Comput Optim Appl (2011) 49:457–491
DOI 10.1007/s10589-009-9301-2

The penalized Fischer-Burmeister SOC complementarity function

Shaohua Pan · Jein-Shan Chen · Sangho Kum · Yongdo Lim

Received: 23 May 2008 / Published online: 20 November 2009

Abstract In this paper, we study the properties of the penalized Fischer-Burmeister (FB) second-order cone (SOC) complementarity function. We show that the function possesses similar desirable properties of the FB SOC complementarity function for local convergence; for example, with the function the second-order cone complementarity problem (SOCCP) can be reformulated as a (strongly) semismooth system of equations, and the corresponding nonsmooth Newton method has local quadratic convergence without strict complementarity of solutions. In addition, the penalized FB merit function has bounded level sets under a rather weak condition which can be satisfied by strictly feasible monotone SOCCPs or SOCCPs with the Cartesian R_{01}-property, although it is not continuously differentiable. Numerical results are included to illustrate the theoretical considerations.

Work of S. Pan is supported by National Young Natural Science Foundation (No. 10901058) and Guangdong Natural Science Foundation (No. 9251802902000001).
J.-S. Chen is a member of Mathematics Division, National Center for Theoretical Sciences, Taipei Office. The author's work is partially supported by National Science Council of Taiwan.

S. Pan
School of Mathematical Sciences, South China University of Technology, Guangzhou 510640, China
e-mail: shhpan@scut.edu.cn

J.-S. Chen (✉)
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
e-mail: jschen@math.ntnu.edu.tw

S. Kum
Department of Mathematics Education, Chungbuk National University, Cheongju 361-763, Korea

Y. Lim
Department of Mathematics, Kyungpook National University, Taegu 702-701, Korea

Keywords Second-order cone complementarity problem · Penalized Fischer-Burmeister function · Nonsmooth Newton method · B-subdifferential · Coerciveness

1 Introduction

Let F and G be continuously differentiable mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$. We consider the second-order cone complementarity problem (SOCCP) which is to *find a* $\zeta \in \mathbb{R}^n$ *such that*

$$F(\zeta) \in \mathcal{K}, \quad G(\zeta) \in \mathcal{K}, \qquad \langle F(\zeta), G(\zeta)\rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, and $\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_r}$ is the Cartesian product of second-order cones (SOCs) with $r, n_1, \ldots, n_r \geq 1$, $n_1 + \cdots + n_r = n$, and

$$\mathcal{K}^{n_i} := \left\{(x_{i1}, x_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid x_{i1} \geq \|x_{i2}\|\right\}.$$

In the rest of this paper, corresponding to the Cartesian structure of $\mathcal{K}$, we write $F = (F_1, \ldots, F_r)$ and $G = (G_1, \ldots, G_r)$ with $F_i, G_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ for $i = 1, 2, \ldots, r$.

An important special case of the SOCCP corresponds to $G(\zeta) \equiv \zeta$. Then (1) reduces to

$$F(\zeta) \in \mathcal{K}, \quad \zeta \in \mathcal{K}, \qquad \langle F(\zeta), \zeta \rangle = 0, \tag{2}$$

which is a natural extension of the nonlinear complementarity problem (NCP) over the nonnegative orthant cone $\mathcal{K} = \mathbb{R}^n_+$; see [9, 11]. Another important special case corresponds to the optimality conditions of the second-order cone program (SOCP):

$$\begin{array}{ll} \min & g(x) \\ \text{s.t.} & Ax = b, \quad x \in \mathcal{K}, \end{array} \tag{3}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a twice continuously differentiable function, $A \in \mathbb{R}^{m \times n}$ has full row rank and $b \in \mathbb{R}^m$. From [6], the KKT conditions of (3) can be reformulated as (1) with

$$F(\zeta) := \bar{x} + (I - A^T(AA^T)^{-1}A)\zeta, \qquad G(\zeta) := \nabla g(F(\zeta)) - A^T(AA^T)^{-1}A\zeta, \tag{4}$$

where $\bar{x} \in \mathbb{R}^n$ satisfies $A\bar{x} = b$. The SOCP (3) has numerous applications arising from the fields of engineering design, finance, robust optimization, combinatorial optimization, and includes as special cases convex quadratically constrained quadratic programs; see [1, 18].

In the past ten years, there have proposed various methods for SOCPs and SOCCPs. They include the interior-point methods [2, 19, 27, 28], the smoothing Newton methods [7, 13, 14], the merit function method [6], and the semismooth Newton method [16]. Among others, SOC complementarity functions play a key role in the last three ones. Specifically, we call $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ an *SOC complementarity function* associated with $\mathcal{K}^n$ if

$$\phi(x, y) = 0 \iff x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n, \quad \langle x, y\rangle = 0. \tag{5}$$

Clearly, when $n = 1$, an SOC complementarity function reduces to an NCP function.

A popular choice of ϕ is the vector-valued FB function $\phi_{\mathrm{FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\phi_{\mathrm{FB}}(x, y) := (x + y) - (x^2 + y^2)^{1/2}, \tag{6}$$

where $x^2 = x \circ x$ denotes the Jordan product of x and itself, $x^{1/2}$ is a vector such that $(x^{1/2})^2 = x$ for any $x \in \mathcal{K}^n$, and $x + y$ means the usual componentwise addition of vectors. This function was shown to be strongly semismooth in [4, 26] via different ways, and its squared norm $\psi_{\mathrm{FB}} = \frac{1}{2}\|\phi_{\mathrm{FB}}\|^2$ induces a continuously differentiable merit function [6]. Recently, we analyze that, to guarantee the boundedness of the level sets of the function

$$\Psi_{\mathrm{FB}}(\zeta) := \sum_{i=1}^{r} \psi_{\mathrm{FB}}(F_i(\zeta), G_i(\zeta)),$$

it requires that F and G at least have the uniform Cartesian P-property; see [21] for details. This means that ϕ_{FB} has a limitation in handling monotone SOCCPs. In fact, observing the disadvantage of the FB merit function Ψ_{FB}, Chen [5] extended two classes of regularized merit functions for the NCPs to deal with the monotone SOCCPs. But, those functions can not be used to design fast Newton-type methods. We notice that in the setting of NCPs, the penalized FB function was proposed in [3] to overcome such shortcoming of the FB function. Thus, it is natural to ask whether the extension of the penalized FB function to the SOCs can become an effective tool in the solution of monotone SOCCPs or not. The main contribution of this paper is to offer partial answers to this question.

The vector-valued penalized FB function $\phi_\rho : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined as follows

$$\phi_\rho(x, y) := \rho\phi_{\mathrm{FB}}(x, y) + (1 - \rho)\left[(x)_+ \circ (y)_+\right], \tag{7}$$

where $\rho \in (0, 1)$ is an arbitrary but fixed parameter, and $(\cdot)_+$ means the minimum Euclidean distance projection on $\mathcal{K}^n$. We show that ϕ_ρ has similar favorable properties of ϕ_{FB} for local convergence. For example, ϕ_ρ is a strongly semismooth SOC complementarity function, by which the SOCCP (1) can be reformulated as a (strongly) semismooth system

$$\Phi_\rho(\zeta) := \begin{pmatrix} \phi_\rho(F_1(\zeta), G_1(\zeta)) \\ \vdots \\ \phi_\rho(F_r(\zeta), G_r(\zeta)) \end{pmatrix} = 0, \tag{8}$$

and consequently one can apply the nonsmooth Newton method in [24, 25], i.e.,

$$\zeta^{k+1} := \zeta^k - W_k^{-1}\Phi_\rho(\zeta^k), \quad W_k \in \partial_B\Phi_\rho(\zeta^k), \quad k = 0, 1, 2, \ldots \tag{9}$$

to solve (1). Particularly, we establish that the nonsmooth method has the local quadratic convergence without strict complementarity of solutions, although their nondegeneracy is still necessary just like the FB semismooth method [20]. This is an advantage compared to interior-point methods where singular Jacobians will occur if strict complementarity is not satisfied. In addition, we also prove that the penalized FB merit function

$$\Psi_\rho(\zeta) := \sum_{i=1}^{r} \psi_\rho\left(F_i(\zeta), G_i(\zeta)\right) \tag{10}$$

with

$$\psi_\rho(x, y) := \frac{1}{2}\|\phi_\rho(x, y)\|^2 \tag{11}$$

has bounded level sets under a rather weak condition, which can be satisfied by strictly feasible monotone SOCCPs or SOCCPs with the Cartesian R_{01}-property. In other words, Ψ_ρ enjoys some nice features of the merit functions studied by [5] for global convergence. However, unlike its counterpart in the NCP setting, Ψ_ρ is not smooth even not differentiable.

Throughout this paper, I represents an identity matrix of suitable dimension, $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors, and $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_r}$ is identified with $\mathbb{R}^{n_1+\cdots+n_r}$. We denote by $\mathrm{int}(\mathcal{K}^n)$ and $\mathrm{bd}(\mathcal{K}^n)$ the interior and the boundary of $\mathcal{K}^n$, respectively. For any $x \in \mathbb{R}^n$, $(x)_+$ and $(x)_-$ denote the Euclidean projection of x onto $\mathcal{K}^n$ and $-\mathcal{K}^n$, respectively. For a differentiable mapping F, $F'(x)$ and $\nabla F(x)$ denotes the Jacobian of F and the transposed Jacobian at x, respectively. For a matrix $B \in \mathbb{R}^{n\times n}$, if $\langle x, Bx\rangle \geq 0$ (> 0) for all $0 \neq x \in \mathbb{R}^n$, we say that B is positive semidefinite (positive definite) and use the symbol $B \succeq 0$ $(\succ 0)$ to mean that B is symmetric and positive semidefinite (positive definite). For any $x \in \mathbb{R}^l$ with $l > 1$, we write $x = (x_1, x_2)$ where x_1 is the first component of x, and x_2 means the vector consisting of the rest $l - 1$ components.

2 Preliminaries

This section recalls some background materials and shows that ϕ_ρ is a strongly semismooth SOC complementarity function. We start with the definition of Jordan product [10]:

$$x \circ y := (\langle x, y\rangle, x_1 y_2 + y_1 x_2) \quad \forall x, y \in \mathbb{R}^n.$$

The Jordan product, unlike matrix multiplication, is not associative in general. The identity element under this product is $e := (1, 0, \ldots, 0)^T \in \mathbb{R}^n$, i.e., $e \circ x = x$ for any $x \in \mathbb{R}^n$. Let

$$L_x := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix},$$

which can be viewed as a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $L_x y = x \circ y$ for any $y \in \mathbb{R}^n$.

From [10] we recall that each $x \in \mathbb{R}^n$ has a spectral factorization associated with $\mathcal{K}^n$

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)},$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values of x and the associated spectral vectors, respectively, defined by

$$\lambda_i(x) := x_1 + (-1)^i \|x_2\|, \qquad u_x^{(i)} := \frac{1}{2}\Big(1, (-1)^i \bar{x}_2\Big),$$

with $\bar{x}_2 = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$ and otherwise $\bar{x}_2$ being any vector from $\mathbb{R}^{n-1}$ with $\|\bar{x}_2\| = 1$. If $x_2 \neq 0$, the factorization is unique. It is easy to verify that the following relation holds between the spectral factorization of x and the eigenvalue decomposition of L_x.

Lemma 2.1 *For any $x \in \mathbb{R}^n$, L_x has two single eigenvalues $\lambda_1(x)$ and $\lambda_2(x)$ with $u_x^{(1)}$ and $u_x^{(2)}$ being the corresponding eigenvectors, and the remaining $n-2$ eigenvalues are identically $x_1 = (\lambda_2(x) + \lambda_1(x))/2$ with the corresponding eigenvectors of the form $(0, \bar{v})$, where $\bar{v}$ lies in the linear subspace of $\mathbb{R}^{n-1}$ orthogonal to x_2.*

To show that ϕ_ρ is an SOC complementarity function, we need the following lemma.

Lemma 2.2 *Let $\phi_{\rm FB}$ and ϕ_ρ be given by (6) and (7), respectively. Then, for any $x, y \in \mathbb{R}^n$,*

$$[\phi_{\rm FB}(x, y)]_1 \geq -2\,(\|(x)_-\| + \|(y)_-\|) \quad \textit{and}$$
$$\|\phi_\rho(x, y)\| \geq \rho \max\{\|(x)_-\|, \|(y)_-\|\}.$$

Proof The first inequality follows from the trace inequality of [17, Theorem 3.1], since

$$\begin{aligned} 2[\phi_{\rm FB}(x, y)]_1 &= \mathrm{tr}(x + y) - \mathrm{tr}\Big((x^2 + y^2)^{1/2}\Big) \\ &\geq \mathrm{tr}(x) + \mathrm{tr}(y) - \mathrm{tr}(|x|) - \mathrm{tr}(|y|) \\ &= 2\,\mathrm{tr}[(x)_-] + 2\,\mathrm{tr}[(y)_-] \\ &\geq -4(\|x_-\| + \|y_-\|). \end{aligned}$$

We next prove the second inequality. Using $x = (x)_+ + (x)_-$, it follows that

$$\begin{aligned} \|\phi_\rho(x, y)\|^2 &= \Big\|\rho[x + y - (x^2 + y^2)^{1/2}] + (1 - \rho)[(x)_+ \circ (y)_+]\Big\|^2 \\ &= \Big\|\rho(x)_- + \rho[(x)_+ + y - (x^2 + y^2)^{1/2}] + (1 - \rho)[(x)_+ \circ (y)_+]\Big\|^2 \\ &= \rho^2\|(x)_-\|^2 \end{aligned}$$

$$
\begin{aligned}
&\quad + \left\| \rho[(x)_+ + y - (x^2+y^2)^{1/2}] + (1-\rho)[(x)_+ \circ (y)_+] \right\|^2 \\
&\quad + 2\rho[(x)_-]^T \left[\rho\left((x)_+ + y - (x^2+y^2)^{1/2}\right) + (1-\rho)(x)_+ \circ (y)_+ \right] \\
&\geq \rho^2\|(x)_-\|^2 + 2\rho^2[(x)_-]^T[(x)_+] + 2\rho^2[(x)_-]^T\left[y - (x^2+y^2)^{1/2}\right] \\
&\quad + 2\rho(1-\rho)[(x)_-]^T\left[(x)_+ \circ (y)_+\right] \\
&\geq \rho^2\|(x)_-\|^2
\end{aligned}
$$

where the last inequality is since $(x)_-,\ y - (x^2+y^2)^{1/2} \in -\mathcal{K}^n$, $\langle (x)_+, (x)_- \rangle = 0$ and

$$
(x \circ y)^T z = (y \circ z)^T x = (z \circ x)^T y \quad \text{for all } x, y, z \in \mathbb{R}^n.
$$

By the symmetry of x and y in ϕ_ρ, similarly, we have $\|\phi_\rho(x, y)\| \geq \rho\|(y)_-\|$. □

Proposition 2.1 *Let ϕ_ρ and Φ_ρ be defined as in* (7) *and* (8), *respectively. Then,*

(a) $\phi_\rho(x, y) = 0 \Leftrightarrow x \in \mathcal{K}^n,\ y \in \mathcal{K}^n$ *and* $\langle x, y \rangle = 0$.
(b) ϕ_ρ *is strongly semismooth.*
(c) Φ_ρ *is semismooth, and strongly semismooth if F', G' are locally Lipschitz continuous.*

Proof (a) The sufficiency is direct by noting that $\langle x, y \rangle = 0 \Leftrightarrow x \circ y = (x)_+ \circ (y)_+ = 0$ and $\phi_{\rm FB}$ is an SOC complementarity function. Now suppose that $\phi_\rho(x, y) = 0$. From the second inequality of Lemma 2.2, we have $(x)_- = 0$ and $(y)_- = 0$, which implies $x, y \in \mathcal{K}^n$, and hence $x^T y \geq 0$. In addition, by the first inequality of Lemma 2.2 and $[\phi_\rho(x, y)]_1 = 0$,

$$
\begin{aligned}
0 &= \rho[\phi_{\rm FB}(x, y)]_1 + (1-\rho)x^T y \geq -2\rho(\|(x)_-\| + \|(y)_-\|) + (1-\rho)x^T y \\
&= (1-\rho)x^T y.
\end{aligned}
$$

The two sides show that $x \in \mathcal{K}^n$, $y \in \mathcal{K}^n$ and $\langle x, y \rangle = 0$.

(b) Since the FB function $\phi_{\rm FB}$ and the projection function $(\cdot)_+$ are strongly semismooth by [26, Corollary 3.3] and [14, Proposition 4.5], respectively, the result follows from the fact given by [12] that the composite of (strongly) semismooth functions is (strongly) semismooth.

(c) The result is immediate by using part (b) and Theorem 19 of [12]. □

To close this section, we review several concepts that will be used in the sequel.

Definition 2.1 (a) Two mappings $F, G : \mathbb{R}^n \to \mathbb{R}^n$ are said to have the jointly Cartesian R_{01}-property if for any sequence $\{\zeta^k\}$ satisfying

$$
\|\zeta^k\| \to +\infty, \qquad \frac{[-G(\zeta^k)]_+}{\|\zeta^k\|} \to 0, \qquad \frac{[-F(\zeta^k)]_+}{\|\zeta^k\|} \to 0, \tag{12}
$$

there exists an index $\nu \in \{1, 2, \ldots, r\}$ such that

$$\liminf_{k \to +\infty} \frac{\langle F_\nu(\zeta^k), G_\nu(\zeta^k) \rangle}{\|\zeta^k\|} > 0.$$

(b) A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to have the Cartesian weak coercive property with respect to an element $\xi \in \mathbb{R}^n$, if there exists an index $\nu \in \{1, 2, \ldots, r\}$ such that

$$\liminf_{\|\zeta\| \to \infty} \frac{\langle \zeta_\nu - \xi_\nu, F_\nu(\zeta) \rangle}{\|\zeta - \xi\|} > 0.$$

Given a mapping $H : \mathbb{R}^n \to \mathbb{R}^m$, if H is locally Lipschitz continuous, then the set

$$\partial_B H(z) := \left\{ V \in \mathbb{R}^{m \times n} \mid \exists \{z^k\} \subseteq D_H : z^k \to z,\ H'(z^k) \to V \right\}$$

is nonempty and called the B-subdifferential of H at z, where $D_H \subseteq \mathbb{R}^n$ is the set of points at which H is differentiable. The convex hull $\partial H(z) := \mathrm{conv}\partial_B H(z)$ is the generalized Jacobian of H at z in the sense of Clarke [8]. We assume that the reader is familiar with the concepts of (strongly) semismooth functions, and refer to [24, 25] for details.

3 B-subdifferential

In this section, we present the representation of the elements in the B-subdifferential of ϕ_ρ at a general point, and then concentrate on the elements of the B-subdifferential of ϕ_ρ at complementarity pairs (x_i^*, y_i^*) with $i \in \{1, 2, \ldots, r\}$, i.e., each pair of x_i^* and y_i^* satisfies

$$x_i^* \in \mathcal{K}^{n_i}, \quad y_i^* \in \mathcal{K}^{n_i}, \qquad \langle x_i^*, y_i^* \rangle = 0. \tag{13}$$

For this purpose, we need the following two lemmas which overestimate the B-subdifferential of the FB function ϕ_{FB} and the projection function $(\cdot)_+$ at a general point, respectively.

Lemma 3.1 [20, Proposition 3.1] *For any given $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, each element $[U\ \ V] \in \partial_B \phi_{\mathrm{FB}}(x, y)$ has the following representation*:

(a) *If $x^2 + y^2 \in \mathrm{int}(\mathcal{K}^n)$, then ϕ_{FB} is continuously differentiable at (x, y) with*

$$U = \nabla_x \phi_{\mathrm{FB}}(x, y)^T = I - L^{-1}_{(x^2+y^2)^{1/2}} L_x,$$
$$V = \nabla_y \phi_{\mathrm{FB}}(x, y)^T = I - L^{-1}_{(x^2+y^2)^{1/2}} L_y.$$

(b) *If $x^2 + y^2 \in \mathrm{bd}(\mathcal{K}^n)$ and $(x, y) \neq (0, 0)$, then*

$$\begin{bmatrix} U & V \end{bmatrix} \in \left\{ \left[I - HL_x - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_2 \end{pmatrix} u^T \quad I - HL_y - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_2 \end{pmatrix} v^T \right] \right.$$

$$
\begin{aligned}
&\textit{for some } u, v \textit{ satisfying } |u_1| \le \|u_2\| \le 1,\ |v_1| \le \|v_2\| \le 1,\\
&(u_1 - v_1) \le \|u_2 - v_2\|,\ (u_1 + v_1) \le \|u_2 + v_2\|,\\
&(u_1 - v_1)^2 + \|u_2 + v_2\|^2 \le 2,\ (u_1 + v_1)^2 + \|u_2 - v_2\|^2 \le 2,\\
&(1, \bar{w}_2^T)u = 0,\ (1, -\bar{w}_2^T)u = 2u_1,\ (1, \bar{w}_2^T)v = 0,\\
&\left.(1, -\bar{w}_2^T)v = 2v_1\right\},
\end{aligned} \tag{14}
$$

where

$$
H = \frac{1}{4\sqrt{x_1^2 + y_1^2}} \begin{pmatrix} 1 & \bar{w}_2^T \\ \bar{w}_2 & 4I - 3\bar{w}_2\bar{w}_2^T \end{pmatrix} \quad \textit{with } \bar{w}_2 = \frac{x_1x_2 + y_1y_2}{\|x_1x_2 + y_1y_2\|}.
$$

(c) *If* $(x, y) = (0, 0)$, *then* $[U \quad V] \in \{[I - L_g \ I - L_h]$ *for some* $g^2 + h^2 = e\}$ *or*

$$
\begin{aligned}
[U \quad V] \in \Bigg\{\Bigg[& I - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_2 \end{pmatrix} u^T - \frac{1}{2}\begin{pmatrix} 1 \\ \bar{w}_2 \end{pmatrix} \xi^T - \begin{pmatrix} 0 & 0 \\ 0 & I - \bar{w}_2\bar{w}_2^T \end{pmatrix} L_s \\
& I - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_2 \end{pmatrix} v^T - \frac{1}{2}\begin{pmatrix} 1 \\ \bar{w}_2 \end{pmatrix} \eta^T - \begin{pmatrix} 0 & 0 \\ 0 & I - \bar{w}_2\bar{w}_2^T \end{pmatrix} L_\omega \Bigg] \\
& \textit{for some } \|\bar{w}_2\| = 1,\ \textit{some } u, v \textit{ satisfying (14) with such } \bar{w}_2,\\
& \textit{some } \xi, \eta \in \mathbb{R}^n \textit{ satisfying } |\xi_1| \le \|\xi_2\| \le 1, |\eta_1| \le \|\eta_2\| \le 1,\\
& (\xi_1 - \eta_1) \le \|\xi_2 - \eta_2\|,\ (\xi_1 + \eta_1) \le \|\xi_2 + \eta_2\|,\\
& (\xi_1 - \eta_1)^2 + \|\xi_2 + \eta_2\|^2 \le 2,\ (\xi_1 + \eta_1)^2 + \|\xi_2 - \eta_2\|^2 \le 2,\\
& (1, \bar{w}_2^T)\xi = 2\xi_1,\\
& (1, -\bar{w}_2^T)\xi = 0,\ (1, \bar{w}_2^T)\eta = 2\eta_1,\ (1, -\bar{w}_2^T)\eta = 0,\\
& \textit{and } s = \sigma u + (1 - \sigma)\xi,\ \omega = \sigma v + (1 - \sigma)\eta\\
& \left.\textit{for } \sigma \in [0, 1/2] \textit{ with } 1/2 \le \|s\|^2 + \|\omega\|^2 \le 2\right\}.
\end{aligned} \tag{15}
$$

Furthermore, all $UV^T + VU^T$ *are symmetric and positive semidefinite.*

Lemma 3.2 [16] *For any* $x \in \mathbb{R}^n$, *each* $X \in \partial_B(x)_+$ *has the following representation*:

(a) *If* $x_1 \neq \pm\|x_2\|$, *then* $(x)_+$ *is continuously differentiable at* x *with*

$$X = (x)'_+ = \begin{cases} 0 & \text{if } x_1 < -\|x_2\| \\ I & \text{if } x_1 > \|x_2\| \\ \frac{1}{2}\begin{pmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & \bar{X} \end{pmatrix} & \text{if } -\|x_2\| < x_1 < \|x_2\|, \end{cases}$$

where

$$\bar{x}_2 := \frac{x_2}{\|x_2\|}, \qquad \bar{X} := \left(\frac{x_1}{\|x_2\|} + 1\right) I - \frac{x_1}{\|x_2\|}\bar{x}_2\bar{x}_2^T.$$

(b) *If* $x_2 \neq 0$ *and* $x_1 = \|x_2\|$, *then*

$$X \in \left\{ I, \ \frac{1}{2}\begin{pmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & \bar{X} \end{pmatrix} \right\}, \quad \text{where } \bar{x}_2 := \frac{x_2}{\|x_2\|} \text{ and } \bar{X} := 2I - \bar{x}_2\bar{x}_2^T.$$

(c) *If* $x_2 \neq 0$ *and* $x_1 = -\|x_2\|$, *then*

$$X \in \left\{ 0, \ \frac{1}{2}\begin{pmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & \bar{X} \end{pmatrix} \right\}, \quad \text{where } \bar{x}_2 := \frac{x_2}{\|x_2\|} \text{ and } \bar{X} := \bar{x}_2\bar{x}_2^T.$$

(d) *If* $x = 0$, *then either* $X = 0$ *or* $X = I$ *or* X *belongs to the set*

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 & \bar{x}_2^T \\ \bar{x}_2 & \bar{X} \end{pmatrix} \,\middle|\, \bar{X} = (x_0+1)I - x_0\bar{x}_2\bar{x}_2^T \text{ for some } |x_0| \le 1 \text{ and } \|\bar{x}_2\| = 1 \right\}.$$

Proposition 3.1 *Let* ϕ_ρ *be defined as in* (7). *Then, for any given* $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, *each element* $[S\ T] \in \partial_B\phi_\rho(x, y)$ *has the following representation*:

$$S = \rho U + (1-\rho)L_{(y)_+}X, \qquad T = \rho V + (1-\rho)L_{(x)_+}Y,$$

where $[U\ \ V] \in \partial_B\phi_{\rm FB}(x, y)$, $X \in \partial_B(x)_+$ *and* $Y \in \partial_B(y)_+$.

Proof From the definition of the B-subdifferential, it is not hard to verify that

$$\partial_B\phi_\rho(x, y) \ \subseteq \ \rho\partial_B\phi_{\rm FB}(x, y) + (1-\rho)\partial_B[(x)_+ \circ (y)_+]. \tag{16}$$

Let $F(x, y) := x \circ y$ and $G(x, y) := \binom{(x)_+}{(y)_+}$. Then, $(x)_+ \circ (y)_+ = F(G(x, y))$. Using Proposition 7 and Lemma 14 of [22] and noting that $\partial_B G(x, y) = \partial_B(x)_+ \times \partial_B(y)_+$ yields

$$\partial_B[(x)_+ \circ (y)_+] = JF(G(x, y))\partial_B G(x, y) = L_{(y)_+}\partial_B(x)_+ \times L_{(x)_+}\partial_B(y)_+,$$

where the set on the right hand side denotes the set of all matrices whose first n columns belong to $L_{(y)_+}\partial_B(x)_+$ and last n columns belong to $L_{(x)_+}\partial_B(y)_+$. Combining the last two equations, we immediately obtain the desired result. $\square$

In the rest of this section, we concentrate on the elements of the B-subdifferential of ϕ_ρ at all complementarity pairs (x_i^*, y_i^*) which satisfies (13). As remarked in [1], the index set $\{1, 2, \ldots, r\}$ can be partitioned as $J_I \cup J_B \cup J_0 \cup J_{B0} \cup J_{0B} \cup J_{00}$ with

$$\begin{aligned} J_I &:= \left\{i \mid x_i^* \in \mathrm{int}(\mathcal{K}^{n_i}),\ y_i^* = 0\right\}, \\ J_B &:= \left\{i \mid x_i^* \in \mathrm{bd}^+(\mathcal{K}^{n_i}),\ y_i^* \in \mathrm{bd}^+(\mathcal{K}^{n_i})\right\}, \\ J_0 &:= \left\{i \mid x_i^* = 0,\ y_i^* \in \mathrm{int}(\mathcal{K}^{n_i})\right\}, \\ J_{B0} &:= \left\{i \mid x_i^* \in \mathrm{bd}^+(\mathcal{K}^{n_i}),\ y_i^* = 0\right\}, \\ J_{0B} &:= \left\{i \mid x_i^* = 0,\ y_i^* \in \mathrm{bd}^+(\mathcal{K}^{n_i})\right\}, \\ J_{00} &:= \left\{i \mid x_i^* = 0,\ y_i^* = 0\right\} \end{aligned} \tag{17}$$

where $\mathrm{bd}^+(\mathcal{K}^{n_i})$ denotes the boundary of $\mathcal{K}^{n_i}$ excluding the origion. First of all, let us pay attention to the elements of $\partial_B \phi_\rho(x_i^*, y_i^*)$ for $i \in J_I \cup J_B \cup J_0$. For convenience, the notation "$\star$" in the sequel always represents some real number from the interval $(0, +\infty)$.

Proposition 3.2 *Let $[S_i\ \ T_i] \in \partial_B \phi_\rho(x_i^*, y_i^*)$ for $i = 1, 2, \ldots, r$. Then, for $i \in J_I \cup J_B \cup J_0$, there exists an orthogonal matrix Q_i such that $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$ where,*

(a) *if $i \in J_I$, then D_i and Λ_i satisfy one of the following cases*:

$$D_i = 0, \qquad \Lambda_i = \rho I, \qquad Q_i = I; \tag{18}$$

$$D_i = 0, \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, \star); \tag{19}$$

$$D_i = 0, \qquad \Lambda_i \text{ is a nonsingular lower triangular matrix.} \tag{20}$$

(b) *If $i \in J_0$, then D_i and Λ_i satisfy one of the following cases*:

$$D_i = \rho I, \qquad \Lambda_i = 0, \qquad Q_i = I; \tag{21}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = 0; \tag{22}$$

$$D_i \text{ is a nonsingular lower triangular matrix}, \qquad \Lambda_i = 0. \tag{23}$$

(c) *If $i \in J_B$, then D_i and Λ_i have one of the following representations*:

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, 0), \qquad \Lambda_i = \mathrm{diag}(0, \star, \ldots, \star, \star); \tag{24}$$

$$D_i = \mathrm{diag}(0, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, 0). \tag{25}$$

Proof For each $i = 1, \ldots, r$, let $[U_i\ \ V_i] \in \partial_B \phi_{\mathrm{FB}}(x_i^*, y_i^*)$, $X_i^* \in \partial_B (x_i^*)_+$ and $Y_i^* \in \partial_B (y_i^*)_+$.

(a) From Lemma 3.1(a) and Lemma 3.2(a) and (d), it follows that

$$U_i = 0, \qquad V_i = I \quad \text{and} \quad X_i^* = I, \qquad Y_i^* = 0,\ I \text{ or } H_i := \frac{1}{2}\begin{pmatrix} 1 & \bar{w}_{i2}^T \\ \bar{w}_{i2} & \bar{H}_i \end{pmatrix},$$

where $\bar{H}_i = (\tau+1)I - \tau\bar{w}_{i2}\bar{w}_{i2}^T$ for some $|\tau|\le 1$ and $\|\bar{w}_{i2}\|=1$. By Proposition 3.1,

$$S_i = 0 \quad \text{and} \quad T_i = \rho I, \quad \rho I + (1-\rho)L_{x_i^*} \text{ or } \rho I + (1-\rho)L_{x_i^*}H_i.$$

In what follows, we proceed the arguments by the possible cases of T_i.

Case 1: $T_i = \rho I$. The result is obvious with $Q_i = I$, $D_i = 0$ and $\Lambda_i = \rho I$.

Case 2: $T_i = \rho I + (1-\rho)L_{x_i^*}$. Under this case, let $Q_i = [q_i\ \hat{q}_1\ \ldots\ \hat{q}_{n_i-2}\ q_i']$ with

$$q_i = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \bar{x}_{i2}^*\end{pmatrix}, \qquad \hat{q}_j = \begin{pmatrix}0\\ \bar{v}_j\end{pmatrix} \quad \text{for } j=1,\ldots,n_i-2, \qquad q_i' = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -\bar{x}_{i2}^*\end{pmatrix},$$

where $\bar{x}_{i2}^* = \frac{x_{i2}^*}{\|x_{i2}^*\|}$ and $\bar{v}_1,\ldots,\bar{v}_{n_i-2}$ are arbitrary unit vectors in $\mathbb{R}^{n_i-1}$ that span the linear subspace $\{\bar{v}\in\mathbb{R}^{n_i-1} \mid \bar{v}^T\bar{x}_{i2}^* = 0\}$. From Lemma 2.1, such orthogonal matrix Q_i satisfies

$$L_{x_i^*} \;=\; Q_i \operatorname{diag}\Big(\lambda_2(x_i^*), x_{i1}^*, \ldots, x_{i1}^*, \lambda_1(x_i^*)\Big) Q_i^T.$$

Therefore, $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i\Lambda_i Q_i^T$ with D_i and Λ_i having the expression of (19).

Case 3: $T_i = \rho I + (1-\rho)L_{x_i^*}H_i$. Now T_i is similar to $\rho I + (1-\rho)(L_{x_i^*})^{1/2}H_i \times (L_{x_i^*})^{1/2}$ since

$$T_i = (L_{x_i^*})^{1/2}[\rho I + (1-\rho)(L_{x_i^*})^{1/2}H_i(L_{x_i^*})^{1/2}](L_{x_i^*})^{-1/2}.$$

Since $H_i \succeq 0$ by Lemma 2.7 of [16], all eigenvalues of T_i are positive. Using Theorem 2.3.1 of [15], there exists an orthogonal Q_i such that $T_i = Q_i\Lambda_i Q_i^T$, where Λ_i is lower triangular with diagonal entries being the eigenvalues of T_i. Such Q_i clearly satisfies the result of (20).

(b) The desired result is due to part (a) and the symmetry of x_i^* and y_i^* in $\phi_\rho(x_i^*, y_i^*)$.

(c) In this case, from Lemma 3.2(b) and Proposition 3.1, it follows that

$$\begin{aligned} S_i &= \rho U_i + (1-\rho)L_{y_i^*} \quad \text{or} \quad S_i = \rho U_i + (1-\rho)L_{y_i^*}X_i^*,\\ T_i &= \rho V_i + (1-\rho)L_{x_i^*} \quad \text{or} \quad T_i = \rho V_i + (1-\rho)L_{x_i^*}Y_i^*, \end{aligned} \tag{26}$$

where

$$X_i^* = \frac{1}{2}\begin{pmatrix}1 & (\bar{x}_{i2}^*)^T\\ \bar{x}_{i2}^* & \bar{X}_i^*\end{pmatrix} \quad \text{with } \bar{x}_{i2}^* = \frac{x_{i2}^*}{\|x_{i2}^*\|} \text{ and } \bar{X}_i^* = 2I - \bar{x}_{i2}^*(\bar{x}_{i2}^*)^T,$$

$$Y_i^* = \frac{1}{2}\begin{pmatrix}1 & (\bar{y}_{i2}^*)^T\\ \bar{y}_{i2}^* & \bar{Y}_i^*\end{pmatrix} \quad \text{with } \bar{y}_{i2}^* = \frac{y_{i2}^*}{\|y_{i2}^*\|} \text{ and } \bar{Y}_i^* = 2I - \bar{y}_{i2}^*(\bar{y}_{i2}^*)^T.$$

Since x_i^* and y_i^* share with the same Jordan frame, without loss of generality we assume

$$x_i^* = \sigma_1 q_i + \sigma_2 q_i' \quad \text{and} \quad y_i^* = \mu_1 q_i + \mu_2 q_i'$$

for some $\sigma_1, \sigma_2, \mu_1, \mu_2 \in \mathbb{R}$, where the Jordan frame $\{q_i, q_i'\}$ has the form of

$$q_i = \frac{1}{2}\begin{pmatrix} 1 \\ \bar{q}_i \end{pmatrix} \quad \text{and} \quad q_i' = \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{q}_i \end{pmatrix} \quad \text{with } \bar{q}_i \in \mathbb{R}^{n_i-1} \text{ satisfying } \|\bar{q}_i\| = 1.$$

Note that $x_i^* + y_i^* \in \operatorname{int}(\mathcal{K}^{n_i})$ holds for this case, which along with $x_i^*, y_i^* \in \operatorname{bd}^+(\mathcal{K}^{n_i})$ implies

$$\sigma_1 + \sigma_2 = |\sigma_1 - \sigma_2|, \qquad \mu_1 + \mu_2 = |\mu_1 - \mu_2|,$$
$$\sigma_1 + \sigma_2 + \mu_1 + \mu_2 > |\sigma_1 - \sigma_2 + \mu_1 - \mu_2|.$$

From this, we deduce that $\sigma_1 = 0, \sigma_2 > 0, \mu_1 > 0, \mu_2 = 0$ or $\sigma_1 > 0, \sigma_2 = 0$, $\mu_1 = 0, \mu_2 > 0$. Let $Q_i = [\sqrt{2}q_i \ \ \hat{q}_1 \ \ \ldots \ \ \hat{q}_{n_i-2} \ \ \sqrt{2}q_i']$ with $\hat{q}_j = \binom{0}{\bar{v}_j}$ for $j = 1, \ldots, n_i - 2$, where $\bar{v}_1, \ldots, \bar{v}_{n_i-2}$ are arbitrary unit vectors that span the linear subspace $\{\bar{v} \in \mathbb{R}^{n_i-1} \mid \bar{v}^T \bar{q}_i = 0\}$. Clearly, such Q_i is an orthogonal matrix. We proceed the arguments by two cases.

Case 1: $\sigma_1 = 0, \sigma_2 > 0, \mu_1 > 0, \mu_2 = 0$. From the proof of Proposition 4.1(c) of [20],

$$U_i = Q_i \operatorname{diag}(1, \star, \ldots, \star, 0) Q_i^T, \qquad V_i = Q_i \operatorname{diag}(0, \star, \ldots, \star, 1) Q_i^T.$$

By Lemma 2.1 and the expressions of x_i^* and y_i^*, it is not difficult to verify that

$$L_{x_i^*} = Q_i \operatorname{diag}\left(\lambda_1(x_i^*), x_{i1}^*, \ldots, x_{i1}^*, \lambda_2(x_i^*)\right) Q_i^T = Q_i \operatorname{diag}\left(0, \frac{\sigma_2}{2}, \ldots, \frac{\sigma_2}{2}, \sigma_2\right) Q_i^T,$$
$$\begin{aligned} L_{y_i^*} &= Q_i \operatorname{diag}\left(\lambda_2(y_i^*), y_{i1}^*, \ldots, y_{i1}^*, \lambda_1(y_i^*)\right) Q_i^T \\ &= Q_i \operatorname{diag}\left(\mu_1, \frac{\mu_1}{2}, \ldots, \frac{\mu_1}{2}, 0\right) Q_i^T. \end{aligned}$$

In addition, by Lemma 2.7 of [16] and the expressions of x_i^* and y_i^*, we have

$$X_i^* = Q_i \operatorname{diag}(0, 1, \ldots, 1, 1) Q_i^T, \qquad Y_i^* = Q_i \operatorname{diag}(1, 1, \ldots, 1, 0) Q_i^T.$$

Combining the last three equations with (26) yields the desired result in (24).

Case 2: $\sigma_1 > 0, \sigma_2 = 0, \mu_1 = 0, \mu_2 > 0$. By the proof of Proposition 4.1(c) of [20],

$$U_i = Q_i \operatorname{diag}(0, \star, \ldots, \star, 1) Q_i^T, \qquad V_i = Q_i \operatorname{diag}(1, \star, \ldots, \star, 0) Q_i^T.$$

Also, by Lemma 2.1, Lemma 2.7 of [16], and the expressions of x_i^* and y_i^*, we verify that

$$L_{x_i^*} = Q_i \operatorname{diag}\left(\lambda_2(x_i^*), x_{i1}^*, \ldots, x_{i1}^*, \lambda_1(x_i^*)\right) Q_i^T = Q_i \operatorname{diag}\left(\sigma_1, \frac{\sigma_1}{2}, \ldots, \frac{\sigma_1}{2}, 0\right) Q_i^T,$$
$$\begin{aligned} L_{y_i^*} &= Q_i \operatorname{diag}\left(\lambda_1(y_i^*), y_{i1}^*, \ldots, y_{i1}^*, \lambda_2(y_i^*)\right) Q_i^T \\ &= Q_i \operatorname{diag}\left(0, \frac{\mu_2}{2}, \ldots, \frac{\mu_2}{2}, \frac{\mu_2}{2}\right) Q_i^T, \end{aligned}$$
$$X_i^* = Q_i \operatorname{diag}(1, 1, \ldots, 1, 0) Q_i^T, \qquad Y_i^* = Q_i \operatorname{diag}(0, 1, \ldots, 1, 1) Q_i^T.$$

The last two equations and (26) gives the result, with D_i, Λ_i given by (25). $\square$

In view of Proposition 3.2, we may partition the index sets J_I, J_0 and J_B as $J_I = J_I^1 \cup J_I^2$, $J_0 = J_0^1 \cup J_0^2$ and $J_B = J_B^1 \cup J_B^2$, respectively, with

$$J_I^1 := \{i \mid S_i = D_i,\ T_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (18)}\},$$

$$J_I^2 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (19) or (20)}\right\},$$

$$J_0^1 := \{i \mid S_i = D_i,\ T_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (21)}\},$$

$$J_0^2 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (22) or (23)}\right\},$$

$$J_B^1 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (24)}\right\},$$

$$J_B^2 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (25)}\right\}.$$

Next we take a look at the elements of the B-subdifferential $\partial_B \phi_\rho(x_i^*, y_i^*)$ for $i \in J_{B0} \cup J_{0B}$.

Proposition 3.3 *Let $[S_i\ T_i] \in \partial_B \phi_\rho(x_i^*, y_i^*)$ for $i = 1, 2, \ldots, r$. Then, for $i \in J_{B0} \cup J_{0B}$, there exists an orthogonal $Q_i = [q_i\ \hat{Q}_i\ q_i']$ such that $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$ where,*

(a) *if $i \in J_{B0}$, then D_i and Λ_i exactly have one of the following representations*:

$$D_i = 0, \qquad \Lambda_i = \rho I, \qquad Q_i = I; \tag{27}$$

$$D_i = 0, \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, \star); \tag{28}$$

$$D_i = \mathrm{diag}(0, 0, \ldots, 0, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, 0); \tag{29}$$

$$D_i = \mathrm{diag}\,(0, 0, \ldots, 0, \star), \qquad \Lambda_i = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star I & 0 \\ 0 & -\frac{\rho}{\sqrt{2}} v_i^T \hat{Q}_i & \star \end{pmatrix}; \tag{30}$$

$$D_i = \mathrm{diag}\,(0, 0, \ldots, 0, 0), \qquad \Lambda_i = \begin{pmatrix} \hat{Z}_i & \hat{z}_i \\ 0 & \rho \end{pmatrix}; \tag{31}$$

$$D_i = \mathrm{diag}\,(0, 0, \ldots, 0, \rho), \qquad \Lambda_i = \begin{pmatrix} \hat{Z}_i & \hat{z}_i \\ 0 & 0 \end{pmatrix}; \tag{32}$$

$$D_i = \mathrm{diag}\,(0, 0, \ldots, 0, \star), \qquad \begin{pmatrix} \Lambda_i = \hat{Z}_i & \hat{z}_i \\ (0 - \frac{\rho}{\sqrt{2}} v_i^T \hat{Q}_i) & \star \end{pmatrix} \tag{33}$$

where v_i is same as in (41) and every eigenvalue of $\hat{Z}_i \in \mathbb{R}^{(n_i-1)\times(n_i-1)}$ is positive.

(b) *If $i \in J_{0B}$, then D_i and Λ_i have one of the following representations*:

$$D_i = \rho I, \qquad \Lambda_i = 0, \qquad Q_i = I; \tag{34}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = 0; \tag{35}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, 0), \qquad \Lambda_i = \mathrm{diag}(0, 0, \ldots, 0, \star); \tag{36}$$

$$D_i = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star I & 0 \\ 0 & -\frac{\rho}{\sqrt{2}} u_i^T \hat{Q}_i & \star \end{pmatrix}, \qquad \Lambda_i = \mathrm{diag}(0, 0, \ldots, 0, \star); \tag{37}$$

$$D_i = \begin{pmatrix} \bar{Z}_i & \bar{z}_i \\ 0 & \rho \end{pmatrix}, \qquad \Lambda_i = \mathrm{diag}\,(0, 0, \ldots, 0, 0)\,; \tag{38}$$

$$D_i = \begin{pmatrix} \bar{Z}_i & \bar{z}_i \\ 0 & 0 \end{pmatrix}, \qquad \Lambda_i = \mathrm{diag}\,(0, 0, \ldots, 0, \rho)\,; \tag{39}$$

$$D_i = \begin{pmatrix} \bar{Z}_i & \bar{z}_i \\ (0 - \frac{\rho}{\sqrt{2}} u_i^T \hat{Q}_i) & \star \end{pmatrix}, \qquad \Lambda_i = \mathrm{diag}\,(0, 0, \ldots, 0, \star) \tag{40}$$

where u_i is same as in (41) and every eigenvalue of $\bar{Z}_i \in \mathbb{R}^{(n_i-1)\times(n_i-1)}$ is positive.

Proof For each $i = 1, \ldots, r$, let $[U_i \;\; V_i] \in \partial_B \phi_{\mathrm{FB}}(x_i^*, y_i^*)$, $X_i^* \in \partial_B (x_i^*)_+$ and $Y_i^* \in \partial_B (y_i^*)_+$.

(a) By Proposition 3.1, $S_i = \rho U_i$ and $T_i = \rho V_i + (1-\rho) L_{x_i^*} Y_i^*$ with $Y_i^* = 0$, I or H_i, where H_i is defined as in Proposition 3.3(a). Let $Q_i = [q_i \;\; \hat{q}_1 \;\; \ldots \;\; \hat{q}_{n_i-2} \;\; q_i'] = [q_i \;\; \hat{Q}_i \;\; q_i']$ with

$$q_i = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \bar{x}_{i2}^* \end{pmatrix}, \qquad \hat{q}_j = \begin{pmatrix} 0 \\ \bar{v}_j \end{pmatrix} \quad \text{for } j = 1, \ldots, n_i - 2, \qquad q_i' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\bar{x}_{i2}^* \end{pmatrix},$$

where $\bar{x}_{i2}^* = \frac{x_{i2}^*}{\|x_{i2}^*\|}$ and $\bar{v}_1, \ldots, \bar{v}_{n_i-2}$ are arbitrary unit vectors in $\mathbb{R}^{n_i-1}$ that span the linear subspace $\{\bar{v} \in \mathbb{R}^{n_i-1} \mid \bar{v}^T \bar{x}_{i2}^* = 0\}$. From the proof of [20, Prop. 4.2(a)], such orthogonal matrix Q_i is such that $U_i = Q_i \Sigma_i Q_i^T$ and $V_i = Q_i \Gamma_i Q_i^T$ with

$$\Sigma_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} u_i^T \hat{Q}_i & 1 - u_{i1} \end{pmatrix}, \qquad \Gamma_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & -\frac{1}{\sqrt{2}} v_i^T \hat{Q}_i & 1 - v_{i1} \end{pmatrix}$$

for some $u_i = (u_{i1}, u_{i2}), v_i = (v_{i1}, v_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1}$ satisfying

$$\begin{aligned} &|u_{i1}| \le \|u_{i2}\| \le 1, \qquad |v_{i1}| \le \|v_{i2}\| \le 1, \\ &(u_{i1} - v_{i1}) \le \|u_{i2} - v_{i2}\|, \qquad (u_{i1} + v_{i1}) \le \|u_{i2} + v_{i2}\|, \\ &(u_{i1} - v_{i1})^2 + \|u_{i2} + v_{i2}\|^2 \le 2, \qquad (u_{i1} + v_{i1})^2 + \|u_{i2} - v_{i2}\|^2 \le 2, \\ &u_i^T q_i = 0, \qquad u_i^T q_i' = \sqrt{2} u_{i1}, \qquad v_i^T q_i = 0, \qquad v_i^T q_i' = \sqrt{2} v_{i1}. \end{aligned} \tag{41}$$

Moreover, the above Σ_i and Γ_i may reduce to one of the following three cases:

$$
\begin{aligned}
&\Sigma_i = 0, \qquad \Gamma_i = I, \qquad Q_i = I;\\
&\Sigma_i = \operatorname{diag}(0, 0, \ldots, 0, 1), \qquad \Gamma_i = \operatorname{diag}(1, 1, \ldots, 1, 0);\\
&\Sigma_i = \operatorname{diag}(0, 0, \ldots, 0, \star), \qquad \Gamma_i = \begin{pmatrix} 1 & 0 & 0\\ 0 & I & 0\\ 0 & -\frac{1}{\sqrt{2}} v_i^T \hat{Q}_i & \star \end{pmatrix}.
\end{aligned}
\tag{42}
$$

In the following, we proceed the arguments by the possible values of Y_i^*.

Case 1: $Y_i^* = 0$. Now, we have $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$ with $D_i = \rho\Sigma_i$ and $\Lambda_i = \rho\Gamma_i$. From (42), clearly, D_i and Λ_i have one of the representations given by (27), (29) and (30).

Case 2: $Y_i^* = I$. Under this case , $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$ with

$$D_i = \rho\Sigma_i \quad \text{and} \quad \Lambda_i = \rho\Gamma_i + (1-\rho) Q_i^T L_{x_i^*} Q_i.$$

From Lemma 2.1, the orthogonal matrix Q_i is also such that

$$L_{x_i^*} = Q_i \operatorname{diag}\left(\lambda_2(x_i^*), x_{i1}^*, \ldots, x_{i1}^*, 0\right) Q_i^T. \tag{43}$$

Combining with (42), D_i and Λ_i have one of representations given by (28), (29) and (30).

Case 3: $Y_i^* = H_i$. For this case, we have $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$ with

$$D_i = \rho\Sigma_i \quad \text{and} \quad \Lambda_i = \rho\Gamma_i + (1-\rho) Q_i^T L_{x_i^*} H_i Q_i.$$

Using (43) and an elementary calculation gives

$$Q_i^T L_{x_i^*} H_i Q_i = \begin{pmatrix} \lambda_2(x_i^*) & 0 & 0\\ 0 & x_{i1}^* I & 0\\ 0 & 0 & 0 \end{pmatrix} Q_i^T H_i Q_i = \begin{pmatrix} Z_i & z_i\\ 0 & 0 \end{pmatrix},$$

where

$$Z_i = \begin{pmatrix} \lambda_2(x_i^*) & 0\\ 0 & x_{i1}^* I \end{pmatrix} \tilde{Q}_i^T H_i \tilde{Q}_i, \qquad z_i = \begin{pmatrix} \lambda_2(x_i^*) q_i^T H_i q_i'\\ x_{i1}^* \hat{Q}_i^T H_i q_i' \end{pmatrix}$$

with $\tilde{Q}_i = [q_i \ \ \hat{Q}_i] \in \mathbb{R}^{n_i \times (n_i-1)}$. Together with the expressions of Σ_i and Γ_i, the matrices D_i and Λ_i exactly have one of the expressions in (31)–(33) with $\hat{Z}_i = \rho I + (1-\rho) Z_i$ and $\hat{z}_i = (1-\rho) z_i$. Since $\left(\begin{smallmatrix} \lambda_2(x_i^*) & 0\\ 0 & x_{i1}^* I \end{smallmatrix}\right) \succ 0$ and $\tilde{Q}_i^T H_i \tilde{Q}_i \succeq 0$, Z_i is diagonalizable and all eigenvalues are real and nonnegative by the result of Problem 3 in [15, p. 468]. From the expression of $\hat{Z}_i$, it then follows that every eigenvalue of $\hat{Z}_i$ is positive.

(b) The result is direct by part (a) and the symmetry of x_i^* and y_i^* in $\phi_\rho(x_i^*, y_i^*)$. □

By Proposition 3.3, we partition the index sets J_{B0} and J_{0B} into $J_{B0} = J_{B0}^1 \cup \cdots \cup J_{B0}^4$ and $J_{0B} = J_{0B}^1 \cup \cdots \cup J_{0B}^4$, respectively, with

$$J_{B0}^1 := \{i \mid S_i = D_i,\ T_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (27)}\},$$

$$J_{B0}^2 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (28) or (31)}\right\},$$

$$J_{B0}^3 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (29) or (32)}\right\},$$

$$J_{B0}^4 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (30) or (33)}\right\},$$

$$J_{0B}^1 := \{i \mid S_i = D_i,\ T_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (34)}\},$$

$$J_{0B}^2 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (35) or (38)}\right\},$$

$$J_{0B}^3 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (36) or (39)}\right\},$$

$$J_{0B}^4 := \left\{i \mid S_i = Q_i D_i Q_i^T,\ T_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (37) or (40)}\right\}.$$

Finally, we come to look at $[S_i\ \ T_i] \in \partial_B \phi_\rho(x_i^*, y_i^*)$ for $i \in J_{00}$. From Proposition 3.1, $S_i = \rho U_i$ and $T_i = \rho V_i$ with $[U_i\ \ V_i] \in \partial_B \phi_{\mathrm{FB}}(x_i^*, y_i^*)$, where by Lemma 3.1(c)

$$U_i = I - L_{g_i} \quad \text{and} \quad V_i = I - L_{h_i} \tag{44}$$

for some $g_i = (g_{i1}, g_{i2}), h_i = (h_{i1}, h_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1}$ satisfying $g_i^2 + h_i^2 = e$, or

$$\begin{aligned} U_i &= I - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_{i2} \end{pmatrix} u_i^T - \frac{1}{2}\begin{pmatrix} 1 \\ \bar{w}_{i2} \end{pmatrix} \xi_i^T - \begin{pmatrix} 0 & 0 \\ 0 & I - \bar{w}_{i2}\bar{w}_{i2}^T \end{pmatrix} L_{s_i}, \\ V_i &= I - \frac{1}{2}\begin{pmatrix} 1 \\ -\bar{w}_{i2} \end{pmatrix} v_i^T - \frac{1}{2}\begin{pmatrix} 1 \\ \bar{w}_{i2} \end{pmatrix} \eta_i^T - \begin{pmatrix} 0 & 0 \\ 0 & I - \bar{w}_{i2}\bar{w}_{i2}^T \end{pmatrix} L_{\omega_i} \end{aligned} \tag{45}$$

for some $\bar{w}_{i2} \in \mathbb{R}^{n_i-1}$ with $\|\bar{w}_{i2}\| = 1$, some u_i, v_i satisfying all the inequalities of (41) and

$$(1, \bar{w}_{i2}^T)u_i = 0, \qquad (1, -\bar{w}_{i2}^T)u_i = 2u_{i1}, \qquad (1, \bar{w}_{i2}^T)v_i = 0,$$
$$(1, -\bar{w}_{i2}^T)v_i = 2v_{i1},$$

some $\xi_i = (\xi_{i1}, \xi_{i2}), \eta_i = (\eta_{i1}, \eta_{i2}) \in \mathbb{R} \times \mathbb{R}^{n_i-1}$ such that

$$|\xi_{i1}| \le \|\xi_{i2}\| \le 1, \qquad |\eta_{i1}| \le \|\eta_{i2}\| \le 1$$
$$(\xi_{i1} - \eta_{i1}) \le \|\xi_{i2} - \eta_{i2}\|, \qquad (\xi_{i1} + \eta_{i1}) \le \|\xi_{i2} + \eta_{i2}\|;$$
$$(\xi_{i1} - \eta_{i1})^2 + \|\xi_{i2} + \eta_{i2}\|^2 \le 2, \qquad (\xi_{i1} + \eta_{i1})^2 + \|\xi_{i2} - \eta_{i2}\|^2 \le 2,$$
$$(1, \bar{w}_{i2}^T)\xi_i = 2\xi_{i1}, \qquad (1, -\bar{w}_{i2}^T)\xi_i = 0, \qquad (1, \bar{w}_{i2}^T)\eta_i = 2\eta_{i1},$$
$$(1, -\bar{w}_{i2}^T)\eta_i = 0,$$

and $s_i = \sigma u_i + (1-\sigma)\xi_i$, $\omega_i = \sigma v_i + (1-\sigma)\eta_i$ for some $\sigma \in [0, 1/2]$, with $(1/2) \le \|s_i\|^2 + \|\omega_i\|^2 \le 2$. Furthermore, all $U_i V_i^T + V_i U_i^T \succeq 0$. Using Proposition 4.3 of [20], we readily have the following result, where $\tilde{D}_i$ and $\tilde{\Lambda}_i$ denote the submatrices consisting of the first $n_i - 1$ rows and $n_i - 1$ columns of D_i and Λ_i, respectively, and $\bar{D}_i$ and $\bar{\Lambda}_i$ denote the submatrices consisting of the last $n_i - 1$ rows and $n_i - 1$ columns of D_i and Λ_i.

Proposition 3.4 *Let $[S_i \;\; T_i] \in \partial_B \phi_\rho(x_i^*, y_i^*)$ for $i = 1, 2, \ldots, r$. Then, for $i \in J_{00}$, there exists an orthogonal matrix $Q_i = [q_i \;\; \hat{Q}_i \;\; q_i']$ such that $S_i = Q_i D_i Q_i^T$ and $T_i = Q_i \Lambda_i Q_i^T$. If U_i and V_i are given by (44), then D_i and Λ_i have one of the following expressions*:

$$Q_i = I, \qquad D_i = 0, \qquad \Lambda_i = \rho I; \tag{46}$$

$$Q_i = I, \qquad D_i = \rho I, \qquad \Lambda_i = 0; \tag{47}$$

$$Q_i = I, \qquad D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, \star); \tag{48}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, \star); \tag{49}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(0, \star, \ldots, \star, \star); \tag{50}$$

$$D_i = \mathrm{diag}(0, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, \star); \tag{51}$$

$$D_i = \mathrm{diag}(\star, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, 0); \tag{52}$$

$$D_i = \mathrm{diag}(0, \star, \ldots, \star, \star), \qquad \Lambda_i = \mathrm{diag}(\star, \star, \ldots, \star, 0); \tag{53}$$

$$Q_i = I, \qquad D_i = \rho(I - L_{g_i}), \qquad \Lambda_i = \rho(I - L_{h_i}) \\ \quad \text{with } \Lambda_i^{-1} D_i,\, D_i^{-1}\Lambda_i \text{ positive definite.} \tag{54}$$

If U_i and V_i are given by (45), then D_i and Λ_i have one of the following expressions:

$$Q_i = I, \qquad D_i = 0, \qquad \Lambda_i = \rho I; \tag{55}$$

$$Q_i = I, \qquad D_i = \rho I, \qquad \Lambda_i = 0; \tag{56}$$

$$D_i = \mathrm{diag}\,(\star, \star, \ldots, \star, 0)\,, \qquad \Lambda_i = \mathrm{diag}\,(0, 0, \ldots, 0, \star)\,; \tag{57}$$

$$D_i = \mathrm{diag}\,(\star, \star, \ldots, \star, 0)\,, \qquad \Lambda_i = \mathrm{diag}\,(0, \star, \ldots, \star, \star)\,; \tag{58}$$

$$D_i = \mathrm{diag}\,(0, 0, \ldots, 0, \star)\,, \qquad \Lambda_i = \mathrm{diag}\,(\star, \star, \ldots, \star, 0)\,; \tag{59}$$

$$D_i = \mathrm{diag}\,(0, \star, \ldots, \star, \star)\,, \qquad \Lambda_i = \mathrm{diag}\,(\star, \star, \ldots, \star, 0)\,; \tag{60}$$

$$D_i = \mathrm{diag}(0, 0, \ldots, 0, \star), \qquad \Lambda_i = \rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & -\frac{1}{\sqrt{2}} v_i^T \hat{Q}_i & \star \end{pmatrix}; \tag{61}$$

$$D_i = \rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & -\frac{1}{\sqrt{2}} u_i^T \hat{Q}_i & \star \end{pmatrix}, \qquad \Lambda_i = \mathrm{diag}(0, 0, \ldots, 0, \star); \tag{62}$$

$$D_i = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star I & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_i = \rho \begin{pmatrix} 1-\eta_{i1} & -\frac{1}{\sqrt{2}}\eta_i^T \hat{Q}_i & 0 \\ \frac{\sigma-1}{\sqrt{2}}\hat{Q}_i^T \eta_i & (1-(1-\sigma)\eta_{i1})I & \frac{\sigma-1}{\sqrt{2}}\hat{Q}_i^T \eta_i \\ 0 & 0 & 1 \end{pmatrix} \tag{63}$$

where $\sigma \in [0, 1/2]$ *and* $\tilde{D}_i^{-1}\tilde{\Lambda}_i$ *is positive semidefinite*;

$$D_i = \begin{pmatrix} 1-\xi_{i1} & -\frac{1}{\sqrt{2}}\xi_i^T \hat{Q}_i & 0 \\ \frac{\sigma-1}{\sqrt{2}}\hat{Q}_i^T \xi_i & (1-(1-\sigma)\xi_{i1})I & \frac{\sigma-1}{\sqrt{2}}\hat{Q}_i^T \xi_i \\ 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_i = \begin{pmatrix} \star & 0 & 0 \\ 0 & \star I & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{64}$$

where $\sigma \in [0, 1/2]$ *and* $\tilde{\Lambda}_i^{-1}\tilde{D}_i$ *is positive semidefinite*;

$$D_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \star I & 0 \\ 0 & 0 & \star \end{pmatrix}, \qquad \Lambda_i = \rho \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\sigma}{\sqrt{2}}\hat{Q}_i^T v_i & \star I & \frac{-\sigma}{\sqrt{2}}\hat{Q}_i^T v_i \\ 0 & \frac{-1}{\sqrt{2}}v_i^T \hat{Q}_i & \star \end{pmatrix} \tag{65}$$

where $\sigma \in (0, 1/2]$ *and* $\bar{D}_i^{-1}\bar{\Lambda}_i$ *is positive semidefinite*;

$$D_i = \rho \begin{pmatrix} 1 & 0 & 0 \\ \frac{-\sigma}{\sqrt{2}}\hat{Q}_i^T u_i & \star I & \frac{-\sigma}{\sqrt{2}}\hat{Q}_i^T u_i \\ 0 & \frac{-1}{\sqrt{2}}u_i^T \hat{Q}_i & \star \end{pmatrix}, \qquad \Lambda_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \star I & 0 \\ 0 & 0 & \star \end{pmatrix} \tag{66}$$

where $\sigma \in (0, 1/2]$ *and* $\bar{\Lambda}_i^{-1}\bar{D}_i$ *is positive semidefinite*;

$$D_i = \rho \begin{pmatrix} 1-\xi_{i1} & \frac{-1}{\sqrt{2}}\xi_i^T \hat{Q}_i & 0 \\ \frac{-1}{\sqrt{2}}\hat{Q}_i^T s_i & (1-s_{i1})I & \frac{-1}{\sqrt{2}}\hat{Q}_i^T s_i \\ 0 & \frac{-1}{\sqrt{2}}u_i^T \hat{Q}_i & 1-u_{i1} \end{pmatrix}, \quad \Lambda_i = \rho \begin{pmatrix} 1-\eta_{i1} & \frac{-1}{\sqrt{2}}\eta_i^T \hat{Q}_i & 0 \\ \frac{-1}{\sqrt{2}}\hat{Q}_i^T \omega_i & (1-\omega_{i1})I & \frac{-1}{\sqrt{2}}\hat{Q}_i^T \omega_i \\ 0 & \frac{-1}{\sqrt{2}}v_i^T \hat{Q}_i & 1-v_{i1} \end{pmatrix} \tag{67}$$

where $u_{i1} < 1$, $v_{i1} < 1$, $\xi_{i1} < 1$ *and* $\eta_{i1} < 1$.

By Proposition 3.4, we may partition the index set J_{00} as $J_{00} = J_{00}^1 \cup \cdots \cup J_{00}^{18}$ with

$$
\begin{aligned}
J_{00}^1 &:= \{i \mid U_i = D_i,\ V_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (46) or (55)}\},\\
J_{00}^2 &:= \{i \mid U_i = D_i,\ V_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (47) or (56)}\},\\
J_{00}^3 &:= \{i \mid U_i = D_i,\ V_i = \Lambda_i \text{ with } D_i, \Lambda_i \text{ given by (48) or (54)}\},\\
J_{00}^4 &:= \left\{i \mid U_i = Q_i D_i Q_i^T, V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (49)}\right\},\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
J_{00}^7 &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (52)}\right\},\\
J_{00}^8 &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (53) or (60)}\right\},\\
J_{00}^9 &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (57)}\right\},\\
J_{00}^{10} &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (58)}\right\},\\
J_{00}^{11} &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (59)}\right\},\\
J_{00}^{12} &:= \left\{i \mid U_i = Q_i D_i Q_i^T,\ V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (61)}\right\},\\
&\ \ \vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
J_{00}^{18} &:= \left\{i \mid U_i = Q_i D_i Q_i^T, V_i = Q_i \Lambda_i Q_i^T \text{ with } D_i, \Lambda_i \text{ given by (67)}\right\}.
\end{aligned}
$$

Taking account into the structure of D_i and Λ_i in Propositions 3.2–3.4, in the subsequent section we sometimes partition the corresponding Q_i as

$$
Q_i = [\tilde{Q}_i \quad q_i'] = [q_i \quad \hat{Q}_i \quad q_i'] = [q_i \quad \bar{Q}_i], \qquad i = 1, 2, \ldots, r
$$

where $q_i, q_i' \in \mathbb{R}^{n_i}$ denote the first column and the last column of Q_i, respectively, $\hat{Q}_i \in \mathbb{R}^{n_i \times (n_i - 2)}$ is composed of the middle $n_i - 2$ columns of Q_i, and $\tilde{Q}_i, \bar{Q}_i \in \mathbb{R}^{n_i \times (n_i - 1)}$ are composed of the first $n_i - 1$ columns and the last $n_i - 1$ columns of Q_i, respectively.

4 Nonsingularity conditions

In this section, we give suitable conditions to guarantee the nonsingularity of all elements of the B-subdifferential of Φ_ρ at solutions which do not necessarily satisfy strict complementarity. The following technical lemma will play an important role in the analysis.

Lemma 4.1 *Let $A_1, A_2 \in \mathbb{R}^{n\times n}$ be given. Let $V^a, V^b \in \mathbb{R}^{n\times n}$ be matrices such that there is an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and block diagonal matrices $D^a = \mathrm{diag}(D_1^a, \dots, D_k^a)$ and $D^b = \mathrm{diag}(D_1^b, \dots, D_k^b)$ satisfying $V^a = QD^aQ^T$ and $V^b = QD^bQ^T$, where each pair of D_i^a and D_i^b belongs to one of the following cases: $D_i^a = 0$ and D_i^b is nonsingular; D_i^a, D_i^b are nonsingular and $(D_i^b)^{-1}D_i^a$, $(D_i^a)^{-1}D_i^b$ are positive definite; D_i^a is nonsingular and $D_i^b = 0$; D_i^a is nonsingular and $(D_i^a)^{-1}D_i^b$ is positive semidefinite; D_i^b is nonsingular and $(D_i^b)^{-1}D_i^a$ is positive semidefinite. Let the set $\{1, 2, \dots, k\}$ be partitioned as $\alpha \cup \beta \cup \gamma \cup \delta \cup \theta$ with*

$$\alpha := \{i \mid D_i^a = 0,\ D_i^b \text{ is nonsingular}\}, \qquad \gamma := \{i \mid D_i^a \text{ is nonsingular},\ D_i^b = 0\},$$

$$\beta := \{i \mid D_i^a, D_i^b \text{ are nonsingular and } (D_i^a)^{-1}D_i^b, (D_i^b)^{-1}D_i^a \text{ are positive definite}\},$$

$$\delta := \{i \mid D_i^a \text{ is nonsingular}, (D_i^a)^{-1}D_i^b \text{ is positive semidefinite}\},$$

$$\theta := \{i \mid D_i^b \text{ is nonsingular}, (D_i^b)^{-1}D_i^a \text{ is positive semidefinite}\},$$

and $Q_\alpha, Q_\beta, Q_\gamma, Q_\delta$ and Q_θ denote the submatrices consisting of those columns from Q corresponding to $\alpha, \beta, \gamma, \delta$ and θ, respectively. Assume that the following conditions hold:

(a) *The matrix $A_1^T A_2$ is positive semidefinite.*
(b) *The matrix $[A_2^T\ Q_\alpha\ A_2^T\ Q_\beta]$ has full row rank.*

Then the matrix $V^aA_1 + V^bA_2$ is nonsingular. If, in addition, A_2 is invertible, then the matrix $V^aA_1 + V^bA_2$ is nonsingular under the following conditions:

(a1) *The matrix $[Q_\beta\ Q_\gamma\ Q_\delta\ Q_\theta]^T\ A_1A_2^{-1}[Q_\beta\ Q_\gamma\ Q_\delta\ Q_\theta]$ is positive semidefinite.*
(b1) *The matrix $[Q_\gamma\ Q_\delta\ Q_\theta]^T\ A_1A_2^{-1}[Q_\gamma\ Q_\delta\ Q_\theta]$ is positive definite.*

Proof Note that the nonsingularity of $V^aA_1 + V^bA_2$ is equivalent to that of the matrix

$$W := \begin{pmatrix} -I & A_1 & 0 \\ 0 & A_2 & -I \\ V^a & 0 & V^b \end{pmatrix}.$$

An elementary calculation shows that W is nonsingular if and only if the matrix

$$\widetilde{W} := \begin{pmatrix} -I & Q^TA_1 & 0 \\ 0 & A_2 & -Q \\ D^a & 0 & D^b \end{pmatrix}$$

is nonsingular. Let $\widetilde{W}z = 0$ for a suitably partitioned vector $z = (w, d, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ with $w = (w_\alpha, w_\beta, w_\gamma, w_\delta, w_\theta)$ and $s = (s_\alpha, s_\beta, s_\gamma, s_\delta, s_\theta)$. It suffices to prove $z = 0$. Since

$$D^a = \mathrm{diag}\left(0, D_\beta^a, D_\gamma^a, D_\delta^a, D_\theta^a\right) \quad \text{and} \quad D^b = \mathrm{diag}\left(D_\alpha^b, D_\beta^b, 0, D_\delta^b, D_\theta^b\right),$$

the system $\widetilde{W}z = 0$ can be rewritten as

$$\begin{aligned}
Q^T A_1 d = w, \qquad & A_2 d = Qs, \\
w_\gamma = 0, \qquad & s_\alpha = 0, \\
w_\beta + \left(D_\beta^a\right)^{-1} D_\beta^b s_\beta = 0,& \\
w_\delta + \left(D_\delta^a\right)^{-1} D_\delta^b s_\delta = 0,& \\
\left(D_\theta^b\right)^{-1} D_\theta^a w_\theta + s_\theta = 0.&
\end{aligned} \tag{68}$$

From the first two equations of (68), $d^T A_1^T A_2 d - s^T w = 0$, which can be rewritten as

$$d^T A_1^T A_2 d + s_\beta^T \left(D_\beta^a\right)^{-1} D_\beta^b s_\beta + s_\delta^T \left(D_\delta^a\right)^{-1} D_\delta^b s_\delta + w_\theta^T \left(D_\theta^b\right)^{-1} D_\theta^a w_\theta = 0.$$

Since $(D_\beta^a)^{-1} D_\beta^b$ is positive definite, $(D_\delta^a)^{-1} D_\delta^b$ and $(D_\theta^b)^{-1} D_\theta^a$ are positive semidefinite, the second term on the left hand side of last equality is strictly positive whenever $s_\beta \neq 0$ and the last two terms are always nonnegative. Therefore, assumption (a) implies $s_\beta = 0$. Using $s_\alpha = 0$, $s_\beta = 0$ and $A_2 d = Qs$, we obtain $Q_\alpha^T A_2 d = 0$ and $Q_\beta^T A_2 d = 0$, which by assumption (b) implies $d = 0$. Hence, $w = 0$ and $s = 0$ follow from the first equation and the second equation of (68), respectively. Thus, we prove $z = 0$.

Suppose that A_2 is invertible. From the first two equations in (68), it follows that

$$Q^T A_1 A_2^{-1} Q s - w = 0.$$

Premultiplying with s^T and using $w_\gamma = 0$ and $s_\alpha = 0$, it then follows that

$$\begin{pmatrix} s_\beta \\ s_\gamma \\ s_\delta \\ s_\theta \end{pmatrix}^T \begin{bmatrix} Q_\beta & Q_\gamma & Q_\delta & Q_\theta \end{bmatrix}^T A_1 A_2^{-1} \begin{bmatrix} Q_\beta & Q_\gamma & Q_\delta & Q_\theta \end{bmatrix} \begin{pmatrix} s_\beta \\ s_\gamma \\ s_\delta \\ s_\theta \end{pmatrix}$$
$$- s_\beta^T w_\beta - s_\delta^T w_\delta - s_\theta^T w_\theta = 0,$$

which along with the last four equations of (68) yields

$$\begin{pmatrix} s_\beta \\ s_\gamma \\ s_\delta \\ s_\theta \end{pmatrix}^T \begin{bmatrix} Q_\beta & Q_\gamma & Q_\delta & Q_\theta \end{bmatrix}^T A_1 A_2^{-1} \begin{bmatrix} Q_\beta & Q_\gamma & Q_\delta & Q_\theta \end{bmatrix} \begin{pmatrix} s_\beta \\ s_\gamma \\ s_\delta \\ s_\theta \end{pmatrix}$$
$$+ s_\beta^T \left(D_\beta^a\right)^{-1} D_\beta^b s_\beta + s_\delta^T \left(D_\delta^a\right)^{-1} D_\delta^b s_\delta + w_\theta^T \left(D_\theta^b\right)^{-1} D_\theta^a w_\theta = 0. \tag{69}$$

Since $(D_\beta^a)^{-1} D_\beta^b$ is positive definite, $(D_\delta^a)^{-1} D_\delta^b$ and $(D_\theta^b)^{-1} D_\theta^a$ are positive semidefinite, assumption (a1) implies $s_\beta = 0$. Plugging $s_\beta = 0$ into (69) and using assumption (b1), we obtain $s_\gamma = 0$, $s_\delta = 0$ and $s_\theta = 0$, and so $s = 0$. From $A_2 d = Qs$ and the nonsingularity of A_2, we get $d = 0$, and $w = 0$ follows by the first equation of (68). Thus, $z = 0$. □

In what follows, we employ Lemma 4.1 to establish the nonsingularity result for the B-subdifferential of Φ_ρ at a solution ζ^*. We first consider that ζ^* is nondegenerate, i.e.,

$$(F_i(\zeta^*), G_i(\zeta^*)) \neq (0,0) \quad \text{for all } i = 1, 2, \ldots, r.$$

Obviously, such a solution does not necessarily satisfy strict complementarity which requires

$$F_i(\zeta^*) + G_i(\zeta^*) \in \operatorname{int}(\mathcal{K}^{n_i}) \quad \text{for all } i = 1, 2, \ldots, r.$$

Theorem 4.1 *Let ζ^* be a nondegenerate solution of (1), and $J_I, J_B, J_0, J_{B0}, J_{0B}$ be defined by (17) with $x_i^* = F_i(\zeta^*)$ and $y_i^* = G_i(\zeta^*)$. Then all matrices $W \in \partial_B \Phi_\rho(\zeta^*)$ are nonsingular if, for the partitions $J_I = J_I^1 \cup J_I^2$, $J_0 = J_0^1 \cup J_0^2$, $J_B = J_B^1 \cup J_B^2$, $J_{B0} = J_{B0}^1 \cup \cdots \cup J_{B0}^4$ and $J_{0B} = J_{0B}^1 \cup \cdots \cup J_{0B}^4$, the following two conditions hold:*

(a) *The matrix $\nabla F(\zeta^*) G'(\zeta^*) \in \mathbb{R}^{n\times n}$ is positive semidefinite.*
(b) *The matrix $[EQ_\alpha \;\; EQ_\beta] \in \mathbb{R}^{n\times(N_1+N_2)}$ has full row rank, where $E = \nabla G(\zeta^*)$ and*

$$N_1 := \sum_{i\in J_I \cup J_{B0}^1 \cup J_{B0}^2} n_i + \sum_{i \in J_{B0}^3 \cup J_{B0}^4} (n_i - 1) + |J_B \cup J_{0B}^3|,$$

$$N_2 := \sum_{i\in J_B} (n_i - 2) + |J_{B0}^4 \cup J_{0B}^4|,$$

$$\begin{aligned} EQ_\alpha := \big[& E_i (i \in J_I^1 \cup J_{B0}^1) \quad E_i Q_i (i \in J_I^2 \cup J_{B0}^2) \quad E_i \tilde{Q}_i (i \in J_{B0}^3 \cup J_{B0}^4) \\ & E_i q_i (i \in J_B^2) \quad E_i q_i' (i \in J_B^1 \cup J_{0B}^3) \big], \\ EQ_\beta := \Big[& E_i \hat{Q}_i (i \in J_B) \quad E_i q_i' (i \in J_{B0}^4 \cup J_{0B}^4) \Big]. \end{aligned}$$

If $G'(\zeta^*)$ is invertible, then all $W \in \partial_B \Phi_\rho(\zeta^*)$ are nonsingular under the conditions that

(a1) *the block matrix $\left[\begin{smallmatrix} C_1 & C_2 \\ C_3 & C_4 \end{smallmatrix}\right] \in \mathbb{R}^{(N_2+N_3)\times(N_2+N_3)}$ is positive semidefinite with*

$$N_3 := \sum_{i \in J_0 \cup J_{0B}^1 \cup J_{0B}^2} n_i + \sum_{i\in J_{0B}^3 \cup J_{0B}^4} (n_i - 1) + |J_B \cup J_{B0}^3|,$$

where the expressions of C_1, C_2, C_3 and C_4 are given in appendix;
(b1) *the matrix $C_4 \in \mathbb{R}^{N_3 \times N_3}$ is positive definite.*

Proof Choose $W \in \partial_B \Phi_\rho(\zeta^*)$ arbitrarily. A calculation gives $W = SF'(\zeta^*) + TG'(\zeta^*)$ for suitable block diagonal matrices $S = \operatorname{diag}(S_1, \ldots, S_r)$ and $T = \operatorname{diag}(T_1, \ldots, T_r)$ with $[S_i \;\; T_i] \in \partial_B \phi_\rho(x_i^*, y_i^*)$ for $i = 1, 2, \ldots, r$. Since ζ^* is a nondegenerate solution, the index set $\{1, 2, \ldots, r\}$ can be partitioned as $J_I \cup J_B \cup J_0 \cup J_{B0} \cup J_{0B}$. By Propositions 3.2–3.3, there exists an orthogonal matrix $Q = \operatorname{diag}(Q_1, \ldots, Q_r) \in \mathbb{R}^{n\times n}$ such that $S = QDQ^T$ and $T = Q\Lambda Q^T$ with the block diagonals $D = \operatorname{diag}(D_1, \ldots, D_r)$ and $\Lambda = \operatorname{diag}(\Lambda_1, \ldots, \Lambda_r)$, and we are in a position to apply Lemma 4.1 with $D^a = D$, $D^b = \Lambda$ and $A_1 = F'(\zeta^*)$, $A_2 = G'(\zeta^*)$. To apply this result, we need identify the index sets α, β and γ since $\delta = \emptyset$ and $\theta = \emptyset$, and the structure of the orthogonal matrix Q. From Propositions 3.2–3.3 and the partition of $J_I, J_B, J_0, J_{B0}, J_{0B}$, we see that the following indices belong to α of Lemma 4.1:

- every block index $i \in J_I^1 \cup J_I^2 \cup J_{B0}^1 \cup J_{B0}^2$, with $Q_i = I$ for $i \in J_I^1 \cup J_{B0}^1$ and Q_i for $i \in J_I^2 \cup J_{B0}^2$ being the corresponding orthogonal matrix;
- every block index $i \in J_{B0}^3 \cup J_{B0}^4$, with $\tilde{Q}_i$ consisting of the first $n_i - 1$ columns of the corresponding orthogonal matrix Q_i;
- each block index $i \in J_B^2$, with q_i being the first column of the corresponding Q_i;
- each block index $i \in J_B^1 \cup J_{0B}^3$, with q_i' being the last column of the corresponding Q_i;

and the following indices belong to β of Lemma 4.1:

- every block index $i \in J_B$, with $\hat{Q}_i$ consisting of the middle $n_i - 2$ columns of the corresponding orthogonal matrix Q_i;
- every block index $i \in J_{B0}^4 \cup J_{0B}^4$, with q_i' being the last column of the corresponding orthogonal matrix Q_i;

and the following indices belong to γ of Lemma 4.1:

- every block index $i \in J_0^1 \cup J_0^2 \cup J_{0B}^1 \cup J_{0B}^2$, with $Q_i = I$ for $i \in J_0^1 \cup J_{0B}^1$ and Q_i for $i \in J_0^2 \cup J_{0B}^2$ being the corresponding orthogonal matrix;
- every block index $i \in J_{0B}^3 \cup J_{0B}^4$, with $\tilde{Q}_i$ consisting of the first $n_i - 1$ columns of the corresponding orthogonal matrix Q_i;
- each block index $i \in J_B^1$, with q_i being the first column of the corresponding Q_i;
- each block index $i \in J_B^2 \cup J_{B0}^3$, with q_i' being the last column of the corresponding Q_i.

The above observations show that the conditions (a)–(b) correspond to the conditions (a)–(b), respectively, of Lemma 4.1. When $G'(\zeta^*)$ is invertible, the expressions of C_1, C_2, C_3 and C_4 in Appendix show that the conditions (a1)–(b1) correspond to the conditions (a1)–(b1), respectively, of Lemma 4.1. The result then follows from Lemma 4.1. □

Next, we come to the case that ζ^* is degenerate. For such ζ^*, we can establish the nonsingularity for the B-subdifferential $\partial_B \Phi_\rho(\zeta^*)$ under the requirement of $J_{00}^{18} = \emptyset$. From Proposition 4.3(b9) of [20], this condition can be satisfied only when one of u_{i1}, v_{i1}, ξ_{i1} and η_{i1} equals 1. In other words, this condition can be satisfied only for some elements of the B-subdifferential $\partial_B \Phi_\rho(\zeta^*)$. Since the proof is similar to that of Theorem 4.1, we omit it.

Theorem 4.2 *Let ζ^* be an arbitrary solution of* (1). *Let $J_I, J_B, J_0, J_{B0}, J_{0B}$ and J_{00} be given by* (17) *with $x_i^* = F_i(\zeta^*)$, $y_i^* = G_i(\zeta^*)$, and the partitions $J_I = J_I^1 \cup J_I^2$, $J_0 = J_0^1 \cup J_0^2$, $J_B = J_B^1 \cup J_B^2$, $J_{B0} = J_{B0}^1 \cup \cdots \cup J_{B0}^4$, $J_{0B} = J_{0B}^1 \cup \cdots \cup J_{0B}^4$ and $J_{00} = J_{00}^1 \cup \cdots \cup J_{00}^{18}$. If $J_{00}^{18} = \emptyset$, then all $W \in \partial_B \Phi_\rho(\zeta^*)$ are nonsingular under the following two conditions*:

(a) *The matrix* $\nabla F(\zeta^*)G'(\zeta^*) \in \mathbb{R}^{n\times n}$ *is positive semidefinite.*
(b) *The matrix* $[EQ_\alpha \ \ EQ_\beta] \in \mathbb{R}^{n\times(N_1+N_2)}$ *has full row rank, where* $E = \nabla G(\zeta^*)$ *and*

$$N_1 := \sum_{i\in J_I\cup J_{B0}^1\cup J_{B0}^2\cup J_{00}^1} n_i + \sum_{i\in J_{B0}^3\cup J_{B0}^4\cup J_{00}^{11}\cup J_{00}^{12}} (n_i-1) + |J_B \cup J_{0B}^3 \cup J_{00}^6 \cup J_{00}^8 \cup J_{00}^9 \cup J_{00}^{10} \cup J_{00}^{14} \cup J_{00}^{16}|,$$

$$N_2 := \sum_{i\in J_{00}^3\cup J_{00}^4} n_i + \sum_{i\in J_{00}^5\cup J_{00}^6\cup J_{00}^7} (n_i-1) + \sum_{i\in J_B\cup J_{00}^8\cup J_{00}^{10}} (n_i-2) + |J_{B0}^4 \cup J_{0B}^4 \cup J_{00}^{12} \cup J_{00}^{13}|,$$

$$EQ_\alpha := \big[E_i(i\in J_I^1\cup J_{B0}^1\cup J_{00}^1) \quad E_iQ_i(i\in J_I^2\cup J_{B0}^2) \quad E_i\tilde{Q}_i(i\in J_{B0}^3\cup J_{B0}^4\cup J_{00}^{11}\cup J_{00}^{12}) \quad E_iq_i(i\in J_B^2\cup J_{00}^6\cup J_{00}^8\cup J_{00}^{16}) \quad E_iq_i'(i\in J_B^1\cup J_{0B}^3\cup J_{00}^9\cup J_{00}^{10}\cup J_{00}^{14})\big],$$

$$EQ_\beta := \big[E_i(i\in J_{00}^3) \quad E_iQ_i(i\in J_{00}^4) \quad E_i\tilde{Q}_i(i\in J_{00}^7) \quad E_i\hat{Q}_i(i\in J_B\cup J_{00}^8\cup J_{00}^{10}) \quad E_i\bar{Q}_i(i\in J_{00}^5\cup J_{00}^6) \quad E_iq_i'(i\in J_{B0}^4\cup J_{0B}^4\cup J_{00}^{12}\cup J_{00}^{13})\big].$$

If $G'(\zeta^*)$ *is invertible, then all* $W \in \partial_B\Phi_\rho(\zeta^*)$ *are nonsingular under the conditions:*

(a1) *The block matrix*

$$\begin{bmatrix} A_1 & C_2 & C_3 & C_4 \\ B_2 & A_2 & C_5 & C_6 \\ B_3 & B_4 & A_3 & C_7 \\ B_5 & B_6 & B_7 & A_4 \end{bmatrix} \in \mathbb{R}^{N_3\times N_3}$$

is positive semidefinite with

$$N_3 := N_2 + \sum_{i\in J_0\cup J_{0B}^1\cup J_{0B}^2\cup J_{00}^2} n_i + \sum_{i\in J_{0B}^3\cup J_{0B}^4\cup J_{00}^9\cup J_{00}^{13}} (n_i-1) + \sum_{i\in J_{00}^{14}\cup\cdots\cup J_{00}^{17}} (n_i-1) + |J_B \cup J_{B0}^3 \cup J_{00}^5 \cup J_{00}^7 \cup J_{00}^8 \cup J_{00}^{10} \cup J_{00}^{11} \cup J_{00}^{15} \cup J_{00}^{17}|,$$

where the expressions of the submatrices A_i, B_i *and* C_i *are given in Appendix.*

(b1) *The block matrix*

$$\begin{bmatrix} A_2 & C_5 & C_6 \\ B_4 & A_3 & C_7 \\ B_6 & B_7 & A_4 \end{bmatrix} \in \mathbb{R}^{(N_3-N_2)\times(N_3-N_2)}$$

is positive definite.

Note that condition (a1) of Theorem 4.2 is satisfied when $F'(\zeta^*)G'(\zeta^*)^{-1}$ is positive semidefinite. For the SOCCP (2), this condition is equivalent to the monotonicity of the mapping F, and moreover, conditions (a1) and (b1) of Theorem 4.2 are satisfied if $F'(\zeta^*)$ is positive definite. In addition, we want to point out that it is impossible for the SOCCP reformulation of the linear SOCPs to satisfy conditions (a1) and (b1) of Theorem 4.2, since $G'(\zeta) = -A^T(AA^T)^{-1}A$ is not invertible. However, such SOCCPs always satisfy condition (a) of Theorem 4.2 since $\nabla F(\zeta^*)G'(\zeta^*)$ is a zero matrix. Of course, for the linear and nonlinear SOCPs, we are able to deal with their KKT conditions directly via the penalized FB function, i.e., we apply the nonsmooth Newton methods [24, 25] to

$$\bar{\Phi}_\rho(z) := \begin{pmatrix} \nabla g(x) - A^T\xi - y \\ Ax - b \\ \phi_\rho(x_1, y_1) \\ \vdots \\ \phi_\rho(x_r, y_r) \end{pmatrix} = 0. \tag{70}$$

Using Propositions 3.2–3.4 and following the arguments in [16] and [20], similar conditions can be provided for the nonsingularity of all elements of $\partial_B\bar{\Phi}_\rho(z)$ at KKT points.

Based on Theorem 4.2 and Proposition 2.1(c), from [25] we get the following result.

Theorem 4.3 *Let ζ^* be a (not necessarily strict complementary) solution of (1). Suppose that the assumptions of Theorem 4.2 hold at ζ^*. Then, the nonsmooth Newton method (9) applied to the system $\Phi_\rho(\zeta) = 0$ is locally superlinearly convergent. If, in addition, F' and G' are locally Lipschitz continuous, then it is quadratically convergent.*

5 The penalized FB merit function

The last section shows that the penalized FB SOC complementarity function ϕ_ρ inherits the desirable properties of the FB SOC complementarity function for local convergence. But, unlike its counterpart in the NCP setting, the squared norm of ϕ_ρ, i.e. the penalized FB merit function ψ_ρ, is not continuously differentiable; see the counterexample below.

Example 5.1 Consider the point $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $x = (2, -2, 0)^T$ and $y = (2, 0, 2)^T$. Since $x^2 + y^2 \in \text{int}(\mathcal{K}^3)$, ϕ_{FB} is continuously differentiable at such point. This means that

$$\partial_B\phi_\rho(x, y) = \partial_B\phi_{\text{FB}}(x, y) + \partial_B[(x)_+ \circ (y)_+].$$

By the proof of Proposition 3.1, $\partial_B[(x)_+ \circ (y)_+] = L_{(y)_+}\partial_B(x)_+ + L_{(x)_+}\partial_B(y)_+$. Noting that $x_1 = \|x_2\|$, $y_1 = \|y_2\|$ and $x_2^T y_2 \neq \pm\|x_2\|\|y_2\|$, we can verify that $\partial_B[(x)_+ \circ (y)_+]^T\phi_\rho(x, y)$ contains more than one element by Lemma 3.2. This along with the last equality implies that $\partial\phi_\rho(x,y)^T\phi_\rho(x,y)$ contains more than one element, and so does $\partial\psi_\rho(x,y)$ since

$$\partial\psi_\rho(x,y)^T = \partial\phi_\rho(x,y)^T\phi_\rho(x,y) \supset \partial_B\phi_\rho(x,y)^T\phi_\rho(x,y).$$

Applying Proposition 2.2.4 of [8] shows that ψ_ρ is not strictly differentiable at (x,y), and consequently ψ_ρ is not continuously differentiable at such point.

Next, we argue that the function Ψ_ρ has bounded level sets under a very weak condition.

Condition 5.1 *For any sequence $\{\zeta^k\} \subset \mathbb{R}^n$ such that $\|\zeta^k\| \to +\infty$, $\|[F(\zeta^k)]_-\| < +\infty$ and $\|[G(\zeta^k)]_-\| < +\infty$, there holds that*

$$\limsup_{k\to\infty} \max_{1\le i\le r} \left\langle [F_i(\zeta^k)]_+,\ [G_i(\zeta^k)]_+ \right\rangle = +\infty. \tag{71}$$

Proposition 5.1 *If the mappings $F, G : \mathbb{R}^n \to \mathbb{R}^n$ satisfy Condition 5.1, then the level sets $L_\gamma(\zeta) := \{\zeta \in \mathbb{R}^n \mid \Psi_\rho(\zeta) \le \gamma\}$ are bounded for all $\gamma \ge 0$.*

Proof Assume on the contrary that there is an unbounded sequence $\{\zeta^k\} \subseteq L_\gamma(\zeta)$ for some $\gamma \ge 0$. Since $\Psi_\rho(\zeta^k) \le \gamma$ for each k, we have $\|[F(\zeta^k)]_-\| < +\infty$ and $\|[G(\zeta^k)]_-\| < +\infty$ from Lemma 2.2. By Condition 5.1, there exists a subsequence $\{\zeta^k\}_{k\in\hat{K}}$ such that

$$\left\langle [F_\nu(\zeta^k)]_+,\ [G_\nu(\zeta^k)]_+ \right\rangle_{k\in\hat{K}} \to +\infty \tag{72}$$

for some $\nu \in \{1, 2, \ldots, r\}$. In addition, from Lemma 2.2 it follows that for each k,

$$\left[\phi_{\rm FB}(F_\nu(\zeta^k), G_\nu(\zeta^k))\right]_1 \ge -2\left(\|[F_\nu(\zeta^k)]_-\| + \|[G_\nu(\zeta^k)]_-\|\right) > -\infty.$$

Combining the last two equations with the following inequality

$$\begin{aligned}
\|\phi_\rho(F_\nu(\zeta^k), G_\nu(\zeta^k))\| &\ge \Big|\rho\left[\phi_{\rm FB}(F_\nu(\zeta^k), G_\nu(\zeta^k))\right]_1 \\
&\qquad - (1-\rho)\langle [F_\nu(\zeta^k)]_+, [G_\nu(\zeta^k)]_+\rangle\Big| \\
&\ge (1-\rho)\langle [F_\nu(\zeta^k)]_+, [G_\nu(\zeta^k)]_+\rangle \\
&\qquad - \rho\left[\phi_{\rm FB}(F_\nu(\zeta^k), G_\nu(\zeta^k))\right]_1,
\end{aligned}$$

we get $\{\|\phi_\rho(F_\nu(\zeta^k), G_\nu(\zeta^k))\|\}_{k\in\hat{K}} \to +\infty$. This is a contradiction to $\{\zeta^k\} \subseteq L_\gamma(\zeta)$. □

As will be shown in the following proposition, Condition 5.1 is rather weak which can be satisfied by the strictly feasible monotone SOCCP or the SOCCP with the Cartesian R_{01}-property, or the SOCCP (2) with the Cartesian weak coercive property.

Proposition 5.2 *Condition 5.1 is satisfied if one of the following assumptions holds*:

(a) *The mappings F and G are jointly monotone with* $\lim_{\|\zeta\|\to\infty}\|F(\zeta)\|+\|G(\zeta)\|=+\infty$, *and there exists* $\bar\zeta\in\mathbb{R}^n$ *such that* $F(\bar\zeta),G(\bar\zeta)\in\operatorname{int}(\mathcal{K}^n)$;
(b) *The mappings F and G have the jointly Cartesian R_{01}-property.*
(c) *F has the Cartesian weak coercive property with respect to $\xi\in\mathcal{K}$ for the SOCCP (2).*

Proof Let $\{\zeta^k\}$ be such that $\|\zeta^k\|\to+\infty$, $\|[F(\zeta^k)]_-\|<+\infty$ and $\|[G(\zeta^k)]_-\|<+\infty$.

(a) From the joint monotonicity of the mappings F and G, it follows that for each k,

$$\langle F(\zeta^k),G(\bar\zeta)\rangle+\langle F(\bar\zeta),G(\zeta^k)\rangle\le\langle F(\zeta^k),G(\zeta^k)\rangle+\langle F(\bar\zeta),G(\bar\zeta)\rangle.$$

Since $\|[F(\zeta^k)]_-\|<+\infty$ and $\|[G(\zeta^k)]_-\|<+\infty$ imply the lower boundedness of $\{\lambda_1[F(\zeta^k)]\}$ and $\{\lambda_1[G(\zeta^k)]\}$, from the given condition $\|F(\zeta^k)\|+\|G(\zeta^k)\|\to+\infty$ we may deduce that $\lambda_2[F(\zeta^k)]\to+\infty$ or $\lambda_2[G(\zeta^k)]\to+\infty$. Using Lemma 9(b) of [6] then yields that

$$\langle F(\zeta^k),G(\bar\zeta)\rangle+\langle F(\bar\zeta),G(\zeta^k)\rangle\to+\infty.$$

The last two equations imply $\langle F(\zeta^k),G(\zeta^k)\rangle\to+\infty$. Now from the following inequality

$$\begin{aligned}\langle F(\zeta^k),G(\zeta^k)\rangle&=\langle [F(\zeta^k)]_++[F(\zeta^k)]_-,[G(\zeta^k)]_++[G(\zeta^k)]_-\rangle\\&\le\langle [F(\zeta^k)]_+,[G(\zeta^k)]_+\rangle+\langle [F(\zeta^k)]_-,[G(\zeta^k)]_-\rangle,\end{aligned}\tag{73}$$

it follows that $\langle [F(\zeta^k)]_+,[G(\zeta^k)]_+\rangle\to+\infty$, and inequality (71) then follows.

(b) The result is direct by Definition 2.1(a) and the following implications:

$$\begin{aligned}\liminf_{k\to\infty}\frac{\langle F_\nu(\zeta^k),G_\nu(\zeta^k)\rangle}{\|\zeta^k\|}>0\quad&\Longrightarrow\quad\liminf_{k\to\infty}\frac{\max_{1\le i\le r}\langle F_i(\zeta^k),G_i(\zeta^k)\rangle}{\|\zeta^k\|}>0\\&\Longrightarrow\quad\max_{1\le i\le r}\langle F_i(\zeta^k),G_i(\zeta^k)\rangle\to+\infty\\&\Longrightarrow\quad\max_{1\le i\le r}\langle [F_i(\zeta^k)]_+,[G_i(\zeta^k)]_+\rangle\to+\infty\end{aligned}$$

where the last implication holds due to (73) with F and G replaced by F_i and G_i, respectively, and the boundedness of $\langle [F_i(\zeta^k)]_-,[G_i(\zeta^k)]_-\rangle$.

(c) By Definition 2.1(b), there exists an index $\nu\in\{1,2,\ldots,r\}$ such that for each k

$$\begin{aligned}\frac{\langle\zeta_\nu^k,F_\nu(\zeta^k)\rangle}{\|\zeta^k\|}&=\left[\frac{\langle\zeta_\nu^k-\xi_\nu,F_\nu(\zeta^k)\rangle}{\|\zeta^k-\xi\|}+\frac{\langle\xi_\nu,F_\nu(\zeta^k)\rangle}{\|\zeta^k-\xi\|}\right]\frac{\|\zeta^k-\xi\|}{\|\zeta^k\|}\\&\ge\left[\frac{\langle\zeta_\nu^k-\xi_\nu,F_\nu(\zeta^k)\rangle}{\|\zeta^k-\xi\|}+\frac{\langle\xi_\nu,[F_\nu(\zeta^k)]_-\rangle}{\|\zeta^k-\xi\|}\right]\frac{\|\zeta^k-\xi\|}{\|\zeta^k\|},\end{aligned}\tag{74}$$

where the inequality is due to $F_\nu(\zeta^k) = [F_\nu(\zeta^k)]_+ + [F_\nu(\zeta^k)]_-$ and $\langle \xi_\nu, [F_\nu(\zeta^k)]_+\rangle \geq 0$. Since $\|[F_\nu(\zeta^k)]_-\| < +\infty$ and $\|\zeta^k\| \to +\infty$, it follows that

$$\lim_{k\to\infty} \frac{\langle \xi_\nu, [F_\nu(\zeta^k)]_-\rangle}{\|\zeta^k - \xi\|} = 0 \quad \text{and} \quad \lim_{k\to\infty} \frac{\|\zeta^k - \xi\|}{\|\zeta^k\|} = 1.$$

Therefore, from (74) we immediately obtain that

$$\begin{aligned}\liminf_{k\to\infty} \frac{\langle \zeta_\nu^k, F_\nu(\zeta^k)\rangle}{\|\zeta^k\|} &\geq \liminf_{k\to\infty} \left[\frac{\langle \zeta_\nu^k - \xi_\nu, F_\nu(\zeta^k)\rangle}{\|\zeta^k - \xi\|} + \frac{\langle \xi_\nu, [F_\nu(\zeta^k)]_-\rangle}{\|\zeta^k - \xi\|}\right] \frac{\|\zeta^k - \xi\|}{\|\zeta^k\|} \\ &\geq \liminf_{k\to\infty} \frac{\langle \zeta_\nu^k - \xi_\nu, F_\nu(\zeta^k)\rangle}{\|\zeta^k - \xi\|} \frac{\|\zeta^k - \xi\|}{\|\zeta^k\|} > 0.\end{aligned}$$

This implies that $\langle \zeta_\nu^k, F_\nu(\zeta^k)\rangle \to +\infty$, and then (71) follows. □

Proposition 5.2(a) and Proposition 5.1 indicate that Ψ_ρ possesses the nice features of the merit functions proposed by Luo and Tseng for the NCPs and extended to the SOCCP by Chen [5]. In addition, it is not hard to verify that the uniformly Cartesian P-property implies the Cartesian weak coercive property, and hence Proposition 5.2(b) is weak than the coerciveness condition of the FB merit function $\Psi_{\rm FB}$.

6 Numerical experiments

In this section, we apply the nonsmooth Newton method (9) for solving the SOCCP. Since the benchmark for the SOCCP is not available, we utilize the SOCCP reformulations of the standard SOCPs from DIMACS collection [23] as test examples, whose KKT conditions can be rewritten as the SOCCP (1) with F and G given by (4). Note that the method is only locally convergent in theory and the aim of our numerical experiments is to demonstrate the theoretical results in the previous sections by examining the local behavior of the method. Also, we compare the method with the following nonsmooth Newton method

$$z^{k+1} := z^k - W_k^{-1}\bar{\Phi}_\rho(z^k), \quad W_k \in \partial_B \bar{\Phi}_\rho(z^k), \quad k = 0, 1, 2, \ldots \tag{75}$$

where the mapping $\bar{\Phi}_\rho : \mathbb{R}^{n+m+n} \to \mathbb{R}^{n+m+n}$ is defined as in (70).

During our tests, the vector $\bar{x}$ in F was computed as a solution of $\min_x \|Ax - b\|$ with Matlab's least square solver, and F and G were evaluated via the Cholesky factorization of AA^T. All experiments were done with a PC of Pentium Dual CPU E2200 and 2047MB memory. The computer codes were written in Matlab 7.0. We started with the initial point $\zeta^0 = 0$ for the method (9), and $z^0 = (0, 0, 0)$ for the method (75). The two methods were terminated once $\|\Phi_\rho(\zeta^k)\|$ ($\|\bar{\Phi}_\rho(\zeta^k)\|$) is less than 10^{-9} or the number of iteration is over 150.

In addition, the parameter ρ is chosen as 0.9 throughout the testing. We want to point out that a smaller ρ is not advisable since the penalized FB nonsmooth method may suffer from the singularity of the B-subdifferential of Φ_ρ, noting that Proposition 3.1 and Lemma 3.2 imply that the B-subdifferential of Φ_ρ will be singular if

Table 1 Numerical results for the linear SOCP nb_L2_bessel

Nonsmooth Newton method (9)		Nonsmooth Newton method (75)			
Iter	$\Phi_\rho(\zeta^k)$	Iter	$\bar{\Phi}_\rho(z^k)$	$\\|Ax-b\\|$	$\phi_\rho(x^k, y^k)$
1	2.829000e+1	1	3.866252e+0	7.090517e−15	3.866252e+0
2	3.307341e−1	2	1.325962e+0	1.344775e−14	1.325962e+0
3	1.856300e−1	3	4.731416e−1	8.116507e−15	4.731416e−1
4	2.440313e−1	4	1.024512e+0	5.736403e−15	1.024512e+0
5	3.396525e−2	5	9.246263e−1	4.138420e−15	9.246263e−1
6	1.380350e−2	6	5.472736e−1	4.394618e−15	5.472736e−1
7	4.972790e−3	7	4.559772e−1	3.282193e−15	4.559772e−1
8	6.126294e−4	8	8.593337e−1	2.747643e−15	8.593337e−1
9	1.818880e−5	9	6.716963e−2	2.211445e−15	6.716963e−2
10	1.815863e−8	10	1.147488e−2	1.771333e−15	1.147488e−2
11	1.922856e−14	11	4.199352e−4	1.516066e−15	4.199352e−4
		12	6.516191e−7	1.450563e−15	6.516191e−7
		13	1.599536e−12	8.048233e−15	1.599514e−12

Table 2 The last ten iterations for the linear SOCP nb

Nonsmooth Newton method (9)		Nonsmooth Newton method (75)			
Iter	$\Phi_\rho(\zeta^k)$	Iter	$\bar{\Phi}_\rho(z^k)$	$\\|Ax-b\\|$	$\phi_\rho(x^k, y^k)$
52	6.303561e−7	66	8.151580e−5	2.861723e−15	8.151580e−5
53	8.390756e−6	67	1.643593e−5	3.172674e−15	1.643593e−5
54	2.759474e−6	68	7.686683e−4	9.392922e−16	7.686683e−4
55	1.677546e−4	69	1.510727e−3	1.259316e−15	1.510727e−3
56	8.744115e−7	70	1.195072e−7	9.559855e−16	1.195072e−7
57	2.331141e−7	71	3.457169e−6	1.226556e−15	3.457169e−6
58	6.007356e−8	72	1.090880e−8	7.028293e−16	1.090880e−8
59	1.545525e−8	73	3.070186e−9	8.435887e−16	3.070186e−9
60	4.005365e−9	74	3.533215e−8	1.448009e−15	3.533215e−8
61	9.720731e−10	75	7.379139e−10	1.133837e−15	7.379139e−10

$\rho = 0$. In fact, such problem also exists for the penalized natural residual SOC complementarity function in [7].

Table 1 reports the iterations of the nonsmooth methods (9) and (75) for **nb_L2_bessel**. We see that the two methods exhibit nice quadratic convergence for this example, and yield the result within 13 iterations. Table 2 lists the last ten iterations of the two methods for problem **nb**. They are able to yield the desired results for this example within 75 iterations. Table 3 indicates the two methods can not give the result for **nb_L1** within 150 iterations. We have checked that the solutions of the three linear SOCPs do not satisfy strict complementarity. So, the numerical results above illustrate that the two nonsmooth methods may have fast local convergence even if the solutions do not satisfy strict complementarity. In addition, it is worth

Table 3 Ten iterations for the linear SOCP nb_L1

Nonsmooth Newton method (9)		Nonsmooth Newton method (75)			
Iter	$\Phi_\rho(\zeta^k)$	Iter	$\bar{\Phi}_\rho(z^k)$	$\|Ax-b\|$	$\phi_\rho(x^k, y^k)$
141	3.747671e−4	141	9.413686e−4	5.298547e−13	9.413686e−4
142	6.054937e−5	142	2.563409e−4	4.846449e−13	2.563409e−4
143	4.799793e−3	143	3.839093e−3	5.078459e−13	3.839093e−3
144	1.165414e−4	144	4.197280e−5	5.128850e−13	4.197280e−5
145	8.773496e−3	145	9.177876e−4	5.503794e−13	9.177876e−4
146	4.655379e−4	146	1.642809e−5	4.810073e−13	1.642809e−5
147	1.791503e−5	147	9.064763e−3	5.317255e−13	9.064763e−3
148	1.231654e−3	148	2.305888e−4	5.409102e−13	2.305888e−4
149	6.150899e−5	149	3.353775e−5	5.256408e−13	3.353775e−5
150	8.041550e−5	150	2.470391e−4	4.526319e−13	2.470391e−4

to mention that the method (9) has a disadvantage of destroying the sparsity of the problems, although it has a little better performance.

7 Conclusions

We have extended the penalized Fischer-Burmeister function [3] to the SOCCP. The nonsmooth Newton method based on the penalized FB SOC function is shown to have fast local convergence without strict complementarity of solutions, but, analogous to the FB semismooth Newton method, their nondegeneracy is still necessary. We also demonstrated that the penalized merit function Ψ_ρ has some but not all nice features of its counterpart for the NCPs; for example, it is even not differentiable. This brings a difficulty to the globalization of semismooth Newton methods based on ϕ_ρ, which will be left as our future research work.

Acknowledgements The authors would like to thank the anonymous referees for their helpful suggestions on the improvements of this paper.

Appendix

The submatrices C_1, C_2, C_3 and C_4 in Theorem 4.1 are defined as follows:

$$C_1 := \begin{pmatrix} C_1^{11} & C_1^{12} \\ C_1^{21} & C_1^{22} \end{pmatrix}, \qquad C_2 := \begin{pmatrix} C_2^{11} & C_2^{12} & C_2^{13} & C_2^{14} & C_2^{15} \\ C_2^{21} & C_2^{22} & C_2^{23} & C_2^{24} & C_2^{25} \end{pmatrix},$$

$$C_3 := \begin{pmatrix} C_3^{11} & C_3^{12} \\ C_3^{21} & C_3^{22} \\ C_3^{31} & C_3^{32} \\ C_3^{41} & C_3^{42} \\ C_3^{51} & C_3^{52} \end{pmatrix}, \qquad C_4 := \begin{pmatrix} C_4^{11} & C_4^{12} & C_4^{13} & C_4^{14} & C_4^{15} \\ C_4^{21} & C_4^{22} & C_4^{23} & C_4^{24} & C_4^{24} \\ C_4^{31} & C_4^{32} & C_4^{33} & C_4^{34} & C_4^{35} \\ C_4^{41} & C_4^{42} & C_4^{43} & C_4^{44} & C_4^{45} \\ C_4^{51} & C_4^{52} & C_4^{53} & C_4^{54} & C_4^{55} \end{pmatrix}.$$

Here, we give the explicit expressions for the second row block submatrices of C_1–C_4 (the other row block submatrices can be given in a similar way), where $E = F'(\zeta^*)(G'(\zeta^*))^{-1}$:

$$C_1^{21} := \left[q_i'^T E_{ij} \hat{Q}_j \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_B)\right],$$

$$C_1^{22} := \left[q_i'^T E_{ij} q_j' \ (i, j \in J_{B0}^4 \cup J_{0B}^4)\right];$$

$$C_2^{21} := \left[q_i'^T E_{ij} \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_0^1 \cup J_{0B}^1)\right],$$

$$C_2^{22} := \left[q_i'^T E_{ij} Q_j \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_0^2 \cup J_{0B}^2)\right],$$

$$C_2^{23} := \left[q_i'^T E_{ij} \tilde{Q}_j \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_{B0}^3 \cup J_{B0}^4)\right],$$

$$C_2^{24} := \left[q_i'^T E_{ij} q_j \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_B^2)\right],$$

$$C_2^{25} := \left[q_i'^T E_{ij} q_j' \ (i \in J_{B0}^4 \cup J_{0B}^4, j \in J_B^1 \cup J_{0B}^3)\right];$$

$$C_3^{21} := \left[Q_i^T E_{ij} \hat{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B)\right],$$

$$C_3^{22} := \left[Q_i^T E_{ij} q_j' \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{B0}^4 \cup J_{0B}^4)\right];$$

$$C_4^{21} := \left[Q_i^T E_{ij} \ (i \in J_0^2 \cup J_{0B}^2, j \in J_0^1 \cup J_{0B}^1)\right],$$

$$C_4^{22} := \left[Q_i^T E_{ij} Q_j \ (i, j \in J_0^2 \cup J_{0B}^2)\right],$$

$$C_4^{23} := \left[Q_i^T E_{ij} \tilde{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{0B}^3 \cup J_{0B}^4)\right],$$

$$C_4^{24} := \left[Q_i^T E_{ij} q_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B^1)\right],$$

$$C_4^{25} := \left[Q_i^T E_{ij} q_j' \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B^2 \cup J_{B0}^3)\right].$$

The block matrices A_1–A_4, C_2–C_7 and B_2–B_7 in Theorem 4.2 have the following form:

$$A_1 := \begin{pmatrix} A_1^{11} & A_1^{12} & A_1^{13} & A_1^{14} & A_1^{15} & A_1^{16} \\ A_1^{21} & A_1^{22} & A_1^{23} & A_1^{24} & A_1^{25} & A_1^{26} \\ A_1^{31} & A_1^{32} & A_1^{33} & A_1^{34} & A_1^{35} & A_1^{36} \\ A_1^{41} & A_1^{42} & A_1^{43} & A_1^{44} & A_1^{45} & A_1^{46} \\ A_1^{51} & A_1^{52} & A_1^{53} & A_1^{54} & A_1^{55} & A_1^{56} \\ A_1^{61} & A_1^{62} & A_1^{63} & A_1^{64} & A_1^{65} & A_1^{66} \end{pmatrix},$$

$$A_2 := \begin{pmatrix} A_2^{11} & A_2^{12} & A_2^{13} & A_2^{14} & A_2^{15} \\ A_2^{21} & A_2^{22} & A_2^{23} & A_2^{24} & A_2^{25} \\ A_2^{31} & A_2^{32} & A_2^{33} & A_2^{34} & A_2^{35} \\ A_2^{41} & A_2^{42} & A_2^{43} & A_2^{44} & A_2^{45} \\ A_2^{51} & A_2^{52} & A_2^{53} & A_2^{54} & A_2^{55} \end{pmatrix},$$

$$A_3 := \begin{pmatrix} A_3^{11} & A_3^{12} \\ A_3^{21} & A_3^{22} \end{pmatrix}, \qquad A_4 := \begin{pmatrix} A_4^{11} & A_4^{12} \\ A_4^{21} & A_4^{22} \end{pmatrix};$$

$$C_2 := \begin{pmatrix} C_2^{11} & C_2^{12} & C_2^{13} & C_2^{14} & C_2^{15} \\ C_2^{21} & C_2^{22} & C_2^{23} & C_2^{24} & C_2^{25} \\ C_2^{31} & C_2^{32} & C_2^{33} & C_2^{34} & C_2^{35} \\ C_2^{41} & C_2^{42} & C_2^{43} & C_2^{44} & C_2^{45} \\ C_2^{51} & C_2^{52} & C_2^{53} & C_2^{54} & C_2^{55} \\ C_2^{61} & C_2^{62} & C_2^{63} & C_2^{64} & C_2^{65} \end{pmatrix},$$

$$C_3 := \begin{pmatrix} C_3^{11} & C_3^{12} \\ C_3^{21} & C_3^{22} \\ C_3^{31} & C_3^{32} \\ C_3^{41} & C_3^{42} \\ C_3^{51} & C_3^{52} \\ C_3^{61} & C_3^{62} \end{pmatrix}, \qquad C_4 := \begin{pmatrix} C_4^{11} & C_4^{12} \\ C_4^{21} & C_4^{22} \\ C_4^{31} & C_4^{32} \\ C_4^{41} & C_4^{42} \\ C_4^{51} & C_4^{52} \\ C_4^{61} & C_4^{62} \end{pmatrix},$$

$$C_5 := \begin{pmatrix} C_5^{11} & C_5^{12} \\ C_5^{21} & C_5^{22} \\ C_5^{31} & C_5^{32} \\ C_5^{41} & C_5^{42} \\ C_5^{51} & C_5^{52} \end{pmatrix}, \qquad C_6 := \begin{pmatrix} C_6^{11} & C_6^{12} \\ C_6^{21} & C_6^{22} \\ C_6^{31} & C_6^{32} \\ C_6^{41} & C_6^{42} \\ C_6^{51} & C_6^{52} \end{pmatrix},$$

$$C_7 := \begin{bmatrix} C_7^{11} & C_7^{12} \\ C_7^{21} & C_7^{22} \end{bmatrix};$$

$$B_2 := \begin{pmatrix} B_2^{11} & B_2^{12} & B_2^{13} & B_2^{14} & B_2^{15} & B_2^{16} \\ B_2^{21} & B_2^{22} & B_2^{23} & B_2^{24} & B_2^{25} & B_2^{26} \\ B_2^{31} & B_2^{32} & B_2^{33} & B_2^{34} & B_2^{35} & B_2^{36} \\ B_2^{41} & B_2^{42} & B_2^{43} & B_2^{44} & B_2^{45} & B_2^{46} \\ B_2^{51} & B_2^{52} & B_2^{53} & B_2^{54} & B_2^{55} & B_2^{56} \end{pmatrix},$$

$$B_3 := \begin{pmatrix} B_3^{11} & B_3^{12} & B_3^{13} & B_3^{14} & B_3^{15} & B_3^{16} \\ B_3^{21} & B_3^{22} & B_3^{23} & B_3^{24} & B_3^{25} & B_3^{26} \end{pmatrix},$$

$$B_4 := \begin{pmatrix} B_4^{11} & B_4^{12} & B_4^{13} & B_4^{14} & B_4^{15} \\ B_4^{21} & B_4^{22} & B_4^{23} & B_4^{24} & B_4^{25} \end{pmatrix},$$

$$B_5 := \begin{pmatrix} B_5^{11} & B_5^{12} & B_5^{13} & B_5^{14} & B_5^{15} & B_5^{16} \\ B_5^{21} & B_5^{22} & B_5^{23} & B_5^{24} & B_5^{25} & B_5^{26} \end{pmatrix},$$

$$B_6 := \begin{pmatrix} B_6^{11} & B_6^{12} & B_6^{13} & B_6^{14} & B_6^{15} \\ B_6^{21} & B_6^{22} & B_6^{23} & B_6^{24} & B_6^{25} \end{pmatrix}, \qquad B_7 := \begin{pmatrix} B_7^{11} & B_7^{12} \\ B_7^{21} & B_7^{22} \end{pmatrix}.$$

Here, we give the explicit expressions only for the second row block submatrices of B_2, A_2, C_5 and C_6, where $E = F'(\zeta^*)(G'(\zeta^*))^{-1}$, and the expressions of other row block submatrices can be given in a similar way:

$$B_2^{21} := \left[Q_i^T E_{ij} \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^3) \right],$$

$$B_2^{22} := \left[Q_i^T E_{ij} Q_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^4) \right],$$

$$B_2^{23} := \left[Q_i^T E_{ij} \tilde{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^7) \right],$$

$$B_2^{24} := \left[Q_i^T E_{ij} \bar{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^5 \cup J_{00}^6) \right],$$

$$B_2^{25} := \left[Q_i^T E_{ij} \hat{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B \cup J_{00}^8 \cup J_{00}^{10}) \right],$$

$$B_2^{26} := \left[Q_i^T E_{ij} q_j' \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{B0}^4 \cup J_{0B}^4 \cup J_{00}^{12} \cup J_{00}^{13}) \right];$$

$$A_2^{21} := \left[Q_i^T E_{ij} \ (i \in J_0^2 \cup J_{0B}^2, j \in J_0^1 \cup J_{0B}^1 \cup J_{00}^2) \right],$$

$$A_2^{22} := \left[Q_i^T E_{ij} Q_j \ (i, j \in J_0^2 \cup J_{0B}^2) \right],$$

$$A_2^{23} := \left[Q_i^T E_{ij} \tilde{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{0B}^3 \cup J_{0B}^4 \cup J_{00}^9 \cup J_{00}^{13}) \right],$$

$$A_2^{24} := \left[Q_i^T E_{ij} q_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B^1 \cup J_{00}^5 \cup J_{00}^{10} \cup J_{00}^{17}) \right],$$

$$A_2^{25} := \left[Q_i^T E_{ij} q_j' \ (i \in J_0^2 \cup J_{0B}^2, j \in J_B^2 \cup J_{B0}^3 \cup J_{00}^7 \cup J_{00}^8 \cup J_{00}^{11} \cup J_{00}^{15}) \right];$$

$$C_5^{21} := \left[Q_i^T E_{ij} \tilde{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^{14}) \right],$$

$$C_5^{22} := \left[Q_i^T E_{ij} \bar{Q}_j \ (i \in i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^{16}) \right];$$

$$C_6^{21} := \left[Q_i^T E_{ij} \tilde{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^{15}) \right],$$

$$C_6^{22} := \left[Q_i^T E_{ij} \bar{Q}_j \ (i \in J_0^2 \cup J_{0B}^2, j \in J_{00}^{17}) \right].$$

References

1. Alizadeh, F., Goldfarb, D.: Second-order cone programming. Math. Program. **95**, 3–51 (2003)
2. Andersen, E.D., Roos, C., Terlaky, T.: On implementing a primal-dual interior-point method for conic quadratic optimization. Math. Program. **95**, 249–277 (2003)
3. Chen, B., Chen, X., Kanzow, C.: A penalized Fischer-Burmeister NCP-function. Math. Program. **88**, 211–216 (2000)
4. Chen, J.-S.: Alternative proofs for some results of vector-valued functions associated with second-order cone. J. Nonlinear Convex Anal. **6**, 297–325 (2005)
5. Chen, J.-S.: Two classes of merit functions for the second-order cone complementarity problem. Math. Methods Oper. Res. **64**, 495–519 (2006)
6. Chen, J.-S., Tseng, P.: An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. Math. Program. **104**, 293–327 (2005)
7. Chen, X.-D., Sun, D., Sun, J.: Complementarity functions and numerical experiments for second-order cone complementarity problems. Comput. Optim. Appl. **25**, 39–56 (2003)
8. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983) (reprinted by SIAM, Philadelphia, 1990)
9. Facchinei, F., Pang, J.-S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vols. I and II. Springer, New York (2003)
10. Faraut, J., Koranyi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs, Oxford University Press, New York (1994)
11. Ferris, M.C., Mangasarian, O.L., Pang, J.-S. (eds.): Complementarity: Applications, Algorithms and Extensions. Kluwer Academic, Norwell (2001)
12. Fischer, A.: Solution of the monotone complementarity problem with locally Lipschitzian functions. Math. Program. **76**, 513–532 (1997)
13. Fukushima, M., Luo, Z.-Q., Tseng, P.: Smoothing functions for second-order cone complementarity problems. SIAM J. Optim. **12**, 436–460 (2002)
14. Hayashi, S., Yamashita, N., Fukushima, M.: A combined smoothing and regularization method for monotone second-order cone complementarity problems. SIAM J. Optim. **15**, 593–615 (2005)
15. Horn, R.A., Johnson, C.R.: Matrix Analysis. Cambridge University Press, Cambridge (1985)
16. Kanzow, C., Ferenczi, I., Fukushima, M.: On the local convergence of semismooth Newton methods for linear and nonlinear second-order cone programs without strict complementarity. SIAM J. Optim. **20**, 297–320 (2009)
17. Kum, S.H., Lim, Y.D.: Coercivity and strong semismoothness of the penalized Fischer-Burmeister function for the symmetric cone complementarity problem. J. Optim. Theory Appl. **142**, 377–383 (2009)
18. Lobo, M.S., Vandenberghe, L., Boyd, S., Lebret, H.: Application of second-order cone programming. Linear Algebra Appl. **284**, 193–228 (1998)
19. Monteiro, R.D.C., Tsuchiya, T.: Polynomial convergence of primal-dual algorithms for the second-order cone programs based on the MZ-family of directions. Math. Program. **88**, 61–83 (2000)
20. Pan, S.-H., Chen, J.-S.: A damped Gauss–Newton method for the second-order conecomplementarity problem. Appl. Math. Optim. **59**, 293–318 (2009)
21. Pan, S.-H., Chen, J.-S.: Quadratic convergence of semismooth methods based on the FB function for SOCPs without strict complementarity condition. IMA J. Numer. Anal. (2009, submitted)

22. Pang, J.-S., Sun, D., Sun, J.: Semismooth homeomorphisms and strong stability of semidefinite and Lorentz complementarity problems. Math. Oper. Res. **284**, 193–228 (2002)
23. Pataki, G., Schmieta, S.: The DIMACS library of semidefinite-quadratic-linear programs. Preliminary draft, Computational Optimization Research Center, Columbia University, New York (2002). http://dimacs.rutgers.edu/Challenges/Seventh/Instances/
24. Qi, L.: Convergence analysis of some algorithms for solving nonsmooth equations. Math. Oper. Res. **18**, 227–244 (1993)
25. Qi, L., Sun, J.: A nonsmooth version of Newton's method. Math. Program. **58**, 353–367 (1993)
26. Sun, D., Sun, J.: Strong semismoothness of the Fischer-Burmeister SDC and SOC complementarity functions. Math. Program. **103**, 575–581 (2005)
27. Tsuchiya, T.: A convergence analysis of the scaling-invariant primal-dual path-following algorithms for second-order cone programming. Optim. Methods Softw. **11**, 141–182 (1999)
28. Yoshise, A.: Interior point trajectories and a homogenous model for nonlinear complementarity problems over symmetric cones. SIAM J. Optim. **17**, 1129–1153 (2006)