Numerical Functional Analysis and Optimization, vol. 31, pp. 387-413, 2010

Two classes of merit function for infinite-dimensional SOCCPs

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November 16, 2009

Abstract. In this paper, we extend two classes of merit functions for the second-order complementarity problem (SOCCP) to infinite-dimensional SOCCP. These two classes of merit functions include several popular merit functions, which are used in NCP (nonlinear complementarity problem), SDCP (semidefinite complementarity problem), and SOCCP, as special cases. We give conditions under which the infinite-dimensional SOCCP has a unique solution and show that all these merit functions provide an error bound for infinite-dimensional SOCCP and have bounded level sets. These results are very useful for designing solution methods for infinite-dimensional SOCCP.

Key words. Hilbert space, second-order cone, merit functions, fixed point, error bound, level set.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, and write the norm induced by $\langle \cdot, \cdot \rangle$ as $\| \cdot \|$. The conic complementarity problem $CP(\mathcal{K}, F, G)$ in $\mathcal{H}$ is, for any given

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closed convex cone $\mathcal{K} \subset \mathcal{H}$ and functions $F, G : \mathcal{H} \to \mathcal{H}$, to find points $x, y, \zeta \in \mathcal{H}$ such that

$$\begin{array}{c} \langle x, y\rangle = 0,\ x \in \mathcal{K},\ y \in \mathcal{K}^*, \\ x = F(\zeta),\ y = G(\zeta), \end{array}$$

where $\mathcal{K}^* := \{x \in \mathcal{H} |\ \langle x, y\rangle \geq 0, \forall y \in \mathcal{K}\}$ is the dual cone of $\mathcal{K}$. A closed convex cone $\mathcal{K} \subset \mathcal{H}$ is called *self-dual* if $\mathcal{K}$ coincides with its dual cone $\mathcal{K}^*$, for example, the nonnegative orthant cone $\mathbb{R}^n_+ := \{(x_1, \cdots, x_n) \in \mathbb{R}^n |\ x_j \geq 0, j = 1, 2, \ldots, n\}$ and the second-order cone (also called Lorentz cone) $\mathbb{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} |\ x_1 \geq \|x_2\|\}$. This paper focuses on the conic complementarity problem associated with the infinite-dimensional second-order cone $\mathbb{K}$ in $\mathcal{H}$ (will be defined as in (10)) which is closed, convex, and self-dual (see Section 2 for details). Since $\mathbb{K}$ is self-dual, the conic complementarity problem reduces to $CP(\mathbb{K}, F, G)$, which is to find $x, y, \zeta \in \mathcal{H}$ such that

$$\begin{array}{c} \langle x, y\rangle = 0,\ x \in \mathbb{K},\ y \in \mathbb{K}, \\ x = F(\zeta),\ y = G(\zeta). \end{array} \tag{1}$$

For finite-dimensional second-order cone optimization and complementarity problems, there have proposed various methods, including the interior point methods [1, 15, 18], the smoothing and semismooth Newton methods [3, 7, 10, 11, 13], and the merit function method [2, 4]. As far as we know, only very few of aforementioned methods are extended to infinite-dimensional SOCCP case. More precisely, for infinite-dimensional second-order cone optimization and complementarity problems, some particular interior point method was employed in [8], and a merit function method was considered in [5] where its merit function is $\psi_{\rm FB} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ given by

$$\psi_{\rm FB}(x, y) := \frac{1}{2}\|\phi_{\rm FB}(x, y)\|^2, \tag{2}$$

which is induced by the Fischer-Burmeister (FB) function $\phi_{\rm FB} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ defined as

$$\phi_{\rm FB}(x, y) := (x^2 + y^2)^{1/2} - (x + y). \tag{3}$$

Here x^2 means $x \bullet x$, where $\bullet$ will be introduced in Section 2. In this paper, we also concern with the merit function method for (1). In other words, we aim to seek a smooth function $\Psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ such that, for any $x, y \in \mathcal{H}$,

$$\Psi(x, y) = 0 \Longleftrightarrow x \in \mathbb{K},\ y \in \mathbb{K},\ \langle x, y\rangle = 0, \tag{4}$$

and then the problem $CP(\mathbb{K}, F, G)$ can be transformed into a smooth minimization problem:

$$\min_{\zeta \in \mathcal{H}} f(\zeta) := \Psi(F(\zeta), G(\zeta)).$$

Traditionally, such a f or Ψ is called a merit function associated with $\mathbb{K}$.

The two classes of merit functions that we will investigate come intuitively from the finite-dimensional case where $\mathcal{H}$ equals $\mathbb{R}^n$ and is associated with the Lorentz cone $\mathbb{K}^n$, which was studied in [2]. The first class is

$$f_{\rm LT}(\zeta) := \psi_0\left(\langle F(\zeta), G(\zeta)\rangle\right) + \psi(F(\zeta), G(\zeta)), \tag{5}$$

where $\psi_0 : \mathbb{R} \to \mathbb{R}_+$ is any smooth function satisfying

$$\psi_0(t) = 0 \quad \forall t \leq 0 \quad \text{and} \quad \psi_0'(t) > 0 \quad \forall t > 0, \tag{6}$$

and $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ satisfies

$$\psi(x, y) = 0,\ \langle x, y\rangle \leq 0 \Longleftrightarrow (x, y) \in \mathbb{K} \times \mathbb{K},\ \langle x, y\rangle = 0. \tag{7}$$

The second class is

$$\widehat{f_{\rm LT}}(\zeta) := \psi_0^*(F(\zeta) \bullet G(\zeta)) + \psi(F(\zeta), G(\zeta)), \tag{8}$$

where $\psi_0^* : \mathcal{H} \to \mathbb{R}_+$ is given by

$$\psi_0^*(w) = \frac{1}{2}\|(w)_+\|^2 \tag{9}$$

and $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ satisfies (7). The function $f_{\rm LT}$ was originally proposed by Luo and Tseng for NCP case [12] and was extended to the SDCP case by Tseng [14], then to the SOCCP case by Chen [2]. We explore the extension to the infinite-dimensional SOCCPs as will be seen in Sections 3. The second class of merit functions for SDCP case was recently studied by Goes and Oliveira [9] and a variant of $\widehat{f_{\rm LT}}$ was also studied by Chen [2] for SOCCP case.

As mentioned, we will define and study these merit functions associated with $\mathbb{K}$ in Hilbert space $\mathcal{H}$. Three examples of ψ will be studied in Section 3. In Section 4, we will show that, under certain conditions, the infinite-dimensional SOCCP has a unique solution and both $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ provide global error bound, which plays an important role in analyzing the convergence rate of some iterative methods for solving $CP(\mathbb{K}, F, G)$. Besides, under the condition that F and G are jointly monotone and a strictly feasible solution exists, we will prove that both $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ have bounded level sets which will ensure that the sequence generated by a decent algorithm has at least an accumulation point. All these properties will make it possible to construct a decent algorithm for solving the equivalent unconstrained reformulation of $CP(\mathbb{K}, F, G)$. Moreover, we will show that both $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ are Fréchet differentiable and their derivatives have computable formulas.

Throughout this paper, for any given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, let $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ denote the Banach space of all continuous linear mappings from $\mathcal{X}$ into $\mathcal{Y}$. We simply write $\mathcal{L}(\mathcal{X}, \mathcal{Y}) = \mathcal{L}(\mathcal{X})$ and denote $GL(\mathcal{X})$ the set of all invertible mappings in $\mathcal{L}(\mathcal{X})$. The norm of any $l \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is defined by $\|l\| := \sup\{\|l(x)\| \mid x \in \mathcal{X} \text{ and } \|x\| = 1\}$. In addition, for any self-adjoint linear operator l from $\mathcal{X} \to \mathcal{X}$, we write $l \succ 0$ (respectively, $l \succeq 0$) to mean that l is positive definite (respectively, positive semidefinite). For any $x \in \mathcal{H}$, $(x)_+$ denotes the orthogonal projection of x onto $\mathbb{K}$, whereas $(x)_-$ means the orthogonal projection of x onto $-\mathbb{K}$. A sequence of elements $\{x_n\} \subset \mathcal{H} \to x$ means $\lim_{n\to\infty} \|x_n - x\| = 0$. A sequence of operators $\{T_n\} \to T$ means $\lim_{n\to\infty} \|T_n - T\| = 0$.

2 Preliminaries

In this section, we recall some background materials and preliminary results that will be used later. We begin with introducing the infinite-dimensional second-order cone.

Recall that the finite-dimensional second-order cone (also called Lorentz cone) is defined as $\mathbb{K}^n := \{(r, x') \in \mathbb{R} \times \mathbb{R}^{n-1} | \ r \geq \|x'\|\}$. As discussed in [5], this Lorentz cone $\mathbb{K}^n$ can be rewritten as

$$\mathbb{K}^n := \left\{ x \in \mathbb{R}^n \middle| \ \langle x, e \rangle \geq \frac{1}{\sqrt{2}} \|x\| \right\} \text{ with } e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Motivated by this, the following closed convex cone in the Hilbert space $\mathcal{H}$ is considered:

$$K(e, r) := \{x \in \mathcal{H} \mid \langle x, e \rangle \geq r\|x\|\},$$

where $e \in \mathcal{H}$ with $\|e\| = 1$ and $0 < r < 1$. Observe that $K(e, r)$ is pointed, that is, $K(e, r) \cap (-K(e, r)) = \{0\}$. Moreover, by denoting

$$\langle e \rangle^{\perp} := \{x \in \mathcal{H} \mid \langle x, e \rangle = 0\},$$

we may express the closed convex cone $K(e, r)$ as

$$K(e, r) = \left\{ x' + \lambda e \in \mathcal{H} \ \middle| \ x' \in \langle e \rangle^{\perp} \text{ and } \lambda \geq \frac{r}{\sqrt{1 - r^2}} \|x'\| \right\}.$$

When $\mathcal{H} = \mathbb{R}^n$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $K(e, \frac{1}{\sqrt{2}})$ coincides with $\mathbb{K}^n$. By this, we shall call $K(e, \frac{1}{\sqrt{2}})$ the infinite-dimensional second-order cone (or infinite-dimensional Lorentz cone) in $\mathcal{H}$ determined by e. In the rest of this paper, we shall only consider any fixed unit vector $e \in \mathcal{H}$, and denote

$$\mathbb{K} := K\left(e, \frac{1}{\sqrt{2}}\right) \tag{10}$$

since two infinite-dimensional second-order cones $K(e_1, \frac{1}{\sqrt{2}})$ and $K(e_2, \frac{1}{\sqrt{2}})$ associated with different unit elements e_1 and e_2 in $\mathcal{H}$ are isometric.

Unless specifically stated otherwise, we shall alternatively write any point $x \in \mathcal{H}$ as $x = x' + \lambda e$ with $x' \in \langle e \rangle^{\perp}$ and $\lambda = \langle x, e \rangle$. In addition, for any $x, y \in \mathcal{H}$, we shall write

$x \succ_{\mathbb{K}} y$ (respectively, $x \succeq_{\mathbb{K}} y$) if $x - y \in \mathrm{int}(\mathbb{K})$ (respectively, $x - y \in \mathbb{K}$). Now, we introduce the spectral decomposition for any element $x \in \mathcal{H}$. For any $x = x' + \lambda e \in \mathcal{H}$, we can decompose x as

$$x = \alpha_1(x) v_x^{(1)} + \alpha_2(x) v_x^{(2)},$$

where $\alpha_1(x)$, $\alpha_2(x)$ and $v_x^{(1)}$, $v_x^{(2)}$ are the spectral values and the associated spectral vectors of x with respect to $\mathbb{K}$, given by

$$\alpha_i(x) = (-1)^i \|x'\| + \lambda,$$

$$v_x^{(i)} = \begin{cases} \frac{1}{2}\left((-1)^i \dfrac{x'}{\|x'\|} + e\right), & \text{if } x' \neq 0, \\ \frac{1}{2}((-1)^i w + e), & \text{if } x' = 0, \end{cases}$$

for $i = 1, 2$ with w being any vector in $\langle e \rangle^{\perp}$ satisfying $\|w\| = 1$. Its determinant and trace is defined as $\det(x) := \alpha_1(x)\alpha_2(x)$ and $\mathrm{tr}(x) := \alpha_1(x) + \alpha_2(x)$, respectively.

Next, we come to the Jordan product associated with the infinite-dimensional Lorentz cone $\mathbb{K}$. For any $x = x' + \lambda e \in \mathcal{H}$ and $y = y' + \mu e \in \mathcal{H}$, we define the Jordan product of x and y by

$$x \bullet y := (\mu x' + \lambda y') + \langle x, y \rangle e. \tag{11}$$

Clearly, when $\mathcal{H} = \mathbb{R}^n$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, this definition is the same as the one given by [6, Chapter II]. From the definition (11) and direct computation, it is easy to verify that the following properties hold.

Property 2.1 **(a)** *$x \bullet y = y \bullet x$ and $x \bullet e = x$ for all $x, y \in \mathcal{H}$.*

(b) *$(x + y) \bullet z = x \bullet z + y \bullet z$ for all $x, y, z \in \mathcal{H}$.*

(c) *$\langle x, y \bullet z \rangle = \langle y, x \bullet z \rangle = \langle z, x \bullet y \rangle$ for all $x, y, z \in \mathcal{H}$.*

(d) *For any $x = x' + \lambda e \in \mathcal{H}$, $x^2 = x \bullet x = 2\lambda x' + \|x\|^2 e \in \mathbb{K}$ and $\langle x^2, e \rangle = \|x\|^2$.*

(e) *If $x = x' + \lambda e \in \mathbb{K}$, then there is a unique $x^{1/2} \in \mathbb{K}$ such that $(x^{1/2})^2 = (x^{1/2}) \bullet (x^{1/2}) = x$, where*

$$x^{1/2} = \sqrt{\alpha_1(x)}\, v_x^{(1)} + \sqrt{\alpha_2(x)}\, v_x^{(2)} = \begin{cases} 0, & \text{if } x = 0 \\ \dfrac{x'}{2\tau} + \tau e, & \text{otherwise} \end{cases}$$

with $\tau = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - \|x'\|^2}}{2}}$.

(f) *Every $x = x' + \lambda e \in \mathcal{H}$ with $\lambda^2 - \|x'\|^2 \neq 0$ is invertible with respect to the Jordan product, i.e., there is a unique point $x^{-1} \in \mathcal{H}$ such that $x \bullet x^{-1} = e$, where*

$$x^{-1} = \alpha_1(x)^{-1} v_x^{(1)} + \alpha_2(x)^{-1} v_x^{(2)} = \frac{-x' + \lambda e}{\det(x)} = \frac{-x' + \lambda e}{\lambda^2 - \|x'\|^2}.$$

Moreover, $x \in \mathrm{int}(\mathbb{K})$ if and only if $x^{-1} \in \mathrm{int}(\mathbb{K})$.

Associated with every $x \in \mathcal{H}$, we define a linear mapping L_x from $\mathcal{H}$ to $\mathcal{H}$ by

$$L_x y := x \bullet y \quad \text{for any} \quad y \in \mathcal{H}. \tag{12}$$

It is clear that $L_x \in \mathcal{L}(\mathcal{H})$ and this mapping possesses the following favorable properties.

Property 2.2 *[5, Lemma 2.2] For any $x \in \mathcal{H}$, let $L_x \in \mathcal{L}(\mathcal{H})$ be defined as in (12). Then, we have*

(a) $x \succ_{\mathbb{K}} 0 \Longleftrightarrow L_x \succ 0$ *and* $x \succeq_{\mathbb{K}} 0 \Longleftrightarrow L_x \succeq 0$;

(b) *if $x = x' + \lambda e$ with $\lambda \neq 0$ and $|\lambda| \neq \|x'\|$, then $L_x \in GL(\mathcal{H})$ with the inverse given by*

$$L_x^{-1} y = \frac{1}{\lambda}\left(y' - \langle x^{-1}, y\rangle x'\right) + \langle x^{-1}, y\rangle e \quad \text{for any} \quad y = y' + \mu e \in \mathcal{H}.$$

Property 2.3 *[5, Lemma 5.1] Let $\mathbb{K}$ be the infinite-dimensional Lorentz cone in $\mathcal{H}$ given as in (10). For any $x, y \in \mathcal{H}$ and $z \succ_{\mathbb{K}} 0$, the following implications hold:*

$$\begin{aligned} z^2 \succ_{\mathbb{K}} x^2 + y^2 \quad &\Longrightarrow \quad L_z^2 - L_y^2 - L_x^2 \succ 0, \\ z^2 \succ_{\mathbb{K}} x^2 \quad &\Longrightarrow \quad z \succ_{\mathcal{K}} x. \end{aligned}$$

Moreover, the above implications remain true when " $\succ$ " is replaced by " $\succeq$ ".

The following describes some important relations when $x^2 + y^2$ lies on the boundary of $\mathbb{K}$.

Property 2.4 *[5, Lemma 2.3] For any $x = x' + \lambda e$, $y = y' + \mu e \in \mathcal{H}$ with $x^2 + y^2 \notin \operatorname{int}(\mathbb{K})$, we have*

$$\lambda^2 = \|x'\|^2, \quad \mu^2 = \|y'\|^2, \quad \lambda\mu = \langle x', y'\rangle, \quad \lambda y' = \mu x'.$$

Property 2.5 *Let $\mathcal{K}$ be any closed convex cone in $\mathcal{H}$. For each $x \in \mathcal{H}$, let $x_{\mathcal{K}}^+$ and $x_{\mathcal{K}}^-$ denote the minimum distance projection of x onto $\mathcal{K}$ and $-\mathcal{K}^*$, respectively. The following results hold.*

(a) *For any $x \in \mathcal{H}$, we have $x = x_{\mathcal{K}}^+ + x_{\mathcal{K}}^-$ and $\|x\|^2 = \|x_{\mathcal{K}}^+\|^2 + \|x_{\mathcal{K}}^-\|^2$.*

(b) *For any $x \in \mathcal{H}$ and $y \in \mathcal{K}$, we have $\langle x, y\rangle \leq \langle x_{\mathcal{K}}^+, y\rangle$.*

(c) *If $\mathcal{K}$ is self-dual, then for any $x \in \mathcal{H}$ and $y \in \mathcal{K}$, we have $\|(x+y)_{\mathcal{K}}^+\| \geq \|x_{\mathcal{K}}^+\|$.*

(d) *For any $x \in \mathcal{K}$ and $y \in \mathcal{H}$ with $x^2 - y^2 \in \mathcal{K}$, we have $x - y \in \mathcal{K}$.*

Proof. These results are true for general closed convex cone whose proofs are the same as in [4, Lemma 5.1]. $\square$

To close this section, we review some definitions that will be used in subsequent analysis.

Definition 2.1 *Let $F, G : \mathcal{H} \to \mathcal{H}$ be single-valued mappings.*

(a) *F is said to be η-strongly monotone if there exists a constant $\eta > 0$ satisfying*

$$\langle F(x) - F(y), x - y\rangle \geq \eta \|x - y\|^2, \ \forall x, y \in \mathcal{H}.$$

(b) *F is said to be Lipschitz continuous with constant γ if*

$$\|F(x) - F(y)\| \leq \gamma \|x - y\|, \ \forall x, y \in \mathcal{H}.$$

(c) *F and G are said to be ρ-jointly strongly monotone if there exists a constant $\rho > 0$ satisfying*

$$\langle F(x) - F(y), G(x) - G(y)\rangle \geq \rho \|x - y\|^2, \ \forall x, y \in \mathcal{H}.$$

We also recall the concept of *Fréchet differentiability.* For given Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, a mapping f from a nonempty open subset X of $\mathcal{X}$ into $\mathcal{Y}$ is said to be Fréchet differentiable at $x \in X$ if there exists $l_x \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ such that

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - l_x h}{\|h\|} = 0,$$

and l_x is called the Fréchet differential of f at x, denoted by $Df(x)$. When f is Fréchet differentiable at every point of X, we say that f is Fréchet differentiable on X. If f is Fréchet differentiable on a neighborhood $U \in X$ of a point $x_0 \in X$, and if, as a mapping from U into the Banach space $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, the mapping $x \to Df(x)$ is continuous at x_0, then f is said to be *continuously Fréchet differentiable* at x_0. The mapping f is called continuously Fréchet differentiable on X if it is continuously Fréchet differentiable at every point of X.

3 Two classes of merit functions

In this section, we elaborate more about the two classes of merit functions for (1). We are motivated by a class of merit functions proposed by Luo and Tseng [12] for the NCP case originally which was already extended to the SDP and SOCCP by Tseng [14] and Chen [2], respectively. We introduce them as below. Let f_{LT} be given as (5), i.e.,

$$f_{\text{LT}}(\zeta) := \psi_0(\langle F(\zeta), G(\zeta)\rangle) + \psi(F(\zeta), G(\zeta)),$$

where $\psi_0 : \mathbb{R} \to \mathbb{R}_+$ satisfies (6) and $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ satisfies (7). We notice that ψ_0 is differentiable and strictly increasing on $[0, \infty)$. Let Ψ_+ denote the collection of $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ satisfying (7) that are Fréchet differentiable and their derivatives satisfy the following conditions:

$$\begin{cases} \langle D_x\psi(x,y), D_y\psi(x,y)\rangle \geq 0, \ \forall (x,y) \in \mathcal{H} \times \mathcal{H}. \\ \langle x, D_x\psi(x,y)\rangle + \langle y, D_y\psi(x,y)\rangle \geq 0, \ \forall (x,y) \in \mathcal{H} \times \mathcal{H}. \end{cases} \tag{13}$$

We will give an example of ψ belonging to Ψ_+ in Proposition 3.1. Before that, we need the following three lemmas which will be used for proving Proposition 3.1 and Proposition 3.2.

Lemma 3.1 **(a)** *For any $x \in \mathcal{H}$, $\langle x, (x)_-\rangle = \|(x)_-\|^2$ and $\langle x, (x)_+\rangle = \|(x)_+\|^2$*

(b) *For any $x \in \mathcal{H}$ and $y \in \mathcal{H}$, we have*

$$x \in \mathbb{K} \Longleftrightarrow \langle x, y\rangle \geq 0, \ \forall y \in \mathbb{K}.$$

Proof. The results follow by Property 2.5 and self-duality of $\mathbb{K}$. $\Box$

Lemma 3.2 *[17] For $x \neq 0 \in \mathcal{H}$, the following hold.*

(a) *If $g(x) = \|x\|$, we have $Dg(x)h = \dfrac{\langle x, h\rangle}{\|x\|}$.*

(b) *If $g(x) = \|x\|^2$, we have $Dg(x)h = 2\langle x, h\rangle$.*

(c) *If $g(x) = \dfrac{x}{\|x\|}$, we have $Dg(x)h = \dfrac{h}{\|x\|} - \dfrac{\langle x, h\rangle}{\|x\|^3}x$.*

Proof. The results can be verified by direct computation, also see [17]. $\Box$

Lemma 3.3 *Let ϕ_{FB} and ψ_{FB} be given as in (3) and (2), respectively. Then,*

(a) $\phi_{\text{FB}}(x,y) = 0 \Longleftrightarrow x \in \mathbb{K}, y \in \mathbb{K}, x \bullet y = 0 \Longleftrightarrow x \in \mathbb{K}, y \in \mathbb{K}, \langle x, y\rangle = 0.$

(b) *for any $x, y \in \mathcal{H}$, there holds*

$$4\psi_{\text{FB}}(x,y) \geq 2\|\phi_{\text{FB}}(x,y)_+\|^2 \geq \|(-x)_+\|^2 + \|(-y)_+\|^2.$$

Proof. (a) This is shown in [5, Lemma 3.1].

(b) The first inequality follows from Property 2.5(a). Since $(x^2 + y^2)^{1/2} - x \in \mathcal{K}$, by Property 2.5(c), we can deduce that $\|((x^2 + y^2)^{1/2} - x - y)_+\|^2 \geq \|(-y)_+\|^2$. Similarly, we can get $\|((x^2 + y^2)^{1/2} - x - y)_+\|^2 \geq \|(-x)_+\|^2$. Adding the above two inequalities yields the desired second inequality. This completes the proof. $\square$

Proposition 3.1 *Let* $\psi_1 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ *be given by*

$$\psi_1(x, y) := \frac{1}{2}\left(\|(-x)_+\|^2 + \|(-y)_+\|^2\right) \tag{14}$$

Then, the following results hold.

(a) ψ_1 *satisfies (7).*

(b) ψ_1 *is convex and Fréchet differentiable at every* $(x, y) \in \mathcal{H} \times \mathcal{H}$ *with* $D_x\psi_1(x, y) = (x)_-$, $D_y\psi_1(x, y) = (y)_-$.

(c) *For every* $(x, y) \in \mathcal{H} \times \mathcal{H}$, *we have*

$$\langle D_x\psi_1(x, y), D_y\psi_1(x, y)\rangle \geq 0.$$

(d) *For every* $(x, y) \in \mathcal{H} \times \mathcal{H}$, *we have*

$$\langle x, D_x\psi_1(x, y)\rangle + \langle y, D_y\psi_1(x, y)\rangle = \|(x)_-\|^2 + \|(y)_-\|^2.$$

(e) ψ_1 *belongs to* Ψ_+.

Proof. The proofs are similar to those in [2, Proposition 3.1], so we omit them. $\square$

Next, we consider a further restriction on ψ. Let Ψ_{++} denote the collection of $\psi \in \Psi_+$ satisfying the following conditions:

$$\psi(x, y) = 0,\ \forall (x, y) \in \mathcal{H} \times \mathcal{H} \quad \text{whenever} \quad \langle D_x\psi(x, y), D_y\psi(x, y)\rangle = 0. \tag{15}$$

Two examples of such ψ are given in next two propositions.

Proposition 3.2 *Let* $\psi_{\rm FB}(x, y)$ *be given by (2). Then, the following results hold.*

(a) $\psi_{\rm FB}$ *satisfies (7).*

(b) *$\psi_{\rm FB}$ is Fréchet differentiable at every $(x, y) = (x' + \lambda e, y' + \mu e) \in \mathcal{H} \times \mathcal{H}$. Moreover, $D_x\psi_{\rm FB}(0,0) = D_y\psi_{\rm FB}(0,0) = 0$. If $(x, y) \neq (0,0)$ and $x^2 + y^2 \in \mathrm{int}(\mathbb{K})$, then*

$$\begin{aligned} D_x\psi_{\rm FB}(x,y) &= L_x L^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y) - \phi_{\rm FB}(x,y), \\ D_y\psi_{\rm FB}(x,y) &= L_y L^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y) - \phi_{\rm FB}(x,y). \end{aligned}$$

If $(x, y) \neq (0,0)$ and $x^2 + y^2 \notin \mathrm{int}(\mathbb{K})$, then $\lambda^2 + \mu^2 \neq 0$ and

$$\begin{aligned} D_x\psi_{\rm FB}(x,y) &= \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y), \\ D_y\psi_{\rm FB}(x,y) &= \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y)). \end{aligned}$$

(c) *For every $(x, y) = (x' + \lambda e, y' + \mu e) \in \mathcal{H} \times \mathcal{H}$, we have*

$$\langle D_x\psi_{\rm FB}(x,y), D_y\psi_{\rm FB}(x,y)\rangle \geq 0$$

and the equality holds whenever $\psi_{\rm FB}(x,y) = 0$.

(d) *For every $(x, y) = (x' + \lambda e, y' + \mu e) \in \mathcal{H} \times \mathcal{H}$, we have*

$$\langle x, D_x\psi_{\rm FB}(x,y)\rangle + \langle y, D_y\psi_{\rm FB}(x,y)\rangle = \|\phi_{\rm FB}(x,y)\|^2.$$

(e) *$\psi_{\rm FB}$ belongs to Ψ_{++}.*

Proof. See [5, Theorem 4.1] and [5, Lemma 5.2]. □

Proposition 3.2 tells us that $\psi_{\rm FB}$ defined as (2) belongs to Ψ_{++} which yields a merit function $\psi_{\rm YF} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ given as

$$\psi_{\rm YF}(x,y) := \psi_0(\langle x, y\rangle) + \psi_{\rm FB}(x,y),$$

and studied by Yamashita and Fukushima [16].

Proposition 3.3 *Let $\psi_2 : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ be given by*

$$\psi_2(x,y) = \frac{1}{2}\|\phi_{\rm FB}(x,y)_+\|^2, \tag{16}$$

where $\phi_{\rm FB}$ is defined as (3). Then, the following results hold.

(a) *ψ_2 satisfies (7).*

(b) *ψ_2 is Fréchet differentiable at every $(x,y)=(x'+\lambda e, y'+\mu e)\in \mathcal{H}\times\mathcal{H}$. Moreover, $D_x\psi_{\rm FB}(0,0)=D_y\psi_{\rm FB}(0,0)=0$. If $(x,y)\neq(0,0)$ and $x^2+y^2\in {\rm int}(\mathbb{K})$, then*

$$\begin{aligned} D_x\psi_2(x,y) &= L_xL^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y)_+ - \phi_{\rm FB}(x,y)_+, \\ D_y\psi_2(x,y) &= L_yL^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y)_+ - \phi_{\rm FB}(x,y)_+. \end{aligned}$$

If $(x,y)\neq(0,0)$ and $x^2+y^2\notin {\rm int}(\mathbb{K})$, then $\lambda^2+\mu^2\neq 0$ and

$$\begin{aligned} D_x\psi_2(x,y) &= \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}-1\right)\phi_{\rm FB}(x,y)_+, \\ D_y\psi_2(x,y) &= \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}}-1\right)\phi_{\rm FB}(x,y)_+. \end{aligned}$$

(c) *For every $(x,y)=(x'+\lambda e, y'+\mu e)\in \mathcal{H}\times\mathcal{H}$, we have*

$$\langle D_x\psi_2(x,y), D_y\psi_2(x,y)\rangle \geq 0$$

and the equality holds whenever $\psi_2(x,y)=0$.

(d) *For every $(x,y)=(x'+\lambda e, y'+\mu e)\in \mathcal{H}\times\mathcal{H}$, we have*

$$\langle x, D_x\psi_2(x,y)\rangle + \langle y, D_y\psi_2(x,y)\rangle = \|\phi_{\rm FB}(x,y)_+\|^2.$$

(e) *ψ_2 belongs to Ψ_{++}.*

Proof. (a) Suppose $\psi_2(x,y)=0$ and $\langle x,y\rangle\leq 0$. Let $z:=-\phi_{\rm FB}(x,y)$. Then $(-z)_+=\phi_{\rm FB}(x,y)_+=0$ which says $z\in\mathbb{K}$. Since $x+y=(x^2+y^2)^{1/2}+z$, squaring both sides and simplifying yield

$$2x\bullet y = 2((x^2+y^2)^{1/2}\bullet z)+z^2.$$

Now, taking trace of both sides and using the fact ${\rm tr}(x\bullet y)=2\langle x,y\rangle$, we obtain

$$4\langle x,y\rangle = 4\langle (x^2+y^2)^{1/2}, z\rangle + 2\|z\|^2. \tag{17}$$

Since $(x^2+y^2)^{1/2}\in\mathbb{K}$ and $z\in\mathbb{K}$, then we know $\langle (x^2+y^2)^{1/2}, z\rangle\geq 0$ by Lemma 3.1(b). Thus the right-hand of (17) is nonnegative, which together with $\langle x,y\rangle\leq 0$ implies $\langle x,y\rangle=0$. Therefore, with this, the equation (17) says $z=0$ which is equivalent to $\phi_{\rm FB}(x,y)=0$. Then by Lemma 3.3, we have $x,y\in\mathbb{K}$. Conversely, if $x,y\in\mathbb{K}$ and $\langle x,y\rangle=0$, then again Lemma 3.3 yields $\phi_{\rm FB}(x,y)=0$. Thus, $\psi_2(x,y)=0$ and $\langle x,y\rangle\leq 0$.

(b) For the proof of part (b), we need to discuss three cases.

Case 1: If $(x,y)=(0,0)$, then for any $h=h'+\bar{\lambda}e$, $k=k'+\bar{\mu}e\in\mathcal{H}$, let $\mu_1\le\mu_2$ be the spectral values and let $v^{(1)},v^{(2)}$ be the corresponding spectral vectors of h^2+k^2. Hence, by Property 2.1(e), we have

$$\begin{aligned}\|(h^2+k^2)^{1/2}-h-k\| &= \|\sqrt{\mu_1}v^{(1)}+\sqrt{\mu_2}v^{(2)}-h-k\|\\ &\le \sqrt{\mu_1}\|v^{(1)}\|+\sqrt{\mu_2}\|v^{(2)}\|+\|h\|+\|k\|\\ &= (\sqrt{\mu_1}+\sqrt{\mu_2})/\sqrt{2}+\|h\|+\|k\|.\end{aligned}$$

Also

$$\begin{aligned}\mu_1\le\mu_2 &= \|h\|^2+\|k\|^2+2\|\bar{\lambda}h'+\bar{\mu}k'\|\\ &\le \|h\|^2+\|k\|^2+2|\bar{\lambda}|\|h'\|+2|\bar{\mu}|\|k'\|\\ &\le 2(\|h\|^2+\|k\|^2).\end{aligned}$$

Combining the above two inequalities yields

$$\begin{aligned}\psi_2(h,k)-\psi_2(0,0) &= \frac{1}{2}\|\phi_{\rm FB}(h,k)_+\|^2\\ &\le \|\phi_{\rm FB}(h,k)\|^2\\ &= \|(h^2+k^2)^{1/2}-h-k\|^2\\ &\le ((\sqrt{\mu_1}+\sqrt{\mu_2})/\sqrt{2}+\|h\|+\|k\|)^2\\ &\le (2\sqrt{2\|h\|^2+2\|k\|^2}/\sqrt{2}+\|h\|+\|k\|)^2\\ &= O(\|h\|^2+\|k\|^2),\end{aligned}$$

where the first inequality is from Lemma 3.3. This shows that ψ_2 is differentiable at $(0,0)$ with

$$D_x\psi_2(0,0)=D_y\psi_2(0,0)=0.$$

Case 2: If $(x,y)\neq(0,0)$ and $x^2+y^2\in\mathrm{int}(\mathbb{K})$, let z be factored as $z=\alpha_1(z)u_z^{(1)}+\alpha_2(z)u_z^{(2)}$ for any $z\in\mathcal{H}$. Now, let $g:\mathcal{H}\to\mathcal{H}$ be defined as

$$g(z):=\frac{1}{2}((z)_+)^2=\hat{g}(\alpha_1(z))u_z^{(1)}+\hat{g}(\alpha_2(z))u_z^{(2)},$$

where $\hat{g}:\mathbb{R}\to\mathbb{R}$ is given by $\hat{g}(\alpha):=\frac{1}{2}(\max(0,\alpha))^2$. From the continuous differentiability of $\hat{g}$ and [17], the vector-valued function g is continuously Fréchet differentiable. Hence, the first component $g_1(z)=\frac{1}{2}\|(z)_+\|^2$ of $g(z)$ is continuously Fréchet differentiable as well. By an easy computation, we have $Dg_1(z)=(z)_+$. Since $\psi_2(x,y)=g_1(\phi_{\rm FB}(x,y))$ and $\phi_{\rm FB}$ is Fréchet differentiable at $(x,y)\neq(0,0)$ with $x^2+y^2\in\mathrm{int}(\mathbb{K})$ (see [5]). Hence, the chain rule yields

$$D_x\psi_2(x,y)=D_x\phi_{\rm FB}(x,y)Dg_1(\phi_{\rm FB}(x,y))=L_xL^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y)_+-\phi_{\rm FB}(x,y)_+,$$

$$D_y\psi_2(x,y) = D_y\phi_{\rm FB}(x,y)Dg_1(\phi_{\rm FB}(x,y)) = L_yL^{-1}_{(x^2+y^2)^{1/2}}\phi_{\rm FB}(x,y)_+ - \phi_{\rm FB}(x,y)_+.$$

Case 3: If $(x,y)\neq(0,0)$ and $x^2+y^2\notin \mathrm{int}(\mathbb{K})$, by direct computation, we know $\|x\|^2+\|y\|^2 = 2\|\lambda x'+\mu y'\|$ under this case. Since $(x,y)\neq(0,0)$, this also implies $\lambda x'+\mu y'\neq 0$. We notice that we can not apply the chain rule as in Case 2 because $\phi_{\rm FB}$ is no longer differentiable in this case. By the spectral factorization, we observe that

$$\begin{array}{rcl} \phi_{\rm FB}(x,y)_+ = \phi_{\rm FB}(x,y) & \Longleftrightarrow & \phi_{\rm FB}(x,y)\in\mathbb{K} \\ \phi_{\rm FB}(x,y)_+ = 0 & \Longleftrightarrow & \phi_{\rm FB}(x,y)\in-\mathbb{K} \\ \phi_{\rm FB}(x,y)_+ = \alpha_2 u^{(2)} & \Longleftrightarrow & \phi_{\rm FB}(x,y)\notin\mathbb{K}\cup-\mathbb{K}, \end{array} \tag{18}$$

where α_2 is the bigger spectral value of $\phi_{\rm FB}(x,y)$ and $u^{(2)}$ is the corresponding spectral vector. Indeed, by applying Property 2.4, we can simplify $\phi_{\rm FB}$ as

$$\phi_{\rm FB}(x,y) = \frac{\lambda x'+\mu y'}{\sqrt{\lambda^2+\mu^2}} - (x'+y') + \left(\sqrt{\lambda^2+\mu^2}-(\lambda+\mu)\right)e. \tag{19}$$

Therefore, α_2 and $u^{(2)}$ are given as below:

$$\begin{array}{rcl} \alpha_2 & = & \sqrt{\lambda^2+\mu^2}-(\lambda+\mu)+\|w_2\|, \\ u^{(2)} & = & \dfrac{1}{2}\left(\dfrac{w_2}{\|w_2\|}+e\right), \end{array} \tag{20}$$

where $w_2 = \dfrac{\lambda x'+\mu y'}{\sqrt{\lambda^2+\mu^2}} - (x'+y')$.
To prove the differentiability of ψ_2 under this case, we shall discuss the following three subcases according to the above observation (18).

(i) If $\phi_{\rm FB}(x,y)\notin\mathbb{K}\cup-\mathbb{K}$ then $\phi_{\rm FB}(x,y)_+ = \alpha_2 u^{(2)}$, where α_2 and $u^{(2)}$ are given as in (20). From the fact that $\|u^{(2)}\| = 1/\sqrt{2}$, we obtain

$$\begin{array}{rcl} \psi_2(x,y) & = & \dfrac{1}{2}\|\phi_{\rm FB}(x,y)_+\|^2 = \dfrac{1}{4}\alpha_2^2 \\ & = & \dfrac{1}{4}\Big[(\sqrt{\lambda^2+\mu^2}-(\lambda+\mu))^2 \\ & & +2\left(\sqrt{\lambda^2+\mu^2}-(\lambda+\mu)\right)\|w_2\| + \|w_2\|^2\Big]. \end{array}$$

Since $(x,y)\neq(0,0)$ in this case, ψ_2 is Fréchet differentiable clearly. For all $h\in\mathcal{H}$, we have

$$
\begin{aligned}
& [D_x w_2]h \\
= & \left(\frac{1}{\sqrt{\lambda^2+\mu^2}} - \frac{\lambda^2}{(\lambda^2+\mu^2)\sqrt{\lambda^2+\mu^2}}\right)\langle h, e\rangle x' + \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\big(h - \langle h, e\rangle e\big) \\
& - \frac{\lambda\mu y'}{(\lambda^2+\mu^2)\sqrt{\lambda^2+\mu^2}}\langle h, e\rangle \\
= & \frac{1}{\left(\sqrt{\lambda^2+\mu^2}\right)^3}\Big((\lambda^2+\mu^2)x' - \lambda^2 x' - \lambda\mu y'\Big)\langle h, e\rangle + \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}}\right)\big(h - \langle h, e\rangle e\big) \\
= & \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\big(h - \langle h, e\rangle e\big),
\end{aligned}
$$

where the last equality holds by Property 2.4. Using the product rule and chain rule for differentiation gives

$$
\begin{aligned}
[D_x\psi_2(x,y)]h &= \frac{1}{2}\alpha_2[D_x\alpha_2]h \\
&= \frac{1}{2}\alpha_2\left[\left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\langle h, e\rangle + \frac{\langle w_2, [D_x w_2]h\rangle}{\|w_2\|}\right] \\
&= \frac{1}{2}\alpha_2\left[\left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\langle h, e\rangle + \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\frac{\langle w_2, h\rangle - \langle w_2, e\rangle\langle h, e\rangle}{\|w_2\|}\right] \\
&= \frac{1}{2}\alpha_2\left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\left\langle \frac{w_2}{\|w_2\|} + e, h\right\rangle.
\end{aligned}
$$

It then follows that

$$
D_x\psi_2(x,y) = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\alpha_2 u^{(2)} = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y)_+. \tag{21}
$$

Similarly, we can obtain that

$$
D_y\psi_2(x,y) = \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y)_+.
$$

(ii) If $\phi_{\rm FB}(x,y) \in \mathbb{K}$ then $\phi_{\rm FB}(x,y)_+ = \phi_{\rm FB}(x,y)$, and hence $\psi_2(x,y) = \frac{1}{2}\|\phi_{\rm FB}(x,y)_+\|^2 = \frac{1}{2}\|\phi_{\rm FB}(x,y)\|^2$. Thus, by Proposition 3.1, we know that the derivative of ψ_2 under this subcase is as below:

$$
\begin{aligned}
D_x\psi_2(x,y) &= \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y) = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y)_+, \\
D_y\psi_2(x,y) &= \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y) = \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x,y)_+.
\end{aligned} \tag{22}
$$

If there is (x', y') such that $\phi_{\rm FB}(x', y') \notin \mathbb{K} \cup -\mathbb{K}$ and $\phi_{\rm FB}(x', y') \to \phi_{\rm FB}(x, y) \in \mathbb{K}$ (the neighborhood of point belonging to this subcase). From (21) and (22), it can be seen that

$$D_x\psi_2(x', y') \to D_x\psi_2(x, y), \ D_y\psi_2(x', y') \to D_y\psi_2(x, y).$$

Thus, ψ_2 is differentiable under this subcase.

(iii) If $\phi_{\rm FB}(x, y) \in -\mathbb{K}$ then $\phi_{\rm FB}(x, y)_+ = 0$. Thus, $\psi_2(x, y) = \frac{1}{2}\|\phi_{\rm FB}(x, y)_+\|^2 = 0$ and it is clear that its derivative under this subcase is

$$\begin{aligned} D_x\psi_2(x, y) = 0 = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x, y)_+, \\ D_y\psi_2(x, y) = 0 = \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\rm FB}(x, y)_+. \end{aligned} \tag{23}$$

Again, if there is (x', y') such that $\phi_{\rm FB}(x', y') \notin \mathbb{K} \cup -\mathbb{K}$ and $\phi_{\rm FB}(x', y') \to \phi_{\rm FB}(x, y) \in \mathbb{K}$ (the neighborhood of point belonging to this subcase). From (21) and (23), it can be seen that

$$D_x\psi_2(x', y') \to D_x\psi_2(x, y), \ D_y\psi_2(x', y') \to D_y\psi_2(x, y).$$

Thus, ψ_2 is differentiable under this subcase. From the above, we complete the proof of this case and therefore the argument for part (b) is done.

(c) We wish to show that $\langle D_x\psi_2(x, y), D_y\psi_2(x, y)\rangle \geq 0$ and the equality holds if and only if $\psi_2(x, y) = 0$. We follow the three cases as above.

Case 1: If $(x, y) = (0, 0)$, by part (b), we know $D_x\psi_2(x, y) = D_y\psi_2(x, y) = 0$. Therefore, the desired equality holds.

Case 2: If $(x, y) \neq (0, 0)$ and $x^2 + y^2 \in {\rm int}\mathbb{K}$, by part (b), we have

$$\begin{aligned} &\langle D_x\psi_2(x, y), D_y\psi_2(x, y)\rangle \\ &= \langle (L_xL_z^{-1} - I)(\phi_{\rm FB})_+, (L_yL_z^{-1} - I)(\phi_{\rm FB})_+\rangle \\ &= \langle (L_x - L_z)L_z^{-1}(\phi_{\rm FB})_+, (L_y - L_z)L_z^{-1}(\phi_{\rm FB})_+\rangle \\ &= \langle (L_y - L_z)(L_x - L_z)L_z^{-1}(\phi_{\rm FB})_+, L_z^{-1}(\phi_{\rm FB})_+\rangle, \end{aligned} \tag{24}$$

where $z = \sqrt{x^2 + y^2}$ and $I \in \mathcal{L}(\mathcal{H})$ is an identity mapping. From elementary calculation, we obtain that

$$(L_z - L_x)(L_z - L_y) + (L_z - L_y)(L_z - L_x) = (L_z - L_x - L_y)^2 + (L_z^2 - L_x^2 - L_y^2).$$

Since $z \in \mathbb{K}$ and $z^2 = x^2 + y^2$, Property 2.3 implies $L_z^2 - L_x^2 - L_y^2 \succeq 0$. Then (24) yields

$$\begin{aligned} \langle D_x\psi_2(x, y), D_y\psi_2(x, y)\rangle &\geq \frac{1}{2}\|(L_z - L_x - L_y)L_z^{-1}(\phi_{\rm FB})_+\|^2 \\ &= \frac{1}{2}\|L_{\phi_{\rm FB}}L_z^{-1}(\phi_{\rm FB})_+\|^2, \end{aligned}$$

where the equality uses $L_z - L_x - L_y = L_{z-x-y} = L_{\phi_{\rm FB}}$. If the equality holds, then the above relation yields $\|L_{\phi_{\rm FB}} L_z^{-1}(\phi_{\rm FB})_+\|^2 = 0$ and, by Property 2.1(d),

$$L_{\phi_{\rm FB}} L_z^{-1}(\phi_{\rm FB})_+ = \phi_{\rm FB} \bullet (L_z^{-1}(\phi_{\rm FB})_+) = (L_z^{-1}(\phi_{\rm FB})_+) \bullet \phi_{\rm FB} = 0.$$

Since $z = \sqrt{x^2+y^2} \in {\rm int}(\mathbb{K})$ so that $L_z^{-1} \succ 0$ (see Property 2.1(d)), multiplying L_z^{-1} both side gives $\phi_{\rm FB} \bullet (\phi_{\rm FB})_+ = 0$. From definition of Jordan product and Lemma 3.1(a), it implies $(\phi_{\rm FB})_+ = 0$ and hence $\psi_2 = 0$. Conversely, if $(\phi_{\rm FB})_+ = 0$, then it is clear that $\langle D_x\psi_2(x,y), D_y\psi_2(x,y)\rangle = 0$.

<u>Case 3</u>: If $(x,y) \neq (0,0)$ and $x^2+y^2 \notin {\rm int}(\mathbb{K})$, by part (b), we have

$$\langle D_x\psi_2(x,y), D_y\psi_2(x,y)\rangle = \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\|\phi_{\rm FB}(x,y)_+\|^2 \geq 0.$$

If the equality holds, then either $\phi_{\rm FB}(x,y)_+ = 0$ or $\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} = 1$ or $\frac{\mu}{\sqrt{\lambda^2+\mu^2}} = 1$. In the second case, we have $\mu = 0$ and $\lambda \geq 0$, so that Property 2.4 yields $y' = 0$ and $\lambda = \|x'\|$. In the third case, we have $\lambda = 0$ and $\mu \geq 0$, so that Property 2.4 yields $x' = 0$ and $\mu = \|y'\|$. Thus, in these cases, we have $x \bullet y = 0, x \in \mathbb{K}, y \in \mathbb{K}$. Then, by (7), $\psi_2(x,y) = 0$.

(d) Again, we need to discuss the three cases as below.

<u>Case 1</u>: If $(x,y) = (0,0)$, by part (b), we know $D_x\psi_2(x,y) = D_x\psi_2(x,y) = 0$. Therefore, the desired equality holds.

<u>Case 2</u>: If $(x,y) \neq (0,0)$ and $x^2+y^2 \in {\rm int}(\mathbb{K})$, by part (b), we have

$$\begin{aligned} D_x\psi_2(x,y) &= (L_xL_z^{-1} - I)\phi_{\rm FB}(x,y)_+, \\ D_y\psi_2(x,y) &= (L_yL_z^{-1} - I)\phi_{\rm FB}(x,y)_+, \end{aligned}$$

where we let $z = \sqrt{x^2+y^2}$. Thus,

$$\begin{aligned} & \langle x, D_x\psi_2(x,y)\rangle + \langle y, D_y\psi_2(x,y)\rangle \\ = \ & \langle x, (L_xL_z^{-1} - I)\phi_{\rm FB}(x,y)_+\rangle + \langle y, (L_yL_z^{-1} - I)\phi_{\rm FB}(x,y)_+\rangle \\ = \ & \langle (L_xL_z^{-1} - I)x, \phi_{\rm FB}(x,y)_+\rangle + \langle (L_yL_z^{-1} - I)y, \phi_{\rm FB}(x,y)_+\rangle \\ = \ & \langle L_z^{-1}L_xx + L_z^{-1}L_yy - x - y, \phi_{\rm FB}(x,y)_+\rangle \\ = \ & \langle L_z^{-1}(x^2+y^2) - x - y, \phi_{\rm FB}(x,y)_+\rangle \\ = \ & \langle L_z^{-1}z^2 - x - y, \phi_{\rm FB}(x,y)_+\rangle \\ = \ & \langle z - x - y, \phi_{\rm FB}(x,y)_+\rangle \\ = \ & \|\phi_{\rm FB}(x,y)_+\|^2, \end{aligned}$$

where the next-to-last equality follows from $L_zz = z^2$, so that $L_z^{-1}z^2 = z$ and the last equality is from Lemma 3.1(a).

Case 3: If $(x, y) \neq (0, 0)$ and $x^2 + y^2 \notin \text{int}(\mathbb{K})$, by part (b), we have

$$\begin{aligned} D_x\psi_2(x,y) &= \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\text{FB}}(x,y)_+, \\ D_y\psi_2(x,y) &= \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\phi_{\text{FB}}(x,y)_+. \end{aligned}$$

Thus,

$$\begin{aligned} & \langle x, D_x\psi_2(x,y)\rangle + \langle y, D_y\psi_2(x,y)\rangle \\ = & \left(\frac{\lambda}{\sqrt{\lambda^2+\mu^2}} - 1\right)\langle x, \phi_{\text{FB}}(x,y)_+\rangle + \left(\frac{\mu}{\sqrt{\lambda^2+\mu^2}} - 1\right)\langle y, \phi_{\text{FB}}(x,y)_+\rangle \\ = & \left\langle \frac{\lambda x + \mu y}{\sqrt{\lambda^2+\mu^2}} - x - y, \phi_{\text{FB}}(x,y)_+ \right\rangle \\ = & \langle \phi_{\text{FB}}(x,y), \phi_{\text{FB}}(x,y)_+\rangle \\ = & \|\phi_{\text{FB}}(x,y)_+\|^2, \end{aligned}$$

where the next-to-last equality uses (19) and the last equality is from Lemma 3.1(a) again.

(e) This is an immediate consequence of (a) through (d). □

Proposition 3.4 *Let $f_{\text{LT}} : \mathcal{H} \to \mathbb{R}_+$ be given as (5) with ψ_0 satisfying (6) and ψ satisfying (7). Then, the following results hold.*

(a) *For all $\zeta \in \mathcal{H}$, we have $f_{\text{LT}}(\zeta) \geq 0$ and $f_{\text{LT}}(\zeta) = 0$ if and only if ζ solves the infinite-dimensional SOCCP (1).*

(b) *Let $D\psi_0(\langle F(\zeta), G(\zeta)\rangle) = D_t\psi_0(t)$ with $t = \langle F(\zeta), G(\zeta)\rangle$. If ψ_0, ψ and F, G are Fréchet differentiable, then so is f_{LT} and*

$$\begin{aligned} Df_{\text{LT}}(\zeta) &= D\psi_0(\langle F(\zeta), G(\zeta)\rangle)[(DF(\zeta))^T G(\zeta) + (DG(\zeta))^T F(\zeta)] \\ &\quad + (DF(\zeta))^T D_x\psi(F(\zeta), G(\zeta)) + (DG(\zeta))^T D_y\psi(F(\zeta), G(\zeta)). \end{aligned}$$

(c) *Assume F, G are are Fréchet differentiable mappings on $\mathcal{H}$ and ψ belongs to Ψ_+ (respectively, Ψ_{++}). Then, for every $\zeta \in \mathcal{H}$ where $DF(\zeta)[DG(\zeta)]^{-1}$ is positive definite (respectively, positive semi-definite), either (i) $f_{\text{LT}}(\zeta) = 0$ or (ii) $f_{\text{LT}}(\zeta) \neq 0$ with $\langle d(\zeta), Df_{\text{LT}}(\zeta)\rangle < 0$, where*

$$d(\zeta) := -(DG(\zeta)^{-1})[D\psi_0(\langle F(\zeta), G(\zeta)\rangle)G(\zeta) + D_x\psi(F(\zeta), G(\zeta))].$$

Proof. (a) This consequence follows from (5), (6) and (7).

(b) Fix any $\zeta \in \mathcal{H}$. From Theorem 4.2 in [5] and the Fréchet differentiability of F and G, it follows that $f_{\text{LT}} : \mathcal{H} \to \mathbb{R}_+$ is Fréchet differentiable on $\mathcal{H}$. By the chain rule of differential, we have, for any $v \in \mathcal{H}$,

$$\begin{aligned} Df_{\text{LT}}(\zeta)v &= D\psi_0(\langle F(\zeta), G(\zeta)\rangle)[\langle DF(\zeta)v, G(\zeta)\rangle + \langle DG(\zeta)v, F(\zeta)\rangle] \\ &\quad + \langle D_x\psi(F(\zeta), G(\zeta)), DF(\zeta)v\rangle + \langle D_y\psi(F(\zeta), G(\zeta)), DG(\zeta)v\rangle, \end{aligned}$$

which means

$$\begin{aligned} Df_{\text{LT}}(\zeta) &= D\psi_0(\langle F(\zeta), G(\zeta)\rangle)[(DF(\zeta))^T G(\zeta) + (DG(\zeta))^T F(\zeta)] \\ &\quad + (DF(\zeta))^T D_x\psi(F(\zeta), G(\zeta)) + (DG(\zeta))^T D_y\psi(F(\zeta), G(\zeta)). \end{aligned}$$

(c) First, we consider the case of $\psi \in \Psi_{++}$ and fix $\zeta \in \mathcal{H}$, where $DF(\zeta)[DG(\zeta)]^{-1}$ is positive semi-definite. Let $\alpha := D\psi_0(\langle F(\zeta), G(\zeta)\rangle)$ and drop the argument (ζ) for simplicity. Then

$$\begin{aligned} \langle d, Df_{\text{LT}}\rangle &= \langle -(DG)^{-1}(\alpha G + D_x\psi(F,G)), (DF)^T(\alpha G + D_x\psi(F,G)) \\ &\quad + (DG)^T(\alpha F + D_y\psi(F,G))\rangle \\ &= -\langle \alpha G + D_x\psi(F,G), ((DG)^{-1})^T(DF)^T(\alpha G + D_x\psi(F,G))\rangle \\ &\quad - \langle \alpha G + D_x\psi(F,G), \alpha F + D_y\psi(F,G)\rangle \\ &\leq -\langle \alpha G + D_x\psi(F,G), \alpha F + D_y\psi(F,G)\rangle \\ &= -\alpha^2\langle G, F\rangle - \alpha(\langle F, D_x\psi(F,G)\rangle) + \langle G, D_y\psi(F,G)\rangle) \\ &\quad - \langle D_x\psi(F,G), D_y\psi(F,G)\rangle \\ &\leq -\alpha^2\langle G, F\rangle - \langle D_x\psi(F,G), D_y\psi(F,G)\rangle, \end{aligned}$$

where the first inequality holds since $DF(DG)^{-1}$ is positive semi-definite and the inequality follows from $\alpha \geq 0$ and equation (13). Now, we observe that $tD\psi_0(t) > 0$ if and only if $t > 0$ since ψ_0 is strictly increasing on $[0, \infty)$. Therefore, the first term on the right-hand side is non-positive and equals zero if $\langle F, G\rangle \leq 0$. In addition, by equations (13) and (15), the second term on the right-hand side is non-positive and equals zero if $\psi(F, G) = 0$. Thus, we have $\langle d, Df_{\text{LT}}(\zeta)\rangle \leq 0$ and the equality hold only when $\langle F(\zeta), G(\zeta)\rangle \leq 0$ and $\psi(F(\zeta), G(\zeta)) = 0$, in which equation (7) implies ζ satisfies (1), i.e., $f_{\text{LT}}(\zeta) = 0$.

Similar argument can be applied for the case of $\psi \in \Psi_+$ and $DF(DG)^{-1}$ being positive definite. $\square$

Next, we further consider another class of merit functions given as (8), i.e.,

$$\widehat{f_{\text{LT}}}(\zeta) := \psi_0^*(F(\zeta) \bullet G(\zeta)) + \psi(F(\zeta), G(\zeta)),$$

where $\psi_0^* : \mathcal{H} \to \mathbb{R}_+$ is given as (9) and $\psi : \mathcal{H} \times \mathcal{H} \to \mathbb{R}_+$ satisfies (7). We notice that ψ_0^* possesses the following property:

$$\psi_0^*(w) = 0 \Longleftrightarrow w \preceq_{\mathbb{K}} 0$$

which is a similar feature to (6) in some sense.

By imitating the steps for proving Proposition 3.4 and using the following Lemma 3.4 which has been proved in SOC case by Chen [2], we obtain Proposition 3.5 which is a result analogous to Proposition 3.4. We omit its proof.

Lemma 3.4 *The function* $\psi_0^*(x \bullet y) := \frac{1}{2}\|(x \bullet y)_+\|^2$ *is differentiable for all* $(x, y) \in \mathcal{H} \times \mathcal{H}$. *Moreover,* $D_x\psi_0^*(x \bullet y) = L_y(x \bullet y)_+$*, and* $D_y\psi_0^*(x \bullet y) = L_x(x \bullet y)_+$.

Proof. For any $z \in \mathcal{H}$, we can factor z as $z = \alpha_1(z)u^{(1)} + \alpha_2(z)u^{(2)}$. Now, let $g : \mathcal{H} \to \mathcal{H}$ be defined as

$$g(z) := \frac{1}{2}((z)_+)^2 = \hat{g}(\alpha_1(z))u^{(1)} + \hat{g}(\alpha_2(z))u^{(2)},$$

where $\hat{g} : \mathbb{R} \to \mathbb{R}$ is given by $\hat{g}(\alpha) := \frac{1}{2}(\max(0, \alpha))^2$. From the continuous differentiability of $\hat{g}$ and [17], the vector-valued function g is continuously Fréchet differentiable. Hence, the first component $g_1(z) = \frac{1}{2}\|(z)_+\|^2$ of $g(z)$ is continuously Fréchet differentiable as well. By an easy computation, we have $Dg_1(z) = (z)_+$. Now, let

$$z(x, y) := x \bullet y = (\lambda y' + \mu x') + \langle x, y\rangle e,$$

then we have $\psi_0^*(x \bullet y) = g_1(z(x, y))$. Fix $x = x' + \lambda e \in \mathcal{H}$, $y = y' + \mu e \in \mathcal{H}$ and $h = h' + le \in \mathcal{H}$, then we have

$$\begin{aligned} [D_x z(x, y)]h &= \langle h, e\rangle y' + \mu(h - \langle h, e\rangle e) + \langle h, y\rangle e \\ &= ly' + \mu h' + \langle h, y\rangle e \\ &= y \bullet h \\ &= L_y h. \end{aligned}$$

Similarly, we can obtain $[D_y z(x, y)]h = L_x h$. Hence, applying the chain rule, the desired result follows. □

Proposition 3.5 *Let* $\widehat{f_{\text{LT}}} : \mathcal{H} \to \mathbb{R}_+$ *be given as (8) with* ψ_0^* *satisfying (9) and* ψ *satisfying (7). Then, the following results hold.*

(a) *For all* $\zeta \in \mathcal{H}$*, we have* $\widehat{f_{\text{LT}}}(\zeta) \geq 0$ *and* $\widehat{f_{\text{LT}}}(\zeta) = 0$ *if and only if* ζ *solves the infinite-dimensional SOCCP (1).*

(b) *If ψ_0^*, ψ and F, G are Fréchet differentiable, then so is $\widehat{f_{\rm LT}}$ and*

$$\begin{aligned} D\widehat{f_{\rm LT}}(\zeta) &= [(DF(\zeta))^T L_{G(\zeta)} + (DG(\zeta))^T L_{F(\zeta)}](F(\zeta) \bullet G(\zeta))_+ \\ &\quad + (DF(\zeta))^T D_x\psi(F(\zeta), G(\zeta)) + (DG(\zeta))^T D_y\psi(F(\zeta), G(\zeta)). \end{aligned}$$

4 Solution existence, error bound, and bounded level sets

In this section, using the above merit functions $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$, we obtain error bounds for the solution of infinite-dimensional SOCCP (1). Meanwhile, we study the existence and uniqueness for the solution of $CP(\mathbb{K}, F, G)$. To reach our results, we need some lemmas as below.

Lemma 4.1 *Let $x = x' + \lambda e \in \mathcal{H}$ and $y = y' + \mu e \in \mathcal{H}$. Then, we have*

$$\langle x, y \rangle \leq \sqrt{2}\|(x \bullet y)_+\|. \tag{25}$$

Proof. First, we observe the fact that

$$\begin{aligned} x \in \mathbb{K} &\iff (x)_+ = x, \\ x \in -\mathbb{K} &\iff (x)_+ = 0, \\ x \notin \mathbb{K} \cup -\mathbb{K} &\iff (x)_+ = \alpha_2 u^{(2)}, \end{aligned}$$

where α_2 is the bigger spectral value of x with the corresponding spectral vector $u^{(2)}$ defined as in section 2. Hence, we have three cases.

Case 1: If $x \bullet y \in \mathbb{K}$, then $(x \bullet y)_+ = x \bullet y$. By definition of Jordan product of x and y as (11), i.e., $x \bullet y := (\mu x' + \lambda y') + \langle x, y \rangle e$. It is clear that $\|(x \bullet y)_+\| \geq \langle x, y \rangle$ and hence (25) holds.

Case 2: If $x \bullet y \in -\mathbb{K}$, then $(x \bullet y)_+ = 0$. Since $x \bullet y \in -\mathbb{K}$, by definition of Jordan product again, we have $\langle x, y \rangle \leq 0$. Hence, it is true that $\sqrt{2}\|(x \bullet y)_+\| \geq \langle x, y \rangle$.

Case 3: If $x \bullet y \notin \mathbb{K} \cup -\mathbb{K}$, then $(x \bullet y)_+ = \alpha_2 u^{(2)}$, where

$$\begin{aligned} \alpha_2 &= \langle x, y \rangle + \|\mu x' + \lambda y'\|, \\ u^{(2)} &= \frac{1}{2}\left(\frac{\mu x' + \lambda y'}{\|\mu x' + \lambda y'\|} + e\right). \end{aligned}$$

If $\langle x, y\rangle \leq 0$, then (25) is trivial. Thus, we can assume $\langle x, y\rangle > 0$. In fact, the desired inequality (25) follows from the below.

$$
\begin{aligned}
\|(x \bullet y)_+\|^2 &= \frac{1}{2}\alpha_2^2 \\
&= \frac{1}{2}(\langle x, y\rangle^2 + 2\langle x, y\rangle \|\mu x' + \lambda y'\| + \|\mu x' + \lambda y'\|^2) \\
&\geq \frac{1}{2}\langle x, y\rangle^2.
\end{aligned}
$$

Then, we complete the proof. $\Box$

Lemma 4.2 *Let ψ_{FB}, ψ_1, ψ_2 be given as (2), (14), (16), respectively. Then, ψ_{FB}, ψ_1, and ψ_2 satisfy the following inequality.*

$$\psi_\diamond(x, y) \geq \alpha(\|(-x)_+\|^2 + \|(-x)_+\|^2), \ \forall (x, y) \in \mathcal{H} \times \mathcal{H},$$

for some positive constant α and $\diamond \in \{\text{FB}, 1, 2\}$.

Proof. For ψ_1, it is clear by definition (14) with $\alpha = \frac{1}{2}$. For ψ_{FB} and ψ_2, the inequality is still true due to Lemma 3.3. $\Box$

By the definition of ψ_0^* given as (9), we can obtain the following lemma easily.

Lemma 4.3 *Let ψ_0^* be given as (9). Then, ψ_0^* satisfies*

$$\psi_0^*(\omega) \geq \beta\|(\omega)_+\|^2, \ \forall \omega \in \mathcal{H},$$

for some positive constant β.

Proposition 4.1 *Let F and G are Lipschitz continuous with constants γ and δ, respectively. Suppose that F is η-strongly monotone and F and G are ρ-jointly strongly monotone mapping from $\mathcal{H}$ to $\mathcal{H}$. Let f_{LT} be given by (5) with ψ satisfying (7). If there exists a constant τ such that*

$$\left|\tau - \frac{\rho}{\delta^2}\right| \leq \frac{\sqrt{\rho^2 - \delta^2(\gamma^2 - \sigma^2)}}{\delta^2}, \ \rho > \delta\sqrt{\gamma^2 - \sigma^2}, \tag{26}$$

where $\sigma = 1 - \sqrt{1 - 2\eta + \gamma^2}$. Then, the infinite-dimensional conic complementarity problem $CP(\mathbb{K}, F, G)$ given as in (1) has a unique solution ζ^* and there exists a scalar $\kappa > 0$ such that

$$\kappa\|\zeta - \zeta^*\|^2 \leq \max\{1, \langle F(\zeta), G(\zeta)\rangle\} + \|(-F(\zeta))_+\| + \|(-G(\zeta))_+\|, \ \forall \zeta \in \mathcal{H}.$$

Moreover,

$$\kappa\|\zeta - \zeta^*\|^2 \leq \psi_0^{-1}(f_{\text{LT}}(\zeta)) + \frac{\sqrt{2}}{\sqrt{\alpha}} f_{\text{LT}}(\zeta)^{1/2}, \ \forall \zeta \in \mathcal{H},$$

where α is a positive constant.

Proof. It is not hard to verify that ζ is a solution to (1) if and only if, for any constant $\tau > 0$, the following equation hold:

$$F(\zeta) - P_{\mathbb{K}}(F(\zeta) - \tau G(\zeta)) = 0,$$

where $P_{\mathbb{K}}$ is the projection of $\mathcal{H}$ onto $\mathbb{K}$, that is,

$$\zeta - F(\zeta) + P_{\mathbb{K}}(F(\zeta) - \tau G(\zeta)) = \zeta,$$

Based on this, we define the mapping $\mathcal{F}$ as

$$\mathcal{F}(x) = x - F(x) + P_{\mathbb{K}}(F(x) - \tau G(x)), \ \forall x \in \mathcal{H}. \tag{27}$$

From the above discussion, ζ satisfies (1) if and only if ζ is a fixed point of $\mathcal{F}$. For any $x, y \in \mathcal{H}$, we have that

$$\begin{aligned}
& \|\mathcal{F}(x) - \mathcal{F}(y)\| \\
= \ & \|x - y - (F(x) - G(x)) + (P_{\mathbb{K}}(F(x) - \tau G(x)) - P_{\mathbb{K}}(F(y) - \tau G(y)))\| \\
\leq \ & \|x - y - (F(x) - G(x))\| + \|F(x) - F(y) - \tau(G(x) - G(y))\| \\
\leq \ & \left(\sqrt{1 - 2\eta + \gamma^2} + \sqrt{\gamma^2 - 2\tau\rho + \tau^2\delta^2}\right) \|x - y\|.
\end{aligned}$$

From (26), it follows that $\sqrt{1 - 2\eta + \gamma^2} + \sqrt{\gamma^2 - 2\tau\rho + \tau^2\delta^2} < 1$, and hence the mapping defined by (27) has a unique fixed point belonging to $\mathcal{H}$, which is the solution of (1). Using Property 2.5 and Lemma 4.2, we can complete the rest of the proof. The arguments are similar to those in [2, Proposition 4.1], so we omit them. □

We would like to point out that, if the mapping F is Lipschitz continuous with constant γ and strongly monotone with constant η, then

$$\eta\|x - y\|^2 \leq \langle F(x) - F(y), x - y\rangle \leq \gamma\|x - y\|^2, \ \forall x, y \in \mathcal{H}$$

which implies that $\eta \leq \gamma$ and

$$1 - 2\eta + \gamma^2 \geq (1 - \gamma)^2 \geq 0. \tag{28}$$

Furthermore, if G and F are jointly strongly monotone with constant ρ and G is Lipschitz continuous with constant δ, then we can get $\rho \leq \gamma\delta$ and

$$\gamma^2 - 2\rho + \delta^2 \geq (\gamma - \delta)^2 \geq 0. \tag{29}$$

Inequalities (28) and (29) ensure that the set where (26) to be held is nonempty.

Similarly, we have the following Proposition 4.2 which is an extension of [2, Proposition 4.2].

Proposition 4.2 *Let F and G are Lipschitz continuous with constants γ and δ, respectively. Suppose that F is η-strongly monotone and F and G are ρ-jointly strongly monotone mapping from $\mathcal{H}$ to $\mathcal{H}$. Let $\widehat{f_{\rm LT}}$ be given by (8) with ψ satisfying (7). If there exists a constant τ such that*

$$\left|\tau - \frac{\rho}{\delta^2}\right| \leq \frac{\sqrt{\rho^2 - \delta^2(\gamma^2 - \sigma^2)}}{\delta^2}, \ \rho > \delta\sqrt{\gamma^2 - \sigma^2},$$

where $\sigma = 1 - \sqrt{1 - 2\eta + \gamma^2}$. Then, the infinite-dimensional conic complementarity problem $CP(\mathbb{K}, F, G)$ given as in (1) has a unique solution ζ^* and there exists a scalar $\kappa > 0$ such that

$$\kappa\|\zeta - \zeta^*\|^2 \leq \|(F(\zeta) \bullet G(\zeta))_+\| + \|(-F(\zeta))_+\| + \|(-G(\zeta))_+\|, \ \forall \zeta \in \mathcal{H}.$$

Moreover,

$$\kappa\|\zeta - \zeta^*\|^2 \leq \left(\frac{1}{\sqrt{\beta}} + \frac{\sqrt{2}}{\sqrt{\alpha}}\right) \widehat{f_{\rm LT}}(\zeta)^{1/2}, \ \forall \zeta \in \mathcal{H},$$

where α and β are positive constants.

Now, we give conditions under which $f_{\rm LT}, \widehat{f_{\rm LT}}$ have bounded level sets. We need the next lemma which is key to proving the properties of bounded level sets.

Lemma 4.4 *Let $\psi_{\rm FB}, \psi_1$ and ψ_2 be given by (2), (14) and (16), respectively. For any $\{(x^k, y^k)\}_{k=1}^{\infty} \in \mathcal{H} \times \mathcal{H}$. Let $\alpha_1(x)^k \leq \alpha_2(x)^k$ and $\alpha_1(y)^k \leq \alpha_2(y)^k$ denote the spectral values of x^k and y^k, respectively. Then, the following results hold.*

(a) If $\alpha_1(x)^k \to -\infty$ or $\alpha_1(y)^k \to -\infty$, then $\psi_\diamond(x^k, y^k) \to \infty$, for $\diamond \in \{\rm FB, 1, 2\}$.

(b) Suppose that $\{\alpha_1(x)^k\}$ and $\{\alpha_1(y)^k\}$ are bounded below. If $\alpha_2(x)^k \to \infty$ or $\alpha_2(y)^k \to \infty$, then $\langle x, x^k\rangle + \langle y, y^k\rangle \to \infty$, for any $x, y \in \mathrm{int}(\mathbb{K})$.

Proof. (a) For ψ_1, the proof follows by the fact that

$$\begin{aligned}
2\|(-x^k)_+\|^2 &= 2\|(-\alpha_1(x)^k v_x^{(1)^k} - \alpha_2(x)^k v_x^{(2)^k})_+\|^2 \\
&= 2\|(-\alpha_1(x)^k)_+ v_x^{(1)^k} - (\alpha_2(x)^k)_+ v_x^{(2)^k}\|^2 \\
&= (-\alpha_1(x)^k)_+^2 + (-\alpha_2(x)^k)_+^2 \\
&= \sum_{i=1}^{2} \left(\max\{0, -\alpha_i(x)^k\}\right)^2.
\end{aligned}$$

Similarly, we have $2\|(-y^k)_+\|^2 = \sum_{i=1}^2 (\max\{0, -\alpha_i(y)^k\})^2$. For $\psi_{\rm FB}$ and ψ_2, using the same fact plus Lemma 3.3 leads to the desired result.

(b) For any $x = x' + \lambda e$, $y = y' + \mu e \in \mathcal{H}$ with $\|x'\| \leq \lambda$, $\|y'\| \leq \mu$. Using the spectral decomposition

$$x^k = \frac{\alpha_2(x)^k - \alpha_1(x)^k}{2} w^k + \frac{\alpha_2(x)^k + \alpha_1(x)^k}{2} e \quad \text{with} \quad \|w^k\| = 1,$$

we have

$$\begin{aligned} \langle x, x^k \rangle &= \frac{\alpha_2(x)^k - \alpha_1(x)^k}{2} \langle x', w^k \rangle + \frac{\alpha_2(x)^k + \alpha_1(x)^k}{2} \lambda \\ &= \frac{\alpha_1(x)^k}{2} \left(\lambda - \langle x', w^k \rangle \right) + \frac{\alpha_2(x)^k}{2} \left(\lambda + \langle x', w^k \rangle \right). \end{aligned} \tag{30}$$

Since $\|w^k\| = 1$, we have $\lambda - \langle x', w^k \rangle \geq \lambda - \|x'\| > 0$ and $\lambda + \langle x', w^k \rangle \geq \lambda - \|x'\| > 0$. In addition, $\{\alpha_1(x)^k\}$ is bounded below, the first term on the right-hand side of (30) is bounded below. If $\{\alpha_2(x)^k\} \to \infty$, then the second term on the right-hand side of (30) tends to infinity. Hence $\langle x, x^k \rangle \to \infty$. A similar argument shows that $\langle y, y^k \rangle$ is bounded below. Thus, $\langle x, x^k \rangle + \langle y, y^k \rangle \to \infty$. Symmetrically, the desired results also holds if $\{\alpha_2(y)^k\} \to \infty$. □

Using the above Lemma 4.4 and the same arguments in Proposition 4.3 and Proposition 4.4 in [2], we have the following Proposition 4.3 and Proposition 4.4.

Proposition 4.3 *Let f_{LT} be given by (5) with ψ satisfying the condition of Lemma 4.4(a). Suppose that F, G are Fréchet differentiable, jointly monotone mappings from $\mathcal{H}$ to $\mathcal{H}$ satisfying*

$$\lim_{\|\zeta\| \to \infty} \Big(\|F(\zeta)\| + \|G(\zeta)\| \Big) = \infty.$$

Suppose also that the infinite-dimensional SOCCP is strictly feasible, i.e., there exists $\bar{\zeta} \in \mathcal{H}$ such that $F(\bar{\zeta}), G(\bar{\zeta}) \in \text{int}(\mathbb{K})$. Then, the level set

$$\mathcal{L}(\nu) := \left\{ \zeta \in \mathcal{H} \;\middle|\; f_{\text{LT}}(\zeta) \leq \nu \right\}$$

is bounded for all $\nu \geq 0$.

Proposition 4.4 *Let $\widehat{f_{\text{LT}}}$ be given by (8) with ψ satisfying the condition of Lemma 4.4(a). Suppose that F, G are Fréchet differentiable, jointly monotone mappings from $\mathcal{H}$ to $\mathcal{H}$ satisfying*

$$\lim_{\|\zeta\| \to \infty} \Big(\|F(\zeta)\| + \|G(\zeta)\| \Big) = \infty.$$

Suppose also that the infinite-dimensional SOCCP is strictly feasible, i.e., there exists $\bar{\zeta} \in \mathcal{H}$ such that $F(\bar{\zeta}), G(\bar{\zeta}) \in \text{int}(\mathbb{K})$. Then, the level set

$$\mathcal{L}(\nu) := \left\{ \zeta \in \mathcal{H} \;\middle|\; \widehat{f_{\text{LT}}}(\zeta) \leq \nu \right\}$$

is bounded for all $\nu \geq 0$.

5 Conclusions

In this paper, we extend two classes of merit functions $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ to infinite-dimensional SOCCP. We show analogous properties as in finite-dimensional cases. In particular, under the condition that F and G are jointly monotone and a strictly feasible solution exists, we prove that the infinite-dimensional SOCCP has a unique solution and both $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ have bounded level sets which will ensure that the sequence generated by a decent algorithm has at least an accumulation point. All these will make it possible to construct a decent algorithm for solving the equivalent unconstrained reformulation of the infinite-dimensional SOCCP. In addition, we show that both $f_{\rm LT}$ and $\widehat{f_{\rm LT}}$ are Fréchet differentiable and their derivatives have computable formulas. All the aforementioned features are significant reasons for studying these merit functions.

An interesting issue after this paper is to establish the semismoothness of all merit functions studied here and in [5], which will provide a background brick for analysis of semismooth methods for infinite-dimensional SOCCP. Another direction is to explore other conditions under which $CP(\mathbb{K}, F, G)$ has a unique solution. We leave them as future research topics.

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