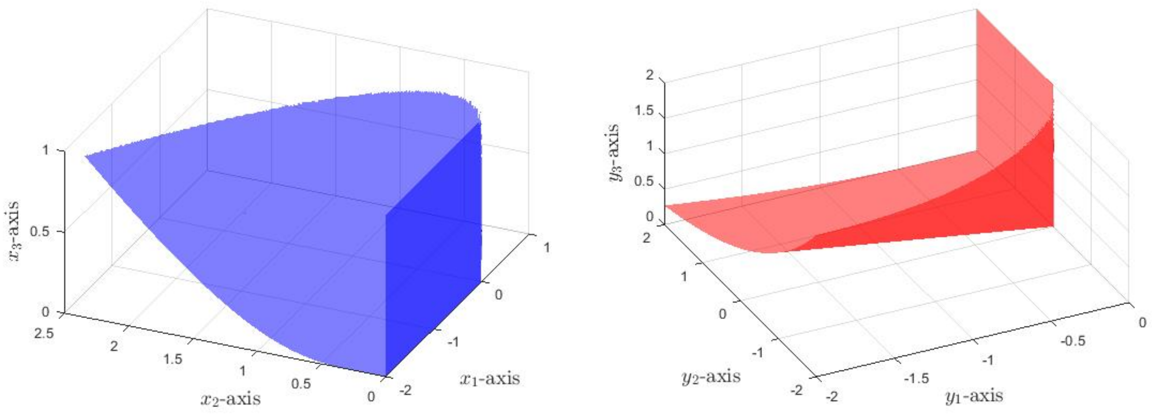

Figure 5: The exponential cone (left) and its dual cone (right) in $\mathbb{R}^3$

Proof. For any $(x, \theta) \in \mathcal{G}^n$ with $x = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n_+$, we have $\sum_{i=1}^n e^{-\frac{x_i}{\theta}} \leq 1$. With this, it is equivalent to say

$$e^{-\frac{x_i}{\theta}} \leq z_i, \quad \text{and} \quad \sum_{i=1}^{n} z_i = 1.$$

Hence, we obtain that

$$(-\frac{x_i}{\theta}, 1, z_i)^T \in \mathcal{K}_e \ \ (i = 1, 2, \cdots, n) \quad \text{and} \quad \sum_{i=1}^{n} z_i = 1.$$

For the above analysis, it is clear to see that the proof is reversible.

Besides, we give another form of transformation for the exponential cone $\mathcal{K}_e$. Indeed, for any $\tilde{x} := (x_1, x_2, x_3)^T := (\hat{x}^T, x_3)^T \in \mathcal{K}_e$ with $\hat{x} := (x_1, x_2)^T$, we have two cases, i.e.,

(a) $x_2 e^{\frac{x_1}{x_2}} \leq x_3$ and $x_2 > 0$, or

(b) $x_1 \leq 0, \ x_2 = 0, \ x_3 \geq 0$.

For the case (a), if $x_2 = x_3$ and $x_1 \leq 0$, it follows that $e^{\frac{x_1}{x_2}} \leq 1$ and $x_2 > 0$, which yields $(-x_1, x_2)^T \in \mathcal{G}^1$. Under the condition $x_2 = x_3$, if $x_1 > 0$, we find that there is no relationship between $\mathcal{K}_e$ and $\mathcal{G}^1$. For the case (b), if $x_2 = x_3$, then, we have $x_1 \leq 0$ and $x_2 = x_3 = 0$. this implies that $e^{\frac{x_1}{0}} = 0$. By this, we have $\hat{x} = (-x_1, 0)^T \in \mathcal{G}^1$. $\square$

We also present the projection of $x \in \mathbb{R}^3$ onto the exponential cone $\mathcal{K}_e$.

Theorem 5.1. *Let $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$. Then the projection of x onto the exponential cone $\mathcal{K}_e$ is given by*

$$\Pi_{\mathcal{K}_e}(x) = \begin{cases} x, & \text{if } x \in \mathcal{K}_e, \\ 0, & \text{if } x \in (\mathcal{K}_e)^\circ = -\mathcal{K}_e^*, \\ v, & \text{otherwise}, \end{cases} \tag{9}$$

where $v = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ has the following form:

(a) *if $x_1 \le 0$ and $x_2 \le 0$, then $v = (x_1, 0, \frac{x_3+|x_3|}{2})^T$.*

(b) *otherwise, the projection $\Pi_{\mathcal{K}_e}(x) = v$ satisfies the equations:*

$$\begin{aligned} v_1 - x_1 + e^{\frac{v_1}{v_2}}\left(v_2 e^{\frac{v_1}{v_2}} - x_3\right) &= 0, \\ v_2(v_2 - x_2) - (v_1 - x_1)(v_2 - v_1) &= 0, \\ v_2 e^{\frac{v_1}{v_2}} &= v_3. \end{aligned}$$

Proof. As the argument of Theorem 4.1, the first two cases of (9) are obvious. Hence, we only need to consider the third case, i.e., $x \notin \mathcal{K}_e \cup (\mathcal{K}_e)^\circ$. For convenience, we denote

$$A := \left\{(x_1, x_2, x_3)^T \,\middle|\, x_2 e^{\frac{x_1}{x_2}} \le x_3, x_2 > 0\right\} \quad \text{and} \quad B := \left\{(x_1, 0, x_3)^T \,\middle|\, x_1 \le 0, x_3 \ge 0\right\}.$$

(a) If $x_1 \le 0$ and $x_2 \le 0$, since the exponential cone $\mathcal{K}_e$ is closed and convex, by Proposition 2.2.1 in [3], we get that v is the projection of x onto $\mathcal{K}_e$ if and only if

$$\langle x - v, y - v \rangle \le 0, \qquad \forall y \in \mathcal{K}_e. \tag{10}$$

From this, we need to verify that $v = (x_1, 0, \frac{x_3+|x_3|}{2})^T$ satisfies (10). For any $y := (y_1, y_2, y_3)^T \in \mathcal{K}_e$, it follows that

$$\begin{aligned} \langle x - v, y - v \rangle &= x_2 y_2 + \frac{x_3 - |x_3|}{2}\left(y_3 - \frac{x_3 + |x_3|}{2}\right) \\ &= x_2 y_2 + y_3 \frac{x_3 - |x_3|}{2}. \end{aligned}$$

If $y \in A$, we have $y_2 > 0$ and $y_3 \ge y_2 e^{\frac{y_1}{y_2}} > 0$, which leads to

$$\langle x - v, y - v \rangle = x_2 y_2 + y_3 \frac{x_3 - |x_3|}{2} \le 0.$$

If $y \in B$, we have $y_2 = 0$ and $y_3 \ge 0$, which implies that

$$\langle x - v, y - v \rangle = y_3 \frac{x_3 - |x_3|}{2} \le 0.$$

Hence, under the conditions of $x_1 \le 0$ and $x_2 \le 0$, we can obtain that $\Pi_{\mathcal{K}_e}(x) = v = (x_1, 0, \frac{x_3+|x_3|}{2})^T$.

(b) If x belongs to other cases, we assert that the projection $\Pi_{\mathcal{K}_e}(x)$ of x onto $\mathcal{K}_e$ lies in the set A. Suppose not, i.e., $\Pi_{\mathcal{K}_e}(x) \in B$. Then, for any $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, it follows that $\Pi_{\mathcal{K}_e}(x) = v = (\min\{x_1, 0\}, 0, \frac{x_3+|x_3|}{2})^T \in B$. By Projection Theorem [3, Prop. 2.2.1], we know that the projection v should satisfy the condition

$$v \in \mathcal{K}_e, \quad x - v \in (\mathcal{K}_e)^\circ, \quad \text{and} \quad \langle x - v, v \rangle = 0.$$

However, we see that there exists $x_1 > 0$ or $x_2 \neq 0$ such that

$$v - x = (\min\{x_1, 0\} - x_1, -x_2, \frac{|x_3| - x_3}{2})^T \notin \mathcal{K}_e^*,$$

i.e., $x - v \notin (\mathcal{K}_e)^\circ$. For example, when $x_1 = 1, x_2 = 0$ and $x_3 = 1$, we have $v - x = (-1, 0, 0)^T \notin \mathcal{K}_e^*$. This contradicts with $x - v \in (\mathcal{K}_e)^\circ$. Hence, the projection $\Pi_{\mathcal{K}_e}(x) \in A$. To obtain the expression of $\Pi_{\mathcal{K}_e}(x)$, we look into the following problem:

$$\begin{array}{ll} \min & f(x) = \frac{1}{2}\|v - x\|^2 \\ \text{s.t.} & v \in A. \end{array} \tag{11}$$

In light of the convexity of the function f and the set A, it is easy to verify that the problem (11) is a convex optimization problem. Moreover, it follows from $v \in A$ that

$$\frac{v_1}{v_2} - \ln v_3 + \ln v_2 \leq 0.$$

Thus, the KKT conditions of the problem (11) are recast as

$$\left\{ \begin{array}{l} v_1 - x_1 + \frac{\mu}{v_2} = 0, \\ v_2 - x_2 + \mu(-\frac{v_1}{v_2^2} + \frac{1}{v_2}) = 0, \\ v_3 - x_3 - \frac{\mu}{v_3} = 0, \\ \mu \geq 0, \quad \frac{v_1}{v_2} - \ln v_3 + \ln v_2 \leq 0, \quad \mu(\frac{v_1}{v_2} - \ln v_3 + \ln v_2) = 0. \end{array} \right. \tag{12}$$

From (12), by the fact that the projection of $x \in\notin \mathcal{K}_e \cup (\mathcal{K}_e^*)^\circ$ must be a point in the boundary, it is not hard to see that $\frac{v_1}{v_2} - \ln v_3 + \ln v_2 = 0$ and $\mu > 0$, i.e., $v_3 = v_2 e^{\frac{v_1}{v_2}}$ and $\mu > 0$. In addition, by the first and third equations in (12), we have

$$v_1 - x_1 + \frac{v_3(v_3 - x_3)}{v_2} = 0.$$

Combining with $v_3 = v_2 e^{\frac{v_1}{v_2}}$, this implies that

$$v_1 - x_1 + e^{\frac{v_1}{v_2}} \left(v_2 e^{\frac{v_1}{v_2}} - x_3 \right) = 0.$$

On the other hand, by the first and second equations in (12), we have

$$v_2(v_2 - x_2) = (v_1 - x_1)(v_2 - v_1).$$

Therefore, we obtain that the projection $\Pi_{\mathcal{K}_e}(x) = v$ satisfies the following equations:

$$\begin{aligned} v_1 - x_1 + e^{\frac{v_1}{v_2}}\left(v_2 e^{\frac{v_1}{v_2}} - x_3\right) &= 0, \\ v_2(v_2 - x_2) - (v_1 - x_1)(v_2 - v_1) &= 0, \\ v_2 e^{\frac{v_1}{v_2}} &= v_3. \end{aligned}$$

Then, the proof is complete. $\square$

Here, we say a few words about Theorem 5.1. Unfortunately, unlike second-order cone or circular cone cases, we do not obtain an explicit formula for the projection onto the exponential cone, since there are nonlinear transcendental equations in Theorem 5.1. For example, when we examine the projection onto the exponential cone $\mathcal{K}_e$. Let $x = (1, -2, 3)$. For the case in Theorem 5.1(b), using the second condition $v_2(v_2 - x_2) - (v_1 - x_1)(v_2 - v_1) = 0$, we have

$$v_2 = \frac{v_1 - 3 + \sqrt{-3v_1^2 - 2v_1 + 9}}{2}.$$

Combining with the first condition $v_1 - x_1 + e^{\frac{v_1}{v_2}}\left(v_2 e^{\frac{v_1}{v_2}} - x_3\right) = 0$ in the case (b), this yields a nonlinear transcendental equations as bellow:

$$v_1 - 1 + e^{\frac{2v_1}{v_1 - 3 + \sqrt{-3v_1^2 - 2v_1 + 9}}}\left(\frac{v_1 - 3 + \sqrt{-3v_1^2 - 2v_1 + 9}}{2} e^{\frac{2v_1}{v_1 - 3 + \sqrt{-3v_1^2 - 2v_1 + 9}}} - 3\right) = 0.$$

From this equation, we do not have the specific expression of v_1. Hence, the explicit formula for the projection onto exponential cone cannot be obtained. Moreover, analogous to the geometric cone $\mathcal{G}^n$, the decomposition for x associated with the exponential cone $\mathcal{K}_e$ and the corresponding nonsmooth analysis for its cone-functions are not established.

6 The power cone

The high dimensional power cone is defined as bellow [25, 39]:

$$\mathcal{K}_{m,n}^{\alpha} := \left\{ (x, z) \in \mathbb{R}_+^m \times \mathbb{R}^n \,\middle|\, \|z\| \le \prod_{i=1}^{m} x_i^{\alpha_i} \right\},$$

where $\alpha_i > 0$, $\sum_{i=1}^m \alpha_i = 1$ and $x = (x_1, \cdots, x_m)^T$. For the power cone, when $m = 2, n = 1$, Truong and Tuncel [39] have discussed the homogeneity of the power cone. However, Hien [25] states that the power cone is not homogeneous in general case, and the power cone is self-dual cone. Moreover, when $m = 2$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$, we see that the power cone $\mathcal{K}_{m,n}^{\alpha}$ is exactly the rotated second-order cone, which has a broad range of applications. In [25], Hien provides the expression of the dual cone of the power cone $\mathcal{K}_{m,n}^{\alpha}$ as below:

$$(\mathcal{K}_{m,n}^{\alpha})^* = \left\{ (s_1, \cdots, s_m, \omega_1, \cdots, \omega_n) \in \mathbb{R}_+^m \times \mathbb{R}^n \,\middle|\, \prod_{i=1}^m \left(\frac{s_i}{\alpha_i} \right)^{\alpha_i} \geq \|\omega\| \right\},$$

where $\omega = (\omega_1, \cdots, \omega_n)^T \in \mathbb{R}^n$. For an application of power cone programming, we refer the readers to [5], in which a lot of practical applications such as location problems and geometric programming can be modelled using $\mathcal{K}_{m,n}^{\alpha}$ and its limiting case $\mathcal{K}_e$. The pictures of the power cone $\mathcal{K}_{m,n}^{\alpha}$ and its dual cone $(\mathcal{K}_{m,n}^{\alpha})^*$ in $\mathbb{R}^3$ are depicted in Figure 6, where the parameters $(m, n) = (2, 1)$ and $(\alpha_1, \alpha_2) = (0.8, 0.2)$.

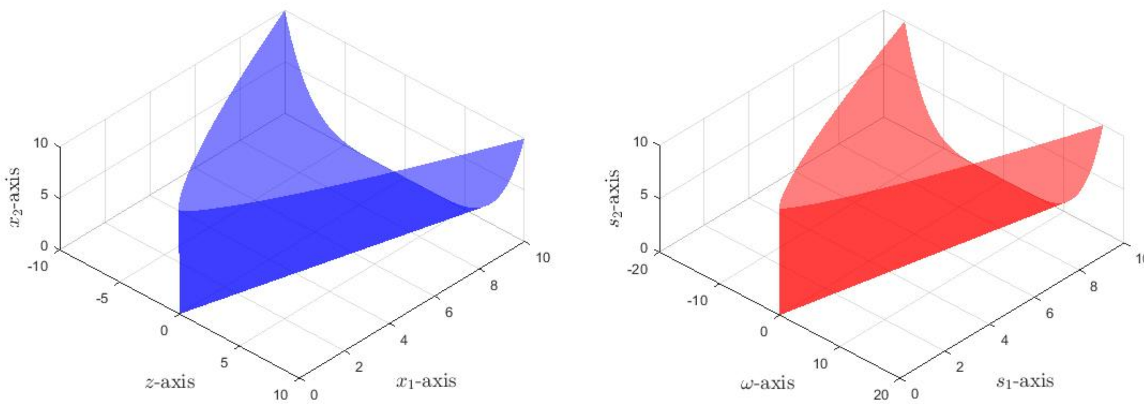

Figure 6: The power cone (left) and its dual cone (right) in $\mathbb{R}^3$.

The projection onto the power cone $\mathcal{K}_{m,n}^{\alpha}$ is already figured out by Hien in [25], which is presented in the following theorem.

Theorem 6.1. ([25, Proposition 2.2]) *Let $(x, z) \in \mathbb{R}^m \times \mathbb{R}^n$ with $x = (x_1, \cdots, x_m)^T \in \mathbb{R}^m$ and $z = (z_1, \cdots, z_n)^T \in \mathbb{R}^n$. Set $(\hat{x}, \hat{z})$ be the projection of (x, z) onto the power cone $\mathcal{K}_{m,n}^{\alpha}$. Denote*

$$\Phi(x, z, r) = \frac{1}{2} \prod_{i=1}^m \left(x_i + \sqrt{x_i^2 + 4\alpha_i r(\|z\| - r)} \right)^{\alpha_i} - r.$$

(a) *If $(x, z) \notin \mathcal{K}_{m,n}^{\alpha} \cup -(\mathcal{K}_{m,n}^{\alpha})^*$ and $z \neq 0$, then its projection onto $\mathcal{K}_{m,n}^{\alpha}$ is*

$$\begin{cases} \hat{x}_i = \frac{1}{2}\left(x_i + \sqrt{x_i^2 + 4\alpha_i r(\|z\| - r)} \right), & i = 1, \cdots, m, \\ \hat{z}_l = z_l \frac{r}{\|z\|}, & l = 1, \cdots, n, \end{cases}$$

where $r = r(x, z)$ is the unique solution of the following system:

$$E(x,z): \quad \begin{cases} \Phi(x,z,r) = 0, \\ 0 < r < \|z\|. \end{cases}$$

(b) *If $(x, z) \notin \mathcal{K}^{\alpha}_{m,n} \cup -(\mathcal{K}^{\alpha}_{m,n})^*$ and $z = 0$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is*

$$\begin{cases} \hat{x}_i = (x_i)_+ = \max\{0, x_i\}, \quad i = 1, \cdots, m, \\ \hat{z}_l = 0, \quad l = 1, \cdots, n. \end{cases}$$

(c) *If $(x, z) \in \mathcal{K}^{\alpha}_{m,n}$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is itself, i.e., $(\hat{x}, \hat{z}) = (x, z)$.*

(d) *If $(x, z) \in -(\mathcal{K}^{\alpha}_{m,n})^*$, then its projection onto $\mathcal{K}^{\alpha}_{m,n}$ is zero vector, i.e., $(\hat{x}, \hat{z}) = 0$.*

Nonetheless, Hein does not obtain an explicit formula for the projection onto the power cone $\mathcal{K}^{\alpha}_{m,n}$ in [25]. Accordingly, analogous to the geometric cone $\mathcal{G}^n$ and the exponential cone $\mathcal{K}_e$, the decomposition for (x, z) associated with the power cone $\mathcal{K}^{\alpha}_{m,n}$ and the corresponding nonsmooth analysis for its cone-functions are not established yet.

7 Conclusion

According to the authors' earlier experience on symmetric cone optimization, we believe that spectral decomposition associated with cones, nonsmooth analysis regarding cone-functions, projections onto cones, and cone-convexity are the bridges between symmetric cone programs and nonsymmetric cone programs. Therefore, in this paper, we survey some related results about circular cone, p-order cone, geometric cone, exponential cone, and the power cone. Although the results are not quite complete due to the difficulty of handling nonsymmetric cones, they are very crucial to subsequent study towards nonsymmetric cone optimization. Further investigations are definitely desirable. We summarize and list out some future topics as below.

1. Exploring more structures and properties for each non-symmetric cone. Also looking for more non-symmetric cones, e.g., EDM cone.

2. For geometric cone, exponential cone, and power cone, etc., figuring out their spectral decompositions, projections, and doing nonsmooth analysis for their corresponding cone-functions like f^{sc}, f^{mat} and f^{circ}. We point out that through appropriate transformations (for example, α-representation and extended α-representation defined in [5]), the aforementioned geometric cone, exponential cone, and power cone can be generated from the 3-dimensional power cone and the exponential cone in Figure 5 and 6. More recently, Lu et al. [31] propose two types of decomposition approaches for these cones. We believe their results yield a possibility to construct the corresponding cone-functions.

3. Designing appropriate algorithms based on the structures of non-symmetric cones.

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