SOC Functions and Their Applications

by

Jein-Shan Chen

Department of Mathematics

National Taiwan Normal University

October 30, 2018

Preface

The second-order cone programs (SOCP) have been an attraction due to plenty of applications in engineering, data science, and finance. To deal with this special type of optimization problems involving second-order cone (SOC). We believe that the following items are crucial concepts: (i) spectral decomposition associated with SOC, (ii) analysis of SOC functions, (iii) SOC-convexity and SOC-monotonicity. In this book, we go through all these concepts and try to provide the readers a whole picture regarding SOC functions and their applications.

As introduced in Chapter 1, the SOC functions are indeed vector-valued functions associated with SOC, which are accompanied by Jordan product. However, unlike the matrix multiplication, the Jordan product associated with SOC is not associative which is the main source of difficulty when we do the analysis. Therefore, the ideas for proofs are usually quite different from those for matrix-valued functions. In other words, although SOC and positive semidefinite cone both belong to symmetric cones, the analysis for them are different. In general, the arguments are more tedious and need subtle arrangements in the SOC setting. This is due to the feature of SOC.

To deal with second-order cone programs (SOCPs) and second-order cone complementarity problems (SOCCPs), many methods rely on some SOC complementarity functions or merit functions to reformulate the KKT optimality conditions as a nonsmooth (or smoothing) system of equations or an unconstrained minimization problem. In fact, such SOC complementarity or merit functions are connected to SOC functions. In other words, the vector-valued functions associated with SOC are heavily used in the solutions methods for SOCP and SOCCP. Therefore, further study on these functions will be helpful for developing and analyzing more solutions methods.

For SOCP, there are still many approaches without using SOC complementarity functions. In this case, the concepts of SOC-convexity and SOC-monotonicity introduced in Chapter 2 play a key to those solution methods. In Chapter 3, we present proximal-type algorithms in which SOC-convexity and SOC-monotonicity are needed in designing solution methods and proving convergence analysis.

In Chapter 4, we pay attention to some other types of applications of SOC-functions, SOC-convexity, and SOC-monotonicity introduced in this monograph. These include so-called SOC means, SOC weighted means, and a few SOC trace versions of Young, Hölder, Minkowski inequalities, and Powers-Størmer's inequality. All these materials are newly discovered and we believe that they will be helpful in convergence analysis of various optimizations involving SOC. Chapter 5 offers a direction for future investigation, although it is not very consummate yet.

This book is based on my series of study regarding second-order cone, SOCP, SOCCP, SOC-functions, etc. during the past fifteen years. It is dedicated to the memory of my supervisor, Prof. Paul Tseng, who guided me into optimization research, especially to second-order cone optimization. Without his encouragement, it is not possible to achieve the whole picture of SOC-functions, which is the main role of this monograph. His attitude towards doing research always remains in my heart, although he got missing in 2009. I would like to thank all my co-authors of the materials that appear in this book, including Prof. Shaohua Pan, Prof. Xin Chen, Prof. Jiawei Zhang, Prof. Yu-Lin Chang, Dr. Chien-Hao Huang, etc.. The collaborations with them are wonderful and enjoyable experiences. I also thank Dr. Chien-Hao Huang, Dr. Yue Lu, Dr. Liguo Jiao, Prof. Xinhe Miao, and Prof. Chu-Chin Hu for their help on proofreading. Final gratitude goes to my family, Vivian, Benjamin, and Ian, who offer me support and stimulate endless strength in pursuing my exceptional academic career.

October 30, 2018
Taipei, Taiwan

Notations

- Throughout this book, an n-dimensional vector $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ means a *column* vector, i.e.,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

  In other words, without ambiguity, we also write the column vector as $x = (x_1, x_2, \cdots, x_n)$.

- $\mathbb{R}^n_+$ means $\{x = (x_1, x_2, \ldots, x_n) \,|\, x_i \geq 0, \ \forall i = 1, 2, \ldots, n\}$, whereas $\mathbb{R}^n_{++}$ denotes $\{x = (x_1, x_2, \ldots, x_n) \,|\, x_i > 0, \ \forall i = 1, 2, \ldots, n\}$.

- $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product.

- ^{T} means transpose.

- $B(x, \delta)$ denotes the neighborhood of x.

- $\mathbb{R}^{n \times n}$ denotes the space of $n \times n$ real matrices.

- I represents an identity matrix of suitable dimension.

- For any symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \succeq B$ (respectively, $A \succ B$) to mean $A - B$ is positive semidefinite (respectively, positive definite).

- $\mathcal{S}^n$ denotes the space of $n \times n$ symmetric matrices; and $\mathcal{S}^n_+$ means the space of $n \times n$ symmetric positive semidefinite matrices.

- $\| \cdot \|$ is the Euclidean norm.

- Given a set S, we denote $\bar{S}$, $\mathrm{int}(S)$ and $\mathrm{bd}(S)$ by the closure, the interior and the boundary of S, respectively.

- For a mapping $f : \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ denotes the gradient of f at x.

- $C^{(i)}(J)$ denotes the family of functions which are defined on $J \subseteq \mathbb{R}^n$ to $\mathbb{R}$ and have continuous i-th derivative.

- For any differentiable mapping $F = (F_1, F_2, \cdots, F_m) : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla F(x) = [\nabla F_1(x) \cdots \nabla F_m(x)]$ is a n by m matrix which denotes the transpose Jacobian of F at x.

- For any $x, y \in \mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$; and write $x \succ_{\mathcal{K}^n} y$ if $x - y \in \mathrm{int}(\mathcal{K}^n)$.

- For a real valued function $f : J \to \mathbb{R}$, $f'(t)$ and $f''(t)$ denote the first derivative and second-order derivative of f at the differentiable point $t \in J$, respectively.
- For a mapping $F : S \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\partial F(x)$ denotes the subdifferential of F at x, while $\partial_B F(x)$ denotes the B-subdifferential of F at x.

Chapter 1

SOC Functions

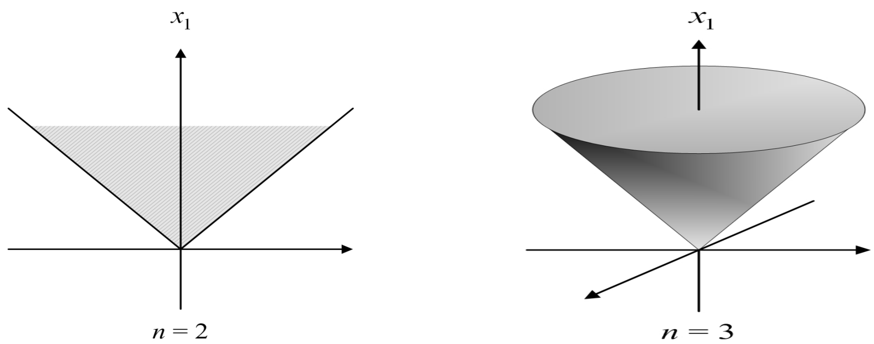
During the past two decades, there have been active research for second-order cone programs (SOCPs) and second-order cone complementarity problems (SOCCPs). Various methods had been proposed which include the interior-point methods [1, 103, 110, 124, 146], the smoothing Newton methods [52, 64, 72], the semismooth Newton methods [87, 121], and the merit function methods [44, 49]. All of these methods are proposed by using some SOC complementarity function or merit function to reformulate the KKT optimality conditions as a nonsmooth (or smoothing) system of equations or an unconstrained minimization problem. In fact, such SOC complementarity functions or merit functions are closely connected to so-called SOC functions. In other words, studying SOC functions is crucial to dealing with SOCP and SOCCP, which is the main target of this chapter.

1.1 On the second-order cone

The second-order cone (SOC) in $\mathbb{R}^n$, also called Lorentz cone, is defined by

$$\mathcal{K}^n = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\right\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, let $\mathcal{K}^n$ denote the set of nonnegative reals $\mathbb{R}_+$. For $n = 2$ and $n = 3$, the pictures of $\mathcal{K}^n$ are depicted in Figure 1.1(a) and Figure 1.1(b), respectively. It is known that $\mathcal{K}^n$ is a pointed closed convex cone so that a partial ordering can be deduced. More specifically, for any x, y in $\mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$; and write $x \succ_{\mathcal{K}^n} y$ if $x - y \in \text{int}(\mathcal{K}^n)$. In other words, we have $x \succeq_{\mathcal{K}^n} 0$ if and only if $x \in \mathcal{K}^n$; whereas $x \succ_{\mathcal{K}^n} 0$ if and only if $x \in \text{int}(\mathcal{K}^n)$. The relation $\succeq_{\mathcal{K}^n}$ is a partial ordering, but not a linear ordering in $\mathcal{K}^n$, i.e., there exist $x, y \in \mathcal{K}^n$ such that neither $x \succeq_{\mathcal{K}^n} y$ nor $y \succeq_{\mathcal{K}^n} x$. To see this, for $n = 2$, let $x = (1, 1) \in \mathcal{K}^2$ and $y = (1, 0) \in \mathcal{K}^2$. Then, we have $x - y = (0, 1) \notin \mathcal{K}^2$ and $y - x = (0, -1) \notin \mathcal{K}^2$.



(a) 2-dimensional SOC (b) 3-dimensional SOC

Figure 1.1: The graphs of SOC

The second-order cone has received much attention in optimization, particularly in the context of applications and solutions methods for second-order cone program (SOCP) [1, 48, 49, 103, 116, 117, 119] and second-order cone complementarity problem (SOCCP), [43, 44, 46, 49, 64, 72, 118]. For those solutions methods, there needs *spectral decomposition* associated with SOC whose basic concept is described below. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, x can be decomposed as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1.2}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $u_x^{(1)}$, $u_x^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \tag{1.3}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\|x_2\|}\right), & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i w\right), & \text{if} \quad x_2 = 0, \end{cases} \tag{1.4}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique.

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (\langle x, y \rangle, \, y_1 x_2 + x_1 y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.5}$$

The Jordan product is not associative. For example, for $n = 3$, let $x = (1, -1, 1)$ and $y = z = (1, 0, 1)$, then we have $(x \circ y) \circ z = (4, -1, 4) \neq x \circ (y \circ z) = (4, -2, 4)$. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$, for all $x \in \mathbb{R}^n$. Thus, without fear of ambiguity, we may write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. The vector $e = (1, 0, \ldots, 0)$ is the unique identity element for the Jordan product, and we define $x^0 = e$ for convenience. In addition, $\mathcal{K}^n$ is not closed under Jordan product. For example, $x = (\sqrt{2}, 1, 1) \in \mathcal{K}^3$, $y = (\sqrt{2}, 1, -1) \in \mathcal{K}^3$, but $x \circ y = (2, 2\sqrt{2}, 0) \notin \mathcal{K}^3$. We point out that lacking associative property of Jordan product and closedness of SOC are the main sources of difficulty when dealing with SOC. We write x^2 to denote $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. Then, "$\circ, +$" together with $e = (1, 0, \ldots, 0) \in \mathbb{R}^n$ have the following basic properties (see [62, 64]):

(1) $e \circ x = x$, for all $x \in \mathbb{R}^n$.

(2) $x \circ y = y \circ x$, for all $x, y \in \mathbb{R}^n$.

(3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, for all $x, y \in \mathbb{R}^n$.

(4) $(x + y) \circ z = x \circ z + y \circ z$, for all $x, y, z \in \mathbb{R}^n$.

For each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the *determinant* and the *trace* of x are defined by

$$\det(x) = x_1^2 - \|x_2\|^2, \quad \mathrm{tr}(x) = 2x_1.$$

In view of the definition of spectral values (1.3), it is clear that the determinant, the trace and the Euclidean norm of x can all be represented in terms of $\lambda_1(x)$ and $\lambda_2(x)$:

$$\det(x) = \lambda_1(x)\lambda_2(x), \quad \mathrm{tr}(x) = \lambda_1(x) + \lambda_2(x), \quad \|x\|^2 = \frac{1}{2}\left(\lambda_1(x)^2 + \lambda_2(x)^2\right).$$

As below, we elaborate more about the determinant and trace by showing some properties.

Proposition 1.1. *For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, the following results hold.*

(a) *If $x \succeq_{\mathcal{K}^n} y$, then $\det(x) \geq \det(y)$ and $\mathrm{tr}(x) \geq \mathrm{tr}(y)$.*

(b) *If $x \succeq_{\mathcal{K}^n} y$, then $\lambda_i(x) \geq \lambda_i(y)$ for $i = 1, 2$.*

Proof. (a) From definition, we know that

$$\begin{aligned} \det(x) &= x_1^2 - \|x_2\|^2, \quad \mathrm{tr}(x) = 2x_1, \\ \det(y) &= y_1^2 - \|y_2\|^2, \quad \mathrm{tr}(y) = 2y_1. \end{aligned}$$

Since $x - y = (x_1 - y_1, x_2 - y_2) \succeq_{\mathcal{K}^n} 0$, we have $\|x_2 - y_2\| \leq x_1 - y_1$. Thus, $x_1 \geq y_1$, and then $\mathrm{tr}(x) \geq \mathrm{tr}(y)$. Besides, using the assumption on x and y gives

$$x_1 - y_1 \geq \|x_2 - y_2\| \geq \big|\, \|x_2\| - \|y_2\| \,\big|, \tag{1.6}$$

which is equivalent to $x_1 - \|x_2\| \geq y_1 - \|y_2\| > 0$ and $x_1 + \|x_2\| \geq y_1 + \|y_2\| > 0$. Hence,

$$\det(x) = x_1^2 - \|x_2\|^2 = (x_1 + \|x_2\|)(x_1 - \|x_2\|) \geq (y_1 + \|y_2\|)(y_1 - \|y_2\|) = \det(y).$$

(b) From definition of spectral values, we know that

$$\lambda_1(x) = x_1 - \|x_2\|,\ \lambda_2(x) = x_1 + \|x_2\| \text{ and } \lambda_1(y) = y_1 - \|y_2\|,\ \lambda_2(y) = y_1 + \|y_2\|.$$

Then, by the inequality (1.6) in the proof of part(a), the results follow immediately. □

We point out that there may have other simpler ways to prove Proposition 1.1. The approach here is straightforward and intuitive by checking definitions. The converse of Proposition 1.1 does not hold, a counterexample occurs when taking $x = (5,3) \in \mathcal{K}^2$ and $y = (3,-1) \in \mathcal{K}^2$. In fact, if $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ serves as a counterexample for $\mathcal{K}^n$, then $(x_1, x_2, 0, \ldots, 0) \in \mathbb{R} \times \mathbb{R}^{m-1}$ is automatically a counterexample for $\mathcal{K}^m$ whenever $m \geq n$. Moreover, for any $x \succeq_{\mathcal{K}^n} y$, there always have $\lambda_i(x) \geq \lambda_i(y)$ and $\mathrm{tr}(x) \geq \mathrm{tr}(y)$ for $i = 1, 2$. There is no need to restrict $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$ as in Proposition 1.1.

Proposition 1.2. *Let $x \succeq_{\mathcal{K}^n} 0$, $y \succeq_{\mathcal{K}^n} 0$ and $e = (1, 0, \cdots, 0)$. Then, the following hold.*

(a) $\det(x + y) \geq \det(x) + \det(y)$.

(b) $\det(x \circ y) \leq \det(x)\det(y)$.

(c) $\det\big(\alpha x + (1-\alpha) y\big) \geq \alpha^2 \det(x) + (1-\alpha)^2 \det(y)$ *for all* $0 < \alpha < 1$.

(d) $\big(\det(e + x)\big)^{1/2} \geq 1 + \det(x)^{1/2}$.

(e) $\det(e + x + y) \leq \det(e + x)\det(e + y)$.

Proof. (a) For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, we know $\|x_2\| \leq x_1$ and $\|y_2\| \leq y_1$, which implies

$$|\langle x_2, y_2 \rangle| \leq \|x_2\|\,\|y_2\| \leq x_1 y_1.$$

Hence, we obtain

$$\begin{aligned}
\det(x+y) &= (x_1 + y_1)^2 - \|x_2 + y_2\|^2 \\
&= \big(x_1^2 - \|x_2\|^2\big) + \big(y_1^2 - \|y_2\|^2\big) + 2\big(x_1 y_1 - \langle x_2, y_2 \rangle\big) \\
&\geq \big(x_1^2 - \|x_2\|^2\big) + \big(y_1^2 - \|y_2\|^2\big) \\
&= \det(x) + \det(y).
\end{aligned}$$

(b) Applying the Cauchy inequality gives

$$\begin{aligned}
\det(x \circ y) &= \langle x, y \rangle^2 - \|x_1 y_2 + y_1 x_2\|^2 \\
&= \big(x_1 y_1 + \langle x_2, y_2 \rangle\big)^2 - \big(x_1^2 \|y_2\|^2 + 2 x_1 y_1 \langle x_2, y_2 \rangle + y_1^2 \|x_2\|^2\big) \\
&= x_1^2 y_1^2 + \langle x_2, y_2 \rangle^2 - x_1^2 \|y_2\|^2 - y_1^2 \|x_2\|^2 \\
&\leq x_1^2 y_1^2 + \|x_2\|^2 \|y_2\|^2 - x_1^2 \|y_2\|^2 - y_1^2 \|x_2\|^2 \\
&= \big(x_1^2 - \|x_2\|^2\big)\big(y_1^2 - \|y_2\|^2\big) \\
&= \det(x)\det(y).
\end{aligned}$$

(c) For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, it is clear that $\alpha x \succeq_{\mathcal{K}^n} 0$ and $(1-\alpha)y \succeq_{\mathcal{K}^n} 0$ for every $0 < \alpha < 1$. In addition, we observe that $\det(\alpha x) = \alpha^2 \det(x)$. Hence,

$$\det\big(\alpha x + (1-\alpha)y\big) \geq \det(\alpha x) + \det((1-\alpha)y) = \alpha^2 \det(x) + (1-\alpha)^2 \det(y),$$

where the inequality is from part(a).

(d) For any $x \succeq_{\mathcal{K}^n} 0$, we know $\det(x) = \lambda_1(x)\lambda_2(x) \geq 0$, where $\lambda_i(x)$ are the spectral values of x. Hence,

$$\det(e+x) = (1+\lambda_1(x))(1+\lambda_2(x)) \geq \left(1+\sqrt{\lambda_1(x)\lambda_2(x)}\right)^2 = \left(1+\det(x)^{1/2}\right)^2.$$

Then, taking square root on both sides yields the desired result.

(e) Again, For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, we have the following inequalities

$$x_1 - \|x_2\| \geq 0, \quad y_1 - \|y_2\| \geq 0, \quad |\langle x_2, y_2\rangle| \leq \|x_2\|\,\|y_2\| \leq x_1 y_1. \tag{1.7}$$

Moreover, we know $\det(e+x+y) = (1+x_1+y_1)^2 - \|x_2+y_2\|^2$, $\det(e+x) = (1+x_1)^2 - \|x_2\|^2$ and $\det(e+y) = (1+y_1)^2 - \|y_2\|^2$. Hence,

$$\begin{aligned}
& \det(e+x)\det(e+y) - \det(e+x+y) \\
= \;& \big((1+x_1)^2 - \|x_2\|^2\big)\big((1+y_1)^2 - \|y_2\|^2\big) - \big((1+x_1+y_1)^2 - \|x_2+y_2\|^2\big) \\
= \;& 2x_1y_1 + 2\langle x_2, y_2\rangle + 2x_1y_1^2 + 2x_1^2y_1 - 2y_1\|x_2\|^2 - 2x_1\|y_2\|^2 \\
& + x_1^2y_1^2 - y_1^2\|x_2\|^2 - x_1^2\|y_2\|^2 + \|x_2\|^2\|y_2\|^2 \\
= \;& 2\big(x_1y_1 + \langle x_2, y_2\rangle\big) + 2x_1\big(y_1^2 - \|y_2\|^2\big) + 2y_1\big(x_1^2 - \|x_2\|^2\big) \\
& + \big(x_1^2 - \|x_2\|^2\big)\big(y_1^2 - \|y_2\|^2\big) \\
\geq \;& 0,
\end{aligned}$$

where we multiply out all the expansions to obtain the second equality and the last inequality holds by (1.7). □

Proposition 1.2(c) can be extended to a more general case:

$$\det\big(\alpha x + \beta y\big) \geq \alpha^2 \det(x) + \beta^2 \det(y) \quad \forall \alpha \geq 0,\ \beta \geq 0.$$

Note that together with Cauchy-Schwartz inequality and properties of determinant, one may achieve other way to verify Proposition 1.2. Again, the approach here is only one choice of proof which is straightforward and intuitive. There are more inequalities about determinant, see Proposition 1.8 and Proposition 2.32, which are established by using the concept of SOC-convexity that will be introduced in Chapter 2. Next, we move to the inequalities about trace.

Proposition 1.3. *For any $x, y \in \mathbb{R}^n$, we have*

(a) $\mathrm{tr}(x+y) = \mathrm{tr}(x) + \mathrm{tr}(y)$ *and* $\mathrm{tr}(\alpha x) = \alpha\,\mathrm{tr}(x)$ *for any* $\alpha \in \mathbb{R}$. *In other words,* $\mathrm{tr}(\cdot)$ *is a linear function on* $\mathbb{R}^n$.

(b) $\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) \le \mathrm{tr}(x \circ y) \le \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)$.

Proof. Part(a) is trivial and it remains to verify part(b). Using the fact that $\mathrm{tr}(x \circ y) = 2\langle x, y\rangle$, we obtain

$$\begin{aligned}
\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) &= (x_1 - \|x_2\|)(y_1 + \|y_2\|) + (x_1 + \|x_2\|)(y_1 - \|y_2\|) \\
&= 2(x_1y_1 - \|x_2\|\|y_2\|) \\
&\le 2(x_1y_1 + \langle x_2, y_2\rangle) \\
&= 2\langle x, y\rangle \\
&= \mathrm{tr}(x \circ y) \\
&\le 2(x_1y_1 + \|x_2\|\|y_2\|) \\
&= (x_1 - \|x_2\|)(y_1 - \|y_2\|) + (x_1 + \|x_2\|)(y_1 + \|y_2\|) \\
&= \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y),
\end{aligned}$$

which completes the proof. □

In general, $\det(x \circ y) \neq \det(x)\det(y)$ unless $x_2 = \alpha y_2$. A vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is said to be *invertible* if $\det(x) \neq 0$. If x is invertible, then there exists a unique $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ satisfying $x \circ y = y \circ x = e$. We call this y the inverse of x and denote it by x^{-1}. In fact, we have

$$x^{-1} = \frac{1}{x_1^2 - \|x_2\|^2}(x_1, -x_2) = \frac{1}{\det(x)}\big(\mathrm{tr}(x)e - x\big).$$

Therefore, $x \in \mathrm{int}(\mathcal{K}^n)$ if and only if $x^{-1} \in \mathrm{int}(\mathcal{K}^n)$. Moreover, if $x \in \mathrm{int}(\mathcal{K}^n)$, then $x^{-k} = (x^k)^{-1} = (x^{-1})^k$ is also well-defined. For any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{1/2}$ (also denoted by $\sqrt{x}$ sometimes) such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s}\right), \quad \text{where} \quad s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2}\right)}.$$

In the above formula, the term $\frac{x_2}{2s}$ is defined to be the zero vector if $s = 0$ (and hence $x_2 = 0$), i.e., $x = 0$.

For any $x \in \mathbb{R}^n$, we always have $x^2 \in \mathcal{K}^n$ (i.e., $x^2 \succeq_{\mathcal{K}^n} 0$). Hence, there exists a unique vector $(x^2)^{1/2} \in \mathcal{K}^n$ denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. It is also known that $|x| \succeq_{\mathcal{K}^n} x$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the projection point of x onto $\mathcal{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP:

$$[x]_+ = \mathrm{argmin}\{\|x - y\| \mid y \in \mathcal{K}^n\}.$$

Here the norm is in Euclidean norm since Jordan product does not induce a norm. Likewise, $[x]_-$ means the projection point of x onto $-\mathcal{K}^n$, which implies $[x]_- = -[-x]_+$. It is well known that $[x]_+ = \frac{1}{2}(x + |x|)$ and $[x]_- = \frac{1}{2}(x - |x|)$, see Property 1.2(f).

The spectral decomposition along with the Jordan algebra associated with SOC entails some basic properties as below. We omit the proofs since they can be found in [62, 64].

Property 1.1. *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with the spectral values $\lambda_1(x)$, $\lambda_2(x)$ and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$ given as in (1.3)-(1.4), we have*

(a) *$u_x^{(1)}$ and $u_x^{(2)}$ are orthogonal under Jordan product and have length $\frac{1}{\sqrt{2}}$, i.e.,*

$$u_x^{(1)} \circ u_x^{(2)} = 0, \quad \|u_x^{(1)}\| = \|u_x^{(2)}\| = \frac{1}{\sqrt{2}}.$$

(b) *$u_x^{(1)}$ and $u_x^{(2)}$ are idempotent under Jordan product, i.e.,*

$$u_x^{(i)} \circ u_x^{(i)} = u_x^{(i)}, \ i = 1, 2.$$

(c) *$\lambda_1(x)$, $\lambda_2(x)$ are nonnegative (positive) if and only if $x \in \mathcal{K}^n$ ($x \in \mathrm{int}(\mathcal{K}^n)$), i.e.,*

$$\begin{aligned} \lambda_i(x) \geq 0 \text{ for } i = 1, 2 \quad &\Longleftrightarrow \quad x \succeq_{\mathcal{K}^n} 0. \\ \lambda_i(x) > 0 \text{ for } i = 1, 2 \quad &\Longleftrightarrow \quad x \succ_{\mathcal{K}^n} 0. \end{aligned}$$

Although the converse of Proposition 1.1(b) does not hold as mentioned earlier, Property 1.1(c) is useful in verifying whether a point x belongs to $\mathcal{K}^n$ or not.

Property 1.2. *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with the spectral values $\lambda_1(x)$, $\lambda_2(x)$ and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$ given as in (1.3)-(1.4), we have*

(a) *$x^2 = \lambda_1(x)^2 u_x^{(1)} + \lambda_2(x)^2 u_x^{(2)}$ and $x^{-1} = \lambda_1^{-1}(x) u_x^{(1)} + \lambda_2^{-1}(x) u_x^{(2)}$.*

(b) *If $x \in \mathcal{K}^n$, then $x^{1/2} = \sqrt{\lambda_1(x)}\, u_x^{(1)} + \sqrt{\lambda_2(x)}\, u_x^{(2)}$.*

(c) *$|x| = |\lambda_1(x)| u_x^{(1)} + |\lambda_2(x)| u_x^{(2)}$.*

(d) *$[x]_+ = [\lambda_1(x)]_+ u_x^{(1)} + [\lambda_2(x)]_+ u_x^{(2)}$ and $[x]_- = [\lambda_1(x)]_- u_x^{(1)} + [\lambda_2(x)]_- u_x^{(2)}$.*

(e) *$|x| = [x]_+ + [-x]_+ = [x]_+ - [x]_-$.*

(f) *$[x]_+ = \frac{1}{2}(x + |x|)$ and $[x]_- = \frac{1}{2}(x - |x|)$.*

Property 1.3. *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Then, the following hold.*

(a) *Any* $x \in \mathbb{R}^n$ *satisfies* $|x| \succeq_{\mathcal{K}^n} x$.

(b) *For any* $x, y \succeq_{\mathcal{K}^n} 0$, *if* $x \succeq_{\mathcal{K}^n} y$, *then* $x^{1/2} \succeq_{\mathcal{K}^n} y^{1/2}$.

(c) *For any* $x, y \in \mathbb{R}^n$, *if* $x^2 \succeq_{\mathcal{K}^n} y^2$, *then* $|x| \succeq_{\mathcal{K}^n} |y|$.

(d) *For any* $x \in \mathbb{R}^n$, $x \succeq_{\mathcal{K}^n} 0$ *if and only if* $\langle x, y\rangle \geq 0$ *for all* $y \succeq_{\mathcal{K}^n} 0$.

(e) *For any* $x \succeq_{\mathcal{K}^n} 0$ *and* $y \in \mathbb{R}^n$, *if* $x^2 \succeq_{\mathcal{K}^n} y^2$, *then* $x \succeq_{\mathcal{K}^n} y$.

Note that for any $x, y \succeq_{\mathcal{K}^n} 0$, if $x \succeq_{\mathcal{K}^n} y$, one can also conclude that $x^{-1} \preceq_{\mathcal{K}^n} y^{-1}$. However, the arguments are not trivial by direct verifications. We present it by other approach, see Proposition 2.3(a).

Property 1.4. *For any* $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *with spectral values* $\lambda_1(x)$, $\lambda_2(x)$ *and any* $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *with spectral values* $\lambda_1(y)$, $\lambda_2(y)$, *we have*

$$|\lambda_i(x) - \lambda_i(y)| \leq \sqrt{2}\|x - y\|, \quad i = 1, 2.$$

Proof. First, we compute that

$$\begin{aligned}
|\lambda_1(x) - \lambda_1(y)| &= |x_1 - \|x_2\| - y_1 + \|y_2\|| \\
&\leq |x_1 - y_1| + |\|x_2\| - \|y_2\|| \\
&\leq |x_1 - y_1| + \|x_2 - y_2\| \\
&\leq \sqrt{2}\left(|x_1 - y_1|^2 + \|x_2 - y_2\|^2\right)^{1/2} \\
&= \sqrt{2}\|x - y\|,
\end{aligned}$$

where the second inequality uses $\|x_2\| \leq \|x_2 - y_2\| + \|y_2\|$ and $\|y_2\| \leq \|x_2 - y_2\| + \|x_2\|$; the last inequality uses the relation between the 1-norm and the 2-norm. A similar argument applies to $|\lambda_2(x) - \lambda_2(y)|$. $\square$

In fact, Property 1.1-1.3 are parallel results analogous to those associated with positive semidefinite cone $\mathcal{S}^n_+$, see [75]. Even though both $\mathcal{K}^n$ and $\mathcal{S}^n_+$ belong to the family of symmetric cones [62] and share similar properties, as we will see, the ideas and techniques for proving these results are quite different. One reason is that the Jordan product is not associative as mentioned earlier.

1.2 SOC function and SOC trace function

In this section, we introduce two types of functions, SOC function and SOC trace function, which are very useful in dealing with optimization involved with SOC. Some inequalities are established in light of these functions.

Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ given as in (1.3) and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$ given as in (1.4). We first define its corresponding SOC function as below. For any real-valued function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ $(n \geq 1)$ was considered [46, 64]:

$$f^{\mathrm{soc}}(x) := f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.8}$$

The definition (1.8) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{\mathrm{soc}}(x) = x^{1/2}$, x^2, $\exp(x)$, which correspond to $f(t) = t^{1/2}$, t^2, e^t, are already discussed in the book [62]. Indeed, the above definition (1.8) is analogous to one associated with the semidefinite cone $\mathcal{S}^n_+$, see [140, 145]. For subsequent analysis, we also need the concept of SOC trace function [47] defined by

$$f^{\mathrm{tr}}(x) := f(\lambda_1(x)) + f(\lambda_2(x)) = \mathrm{tr}(f^{\mathrm{soc}}(x)). \tag{1.9}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} and f^{tr} are defined on the corresponding subset of $\mathbb{R}^n$. More specifically, from Proposition 1.4 shown as below, we see that the corresponding subset for f^{soc} and f^{tr} is

$$S = \{x \in \mathbb{R}^n \mid \lambda_i(x) \in J,\ i = 1, 2.\} \tag{1.10}$$

provided f is defined on a subset of $J \subseteq \mathbb{R}$. In addition, S is open in $\mathbb{R}^n$ whenever J is open in $\mathbb{R}$. To see this assertion, we need the following technical lemma.

Lemma 1.1. *Let $A \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix, $C \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and $B \in \mathbb{R}^{m\times n}$. Then,*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq O \quad \Longleftrightarrow \quad C - B^TA^{-1}B \succeq O \tag{1.11}$$

and

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ O \quad \Longleftrightarrow \quad C - B^TA^{-1}B \succ O. \tag{1.12}$$

Proof. This is indeed the Schur Complement Theorem, please see [22, 23, 75] for a proof. □

Proposition 1.4. *For any given $f : J \subseteq \mathbb{R} \to \mathbb{R}$, let $f^{\mathrm{soc}} : S \to \mathbb{R}^n$ and $f^{\mathrm{tr}} : S \to \mathbb{R}$ be given by (1.8) and (1.9), respectively. Assume that J is open. Then, the following results hold.*

(a) *The domain S of f^{soc} and f^{tr} is also open.*

(b) *If f is (continuously) differentiable on J, then f^{soc} is (continuously) differentiable on S. Moreover, for any $x \in S$, $\nabla f^{\mathrm{soc}}(x) = f'(x_1)I$ if $x_2 = 0$, and otherwise*

$$\nabla f^{\mathrm{soc}}(x) = \begin{bmatrix} b(x) & c(x)\dfrac{x_2^T}{\|x_2\|} \\ c(x)\dfrac{x_2}{\|x_2\|} & a(x)I + (b(x)-a(x))\dfrac{x_2x_2^T}{\|x_2\|^2} \end{bmatrix}, \tag{1.13}$$

where

$$\begin{aligned} a(x) &= \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x)-\lambda_1(x)}, \\ b(x) &= \frac{f'(\lambda_2(x)) + f'(\lambda_1(x))}{2}, \\ c(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{2}. \end{aligned}$$

(c) *If f is (continuously) differentiable, then f^{tr} is (continuously) differentiable on S with $\nabla f^{\mathrm{tr}}(x) = 2(f')^{\mathrm{soc}}(x)$; if f is twice (continuously) differentiable, then f^{tr} is twice (continuously) differentiable on S with $\nabla^2 f^{\mathrm{tr}}(x) = \nabla(f')^{\mathrm{soc}}(x)$.*

Proof. (a) Fix any $x \in S$. Then $\lambda_1(x)$, $\lambda_2(x) \in J$. Since J is an open subset of $\mathbb{R}$, there exist $\delta_1, \delta_2 > 0$ such that $\{t \in \mathbb{R} \,|\, |t-\lambda_1(x)| < \delta_1\} \subseteq J$ and $\{t \in \mathbb{R} \,|\, |t-\lambda_2(x)| < \delta_2\} \subseteq J$. Let $\delta := \min\{\delta_1, \delta_2\}/\sqrt{2}$. Then, for any y satisfying $\|y-x\| < \delta$, we have $|\lambda_1(y) - \lambda_1(x)| < \delta_1$ and $|\lambda_2(y) - \lambda_2(x)| < \delta_2$ by noting that

$$\begin{aligned} & (\lambda_1(x)-\lambda_1(y))^2 + (\lambda_2(x)-\lambda_2(y))^2 \\ = \;& 2(x_1^2 + \|x_2\|^2) + 2(y_1^2 + \|y_2\|^2) - 4(x_1y_1 + \|x_2\|\|y_2\|) \\ \leq \;& 2(x_1^2 + \|x_2\|^2) + 2(y_1^2 + \|y_2\|^2) - 4(x_1y_1 + \langle x_2, y_2\rangle) \\ = \;& 2\left(\|x\|^2 + \|y\|^2 - 2\langle x, y\rangle\right) \\ = \;& 2\|x-y\|^2, \end{aligned}$$

and consequently $\lambda_1(y) \in J$ and $\lambda_2(y) \in J$. Since f is a function from J to $\mathbb{R}$, this means that $\{y \in \mathbb{R}^n \,|\, \|y-x\| < \delta\} \subseteq S$, and therefore the set S is open. In addition, from the above, we see that S is characterized as in (1.10).

(b) The arguments are similar to Proposition 1.13 and Proposition 1.14 in Section 1.3. Please check them for details.

(c) If f is (continuously) differentiable, then from part(b) and $f^{\mathrm{tr}}(x) = 2\left\langle e, f^{\mathrm{soc}}(x)\right\rangle$ it follows that f^{tr} is (continuously) differentiable. In addition, a simple computation yields that $\nabla f^{\mathrm{tr}}(x) = 2\nabla f^{\mathrm{soc}}(x)e = 2(f')^{\mathrm{soc}}(x)$. Similarly, by part(b), the second part follows. $\square$

Proposition 1.5. *For any $f : J \to \mathbb{R}$, let $f^{\text{soc}} : S \to \mathbb{R}^n$ and $f^{\text{tr}} : S \to \mathbb{R}$ be given by (1.8) and (1.9), respectively. Assume that J is open. If f is twice differentiable on J, then*

(a) *$f''(t) \geq 0$ for any $t \in J \iff \nabla(f')^{\text{soc}}(x) \succeq O$ for any $x \in S \iff f^{\text{tr}}$ is convex in S.*

(b) *$f''(t) > 0$ for any $t \in J \iff \nabla(f')^{\text{soc}}(x) \succ O$ for any $x \in S \implies f^{\text{tr}}$ is strictly convex in S.*

Proof. (a) By Proposition 1.4(c), $\nabla^2 f^{\text{tr}}(x) = 2\nabla(f')^{\text{soc}}(x)$ for any $x \in S$, and the second equivalence follows by [21, Prop. B.4(a) and (c)]. We next come to the first equivalence. By Proposition 1.4(b), for any fixed $x \in S$, $\nabla(f')^{\text{soc}}(x) = f''(x_1)I$ if $x_2 = 0$, and otherwise $\nabla(f')^{\text{soc}}(x)$ has the same expression as in (1.13) except that

$$\begin{aligned}
b(x) &= \frac{f''(\lambda_2(x)) + f''(\lambda_1(x))}{2}, \\
c(x) &= \frac{f''(\lambda_2(x)) - f''(\lambda_1(x))}{2}, \\
a(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}.
\end{aligned}$$

Assume that $\nabla(f')^{\text{soc}}(x) \succeq O$ for any $x \in S$. Then, we readily have $b(x) \geq 0$ for any $x \in S$. Noting that $b(x) = f''(x_1)$ when $x_2 = 0$, we particularly have $f''(x_1) \geq 0$ for all $x_1 \in J$, and consequently $f''(t) \geq 0$ for all $t \in J$. Assume that $f''(t) \geq 0$ for all $t \in J$. Fix any $x \in S$. Clearly, $b(x) \geq 0$ and $a(x) \geq 0$. If $b(x) = 0$, then $f''(\lambda_1(x)) = f''(\lambda_2(x)) = 0$, and consequently $c(x) = 0$, which in turn implies that

$$\nabla(f')^{\text{soc}}(x) = \begin{bmatrix} 0 & 0 \\ 0 & a(x)\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right) \end{bmatrix} \succeq O. \tag{1.14}$$

If $b(x) > 0$, then by the first equivalence of Lemma 1.1 and the expression of $\nabla(f')^{\text{soc}}(x)$ it suffices to argue that the following matrix

$$a(x)I + (b(x) - a(x))\frac{x_2 x_2^T}{\|x_2\|^2} - \frac{c^2(x)}{b(x)}\frac{x_2 x_2^T}{\|x_2\|^2} \tag{1.15}$$

is positive semidefinite. Since the rank-one matrix $x_2 x_2^T$ has only one nonzero eigenvalue $\|x_2\|^2$, the matrix in (1.15) has one eigenvalue $a(x)$ of multiplicity $n-1$ and one eigenvalue $\frac{b(x)^2 - c(x)^2}{b(x)}$ of multiplicity 1. Since $a(x) \geq 0$ and $\frac{b(x)^2 - c(x)^2}{b(x)} = f''(\lambda_1(x)) f''(\lambda_2(x)) \geq 0$, the matrix in (1.15) is positive semidefinite. By the arbitrary of x, we have that $\nabla(f')^{\text{soc}}(x) \succeq O$ for all $x \in S$.

(b) The first equivalence is direct by using (1.12) of Lemma 1.1, noting $\nabla(f')^{\text{soc}}(x) \succ O$ implies $a(x) > 0$ when $x_2 \neq 0$, and following the same arguments as part(a). The second part is due to [21, Prop. B.4(b)]. $\square$

Remark 1.1. *Note that the strict convexity of $f^{\rm tr}$ does not necessarily imply the positive definiteness of $\nabla^2 f^{\rm tr}(x)$. Consider $f(t) = t^4$ for $t \in {\rm I\!R}$. We next show that $f^{\rm tr}$ is strictly convex. Indeed, $f^{\rm tr}$ is convex in ${\rm I\!R}^n$ by Proposition 1.5(a) since $f''(t) = 12t^2 \geq 0$. Taking into account that $f^{\rm tr}$ is continuous, it remains to prove that*

$$f^{\rm tr}\left(\frac{x+y}{2}\right) = \frac{f^{\rm tr}(x) + f^{\rm tr}(y)}{2} \implies x = y. \tag{1.16}$$

Since $h(t) = (t_0 + t)^4 + (t_0 - t)^4$ for some $t_0 \in {\rm I\!R}$ is increasing on $[0, +\infty)$, and the function $f(t) = t^4$ is strictly convex in ${\rm I\!R}$, we have that

$$\begin{aligned}
f^{\rm tr}\left(\frac{x+y}{2}\right) &= \left[\lambda_1\left(\frac{x+y}{2}\right)\right]^4 + \left[\lambda_2\left(\frac{x+y}{2}\right)\right]^4 \\
&= \left(\frac{x_1 + y_1 - \|x_2 + y_2\|}{2}\right)^4 + \left(\frac{x_1 + y_1 + \|x_2 + y_2\|}{2}\right)^4 \\
&\leq \left(\frac{x_1 + y_1 - \|x_2\| - \|y_2\|}{2}\right)^4 + \left(\frac{x_1 + y_1 + \|x_2\| + \|y_2\|}{2}\right)^4 \\
&= \left(\frac{\lambda_1(x) + \lambda_1(y)}{2}\right)^4 + \left(\frac{\lambda_2(x) + \lambda_2(y)}{2}\right)^4 \\
&\leq \frac{(\lambda_1(x))^4 + (\lambda_1(y))^4 + (\lambda_2(x))^4 + (\lambda_2(y))^4}{2} \\
&= \frac{f^{\rm tr}(x) + f^{\rm tr}(y)}{2},
\end{aligned}$$

and moreover, the above inequalities become the equalities if and only if

$$\|x_2 + y_2\| = \|x_2\| + \|y_2\|, \ \lambda_1(x) = \lambda_1(y), \ \lambda_2(x) = \lambda_2(y).$$

It is easy to verify that the three equalities hold if and only if $x = y$. Thus, the implication in (1.16) holds, i.e., $f^{\rm tr}$ is strictly convex. However, by Proposition 1.5(b), $\nabla(f')^{\rm soc}(x) \succ O$ does not hold for all $x \in {\rm I\!R}^n$ since $f''(t) > 0$ does not hold for all $t \in {\rm I\!R}$.

We point out that the fact that the strict convexity of f implies the strict convexity of $f^{\rm tr}$ was proved in [7, 16] via the definition of convex function, but here we use the Schur Complement Theorem and the relation between $\nabla(f')^{\rm soc}$ and $\nabla^2 f^{\rm tr}$ to establish the convexity of SOC-trace functions. Next, we illustrate the application of Proposition 1.5 with some SOC trace functions.

Proposition 1.6. *The following functions associated with $\mathcal{K}^n$ are all strictly convex.*

(a) $F_1(x) = -\ln(\det(x))$ *for* $x \in {\rm int}(\mathcal{K}^n)$.

(b) $F_2(x) = {\rm tr}(x^{-1})$ *for* $x \in {\rm int}(\mathcal{K}^n)$.

(c) $F_3(x) = \text{tr}(\phi(x))$ *for* $x \in \text{int}(\mathcal{K}^n)$, *where*

$$\phi(x) = \begin{cases} \frac{x^{p+1}-e}{p+1} + \frac{x^{1-q}-e}{q-1} & \text{if } p \in [0,1],\ q > 1, \\ \frac{x^{p+1}-e}{p+1} - \ln x & \text{if } p \in [0,1],\ q = 1. \end{cases}$$

(d) $F_4(x) = -\ln(\det(e - x))$ *for* $x \prec_{\mathcal{K}^n} e$.

(e) $F_5(x) = \text{tr}((e - x)^{-1} \circ x)$ *for* $x \prec_{\mathcal{K}^n} e$.

(f) $F_6(x) = \text{tr}(\exp(x))$ *for* $x \in \mathbb{R}^n$.

(g) $F_7(x) = \ln(\det(e + \exp(x)))$ *for* $x \in \mathbb{R}^n$.

(h) $F_8(x) = \text{tr}\left(\dfrac{x + (x^2 + 4e)^{1/2}}{2}\right)$ *for* $x \in \mathbb{R}^n$.

Proof. Note that $F_1(x)$, $F_2(x)$ and $F_3(x)$ are the SOC trace functions associated with $f_1(t) = -\ln t$ $(t > 0)$, $f_2(t) = t^{-1}$ $(t > 0)$ and $f_3(t)$ $(t > 0)$, respectively, where

$$f_3(t) = \begin{cases} \frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1} & \text{if } p \in [0,1],\ q > 1, \\ \frac{t^{p+1}-1}{p+1} - \ln t & \text{if } p \in [0,1],\ q = 1; \end{cases}$$

Next, $F_4(x)$ is the SOC trace function associated with $f_4(t) = -\ln(1 - t)$ $(t < 1)$, $F_5(x)$ is the SOC trace function associated with $f_5(t) = \frac{t}{1-t}$ $(t < 1)$ by noting that

$$(e - x)^{-1} \circ x = \frac{\lambda_1(x)}{\lambda_1(e - x)} u_x^{(1)} + \frac{\lambda_2(x)}{\lambda_2(e - x)} u_x^{(2)};$$

In addition, $F_6(x)$ and $F_7(x)$ are the SOC trace functions associated with $f_6(t) = \exp(t)$ $(t \in \mathbb{R})$ and $f_7(t) = \ln(1 + \exp(t))$ $(t \in \mathbb{R})$, respectively, and $F_8(x)$ is the SOC trace function associated with $f_8(t) = \frac{1}{2}\left(t + \sqrt{t^2 + 4}\right)$ $(t \in \mathbb{R})$. It is easy to verify that all the functions f_1-f_8 have positive second-order derivatives in their respective domain, and therefore F_1-F_8 are strictly convex functions by Proposition 1.5(b). $\square$

The functions F_1, F_2 and F_3 are the popular barrier functions which play a key role in the development of interior point methods for SOCPs, see, e.g., [15, 20, 110, 124, 146], where F_3 covers a wide range of barrier functions, including the classical logarithmic barrier function, the self-regular functions and the non-self-regular functions; see [15] for details. The functions F_4 and F_5 are the popular shifted barrier functions [6, 7, 9] for SOCPs, and F_6-F_8 can be used as penalty functions for second-order cone programs (SOCPs), and these functions are added to the objective of SOCPs for forcing the solution to be feasible.

Besides the application in establishing convexity for SOC trace functions, the Schur Complement Theorem can be employed to establish convexity of some compound functions of SOC trace functions and scalar-valued functions, which are usually difficult to achieve by checking the definition of convexity directly. The following proposition presents such an application.

Proposition 1.7. *For any $x \in \mathcal{K}^n$, let $F_9(x) := -[\det(x)]^{1/p}$ with $p > 1$. Then,*

(a) *F_9 is twice continuously differentiable in $\mathrm{int}(\mathcal{K}^n)$.*

(b) *F_9 is convex when $p \geq 2$, and moreover, it is strictly convex when $p > 2$.*

Proof. (a) Note that $-F_9(x) = \exp\left(p^{-1}\ln(\det(x))\right)$ for any $x \in \mathrm{int}(\mathcal{K}^n)$, and $\ln(\det(x)) = f^{\mathrm{tr}}(x)$ with $f(t) = \ln(t)$ for $t \in \mathbb{R}_{++}$. By Proposition 1.4(c), $\ln(\det(x))$ is twice continuously differentiable in $\mathrm{int}(\mathcal{K}^n)$. Hence $-F_9(x)$ is twice continuously differentiable in $\mathrm{int}(\mathcal{K}^n)$. The result then follows.

(b) In view of the continuity of F_9, we only need to prove its convexity over $\mathrm{int}(\mathcal{K}^n)$. By part(a), we next achieve this goal by proving that the Hessian matrix $\nabla^2 F_9(x)$ for any $x \in \mathrm{int}(\mathcal{K}^n)$ is positive semidefinite when $p \geq 2$, and positive definite when $p > 2$. Fix any $x \in \mathrm{int}(\mathcal{K}^n)$. From direct computations, we obtain

$$\nabla F_9(x) = -\frac{1}{p}\begin{bmatrix} (2x_1)\left(x_1^2 - \|x_2\|^2\right)^{\frac{1}{p}-1} \\ (-2x_2)\left(x_1^2 - \|x_2\|^2\right)^{\frac{1}{p}-1} \end{bmatrix}$$

and

$$\nabla^2 F_9(x) = \frac{p-1}{p^2}\left(\det(x)\right)^{\frac{1}{p}-2}\begin{bmatrix} 4x_1^2 - \frac{2p\left(x_1^2 - \|x_2\|^2\right)}{p-1} & -4x_1 x_2^T \\ -4x_1 x_2 & 4x_2 x_2^T + \frac{2p\left(x_1^2 - \|x_2\|^2\right)}{p-1} I \end{bmatrix}.$$

Since $x \in \mathrm{int}(\mathcal{K}^n)$, we have $x_1 > 0$ and $\det(x) = x_1^2 - \|x_2\|^2 > 0$, and therefore

$$a_1(x) := 4x_1^2 - \frac{2p\left(x_1^2 - \|x_2\|^2\right)}{p-1} = \left(4 - \frac{2p}{p-1}\right)x_1^2 + \frac{2p}{p-1}\|x_2\|^2.$$

We next proceed the arguments by the following two cases: $a_1(x) = 0$ or $a_1(x) > 0$.

Case 1: $a_1(x) = 0$. Since $p \geq 2$, under this case we must have $x_2 = 0$, and consequently,

$$\nabla^2 F_9(x) = \frac{p-1}{p^2}(x_1)^{\frac{2}{p}-4}\begin{bmatrix} 0 & 0 \\ 0 & \frac{2p}{p-1}x_1^2 I \end{bmatrix} \succeq O.$$

Case 2: $a_1(x) > 0$. Under this case, we calculate that

$$\begin{aligned} & \left[4x_1^2 - \frac{2p\left(x_1^2 - \|x_2\|^2\right)}{p-1}\right]\left[4x_2 x_2^T + \frac{2p\left(x_1^2 - \|x_2\|^2\right)}{p-1} I\right] - 16x_1^2 x_2 x_2^T \\ = & \frac{4p\left(x_1^2 - \|x_2\|^2\right)}{p-1}\left[\frac{p-2}{p-1}x_1^2 I + \frac{p}{p-1}\|x_2\|^2 I - 2x_2 x_2^T\right]. \end{aligned} \quad (1.17)$$

Since the rank-one matrix $2x_2x_2^T$ has only one nonzero eigenvalue $2\|x_2\|^2$, the matrix in the bracket of the right hand side of (1.17) has one eigenvalue of multiplicity 1 given by

$$\frac{p-2}{p-1}x_1^2 + \frac{p}{p-1}\|x_2\|^2 - 2\|x_2\|^2 = \frac{p-2}{p-1}\left(x_1^2 - \|x_2\|^2\right) \geq 0,$$

and one eigenvalue of multiplicity $n-1$ given by $\frac{p-2}{p-1}x_1^2 + \frac{p}{p-1}\|x_2\|^2 \geq 0$. Furthermore, we see that these eigenvalues must be positive when $p > 2$ since $x_1^2 > 0$ and $x_1^2 - \|x_2\|^2 > 0$. This means that the matrix on the right hand side of (1.17) is positive semidefinite, and moreover, it is positive definite when $p > 2$. Applying Lemma 1.1, we have that $\nabla^2 F_9(x) \succeq O$, and furthermore $\nabla^2 F_9(x) \succ O$ when $p > 2$.

Since $a_1(x) > 0$ must hold when $p > 2$, the arguments above show that $F_9(x)$ is convex over $\mathrm{int}(\mathcal{K}^n)$ when $p \geq 2$, and strictly convex over $\mathrm{int}(\mathcal{K}^n)$ when $p > 2$. $\square$

It is worthwhile to point out that $\det(x)$ is neither convex nor concave on $\mathcal{K}^n$, and it is difficult to argue the convexity of those compound functions involving $\det(x)$ by the definition of convex function. But, our SOC trace function offers a simple way to prove their convexity. Moreover, it helps on establishing more inequalities associated with SOC. Some of these inequalities have been used to analyze the properties of SOC function f^{soc} [42] and the convergence of interior point methods for SOCPs [7].

Proposition 1.8. *For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, the following inequalities hold.*

(a) $\det(\alpha x + (1-\alpha)y) \geq (\det(x))^\alpha (\det(y))^{1-\alpha}$ *for any* $0 < \alpha < 1$.

(b) $\det(x+y)^{1/p} \geq 2^{\frac{2}{p}-1}\left(\det(x)^{1/p} + \det(y)^{1/p}\right)$ *for any* $p \geq 2$.

(c) $\det(\alpha x + (1-\alpha)y) \geq \alpha^2 \det(x) + (1-\alpha)^2 \det(y)$ *for any* $0 < \alpha < 1$.

(d) $[\det(e+x)]^{1/2} \geq 1 + \det(x)^{1/2}$.

(e) $\det(x)^{1/2} = \inf\left\{\frac{1}{2}\mathrm{tr}(x \circ y) \,\middle|\, \det(y) = 1,\ y \succeq_{\mathcal{K}^n} 0\right\}$. *Furthermore, when $x \succ_{\mathcal{K}^n} 0$, the same relation holds with* inf *replaced by* min.

(f) $\mathrm{tr}(x \circ y) \geq 2\det(x)^{1/2}\det(y)^{1/2}$.

Proof. (a) From Proposition 1.6(a), we know that $\ln(\det(x))$ is strictly concave in $\mathrm{int}(\mathcal{K}^n)$. With this, we have

$$\begin{aligned} \ln(\det(\alpha x + (1-\alpha)y)) &\geq \alpha\ln(\det(x)) + (1-\alpha)\ln(\det(y)) \\ &= \ln(\det(x)^\alpha) + \ln(\det(x)^{1-\alpha}) \end{aligned}$$

for any $0 < \alpha < 1$ and $x, y \in \mathrm{int}(\mathcal{K}^n)$. This, together with the increasing of $\ln t$ $(t > 0)$ and the continuity of $\det(x)$, implies the desired result.

(b) By Proposition 1.7(b), $\det(x)^{1/p}$ is concave over $\mathcal{K}^n$. Then, for any $x, y \in \mathcal{K}^n$, we have

$$\begin{aligned}
& \det\left(\frac{x+y}{2}\right)^{1/p} \geq \frac{1}{2}\left[\det(x)^{1/p} + \det(y)^{1/p}\right] \\
\Longleftrightarrow \quad & 2\left[\left(\frac{x_1+y_1}{2}\right)^2 - \left\|\frac{x_2+y_2}{2}\right\|^2\right]^{1/p} \geq \left(x_1^2 - \|x_2\|^2\right)^{1/p} + \left(y_1^2 - \|y_2\|^2\right)^{1/p} \\
\Longleftrightarrow \quad & \left[(x_1+y_1)^2 - \|x_2+y_2\|^2\right]^{1/p} \geq \frac{4^{\frac{1}{p}}}{2}\left[\left(x_1^2 - \|x_2\|^2\right)^{1/p} + \left(y_1^2 - \|y_2\|^2\right)^{1/p}\right] \\
\Longleftrightarrow \quad & \det(x+y)^{1/p} \geq 2^{\frac{2}{p}-1}\left(\det(x)^{1/p} + \det(y)^{1/p}\right),
\end{aligned}$$

which is the desired result.

(c) Using the inequality in part(b) with $p = 2$, we have

$$\det(x+y)^{1/2} \geq \det(x)^{1/2} + \det(y)^{1/2}.$$

Squaring both sides yields

$$\det(x+y) \geq \det(x) + \det(y) + 2\det(x)^{1/2}\det(y)^{1/2} \geq \det(x) + \det(y),$$

where the last inequality is by the nonnegativity of $\det(x)$ and $\det(y)$ since $x, y \in \mathcal{K}^n$. This together with the fact $\det(\alpha x) = \alpha^2 \det(x)$ leads to the desired result.

(d) This inequality is presented in Proposition 1.2(d). Nonetheless, we provide a different approach by applying part(b) with $p = 2$ and the fact that $\det(e) = 1$.

(e) From Proposition 1.3(b), we have

$$\mathrm{tr}(x \circ y) \geq \lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x), \quad \forall x, y \in \mathbb{R}^n.$$

For any $x, y \in \mathcal{K}^n$, this along with the arithmetic-geometric mean inequality implies that

$$\begin{aligned}
\frac{\mathrm{tr}(x \circ y)}{2} &\geq \frac{\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)}{2} \\
&\geq \sqrt{\lambda_1(x)\lambda_2(y)\lambda_1(y)\lambda_2(x)} \\
&= \det(x)^{1/2}\det(y)^{1/2},
\end{aligned}$$

which means that $\inf\left\{\frac{1}{2}\mathrm{tr}(x \circ y) \,\middle|\, \det(y) = 1, y \succeq_{\mathcal{K}^n} 0\right\} = \det(x)^{1/2}$ for a fixed $x \in \mathcal{K}^n$. If $x \succ_{\mathcal{K}^n} 0$, then we can verify that the feasible point $y^* = \frac{x^{-1}}{\sqrt{\det(x)}}$ is such that $\frac{1}{2}\mathrm{tr}(x \circ y^*) = \det(x)^{1/2}$, and the second part follows.

(f) Using part(e), for any $x \in \mathcal{K}^n$ and $y \in \mathrm{int}(\mathcal{K}^n)$, we have

$$\frac{\mathrm{tr}(x \circ y)}{2\sqrt{\det(y)}} = \frac{1}{2}\mathrm{tr}\left(x \circ \frac{y}{\sqrt{\det(y)}}\right) \geq \sqrt{\det(x)},$$

which together with the continuity of $\det(x)$ and $\mathrm{tr}(x)$ implies that

$$\mathrm{tr}(x \circ y) \geq 2 \det(x)^{1/2} \det(y)^{1/2}, \quad \forall x, y \in \mathcal{K}^n.$$

Thus, we complete the proof. $\Box$

We close this section by remarking some extensions. Some of the inequalities in Proposition 1.8 were established with the help of the Schwartz Inequality, see Proposition 1.2, whereas here we achieve the goal easily by using the convexity of SOC functions. In particular, Proposition 1.8(b) has a stronger version shown as in Proposition 2.32 in which $p \geq 2$ is relaxed to $p \geq 1$ and the proof is done by different approach. These inequalities all have their counterparts for matrix inequalities [22, 75, 135]. For example, Proposition 1.8(b) with $p = 2$, i.e., p being equal to the rank of Jordan algebra $(\mathbb{R}^n, \circ)$, corresponds to the Minkowski Inequality of matrix setting:

$$\det(A + B)^{1/n} \geq \det(A)^{1/n} + \det(B)^{1/n}$$

for any $n \times n$ positive semidefinite matrices A and B. Moreover, some inequalities in Proposition 1.8 have been extended to symmetric cone setting [38] by using Euclidean Jordan algebras. Proposition 1.6 and Proposition 1.7 have also generalized versions in symmetric cone setting, see [36]. There will have SOC trace versions of Young, Hölder, and Minkowski inequalities in Chapter 4.

1.3 Nonsmooth analysis of SOC functions

To explore the properties of the aforementioned SOC functions, we review some basic concepts of vector-valued functions, including continuity, (local) Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, as well as (ρ-order) semismoothness. In what follows, we consider a function $F : \mathbb{R}^k \to \mathbb{R}^\ell$. We say F is continuous at $x \in \mathbb{R}^k$ if

$$F(y) \to F(x) \quad \text{as} \quad y \to x;$$

and F is continuous if F is continuous at every $x \in \mathbb{R}^k$. F is strictly continuous (also called 'locally Lipschitz continuous') at $x \in \mathbb{R}^k$ [134, Chap. 9] if there exist scalars $\kappa > 0$ and $\delta > 0$ such that

$$\|F(y) - F(z)\| \leq \kappa \|y - z\| \quad \forall y, z \in \mathbb{R}^k \text{ with } \|y - x\| \leq \delta, \ \|z - x\| \leq \delta;$$

and F is strictly continuous if F is strictly continuous at every $x \in \mathbb{R}^k$. If δ can be taken to be ∞, then F is Lipschitz continuous with Lipschitz constant κ. Define the function $\mathrm{lip}F : \mathbb{R}^k \to [0, \infty]$ by

$$\mathrm{lip}F(x) := \limsup_{\substack{y,z \to x \\ y \neq z}} \frac{\|F(y) - F(z)\|}{\|y - z\|}.$$

Then, F is strictly continuous at x if and only if $\text{lip}F(x)$ is finite.

We say F is directionally differentiable at $x \in \mathbb{R}^k$ if

$$F'(x;h) \;:=\; \lim_{t \to 0^+} \frac{F(x+th)-F(x)}{t} \quad \text{exists} \qquad \forall h \in \mathbb{R}^k;$$

and F is directionally differentiable if F is directionally differentiable at every $x \in \mathbb{R}^k$. F is differentiable (in the Fréchet sense) at $x \in \mathbb{R}^k$ if there exists a linear mapping $\nabla F(x) : \mathbb{R}^k \to \mathbb{R}^\ell$ such that

$$F(x+h) - F(x) - \nabla F(x)h \;=\; o(\|h\|).$$

We say that F is continuously differentiable if F is differentiable at every $x \in \mathbb{R}^k$ and ∇F is continuous.

If F is strictly continuous, then F is almost everywhere differentiable by Rademacher's Theorem, see [54] and [134, Chapter 9J]. In this case, the generalized Jacobian $\partial F(x)$ of F at x (in the Clarke sense) can be defined as the convex hull of the generalized Jacobian $\partial_B F(x)$, where

$$\partial_B F(x) := \left\{ \lim_{x^j \to x} \nabla F(x^j) \,\middle|\, F \text{ is differentiable at } x^j \in \mathbb{R}^k \right\}.$$

The notation ∂_B is adopted from [129]. In [134, Chap. 9], the case of $\ell = 1$ is considered and the notations "$\bar{\nabla}$" and "$\bar{\partial}$" are used instead of, respectively, "∂_B" and "∂".

Assume $F : \mathbb{R}^k \to \mathbb{R}^\ell$ is strictly continuous. We say F is semismooth at x if F is directionally differentiable at x and, for any $V \in \partial F(x+h)$, we have

$$F(x+h) - F(x) - Vh = o(\|h\|).$$

We say F is ρ-order semismooth at x ($0 < \rho < \infty$) if F is semismooth at x and, for any $V \in \partial F(x+h)$, we have

$$F(x+h) - F(x) - Vh = O(\|h\|^{1+\rho}).$$

We say F is semismooth (respectively, ρ-order semismooth) if F is semismooth (respectively, ρ-order semismooth) at every $x \in \mathbb{R}^k$. We say F is strongly semismooth if it is 1-order semismooth. Convex functions and piecewise continuously differentiable functions are examples of semismooth functions. The composition of two (respectively, ρ-order) semismooth functions is also a (respectively, ρ-order) semismooth function. The property of semismoothness plays an important role in nonsmooth Newton methods [129, 130] as well as in some smoothing methods [52, 64, 72]. For extensive discussions of semismooth functions, see [63, 109, 130]. At last, we provide a diagram describing the relation between smooth and nonsmooth functions in Figure 1.2.

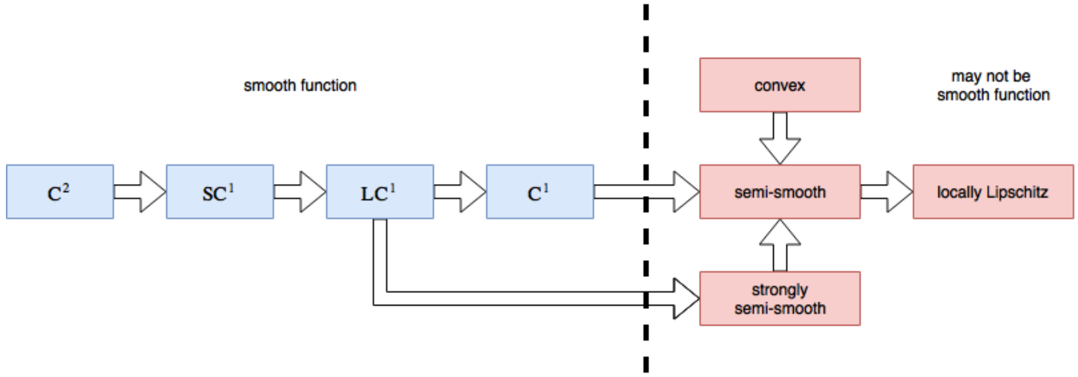


Figure 1.2: Relation between smooth and nonsmooth functions

Let $\mathbb{R}^{n\times n}$ denote the space of $n \times n$ real matrices, equipped with the trace inner product and the Frobenius norm

$$\langle X, Y\rangle_F := \mathrm{tr}[X^T Y], \quad \|X\|_F := \sqrt{\langle X, X\rangle_F},$$

where $X, Y \in \mathbb{R}^{n\times n}$ and $\mathrm{tr}[\cdot]$ denotes the matrix trace, i.e., $\mathrm{tr}[X] = \sum_{i=1}^n X_{ii}$. Let $\emptyset$ denote the set of $P \in \mathbb{R}^{n\times n}$ that are orthogonal, i.e., $P^T = P^{-1}$. Let $\mathcal{S}^n$ denote the subspace comprising those $X \in \mathbb{R}^{n\times n}$ that are symmetric, i.e., $X^T = X$. This is a subspace of $\mathbb{R}^{n\times n}$ with dimension $n(n+1)/2$, which can be identified with $\mathbb{R}^{n(n+1)/2}$. Thus, a function mapping $\mathcal{S}^n$ to $\mathcal{S}^n$ may be viewed equivalently as a function mapping $\mathbb{R}^{n(n+1)/2}$ to $\mathbb{R}^{n(n+1)/2}$. We consider such a function below.

For any $X \in \mathcal{S}^n$, its (repeated) eigenvalues $\lambda_1, \cdots, \lambda_n$ are real and it admits a spectral decomposition of the form:

$$X = P\,\mathrm{diag}[\lambda_1, \cdots, \lambda_n]\,P^T, \tag{1.18}$$

for some orthogonal matrix P, where $\mathrm{diag}[\lambda_1, \cdots, \lambda_n]$ denotes the $n \times n$ diagonal matrix with its ith diagonal entry λ_i. Then, for any function $f : \mathbb{R} \to \mathbb{R}$, we can define a corresponding function $f^{\mathrm{mat}} : \mathcal{S}^n \to \mathcal{S}^n$ [22, 76] by

$$f^{\mathrm{mat}}(X) := P\,\mathrm{diag}[f(\lambda_1), \cdots, f(\lambda_n)]\,P^T. \tag{1.19}$$

It is known that $f^{\mathrm{mat}}(X)$ is well-defined (independent of the ordering of $\lambda_1, \ldots, \lambda_n$ and the choice of P) and belongs to $\mathcal{S}^n$, see [22, Chap. V] and [76, Sec. 6.2]. Moreover, a result of Daleckii and Krein showed that if f is continuously differentiable, then f^{mat} is differentiable and its Jacobian $\nabla f^{\mathrm{mat}}(X)$ has a simple formula, see [22, Theorem V.3.3]; also see [51, Proposition 4.3].

In [50], f^{mat} was used to develop non-interior continuation methods for solving semidefinite programs and semidefinite complementarity problems. A related method was studied in [86]. Further studies of f^{mat} in the case of $f(\xi) = |\xi|$ and $f(\xi) = \max\{0, \xi\}$ are given in [123, 140], obtaining results such as strong semismoothness, formulas for directional derivatives, and necessary/sufficient conditions for strong stability of an isolated solution to semidefinite complementarity problem (SDCP).

The following key results are extracted from [51], which says that f^{mat} inherits from f the property of continuity (respectively, strict continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, semismoothness, ρ-order semismoothness).

Proposition 1.9. *For any $f : \mathbb{R} \to \mathbb{R}$, the following results hold.*

(a) f^{mat} *is continuous at an $X \in \mathcal{S}^n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ if and only if f is continuous at $\lambda_1, \cdots, \lambda_n$.*

(b) f^{mat} *is directionally differentiable at an $X \in \mathcal{S}^n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ if and only if f is directionally differentiable at $\lambda_1, \cdots, \lambda_n$.*

(c) f^{mat} *is differentiable at an $X \in \mathcal{S}^n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ if and only if f is differentiable at $\lambda_1, \cdots, \lambda_n$.*

(d) f^{mat} *is continuously differentiable at an $X \in \mathcal{S}^n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ if and only if f is continuously differentiable at $\lambda_1, \cdots, \lambda_n$.*

(e) f^{mat} *is strictly continuous at an $X \in \mathcal{S}^n$ with eigenvalues $\lambda_1, \cdots, \lambda_n$ if and only if f is strictly continuous at $\lambda_1, \cdots, \lambda_n$.*

(f) f^{mat} *is Lipschitz continuous (with respect to $\|\cdot\|_F$) with constant κ if and only if f is Lipschitz continuous with constant κ.*

(g) f^{mat} *is semismooth if and only if f is semismooth. If $f : \mathbb{R} \to \mathbb{R}$ is ρ-order semismooth ($0 < \rho < \infty$), then f^{mat} is $\min\{1, \rho\}$-order semismooth.*

The SOC function f^{soc} defined as in (1.8) has a connection to the matrix-valued f^{mat} given as in (1.19) via a special mapping. To see this, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ as

$$\begin{array}{rccl} L_x : & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ & y & \longmapsto & L_x y := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} y. \end{array} \tag{1.20}$$

It can be easily verified that $x \circ y = L_x y$ for all $y \in \mathbb{R}^n$, and L_x is positive definite (and hence invertible) if and only if $x \in \mathrm{int}(\mathcal{K}^n)$. However, $L_x^{-1} y \neq x^{-1} \circ y$, for some $x \in \mathrm{int}(\mathcal{K}^n)$ and $y \in \mathbb{R}^n$, i.e., $L_x^{-1} \neq L_{x^{-1}}$. The mapping L_x will be used to relate f^{soc} to f^{mat}. For convenience, in the subsequent contexts, we sometimes omit the variable notion x in $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$.

Proposition 1.10. *Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ given by (1.3) and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$ given by (1.4). We denote $z := x_2$ if $x_2 \neq 0$; otherwise let z be any nonzero vector in $\mathbb{R}^{n-1}$. Then, the following results hold.*

(a) *For any $t \in \mathbb{R}$, the matrix $L_x + tM_z$ has eigenvalues $\lambda_1(x)$, $\lambda_2(x)$, and $x_1 + t$ of multiplicity $n-2$, where*

$$M_z := \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{zz^T}{\|z\|^2} \end{bmatrix} \tag{1.21}$$

(b) *For any $f : \mathbb{R} \to \mathbb{R}$ and any $t \in \mathbb{R}$, we have*

$$f^{\text{soc}}(x) = f^{\text{mat}}(L_x + tM_z)e. \tag{1.22}$$

Proof. (a) It is straightforward to verify that, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the eigenvalues of L_x are $\lambda_1(x)$, $\lambda_2(x)$, as given by (1.3), and x_1 of multiplicity $n-2$. Its corresponding orthonormal set of eigenvectors is

$$\sqrt{2}u_x^{(1)}, \sqrt{2}u_x^{(2)}, u_x^{(i)} = (0, u_2^{(i)}), i = 3, ..., n,$$

where $u_x^{(1)}, u_x^{(2)}$ are the spectral vectors with $w = \frac{z}{\|z\|}$ whenever $x_2 = 0$, and $u_2^{(3)}, \cdots, u_2^{(n)}$ is any orthonormal set of vectors that span the subspace of $\mathbb{R}^{n-2}$ orthogonal to z. Thus,

$$L_x = U\text{diag}[\lambda_1(x), \lambda_2(x), x_1, \cdots, x_1]U^T,$$

where $U := \begin{bmatrix} \sqrt{2}u_x^{(1)} & \sqrt{2}u_x^{(2)} & u_x^{(3)} & \cdots & u_x^{(n)} \end{bmatrix}$. In addition, it is not hard to verify using $u_x^{(i)} = (0, u_2^{(i)})$, $i = 3, ..., n$, that

$$U\,\text{diag}[0, 0, 1, \cdots, 1]\,U^T = \begin{bmatrix} 0 & 0 \\ 0 & \displaystyle\sum_{i=3}^{n} u_2^{(i)}(u_2^{(i)})^T \end{bmatrix}.$$

Since $Q := \begin{bmatrix} \frac{z}{\|z\|} & u_2^{(3)} & \cdots & u_2^{(n)} \end{bmatrix}$ is an orthogonal matrix, we have

$$I = QQ^T = \frac{zz^T}{\|z\|^2} + \sum_{i=3}^{n} u_2^{(i)}(u_2^{(i)})^T$$

and hence $\sum_{i=3}^{n} u_2^{(i)}(u_2^{(i)})^T = I - \frac{zz^T}{\|z\|^2}$. This together with (1.21) shows that

$$U\text{diag}[0, 0, 1, ..., 1]\,U^T = M_z.$$

Thus, we obtain

$$L_x + tM_z = U\text{diag}[\lambda_1(x), \lambda_2(x), x_1 + t, \cdots, x_1 + t]\,U^T, \tag{1.23}$$

which is the desired result.

(b) Using (1.23) yields

$$\begin{aligned} f^{\text{mat}}(L_x + tM_z)e &= U\text{diag}\left[f(\lambda_1(x)), f(\lambda_2(x)), f(x_1+t), \cdots, f(x_1+t)\right]U^T e \\ &= f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \\ &= f^{\text{soc}}(x), \end{aligned}$$

where the second equality uses the special form of U. Then, the proof is complete. □

Of particular interest is the choice of $t = \pm\|x_2\|$, for which $L_x + tM_{x_2}$ has eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ with some multiplicities. More generally, for any $f, g : \mathbb{R} \to \mathbb{R}_+$, any $h : \mathbb{R}_+ \to \mathbb{R}$ and any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have

$$h^{\text{soc}}\left(f^{\text{soc}}(x) + g(\mu)e\right) = h^{\text{mat}}\left(f^{\text{mat}}(L_x) + g(\mu)I\right)e.$$

In particular, the spectral values of $f^{\text{soc}}(x)$ and $g(\mu)e$ are nonnegative, as are the eigenvalues of $f^{\text{mat}}(L_x)$ and $g(\mu)I$, so both sides are well-defined. Moreover, taking

$$f(\xi) = \xi^2, \quad g(\mu) = \mu^2, \quad h(\xi) = \xi^{1/2}$$

leads to

$$\left(x^2 + \mu^2 e\right)^{1/2} = \left(L_x^2 + \mu^2 I\right)^{1/2} e.$$

It was shown in [142] that $(X, \mu) \mapsto (X^2 + \mu^2 I)^{1/2}$ is strongly semismooth. Then, it follows from the above equation that $(x, \mu) \mapsto (x^2 + \mu^2 e)^{1/2}$ is strongly semismooth. This provides an alternative and indeed shorter proof for [52, Theorem 4.2].

Now, we use the results of Proposition 1.9 and Proposition 1.10 to show that if $f : \mathbb{R} \to \mathbb{R}$ has the property of continuity (respectively, strict continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, semismoothness, ρ-order semismoothness), then so does the vector-valued function f^{soc}.

Proposition 1.11. *For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following results hold.*

(a) *f^{soc} is continuous at an $x \in \mathcal{S}$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ if and only if f is continuous at $\lambda_1(x)$, $\lambda_2(x)$.*

(b) *f^{soc} is continuous if and only if f is continuous.*

Proof. (a) Suppose f is continuous at $\lambda_1(x)$, $\lambda_2(x)$. If $x_2 = 0$, then $x_1 = \lambda_1(x) = \lambda_2(x)$ and, by Proposition 1.10(a), L_x has eigenvalue of $\lambda_1(x) = \lambda_2(x)$ of multiplicity n. Then, applying Proposition 1.9(a), f^{mat} is continuous at L_x. Since L_x is continuous in x, Proposition 1.10(b) yields that $f^{\text{soc}}(x) = f^{\text{mat}}(L_x)e$ is continuous at x. If $x_2 \neq 0$, then, by Proposition 1.10(a), $L_x + \|x_2\|M_{x_2}$ has eigenvalue of $\lambda_1(x)$ of multiplicity 1 and $\lambda_2(x)$

of multiplicity $n-1$. Then, by Proposition 1.9(a), f^{mat} is continuous at $L_x + \|x_2\|M_{x_2}$. Since $x \mapsto L_x + \|x_2\|M_{x_2}$ is continuous at x, Proposition 1.10(b) yields that $x \mapsto f^{\text{soc}}(x) = f^{\text{mat}}(L_x + \|x_2\|M_{x_2})e$ is continuous at x.

For the other direction, suppose f^{soc} is continuous at x with spectral values $\lambda_1(x)$, $\lambda_2(x)$, and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$. For any $\mu_1 \in \mathbb{R}$, let

$$y := \mu_1 u_x^{(1)} + \lambda_2(x) u_x^{(2)}.$$

We first claim that the spectral decomposition of y is

$$y = \begin{cases} \mu_1 u_x^{(1)} + \lambda_2(x) u_x^{(2)} & \text{if} \quad \mu_1 \le \lambda_2(x), \\ \lambda_1(x) u_x^{(1)} + \mu_1 u_x^{(2)} & \text{if} \quad \mu_1 > \lambda_2(x). \end{cases}$$

To ratify this assertion, we write out $y = \mu_1 u_x^{(1)} + \lambda_2(x) u_x^{(2)}$ as (y_1, y_2), which means $y_1 = \frac{1}{2}(\lambda_2(x) + \mu_1)$ and $\|y_2\| = \frac{1}{2}|\lambda_2(x) - \mu_1|$. Then, we have $u_y^{(1)} = u_x^{(1)}$, $u_y^{(2)} = u_x^{(2)}$, and

$$\lambda_1(y) = y_1 - \|y_2\| = \begin{cases} \mu_1 & \text{if} \quad \mu_1 \le \lambda_2(x), \\ \lambda_2(x) & \text{if} \quad \mu_1 > \lambda_2(x). \end{cases}$$

$$\lambda_2(y) = y_1 + \|y_2\| = \begin{cases} \lambda_2(x) & \text{if} \quad \mu_1 \le \lambda_2(x), \\ \mu_1 & \text{if} \quad \mu_1 > \lambda_2(x). \end{cases}$$

Thus, the assertion is proved, which says $y \to x$ as $\mu_1 \to \lambda_1(x)$. Since f^{soc} is continuous at x, we have

$$f(\mu_1)u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} = f^{\text{soc}}(y) \to f^{\text{soc}}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}.$$

Due to $u_x^{(1)} \ne 0$, this implies $f(\mu_1) \to f(\lambda_1(x))$ as $\mu_1 \to \lambda_1(x)$. Thus, f is continuous at $\lambda_1(x)$. A similar argument shows that f is continuous at $\lambda_2(x)$.

(b) This is an immediate consequence of part(a). $\square$

The "if" direction of Proposition 1.11(a) can alternatively be proved using the Lipschitzian property of the spectral values (see Property 1.4) and an upper Lipschitzian property of the spectral vectors. However, this alternative proof is more complicated. If f has a power series expansion, then so does f^{soc}, with the same coefficients of expansion, see [64, Proposition 3.1].

By using Proposition 1.10 and Proposition 1.9(b), we have the following directional differentiability result for f^{soc}, together with a computable formula for the directional derivative of f^{soc}. In the special case of $f(\cdot) = \max\{0, \cdot\}$, for which $f^{\text{soc}}(x)$ corresponds to the projection of x onto $\mathcal{K}^n$, an alternative formula expressing the directional derivative as the unique solution to a certain convex program is given in [123, Proposition 13].

Proposition 1.12. *For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following results hold.*

(a) *$f^{\rm soc}$ is directionally differentiable at an $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ if and only if f is directionally differentiable at $\lambda_1(x)$, $\lambda_2(x)$. Moreover, for any nonzero $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have*

$$\left(f^{\rm soc}\right)'(x;h) = f'(x_1;h_1)e$$

if $x_2 = 0$ and $h_2 = 0$;

$$\left(f^{\rm soc}\right)'(x;h) = \frac{1}{2}f'(x_1;h_1-\|h_2\|)\left(1, \frac{-h_2}{\|h_2\|}\right)+\frac{1}{2}f'(x_1;h_1+\|h_2\|)\left(1, \frac{h_2}{\|h_2\|}\right) \quad (1.24)$$

if $x_2 = 0$ and $h_2 \neq 0$; otherwise

$$\begin{aligned}\left(f^{\rm soc}\right)'(x;h) &= \frac{1}{2}f'\left(\lambda_1(x); h_1 - \frac{x_2^T h_2}{\|x_2\|}\right)\left(1, \frac{-x_2}{\|x_2\|}\right) - \frac{f(\lambda_1(x))}{2\|x_2\|}M_{x_2}h \\ &+ \frac{1}{2}f'\left(\lambda_2(x); h_1 + \frac{x_2^T h_2}{\|x_2\|}\right)\left(1, \frac{x_2}{\|x_2\|}\right) + \frac{f(\lambda_2(x))}{2\|x_2\|}M_{x_2}h. \quad (1.25)\end{aligned}$$

(b) *$f^{\rm soc}$ is directionally differentiable if and only if f is directionally differentiable.*

Proof. (a) Suppose f is directionally differentiable at $\lambda_1(x)$, $\lambda_2(x)$. If $x_2 = 0$, then $x_1 = \lambda_1(x) = \lambda_2(x)$ and, by Proposition 1.10(a), L_x has eigenvalue of x_1 of multiplicity n. Then, by Proposition 1.9(b), $f^{\rm mat}$ is directionally differentiable at L_x. Since L_x is differentiable in x, Proposition 1.10(b) yields that $f^{\rm soc}(x) = f^{\rm mat}(L_x)e$ is directionally differentiable at x. If $x_2 \neq 0$, then, by Proposition 1.10(a), $L_x + \|x_2\|M_{x_2}$ has eigenvalue of $\lambda_1(x)$ of multiplicity 1 and $\lambda_2(x)$ of multiplicity $n-1$. Then, by Proposition 1.9(b), $f^{\rm mat}$ is directionally differentiable at $L_x + \|x_2\|M_{x_2}$. Since $x \mapsto L_x + \|x_2\|M_{x_2}$ is differentiable at x, Proposition 1.10(b) yields that $x \mapsto f^{\rm soc}(x) = f^{\rm mat}(L_x + \|x_2\|M_{x_2})e$ is directionally differentiable at x.

Fix any nonzero $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Below we calculate $(f^{\rm soc})'(x;h)$. Suppose $x_2 = 0$. Then, $\lambda_1(x) = \lambda_2(x) = x_1$ and the spectral vectors $u^{(1)}$, $u^{(2)}$ sum to $e = (1,0)$. If $h_2 = 0$, then for any $t > 0$, $x + th$ has the spectral values $\mu_1 = \mu_2 = x_1 + th_1$ and its spectral vectors $v^{(1)}, v^{(2)}$ sum to $e = (1,0)$. Thus,

$$\begin{aligned}&\frac{f^{\rm soc}(x+th) - f^{\rm soc}(x)}{t} \\ =\ &\frac{1}{t}\left(f(\mu_1)v^{(1)} + f(\mu_2)v^{(2)} - f(\lambda_1(x))u^{(1)} - f(\lambda_2(x))u^{(2)}\right) \\ =\ &\frac{f(x_1+th_1) - f(x_1)}{t}e \\ \to\ &f'(x_1;h_1)e \quad \text{as } t \to 0^+.\end{aligned}$$

If $h_2 \neq 0$, then for any $t > 0$, $x + th$ has the spectral values $\mu_i = (x_1 + th_1) + (-1)^i t\|h_2\|$ and spectral vectors $v^{(i)} = \frac{1}{2}(1, (-1)^i h_2/\|h_2\|)$, $i = 1, 2$. Moreover, since $x_2 = 0$, we can choose $u^{(i)} = v^{(i)}$ for $i = 1, 2$. Thus,

$$\begin{aligned}
& \frac{f^{\text{soc}}(x+th) - f^{\text{soc}}(x)}{t} \\
= \; & \frac{1}{t}\left(f(\mu_1)v^{(1)} + f(\mu_2)v^{(2)} - f(\lambda_1)v^{(1)} - f(\lambda_2)v^{(2)}\right) \\
= \; & \frac{f(x_1 + t(h_1 - \|h_2\|)) - f(x_1)}{t} v^{(1)} + \frac{f(x_1 + t(h_1 + \|h_2\|)) - f(x_1)}{t} v^{(2)} \\
\to \; & f'(x_1; h_1 - \|h_2\|)v^{(1)} + f'(x_1; h_1 + \|h_2\|)v^{(2)} \quad \text{as } t \to 0^+.
\end{aligned}$$

This together with $v^{(i)} = \frac{1}{2}(1, (-1)^i h_2/\|h_2\|)$, $i = 1, 2$, yields (1.24). Suppose $x_2 \neq 0$. Then, $\lambda_i(x) = x_1 + (-1)^i\|x_2\|$ and the spectral vectors are $u^{(i)} = \frac{1}{2}(1, (-1)^i x_2/\|x_2\|)$, $i = 1, 2$. For any $t > 0$ sufficiently small so that $x_2 + th_2 \neq 0$, $x + th$ has the spectral values $\mu_i = x_1 + th_1 + (-1)^i\|x_2 + th_2\|$ and spectral vectors $v^{(i)} = \frac{1}{2}(1, (-1)^i(x_2 + th_2)/\|x_2 + th_2\|)$, $i = 1, 2$. Thus,

$$\begin{aligned}
& \frac{f^{\text{soc}}(x+th) - f^{\text{soc}}(x)}{t} \\
= \; & \frac{1}{t}\left(f(\mu_1)v^{(1)} + f(\mu_2)v^{(2)} - f(\lambda_1(x))u^{(1)} - f(\lambda_2(x))u^{(2)}\right) \\
= \; & \frac{1}{t}\left(\frac{1}{2}f(x_1 + th_1 - \|x_2 + th_2\|)(1, -\frac{x_2 + th_2}{\|x_2 + th_2\|}) - \frac{1}{2}f(\lambda_1(x))(1, -\frac{x_2}{\|x_2\|})\right. \\
+ \; & \left.\frac{1}{2}f(x_1 + th_1 + \|x_2 + th_2\|)(1, \frac{x_2 + th_2}{\|x_2 + th_2\|}) - \frac{1}{2}f(\lambda_2(x))(1, \frac{x_2}{\|x_2\|})\right).
\end{aligned} \tag{1.26}$$

We now focus on the individual terms in (1.26). Since

$$\frac{\|x_2 + th_2\| - \|x_2\|}{t} = \frac{\|x_2 + th_2\|^2 - \|x_2\|^2}{(\|x_2 + th_2\| + \|x_2\|)t} = \frac{2x_2^T h_2 + t\|h_2\|^2}{\|x_2 + th_2\| + \|x_2\|} \to \frac{x_2^T h_2}{\|x_2\|} \quad \text{as } t \to 0^+,$$

we have

$$\begin{aligned}
& \frac{1}{t}\Big(f(x_1 + th_1 - \|x_2 + th_2\|) - f(\lambda_1(x))\Big) \\
= \; & \frac{1}{t}\left(f\left(\lambda_1(x) + t\left(h_1 - \frac{\|x_2 + th_2\| - \|x_2\|}{t}\right)\right) - f(\lambda_1(x))\right) \\
\to \; & f'\left(\lambda_1(x); h_1 - \frac{x_2^T h_2}{\|x_2\|}\right) \quad \text{as } t \to 0^+.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
& \frac{1}{t}\Big(f(x_1 + th_1 + \|x_2 + th_2\|) - f(\lambda_2(x))\Big) \\
\to \; & f'\left(\lambda_2(x); h_1 + \frac{x_2^T h_2}{\|x_2\|}\right) \quad \text{as } t \to 0^+.
\end{aligned}$$

Also, letting $\Phi(x_2) = x_2/\|x_2\|$, we have that

$$\frac{1}{t}\left(\frac{x_2+th_2}{\|x_2+th_2\|} - \frac{x_2}{\|x_2\|}\right) = \frac{\Phi(x_2+th_2)-\Phi(x_2)}{t} \to \nabla\Phi(x_2)h_2 \quad \text{as } t \to 0^+.$$

Combining the above relations with (1.26) and using a product rule, we obtain that

$$\begin{aligned} & \lim_{t\to 0^+} \frac{f^{\text{soc}}(x+th) - f^{\text{soc}}(x)}{t} \\ = \;& \frac{1}{2}\left(f'\left(\lambda_1(x); h_1 - \frac{x_2^T h_2}{\|x_2\|}\right)\left(1, \frac{-x_2}{\|x_2\|}\right) - f(\lambda_1(x))(0, \nabla\Phi(x_2)h_2)\right) \\ & + \frac{1}{2}\left(f'\left(\lambda_2(x); h_1 + \frac{x_2^T h_2}{\|x_2\|}\right)\left(1, \frac{x_2}{\|x_2\|}\right) + f(\lambda_2(x))(0, \nabla\Phi(x_2)h_2)\right). \end{aligned}$$

Using $\nabla\Phi(x_2)h_2 = \frac{1}{\|x_2\|}\left(I - \frac{x_2 x_2^T}{\|x_2\|^2}\right)h_2$ so that $(0, \nabla\Phi(x_2)h_2) = \frac{1}{\|x_2\|}M_{x_2}h$ yields (1.25).

Suppose f^{soc} is directionally differentiable at x with spectral eigenvalues $\lambda_1(x)$, $\lambda_2(x)$ and spectral vectors $u_x^{(1)}$, $u_x^{(2)}$. For any direction $d_1 \in \mathbb{R}$, let

$$h := d_1 u_x^{(1)}.$$

Since $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$, this implies $x + th = (\lambda_1(x) + td_1)\,u_x^{(1)} + \lambda_2(x)u_x^{(2)}$, so that

$$\frac{f^{\text{soc}}(x+th) - f^{\text{soc}}(x)}{t} = \frac{f(\lambda_1(x)+td_1) - f(\lambda_1(x))}{t}u^{(1)}.$$

Since f^{soc} is directionally differentiable at x, the above difference quotient has a limit as $t \to 0^+$. Since $u^{(1)} \neq 0$, this implies that

$$\lim_{t\to 0^+} \frac{f(\lambda_1(x)+td_1) - f(\lambda_1(x))}{t} \quad \text{exists.}$$

Hence, f is directionally differentiable at $\lambda_1(x)$. A similar argument shows f is directionally differentiable at $\lambda_2(x)$.

(b) This is an immediate consequence of part(a). □

Proposition 1.13. *Let $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ given by (1.3). For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following results hold.*

(a) *f^{soc} is differentiable at an $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values λ_1, λ_2 if and only if f is differentiable at λ_1, λ_2. Moreover,*

$$\nabla f^{\text{soc}}(x) = f'(x_1)I \tag{1.27}$$

if $x_2 = 0$, and otherwise

$$\nabla f^{\mathrm{soc}}(x) = \begin{bmatrix} b & c\, x_2^T/\|x_2\| \\ c\, x_2/\|x_2\| & aI + (b-a)x_2x_2^T/\|x_2\|^2 \end{bmatrix}, \tag{1.28}$$

where

$$a = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}, \quad b = \frac{1}{2}\left(f'(\lambda_2) + f'(\lambda_1)\right), \quad c = \frac{1}{2}\left(f'(\lambda_2) - f'(\lambda_1)\right). \tag{1.29}$$

(b) *f^{soc} is differentiable if and only if f is differentiable.*

Proof. (a) The proof of the "if" direction is identical to the proof of Proposition 1.12, but with "directionally differentiable" replaced by "differentiable" and with Proposition 1.9(b) replaced by Proposition 1.9(c). The formula for $\nabla f^{\mathrm{soc}}(x)$ is from [64, Proposition 5.2].

To prove the "only if" direction, suppose f^{soc} is differentiable at x. Then, for each $i = 1, 2$,

$$\frac{f^{\mathrm{soc}}(x + tu^{(i)}) - f^{\mathrm{soc}}(x)}{t} = \frac{f(\lambda_i(x) + t) - f(\lambda_i(x))}{t} u^{(i)}$$

has a limit as $t \to 0$. Since $u^{(i)} \neq 0$, this implies that

$$\lim_{t \to 0} \frac{f(\lambda_i(x) + t) - f(\lambda_i(x))}{t} \quad \text{exists.}$$

Hence, f is differentiable at $\lambda_i(x)$ for $i = 1, 2$.

(b) This is an immediate consequence of part(a). $\square$

We next have the following continuous differentiability result for f^{soc} based on Proposition 1.9(d) and Proposition 1.10. Again, we sometimes omit the variable notation x in $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$.

Proposition 1.14. *Let $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ given by (1.3). For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following results hold.*

(a) *f^{soc} is continuously differentiable at an $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values λ_1, λ_2 if and only if f is continuously differentiable at λ_1, λ_2.*

(b) *f^{soc} is continuously differentiable if and only if f is continuously differentiable.*

Proof. (a) The proof of the "if" direction is identical to the proof of Proposition 1.11, but with "continuous" replaced by "continuously differentiable" and with Proposition 1.9(a) replaced by Proposition 1.9(d). Alternatively, we note that (1.28) is continuous at any x with $x_2 \neq 0$. The case of $x_2 = 0$ can be checked by taking $y = (y_1, y_2) \to x$ and considering the two cases: $y_2 = 0$ or $y_2 \neq 0$.

Conversely, suppose f^{soc} is continuously differentiable at an $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$. Then, by Proposition 1.13, f is differentiable in neighborhoods around $\lambda_1(x)$, $\lambda_2(x)$. If $x_2 = 0$, then $\lambda_1(x) = \lambda_2(x) = x_1$ and (1.27) yields $\nabla f^{\mathrm{soc}}(x) = f'(x_1)I$. For any $h_1 \in \mathbb{R}$, let $h := (h_1, 0)$. Then, $\nabla f^{\mathrm{soc}}(x+h) = f'(x_1+h_1)I$. Since ∇f^{soc} is continuous at x, then $\lim_{h_1 \to 0} f'(x_1+h_1)I = f'(x_1)I$, implying $\lim_{h_1 \to 0} f'(x_1+h_1) = f'(x_1)$. Thus, f' is continuous at x_1. If $x_2 \neq 0$, then $\nabla f^{\mathrm{soc}}(x)$ is given by (1.28) with a, b, c given by (1.29). For any $h_1 \in \mathbb{R}$, let $h := (h_1, 0)$. Then, $x + h = (x_1 + h_1, x_2)$ has spectral values $\mu_1 := \lambda_1(x) + h_1$, $\mu_2 := \lambda_2(x) + h_1$. By (1.28),

$$\nabla f^{\mathrm{soc}}(x+h) = \begin{bmatrix} \beta & \chi\, x_2^T/\|x_2\| \\ \chi\, x_2/\|x_2\| & \alpha I + (\beta - \alpha) x_2 x_2^T/\|x_2\|^2 \end{bmatrix},$$

where

$$\alpha = \frac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1}, \quad \beta = \frac{1}{2}\left(f'(\mu_2) + f'(\mu_1)\right), \quad \chi = \frac{1}{2}\left(f'(\mu_2) - f'(\mu_1)\right).$$

Since ∇f^{soc} is continuous at x so that $\lim_{h \to 0} \nabla f^{\mathrm{soc}}(x+h) = \nabla f^{\mathrm{soc}}(x)$ and $x_2 \neq 0$, we see from comparing terms that $\beta \to b$ and $\chi \to c$ as $h \to 0$. This means that

$$f'(\mu_2) + f'(\mu_1) \to f'(\lambda_2) + f'(\lambda_1) \quad \text{and} \quad f'(\mu_2) - f'(\mu_1) \to f'(\lambda_2) - f'(\lambda_1) \text{ as } h_1 \to 0.$$

Adding and subtracting the above two limits and we obtain

$$f'(\mu_1) \to f'(\lambda_1) \quad \text{and} \quad f'(\mu_2) \to f'(\lambda_2) \text{ as } h_1 \to 0.$$

Since $\mu_1 = \lambda_1(x) + h_1$, $\mu_2 = \lambda_2(x) + h_1$, this shows that f' is continuous at $\lambda_1(x)$, $\lambda_2(x)$.

(b) This is an immediate consequence of part(a). □

In the case where $f = g'$ for some differentiable g, Proposition 1.9(d) is a special case of [101, Theorem 4.2]. This raises the question of whether an SOC analog of the second derivative results in [101] holds.

We now study the strict continuity and Lipschitz continuity properties of f^{soc}. The proof is similar to that of [51, Proposition 4.6], but with a different estimation of $\nabla (f^{\nu})^{\mathrm{soc}}$. We begin with the following lemma, which is analogous to a result of Weyl for eigenvalues of symmetric matrices, e.g., [22, page 63], [75, page 367].

We also need the following result of Rockafellar and Wets [134, Theorem 9.67].

Lemma 1.2. *Suppose $f : \mathbb{R}^k \to \mathbb{R}$ is strictly continuous. Then, there exist continuously differentiable functions $f^{\nu} : \mathbb{R}^k \to \mathbb{R}$, $\nu = 1, 2, \ldots$, converging uniformly to f on any compact set C in $\mathbb{R}^k$ and satisfying*

$$\nabla f^{\nu}(x) \leq \sup_{y \in C} \operatorname{lip} f(y) \qquad \forall x \in C, \ \forall \nu.$$

Lemma 1.2 is slightly different from the original version given in [134, Theorem 9.67]. In particular, the second part of Lemma 1.2 is not contained in [134, Theorem 9.67], but is implicit in its proof. This second part is needed to show that strict continuity and Lipschitz continuity are inherited by f^{soc} from f. We note that Proposition 1.9(e),(f) and Proposition 1.10 can be used to give a short proof of strict continuity and Lipschitz continuity of f^{soc}, but the Lipschitz constant would not be sharp. In particular, the constant would be off by a multiplicative factor of $\sqrt{n}$ due to $\|L_x\|_F \leq \sqrt{n}\|x\|$ for all $x \in \mathbb{R}^n$. Also, spectral vectors do not behave in a (locally) Lipschitzian manner, so we cannot use (1.8) directly.

Proposition 1.15. *Let $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x)$, $\lambda_2(x)$ given by (1.3). For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following results hold.*

(a) *f^{soc} is strictly continuous at an $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2 if and only if f is strictly continuous at λ_1, λ_2.*

(b) *f^{soc} is strictly continuous if and only if f is strictly continuous.*

(c) *f^{soc} is Lipschitz continuous (with respect to $\|\cdot\|$) with constant κ if and only if f is Lipschitz continuous with constant κ.*

Proof. (a) "if" Suppose f is strictly continuous at λ_1, λ_2. Then, there exist $\kappa_i > 0$ and $\delta_i > 0$ for $i = 1, 2$, such that

$$|f(\xi) - f(\zeta)| \leq \kappa_i|\xi - \zeta|, \quad \forall \xi, \zeta \in [\lambda_i - \delta_i, \lambda_i + \delta_i].$$

Let $\bar{\delta} := \min\{\delta_1, \delta_2\}$ and

$$C := [\lambda_1 - \bar{\delta}, \lambda_1 + \bar{\delta}] \; \cup \; [\lambda_2 - \bar{\delta}, \lambda_2 + \bar{\delta}] \; .$$

We define $\tilde{f} : \mathbb{R} \to \mathbb{R}$ to be the function that coincides with f on C; and is linearly extrapolated at the boundary points of C on $\mathbb{R} \setminus C$. In other words,

$$\tilde{f}(\xi) = \begin{cases} f(\xi) & \text{if } \xi \in C, \\ (1-t)f(\lambda_1 + \bar{\delta}) + tf(\lambda_2 - \bar{\delta}) & \text{if } \lambda_1 + \bar{\delta} < \lambda_2 - \bar{\delta} \text{ and, for some } t \in (0,1), \\ & \xi = (1-t)(\lambda_1 + \bar{\delta}) + t(\lambda_2 - \bar{\delta}), \\ f(\lambda_1 - \bar{\delta}) & \text{if } \xi < \lambda_1 - \bar{\delta}, \\ f(\lambda_2 + \bar{\delta}) & \text{if } \xi > \lambda_2 + \bar{\delta}. \end{cases}$$

From the above, we see that $\tilde{f}$ is Lipschitz continuous, so that there exists a scalar $\kappa > 0$ such that $\mathrm{lip}\tilde{f}(\xi) \leq \kappa$ for all $\xi \in \mathbb{R}$. Since C is compact, by Lemma 1.2, there exist continuously differentiable functions $f^\nu : \mathbb{R} \to \mathbb{R}$, $\nu = 1, 2, \ldots$, converging uniformly to $\tilde{f}$ and satisfying

$$|(f^\nu)'(\xi)| \leq \kappa \quad \forall \xi \in C, \; \forall \nu \; . \tag{1.30}$$

Let $\delta := \frac{1}{\sqrt{2}}\bar{\delta}$, so by Property 1.4, C contains two spectral values of any $y \in \mathcal{B}(x,\delta)$. Moreover, for any $w \in \mathcal{B}(x,\delta)$ with spectral factorization

$$w = \mu_1 u^{(1)} + \mu_2 u^{(2)} ,$$

we have $\mu_1, \mu_2 \in C$ and

$$\begin{aligned} \left\|(f^\nu)^{\text{soc}}(w) - f^{\text{soc}}(w)\right\|^2 &= \|(f^\nu(\mu_1) - f(\mu_1))u^{(1)} + (f^\nu(\mu_2) - f(\mu_2))u^{(2)}\|^2 \\ &= \frac{1}{2}|f^\nu(\mu_1) - f(\mu_1)|^2 + \frac{1}{2}|f^\nu(\mu_2) - f(\mu_2)|^2 , \end{aligned} \tag{1.31}$$

where we use $\|u^{(i)}\|^2 = 1/2$ for $i = 1, 2$, and $(u^{(1)})^T u^{(2)} = 0$. Since $\{f^\nu\}_{\nu=1}^\infty$ converges uniformly to f on C, equation (1.31) shows that $\{(f^\nu)^{\text{soc}}\}_{\nu=1}^\infty$ converges uniformly to f^{soc} on $\mathcal{B}(x,\delta)$. Moreover, for all $w = (w_1, w_2) \in \mathcal{B}(x,\delta)$ and all ν, we have from Proposition 1.13 that $\nabla(f^\nu)^{\text{soc}}(w) = (f^\nu)'(w_1)I$ if $w_2 = 0$, in which case $\nabla(f^\nu)^{\text{soc}}(w) = |(f^\nu)'(w_1)| \leq \kappa$. Otherwise $w_2 \neq 0$ and

$$\nabla(f^\nu)^{\text{soc}}(w) = \begin{bmatrix} b & c\, w_2^T/\|w_2\| \\ c\, w_2/\|w_2\| & aI + (b-a)w_2 w_2^T/\|w_2\|^2 \end{bmatrix},$$

where a, b, c are given by (1.29) but with λ_1, λ_2 replaced by μ_1, μ_2, respectively. If $c = 0$, the above matrix has the form $bI + (a-b)M_{w_2}$. Since M_{w_2} has eigenvalues of 0 and 1, this matrix has eigenvalues of b and a. Thus,

$$\left\|\nabla(f^\nu)^{\text{soc}}(w)\right\| = \max\{|a|, |b|\} \leq \kappa.$$

If $c \neq 0$, the above matrix has the form $\frac{c}{\|w_2\|}L_z + (a-b)M_{w_2} = \frac{c}{\|w_2\|}\left(L_z + (a-b)\|w_2\|c^{-1}M_{w_2}\right)$, where $z = (b\|w_2\|/c, w_2)$. By Proposition 1.10, this matrix has eigenvalues of $b \pm c$ and a. Thus, $\left\|\nabla(f^\nu)^{\text{soc}}(w)\right\| = \max\{|b+c|, |b-c|, |a|\} \leq \kappa$. In all cases, we have

$$\left\|\nabla(f^\nu)^{\text{soc}}(w)\right\| \leq \kappa. \tag{1.32}$$

Fix any $y, z \in \mathcal{B}(x,\delta)$ with $y \neq z$. Since $\{(f^\nu)^{\text{soc}}\}_{\nu=1}^\infty$ converges uniformly to f^{soc} on $\mathcal{B}(x,\delta)$, for any $\epsilon > 0$ there exists an integer ν_0 such that for all $\nu \geq \nu_0$ we have

$$\|(f^\nu)^{\text{soc}}(w) - f^{\text{soc}}(w)\| \leq \epsilon\|y - z\|, \quad \forall w \in \mathcal{B}(x,\delta).$$

Since f^ν is continuously differentiable, then Proposition 1.14 shows that $(f^\nu)^{\text{soc}}$ is also continuously differentiable for all ν. Thus, by inequality (1.32) and the mean value theorem for continuously differentiable functions, we have

$$\begin{aligned} & \|f^{\text{soc}}(y) - f^{\text{soc}}(z)\| \\ = \;& \|f^{\text{soc}}(y) - (f^\nu)^{\text{soc}}(y) + (f^\nu)^{\text{soc}}(y) - (f^\nu)^{\text{soc}}(z) + (f^\nu)^{\text{soc}}(z) - f^{\text{soc}}(z)\| \\ \leq \;& \|f^{\text{soc}}(y) - (f^\nu)^{\text{soc}}(y)\| + \|(f^\nu)^{\text{soc}}(y) - (f^\nu)^{\text{soc}}(z)\| + \|(f^\nu)^{\text{soc}}(z) - f^{\text{soc}}(z)\| \\ \leq \;& 2\epsilon\|y - z\| + \left\|\int_0^1 \nabla(f^\nu)^{\text{soc}}(z + \tau(y-z))(y-z)d\tau\right\| \\ \leq \;& (\kappa + 2\epsilon)\|y - z\| . \end{aligned}$$

Since $y, z \in \mathcal{B}(x, \delta)$ and ϵ is arbitrary, this yields

$$\left\| f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(z) \right\| \le \kappa \|y - z\| \quad \forall y, z \in \mathcal{B}(x, \delta). \tag{1.33}$$

Hence, f^{soc} is strictly continuous at x.

"only if" Suppose instead that f^{soc} is strictly continuous at x with spectral values λ_1, λ_2 and spectral vectors $u^{(1)}$, $u^{(2)}$. Then, there exist scalars $\kappa > 0$ and $\delta > 0$ such that (1.33) holds. For any $i \in \{1, 2\}$ and any $\psi, \zeta \in [\lambda_i - \delta, \lambda_i + \delta]$, let

$$y := x + (\psi - \lambda_i)u^{(i)}, \ z := x + (\zeta - \lambda_i)u^{(i)}.$$

Then, $\|y - x\| = |\psi - \lambda_i|/\sqrt{2} \le \delta$ and $\|z - x\| = |\zeta - \lambda_i|/\sqrt{2} \le \delta$, so it follows from (1.8) and (1.33) that

$$\begin{aligned} |f(\psi) - f(\zeta)| &= \sqrt{2}\|f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(z)\| \\ &\le \sqrt{2}\kappa\|y - z\| \\ &= \kappa|\psi - \zeta|. \end{aligned}$$

This shows that f is strictly continuous at λ_1, λ_2.

(b) This is an immediate consequence of part(a).

(c) Suppose f is Lipschitz continuous with constant $\kappa > 0$. Then $\mathrm{lip} f(\xi) \le \kappa$ for all $\xi \in \mathbb{R}$. Fix any $x \in \mathbb{R}^n$ with spectral values λ_1, λ_2. For any scalar $\delta > 0$, let

$$C := [\lambda_1 - \delta, \lambda_1 + \delta] \ \cup \ [\lambda_2 - \delta, \lambda_2 + \delta] \ .$$

Then, as in the proof of part (a), we obtain that (1.33) holds. Since the choice of $\delta > 0$ was arbitrary and κ is independent of δ, this implies that

$$\|f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(z)\| \le \kappa\|y - z\| \quad \forall y, z \in \mathbb{R}^n \ .$$

Hence, f^{soc} is Lipschitz continuous with Lipschitz constant κ.

Suppose instead that f^{soc} is Lipschitz continuous with constant $\kappa > 0$. Then, for any $\xi, \zeta \in \mathbb{R}$ we have

$$\begin{aligned} |f(\xi) - f(\zeta)| &= \left\|f^{\mathrm{soc}}(\xi e) - f^{\mathrm{soc}}(\zeta e)\right\| \\ &\le \kappa\|\xi e - \zeta e\| \\ &= \kappa|\xi - \zeta|, \end{aligned}$$

which says f is Lipschitz continuous with constant κ. $\square$

Suppose $f : \mathbb{R} \to \mathbb{R}$ is strictly continuous. Then, by Proposition 1.15, f^{soc} is strictly continuous. Hence, $\partial_B f^{\mathrm{soc}}(x)$ is well-defined for all $x \in \mathbb{R}^n$. The following lemma studies the structure of this generalized Jacobian.

Lemma 1.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be strictly continuous. Then, for any $x \in \mathbb{R}^n$, the generalized Jacobian $\partial_B f^{\text{soc}}(x)$ is well-defined and nonempty. Moreover, if $x_2 \neq 0$, then $\partial_B f^{\text{soc}}(x)$ equals the following set*

$$\left\{ \left[\begin{array}{cc} b & c\, x_2^T/\|x_2\| \\ c\, x_2/\|x_2\| & aI + (b-a)x_2x_2^T/\|x_2\|^2 \end{array} \right] \Big|\ a = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1},\ \begin{array}{l} b + c \in \partial_B f(\lambda_2) \\ b - c \in \partial_B f(\lambda_1) \end{array} \right\}, \tag{1.34}$$

where λ_1, λ_2 are the spectral values of x. If $x_2 = 0$, then $\partial_B f^{\text{soc}}(x)$ is a subset of the following set

$$\left\{ \left[\begin{array}{cc} b & c\, w^T \\ c\, w & aI + (b-a)ww^T \end{array} \right] \Big|\ a \in \partial f(x_1),\ b \pm c \in \partial_B f(x_1),\ \|w\| = 1 \right\}. \tag{1.35}$$

Proof. Suppose $x_2 \neq 0$. For any sequence $\{x^k\}_{k=1}^{\infty} \to x$ with f^{soc} differentiable at x^k, we have from Proposition 1.13 that $\{\lambda_i^k\}_{k=1}^{\infty} \to \lambda_i$ with f differentiable at λ_i^k, $i = 1, 2$, where λ_1^k, λ_2^k are the spectral values of x^k. Since any cluster point of $\{f'(\lambda_i^k)\}_{k=1}^{\infty}$ is in $\partial_B f(\lambda_i)$, it follows from the gradient formula (1.28)-(1.29) that any cluster point of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is an element of (1.34). Conversely, for any b, c with $b - c \in \partial_B f(\lambda_1)$, $b + c \in \partial_B f(\lambda_2)$, there exist $\{\lambda_1^k\}_{k=1}^{\infty} \to \lambda_1$, $\{\lambda_2^k\}_{k=1}^{\infty} \to \lambda_2$ with f differentiable at λ_1^k, λ_2^k and $\{f'(\lambda_1^k)\}_{k=1}^{\infty} \to b - c$, $\{f'(\lambda_2^k)\}_{k=1}^{\infty} \to b + c$. Since $\lambda_2 > \lambda_1$, by taking k large, we can assume that $\lambda_2^k \geq \lambda_1^k$ for all k. Let

$$x_1^k = \frac{1}{2}(\lambda_2^k + \lambda_1^k), \qquad x_2^k = \frac{1}{2}(\lambda_2^k - \lambda_1^k)\frac{x_2}{\|x_2\|}, \qquad x^k = (x_1^k, x_2^k).$$

Then, $\{x^k\}_{k=1}^{\infty} \to x$ and, by Proposition 1.13, f^{soc} is differentiable at x^k. Moreover, the limit of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is an element of (1.34) associated with the given b, c. Thus $\partial_B f^{\text{soc}}(x)$ equals (1.34).

Suppose $x_2 = 0$. Consider any sequence $\{x^k\}_{k=1}^{\infty} = \{(x_1^k, x_2^k)\}_{k=1}^{\infty} \to x$ with f^{soc} differentiable at x^k for all k. By passing to a subsequence, we can assume that either $x_2^k = 0$ for all k or $x_2^k \neq 0$ for all k. If $x_2^k = 0$ for all k, Proposition 1.13 yields that f is differentiable at x_1^k and $\nabla f^{\text{soc}}(x^k) = f'(x_1^k)I$. Hence, any cluster point of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is an element of (1.35) with $a = b \in \partial_B f(x_1) \subseteq \partial f(x_1)$ and $c = 0$. If $x_2^k \neq 0$ for all k, by further passing to a subsequence, we can assume without loss of generality that $\{x_2^k/\|x_2^k\|\}_{k=1}^{\infty} \to w$ for some w with $\|w\| = 1$. Let λ_1^k, λ_2^k be the spectral values of x^k and let a^k, b^k, c^k be the coefficients given by (1.29) corresponding to λ_1^k, λ_2^k. We can similarly prove that $b \pm c \in \partial_B f(x_1)$, where (b, c) is any cluster point of $\{(b^k, c^k)\}_{k=1}^{\infty}$. Also, by a mean-value theorem of Lebourg [54, Proposition 2.3.7],

$$a^k = \frac{f(\lambda_2^k) - f(\lambda_1^k)}{\lambda_2^k - \lambda_1^k} \in \partial f(\hat{\lambda}^k)$$

for some $\hat{\lambda}^k$ in the interval between λ_2^k and λ_1^k. Since f is strictly continuous so that ∂f is upper semicontinuous [54, Proposition 2.1.5] or, equivalently, outer semicontinuous

[134, Proposition 8.7], this together with $\lambda_i^k \to x_1$, $i = 1, 2$, implies that any cluster point of $\{a^k\}_{k=1}^{\infty}$ belongs to $\partial f(x_1)$. Then, the gradient formula (1.28)-(1.29) yields that any cluster point of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is an element of (1.35). $\square$

Below we refine Lemma 1.3 to characterize $\partial_B f^{\text{soc}}(x)$ completely for two special cases of f. In the first case, the directional derivative of f has a one-sided continuity property, and our characterization is analogous to [51, Proposition 4.8] for the matrix-valued function f^{mat}. However, despite Proposition 1.10, our characterization cannot be deduced from [51, Proposition 4.8] and hence is proved directly. The second case is an example from [134, page 304]. Our analysis shows that the structure of $\partial_B f^{\text{soc}}(x)$ depends on f in a complicated way. In particular, in both cases, $\partial_B f^{\text{soc}}(x)$ is a proper subset of (1.35) when $x_2 = 0$.

In what follows we denote the right- and left-directional derivative of $f : \mathbb{R} \to \mathbb{R}$ by

$$f'_+(\xi) := \lim_{\zeta \to \xi^+} \frac{f(\zeta) - f(\xi)}{\zeta - \xi}, \qquad f'_-(\xi) := \lim_{\zeta \to \xi^-} \frac{f(\zeta) - f(\xi)}{\zeta - \xi}.$$

Lemma 1.4. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is strictly continuous and directionally differentiable function with the property that*

$$\lim_{\substack{\zeta,\nu \to \xi^\sigma \\ \zeta \neq \nu}} \frac{f(\zeta) - f(\nu)}{\zeta - \nu} = \lim_{\substack{\zeta \to \xi^\sigma \\ \zeta \in D_f}} f'(\zeta) = f'_\sigma(\xi), \quad \forall \xi \in \mathbb{R},\ \sigma \in \{-, +\}, \tag{1.36}$$

where $D_f = \{\xi \in \mathbb{R} | f$ is differentiable at $\xi\}$. Then, for any $x = (x_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\partial_B f(x_1) = \{f'_-(x_1), f'_+(x_1)\}$, and $\partial_B f^{\text{soc}}(x)$ equals the following set

$$\left\{ \begin{bmatrix} b & c\, w^T \\ c\, w & aI + (b-a)ww^T \end{bmatrix} \middle| \begin{array}{l} \text{either } a = b \in \partial_B f(x_1),\ c = 0 \\ \text{or } a \in \partial f(x_1), b - c = f'_-(x_1), b + c = f'_+(x_1) \end{array}, \|w\| = 1 \right\}. \tag{1.37}$$

Proof. By (1.36), $\partial_B f(x_1) = \{f'_-(x_1), f'_+(x_1)\}$. Consider any sequence $\{x^k\}_{k=1}^{\infty} \to x$ with f^{soc} differentiable at $x^k = (x_1^k, x_2^k)$ for all k. By passing to a subsequence, we can assume that either $x_2^k = 0$ for all k or $x_2^k \neq 0$ for all k.

If $x_2^k = 0$ for all k, Proposition 1.13 yields that f is differentiable at x_1^k and $\nabla f^{\text{soc}}(x^k) = f'(x_1^k)I$. Hence, any cluster point of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is an element of (1.37) with $a = b \in \partial_B f(x_1)$ and $c = 0$.

If $x_2^k \neq 0$ for all k, by passing to a subsequence, we can assume without loss of generality that $\{x_2^k/\|x_2^k\|\}_{k=1}^{\infty} \to w$ for some w with $\|w\| = 1$. Let λ_1^k, λ_2^k be the spectral values of x^k. Then $\lambda_1^k < \lambda_2^k$ for all k and $\lambda_i^k \to x_1$, $i = 1, 2$. By further passing to a subsequence if necessary, we can assume that either (i) $\lambda_1^k < \lambda_2^k \leq x_1$ for all k or (ii) $x_1 \leq \lambda_1^k < \lambda_2^k$ for all k or (iii) $\lambda_1^k < x_1 < \lambda_2^k$ for all k. Let a^k, b^k, c^k be the coefficients given by (1.29) corresponding to λ_1^k, λ_2^k. By Proposition 1.13, f is differentiable at λ_1^k, λ_2^k and

$f'(\lambda_1^k) = b^k - c^k, f'(\lambda_2^k) = b^k + c^k$. Let (a, b, c) be any cluster point of $\{(a^k, b^k, c^k)\}_{k=1}^{\infty}$. In case (i), we see from (1.36) that $b \pm c = a = f'_-(x_1)$, which implies $b = f'_-(x_1)$ and $c = 0$. In case (ii), we obtain similarly that $a = b = f'_+(x_1)$ and $c = 0$. In case (iii), we obtain that $b - c = f'_-(x_1)$, $b + c = f'_+(x_1)$. Also, the directional differentiability of f implies that

$$a^k = \frac{f(\lambda_2^k) - f(\lambda_1^k)}{\lambda_2^k - \lambda_1^k} = \frac{\lambda_2^k - x_1}{\lambda_2^k - \lambda_1^k}\frac{f(\lambda_2^k) - f(x_1)}{\lambda_2^k - x_1} + \frac{x_1 - \lambda_1^k}{\lambda_2^k - \lambda_1^k}\frac{f(x_1) - f(\lambda_1^k)}{x_1 - \lambda_1^k},$$

which yields in the limit that

$$a = (1 - \omega)f'_+(x_1) + \omega f'_-(x_1),$$

for some $\omega \in [0, 1]$. Thus $a \in \partial f(x_1)$. This shows that $\partial_B f^{\mathrm{soc}}(x)$ is a subset of (1.37).

Conversely, for any $a = b \in \partial_B f(x_1)$, $c = 0$ and any $w \in \mathbb{R}^{n-1}$ with $\|w\| = 1$, we can find a sequence $x_1^k \in D_f$, $k = 1, 2, ...,$ such that $x_1^k \to x_1$ and $f'(x_1^k) \to a$. Then, $x^k = (x_1^k, 0) \to x$ and the preceding analysis shows that $\{\nabla f^{\mathrm{soc}}(x^k)\}_{k=1}^{\infty}$ converges to the element of (1.37) corresponding to the given a, b, c, w. For any a, b, c with $b - c = f'_-(x_1)$, $b + c = f'_+(x_1)$, $a \in \partial f(x_1)$, and any $w \in \mathbb{R}^{n-1}$ with $\|w\| = 1$, we have that $a = (1 - \omega)f_+(x_1) + \omega f_-(x_1)$ for some $\omega \in [0, 1]$. Since D_f is dense in $\mathbb{R}$, for any integer $k \geq 1$, there have

$$D_f \cap \left[x_1 - \omega\frac{1}{k} - \frac{1}{k^2}, x_1 - \omega\frac{1}{k}\right] \neq \emptyset, \quad D_f \cap \left[x_1 + (1 - \omega)\frac{1}{k}, x_1 + (1 - \omega)\frac{1}{k} + \frac{1}{k^2}\right] \neq \emptyset.$$

Let λ_1^k be any element of the first set and let λ_2^k be any element of the second set. Then, $x^k = \left(\frac{\lambda_2^k + \lambda_1^k}{2}, \frac{\lambda_2^k - \lambda_1^k}{2}w\right) \to x$ and x^k has spectral values $\lambda_1^k < \lambda_2^k$ which satisfy

$$\lambda_1^k < x_1 < \lambda_2^k \ \ \forall k, \qquad \frac{\lambda_2^k - x_1}{\lambda_2^k - \lambda_1^k} \to 1 - \omega, \qquad \frac{x_1 - \lambda_1^k}{\lambda_2^k - \lambda_1^k} \to \omega.$$

The preceding analysis shows that $\{\nabla f^{\mathrm{soc}}(x^k)\}_{k=1}^{\infty}$ converges to the element of (1.37) corresponding to the given a, b, c, w. $\square$

The assumptions of Lemma 1.4 are satisfied if f is piecewise continuously differentiable, e.g., $f(\cdot) = |\cdot|$ or $f(\cdot) = \max\{0, \cdot\}$. If f is differentiable, but not continuously differentiable, then $\partial_B f^{\mathrm{soc}}(x)$ is more complicated as is shown in the following lemma.

Lemma 1.5. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by*

$$f(\xi) = \begin{cases} \xi^2 \sin(1/\xi) & \text{if } \xi \neq 0, \\ 0 & \text{else.} \end{cases}$$

Then, for any $x = (x_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have that $\partial_B f(x_1) = [-1, 1]$, and $\partial_B f^{\rm soc}(x) = \{f'(x_1)I\}$ if $x_1 \neq 0$ and otherwise $\partial_B f^{\rm soc}(x)$ equals the following set

$$\left\{ \begin{bmatrix} b & c\, w^T \\ c\, w & aI + (b-a)ww^T \end{bmatrix} \,\middle|\, \begin{array}{l} b - c = -\cos(\theta_1),\ b + c = -\cos(\theta_2),\ \|w\| = 1, \\ a = \dfrac{\sin(\theta_1) - \sin(\theta_2)}{\theta_1 - \theta_2 + 2\kappa\pi},\ \kappa \in \{0, 1, ..., \infty\},\ \theta_1, \theta_2 \in [0, 2\pi], \\ \qquad\qquad\qquad\qquad \theta_1 > \theta_2 \text{ if } \kappa = 0 \end{array} \right\}, \tag{1.38}$$

with the convention that $a = 0$ if $\kappa = \infty$ and $a = \cos(\theta_1)$ if $\kappa = 0$ and $\theta_1 = \theta_2$.

Proof. f is differentiable everywhere, with

$$f'(\xi) = \begin{cases} 2\xi \sin(1/\xi) - \cos(1/\xi) & \text{if } \xi \neq 0, \\ 0 & \text{else.} \end{cases} \tag{1.39}$$

Thus $\partial_B f(x_1) = [-1, 1]$. Consider any sequence $\{x^k\}_{k=1}^{\infty} \to x$ with $f^{\rm soc}$ differentiable at $x^k = (x_1^k, x_2^k)$ for all k. By passing to a subsequence, we can assume that either $x_2^k = 0$ for all k or $x_2^k \neq 0$ for all k. Let $\lambda_1^k = x_1^k - \|x_2^k\|$, $\lambda_2^k = x_1^k + \|x_2^k\|$ be the spectral values of x^k.

If $x_2^k = 0$ for all k, Proposition 1.13 yields that f is differentiable at x_1^k and $\nabla f^{\rm soc}(x^k) = f'(x_1^k)I$. Hence, any cluster point of $\{\nabla f^{\rm soc}(x^k)\}_{k=1}^{\infty}$ is of the form bI for some $b \in \partial_B f(x_1)$. If $x_1 \neq 0$, then $b = f'(x_1)$. If $x_1 = 0$, then $b \in [-1, 1]$, i.e., $b = \cos(\theta_1)$ for some $\theta \in [0, 2\pi]$. Then, bI has the form (1.38) with $a = b$, $c = 0$, corresponding to $\theta_1 = \theta_2$, $\kappa = 0$.

If $x_2^k \neq 0$ for all k, by passing to a subsequence, we can assume without loss of generality that $\{x_2^k/\|x_2^k\|\}_{k=1}^{\infty} \to w$ for some w with $\|w\| = 1$. By Proposition 1.13, f is differentiable at λ_1^k, λ_2^k and $f'(\lambda_1^k) = b^k - c^k$, $f'(\lambda_2^k) = b^k + c^k$, where a^k, b^k, c^k are the coefficients given by (1.29) corresponding to λ_1^k, λ_2^k. If $x_1 \neq 0$, then $a^k \to f'(x_1)$, $b^k \to f'(x_1)$ and $c^k \to 0$, so any cluster point of $\{\nabla f^{\rm soc}(x^k)\}_{k=1}^{\infty}$ equals $f'(x_1)I$. Suppose $x_1 = 0$. Then, $\lambda_1^k < \lambda_2^k$ tend to zero. By further passing to a subsequence if necessary, we can assume that either (i) both are nonzero for all k or (ii) $\lambda_1^k = 0$ for all k or (iii) $\lambda_2^k = 0$ for all k. In case (i),

$$\frac{1}{\lambda_1^k} = \theta_1^k + 2\nu_k\pi, \qquad \frac{1}{\lambda_2^k} = \theta_2^k + 2\mu_k\pi \tag{1.40}$$

for some $\theta_1^k, \theta_2^k \in [0, 2\pi]$ and integers ν_k, μ_k tending to ∞ or $-\infty$. By further passing to a subsequence if necessary, we can assume that $\{(\theta_1^k, \theta_2^k)\}_{k=1}^{\infty}$ converges to some $(\theta_1, \theta_2) \in [0, 2\pi]^2$. Then, (1.39) yields

$$\begin{aligned} f'(\lambda_i^k) &= 2\lambda_i^k \sin(\theta_i^k) - \cos(\theta_i^k) \ \to\ -\cos(\theta_i), \quad i = 1, 2, \\ a^k &= \frac{f(\lambda_2^k) - f(\lambda_1^k)}{\lambda_2^k - \lambda_1^k} = \frac{(\lambda_2^k)^2 \sin(\theta_2^k) - (\lambda_1^k)^2 \sin(\theta_1^k)}{\lambda_2^k - \lambda_1^k} \\ &= (\lambda_2^k + \lambda_1^k) \sin(\theta_2^k) + \frac{\sin(\theta_2^k) - \sin(\theta_1^k)}{(\theta_1^k - \theta_2^k + 2(\nu_k - \mu_k)\pi)\lambda_2^k/\lambda_1^k}. \end{aligned}$$

If $|\nu_k - \mu_k|$ is bounded as $k \to \infty$, then $\lambda_2^k/\lambda_1^k \to 1$ and, by (1.40) and $\lambda_1^k < \lambda_2^k$, $\nu_k \geq \mu_k$. In this case, any cluster point (a, b, c) of $\{(a^k, b^k, c^k)\}_{k=1}^{\infty}$ would satisfy

$$b - c = -\cos(\theta_1), \qquad b + c = -\cos(\theta_2), \qquad a = \frac{\sin(\theta_2) - \sin(\theta_1)}{\theta_1 - \theta_2 + 2\kappa\pi} \tag{1.41}$$

for some integer $\kappa \geq 0$. Here, we use the convention that $a = \cos(\theta_1)$ if $\kappa = 0$, $\theta_1 = \theta_2$. Moreover, if $\kappa = 0$, then $\nu_k = \mu_k$ for all k sufficiently large along the corresponding subsequence, so (1.40) and $\lambda_1^k < \lambda_2^k$ yields $\theta_1^k > \theta_2^k > 0$, implying furthermore that $\theta_1 \geq \theta_2$.

If $|\nu_k - \mu_k| \to \infty$ and $|\mu_k/\nu_k|$ is bounded away from zero, then $|\nu_k - \mu_k||\mu_k/\nu_k| \to \infty$. If $|\nu_k - \mu_k| \to \infty$ and $|\mu_k/\nu_k| \to 0$, then $|\nu_k - \mu_k||\mu_k/\nu_k| = |\mu_k(1 - \mu_k/\nu_k)| \to \infty$ due to $|\mu_k| \to \infty$. Thus, if $|\nu_k - \mu_k| \to \infty$, we have $|\nu_k - \mu_k||\lambda_2^k/\lambda_1^k| \to \infty$ and the above equation yields $a^k \to 0$, corresponding to (1.41) with $\kappa = \infty$. In case (ii), we have $f'(\lambda_1^k) = 0$ and $a^k = f(\lambda_2^k)/\lambda_2^k = \lambda_2^k \sin(1/\lambda_2^k)$ for all k, so any cluster point (a, b, c) of $\{(a^k, b^k, c^k)\}_{k=1}^{\infty}$ satisfies $b - c = 0$, $b + c = -\cos(\theta_2)$, $a = 0$. This corresponds to (1.41) with $\theta_1 = \frac{\pi}{2}$, $\kappa = \infty$. In case (iii), we obtain similarly (1.41) with $\theta_2 = \frac{\pi}{2}$, $\kappa = \infty$. This and (1.28)-(1.29) show that any cluster point of $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ is in the set (1.38).

Conversely, if $x_1 \neq 0$, since $\partial_B f^{\text{soc}}(x)$ is a nonempty subset of $\{f'(x_1)I\}$, the two must be equal. If $x_1 = 0$, then for any integer $\kappa \geq 0$ and any $\theta_1, \theta_2 \in [0, 2\pi]$ satisfying $\theta_1 \geq \theta_2$ whenever $\kappa = 0$, and any $w \in \mathbb{R}^{n-1}$ with $\|w\| = 1$, we let, for each integer $k \geq 1$,

$$\lambda_1^k = \frac{1}{\theta_1 + 2(k+\kappa)\pi + 1/k}\ , \qquad \lambda_2^k = \frac{1}{\theta_2 + 2k\pi}.$$

Then, $0 < \lambda_1^k < \lambda_2^k$, $x^k = \left(\frac{\lambda_2^k + \lambda_1^k}{2}, \frac{\lambda_2^k - \lambda_1^k}{2} w\right) \to x$ and x^k has spectral values λ_1^k, λ_2^k which satisfy (1.40) with $\nu_k = k + \kappa$, $\mu_k = k$, $\theta_1^k = \theta_1 + 1/k$, $\theta_2^k = \theta_2$. The preceding analysis shows that $\{\nabla f^{\text{soc}}(x^k)\}_{k=1}^{\infty}$ converges to the element of (1.37) corresponding to the given θ_1, θ_2, κ, w with a given by (1.41). The case of $a = 0$ can be obtained similarly by taking κ to go to ∞ with k. $\square$

The following lemma, proven by Sun and Sun [140, Theorem 3.6] using the definition of generalized Jacobian,[1] enables one to study the semismooth property of f^{soc} by examining only those points $x \in \mathbb{R}^n$ where f^{soc} is differentiable and thus work only with the Jacobian of f^{soc}, rather than the generalized Jacobian.

Lemma 1.6. *Suppose $F : \mathbb{R}^k \to \mathbb{R}^k$ is strictly continuous and directionally differentiable in a neighborhood of $x \in \mathbb{R}^k$. Then, for any $0 < \rho < \infty$, the following two statements (where $O(\cdot)$ depends on F and x only) are equivalent:*

(a) *For any $h \in \mathbb{R}^k$ and any $V \in \partial F(x + h)$,*

$$F(x + h) - F(x) - Vh = o(\|h\|) \quad (\textit{respectively, } O(\|h\|^{1+\rho})).$$

[1] Sun and Sun did not consider the case of $o(\|h\|)$ but their argument readily applies to this case.

(b) *For any $h \in \mathbb{R}^k$ such that F is differentiable at $x+h$,*

$$F(x+h) - F(x) - \nabla F(x+h)h = o(\|h\|) \quad (\textit{respectively, } O(\|h\|^{1+\rho})).$$

By using Propositions 1.10, 1.6 and Propositions 1.9, 1.12, 1.15, 1.13, we can now state and prove the last result of this section, on the semismooth property of f^{soc}. This result generalizes [52, Thmeorem 4.2] for the cases of $f(\xi) = |\xi|$, $f(\xi) = \max\{0, \xi\}$.

Proposition 1.16. *For any $f : \mathbb{R} \to \mathbb{R}$, let f^{soc} be its corresponding SOC function defined as in (1.8). Then, the following hold.*

(a) *The vector-valued function f^{soc} is semismooth if and only if f is semismooth.*

(b) *If f is ρ-order semismooth ($0 < \rho < \infty$), then f^{soc} is $\min\{1, \rho\}$-order semismooth.*

Proof. Suppose f is semismooth. Then f is strictly continuous and directionally differentiable. By Propositions 1.12 and 1.15, f^{soc} is strictly continuous and directionally differentiable. By Proposition 1.10(b), $f^{\mathrm{soc}}(x) = f^{\mathrm{mat}}(L_x)e$ for all x. By Proposition 1.9(g), f^{mat} is semismooth. Since L_x is continuously differentiable in x, $f^{\mathrm{soc}}(x) = f^{\mathrm{mat}}(L_x)e$ is semismooth in x. If f is ρ-order semismooth ($0 < \rho < \infty$), then, by Proposition 1.9(g), f^{mat} is $\min\{1, \rho\}$-order semismooth. Since L_x is continuously differentiable in x, $f^{\mathrm{soc}}(x) = f^{\mathrm{mat}}(L_x)e$ is $\min\{1, \rho\}$-order semismooth in x.

Suppose f^{soc} is semismooth. Then f^{soc} is strictly continuous and directionally differentiable. By Propositions 1.12 and 1.15, f is strictly continuous and directionally differentiable. For any $\xi \in \mathbb{R}$ and any $\eta \in \mathbb{R}$ such that f is differentiable at $\xi + \eta$, Proposition 1.13 yields that f^{soc} is differentiable at $x+h$, where we denote $x := \xi e$ and $h := \eta e$. Since f^{soc} is semismooth, it follows from Lemma 1.6 that

$$f^{\mathrm{soc}}(x+h) - f^{\mathrm{soc}}(x) - \nabla f^{\mathrm{soc}}(x+h)h = o(\|h\|),$$

which, by (1.8) and (1.27), is equivalent to

$$f(\xi+\eta) - f(\xi) - f'(\xi+\eta)\eta = o(|\eta|).$$

Then, Lemma 1.6 yields that f is semismooth. □

For each of the preceding global results there is a corresponding local result and there is also an alternative way to prove each result by using the structure of SOC and the spectral decomposition. Please refer to [41] for more details. We point out that both $\mathcal{S}^n_+$ and $\mathcal{K}^n$ belong to the class of symmetric cones [62], hence there holds a unified framework for f^{mat} and f^{soc}, which is called Löwner operator. Almost parallel analysis are extended to the setting of Löwner operator associated with symmetric cone by Sun and Sun in [141]. Recently, another generalization of $\mathcal{S}^n_+$ is done by Ding et al. [56, 57]. They introduce the so-called matrix cones and a class of matrix-valued functions, which is called spectral operator of matrices. This class of functions not only generalizes the well known Löwner operator, but also has been used in many applications related to structured low rank matrices and other matrix optimization problems in machine learning and statistics. Some parallel results like the continuity, directional differentiability and Frechet-differentiability of spectral operator are also analyzed, see [57, Theorems 3-5].

Chapter 2

SOC-convexity and SOC-monotonity

In this chapter, we introduce the SOC-convexity and SOC-monotonicity which are natural extensions of traditional convexity and monotonicity. These kinds of SOC-convex and SOC-monotone functions are also parallel to matrix-convex and matrix-monotone functions, see [22, 75]. We start with studying the SOC-convexity and SOC-monotonicity for some simple functions, e.g., $f(t) = t^2, t^3, 1/t, t^{1/2}, |t|$, and $[t]_+$. Then, we explore characterizations of SOC-convex and SOC-monotone functions.

2.1 Motivations and Examples

Definition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function.*

(a) *f is said to be SOC-monotone of order n if the corresponding vector-valued function f^{soc} satisfies the following:*

$$x \succeq_{\mathcal{K}^n} y \implies f^{\mathrm{soc}}(x) \succeq_{\mathcal{K}^n} f^{\mathrm{soc}}(y). \tag{2.1}$$

We say f is SOC-monotone if f is SOC-monotone of all order n.

(b) *f is said to be SOC-convex of order n if the corresponding vector-valued function f^{soc} satisfies the following:*

$$f^{\mathrm{soc}}\big((1-\lambda)x + \lambda y\big) \preceq_{\mathcal{K}^n} (1-\lambda) f^{\mathrm{soc}}(x) + \lambda f^{\mathrm{soc}}(y), \tag{2.2}$$

for all $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$. We say f is SOC-convex if f is SOC-convex of all order n.

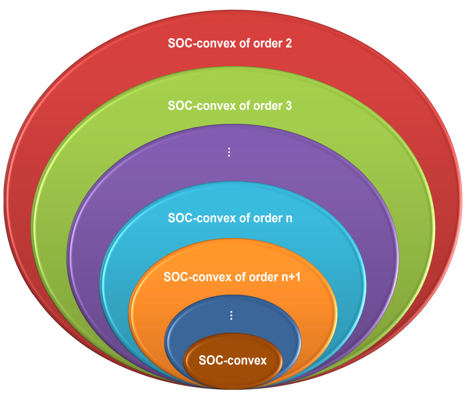
Remark 2.1. *We elaborate more about the concepts of SOC-convexity and SOC-monotonicity in this remark.*

1. *A function f is SOC-convex of order 1 is the same as f being a convex function. If a function f is SOC-convex of order n, then f is SOC-convex of order m for any $m \leq n$, see Figure 2.1(a).*

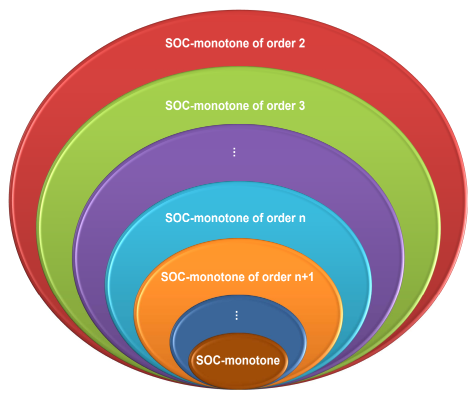
2. *A function f is SOC-monotone of order 1 is the same as f being an increasing function. If a function f is SOC-monotone of order n, then f is SOC-monotone of order m for any $m \leq n$, see Figure 2.1(b).*

3. *If f is continuous, then the condition (2.2) can be replaced by the more special condition:*
$$f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) \preceq_{\mathcal{K}^n} \frac{1}{2}\left(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\right). \tag{2.3}$$

4. *It is clear that the set of SOC-monotone functions and the set of SOC-convex functions are both closed under positive linear combinations and under pointwise limits.*



(a) SOC-convex functions



(b) SOC-monotone functions

Figure 2.1: The concepts of SOC-convex and SOC-monotone functions

Proposition 2.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = \alpha + \beta t$. Then,*

(a) *f is SOC-monotone on $\mathbb{R}$ for every $\alpha \in \mathbb{R}$ and $\beta \geq 0$;*

(b) *f is SOC-convex on $\mathbb{R}$ for all $\alpha, \beta \in \mathbb{R}$.*

Proof. The proof is straightforward by checking that Definition 2.1 is satisfied. □

Proposition 2.2. **(a)** *Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = t^2$, then f is SOC-convex on $\mathbb{R}$.*

(b) *Hence, the function $g(t) = \alpha + \beta t + \gamma t^2$ is SOC-convex on $\mathbb{R}$ for all $\alpha, \beta \in \mathbb{R}$ and $\gamma \geq 0$.*

Proof. (a) For any $x, y \in \mathbb{R}^n$, we have

$$\frac{1}{2}\big(f^{\rm soc}(x) + f^{\rm soc}(y)\big) - f^{\rm soc}\left(\frac{x+y}{2}\right) = \frac{x^2+y^2}{2} - \left(\frac{x+y}{2}\right)^2 = \frac{1}{4}(x-y)^2 \succeq_{\mathcal{K}^n} 0,$$

which says (2.3) is satisfied. Since f is continuous, it implies that f is SOC-convex.
(b) It is an immediate consequence of part(a). □

Example 2.1. *The function $f(t) = t^2$ is not SOC-monotone on* $\mathbb{R}$.

Solution. Taking $x = (1, 0)$, $y = (-2, 0)$, then $x - y = (3, 0) \succeq_{\mathcal{K}^n} 0$. But,

$$x^2 - y^2 = (1, 0) - (4, 0) = (-3, 0) \not\succeq_{\mathcal{K}^n} 0,$$

which violates (2.1). ■

As mentioned in Section 1.2, if f is defined on a subset $J \subseteq \mathbb{R}$, $f^{\rm soc}$ is defined on its corresponding set given as in (1.10), i.e.,

$$S = \{x \in \mathbb{R}^n \,|\, \lambda_i(x) \in J,\ i = 1, 2.\} \subseteq \mathbb{R}^n.$$

In addition, from Proposition 2.2(a), it indicates that $f(t) = t^2$ is also SOC-convex on the smaller interval $[0, \infty)$. These observations raise a natural question. Is $f(t) = t^2$ SOC-monotone on the interval $[0, \infty)$ although it is not SOC-monotone on $\mathbb{R}$? The answer is no! Indeed, it is true only for $n = 2$, but, false for $n \geq 3$. We illustrate this in the next example.

Example 2.2. (a) *The function $f(t) = t^2$ is SOC-monotone of order 2 on $[0, \infty)$.*

(b) *However, $f(t) = t^2$ is not SOC-monotone of order $n \geq 3$ on $[0, \infty)$.*

Solution. (a) Suppose that $x = (x_1, x_2) \succeq_{\mathcal{K}^2} y = (y_1, y_2) \succeq_{\mathcal{K}^2} 0$. Then, we have the following inequalities:

$$|x_2| \leq x_1, \quad |y_2| \leq y_1, \quad |x_2 - y_2| \leq x_1 - y_1,$$

which implies

$$\begin{cases} x_1 - x_2 \geq y_1 - y_2 \geq 0, \\ x_1 + x_2 \geq y_1 + y_2 \geq 0. \end{cases} \tag{2.4}$$

The goal is to show that $f^{\rm soc}(x) - f^{\rm soc}(y) = (x_1^2 + x_2^2 - y_1^2 - y_2^2, 2x_1x_2 - 2y_1y_2) \succeq_{\mathcal{K}^2} 0$, which suffices to verify that $x_1^2 + x_2^2 - y_1^2 - y_2^2 \geq |2x_1x_2 - 2y_1y_2|$. This can be seen by

$$\begin{aligned}
& x_1^2 + x_2^2 - y_1^2 - y_2^2 - \big|2x_1x_2 - 2y_1y_2\big| \\
= & \begin{cases} x_1^2 + x_2^2 - y_1^2 - y_2^2 - (2x_1x_2 - 2y_1y_2), & \text{if} \quad x_1x_2 - y_1y_2 \geq 0 \\ x_1^2 + x_2^2 - y_1^2 - y_2^2 - (2y_1y_2 - 2x_1x_2), & \text{if} \quad x_1x_2 - y_1y_2 \leq 0 \end{cases} \\
= & \begin{cases} (x_1 - x_2)^2 - (y_1 - y_2)^2, & \text{if} \quad x_1x_2 - y_1y_2 \geq 0 \\ (x_1 + x_2)^2 - (y_1 + y_2)^2, & \text{if} \quad x_1x_2 - y_1y_2 \leq 0 \end{cases} \\
\geq & \ 0,
\end{aligned}$$

where the inequalities are true due to the inequalities (2.4).

(b) From Remark 2.1, we only need to provide a counterexample for case of $n = 3$ to show that $f(t) = t^2$ is not SOC-monotone on the interval $[0, \infty)$. Take $x = (3, 1, -2) \in \mathcal{K}^3$ and $y = (1, 1, 0) \in \mathcal{K}^3$. It is clear that $x - y = (2, 0, -2) \succeq_{\mathcal{K}^3} 0$. But, $x^2 - y^2 = (14, 6, -12) - (2, 2, 0) = (12, 4, -12) \not\succeq_{\mathcal{K}^3} 0$. ■

Now we look at the function $f(t) = t^3$. As expected, $f(t) = t^3$ is not SOC-convex. However, it is true that $f(t) = t^3$ is SOC-convex on $[0, \infty)$ for $n = 2$, whereas false for $n \geq 3$. Besides, we will see $f(t) = t^3$ is neither SOC-monotone on $\mathbb{R}$ nor SOC-monotone on the interval $[0, \infty)$. Nonetheless, it is true that it is SOC-monotone on the interval $[0, \infty)$, for $n = 2$. The following two examples demonstrate what we have just said.

Example 2.3. **(a)** *The function $f(t) = t^3$ is not SOC-convex on $\mathbb{R}$.*

(b) *However, $f(t) = t^3$ is SOC-convex of order 2 on $[0, \infty)$.*

(c) *Moreover, $f(t) = t^3$ is not SOC-convex of order $n \geq 3$ on $[0, \infty)$.*

Solution. (a) Taking $x = (0, -2), y = (1, 0)$ gives

$$\frac{1}{2}\left(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\right) - f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) = \left(-\frac{9}{8}, -\frac{9}{4}\right) \not\succeq_{\mathcal{K}^2} 0,$$

which says $f(t) = t^3$ is not SOC-convex on $\mathbb{R}$.

(b) It suffices to show that $f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) \preceq_{\mathcal{K}^2} \frac{1}{2}\left(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\right)$, for any $x, y \succeq_{\mathcal{K}^2} 0$. Suppose that $x = (x_1, x_2) \succeq_{\mathcal{K}^2} 0$ and $y = (y_1, y_2) \succeq_{\mathcal{K}^2} 0$, then we have

$$\begin{cases} x^3 &= \left(x_1^3 + 3x_1x_2^2, 3x_1^2x_2 + x_2^3\right), \\ y^3 &= \left(y_1^3 + 3y_1y_2^2, 3y_1^2y_2 + y_2^3\right), \end{cases}$$

which yields

$$\begin{cases} f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) = \frac{1}{8}\left((x_1+y_1)^3 + 3(x_1+y_1)(x_2+y_2)^2, 3(x_1+y_1)^2(x_2+y_2) + (x_2+y_2)^3\right), \\ \frac{1}{2}\left(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\right) = \frac{1}{2}\left(x_1^3 + y_1^3 + 3x_1x_2^2 + 3y_1y_2^2, x_2^3 + y_2^3 + 3x_1^2x_2 + 3y_1^2y_2\right). \end{cases}$$

After simplifications, we denote $\frac{1}{2}\left(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\right) - f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) := \frac{1}{8}(\Xi_1, \Xi_2)$, where

$$\begin{cases} \Xi_1 &= 4x_1^3 + 4y_1^3 + 12x_1x_2^2 + 12y_1y_2^2 - (x_1+y_1)^3 - 3(x_1+y_1)(x_2+y_2)^2, \\ \Xi_2 &= 4x_2^3 + 4y_2^3 + 12x_1^2x_2 + 12y_1^2y_2 - (x_2+y_2)^3 - 3(x_1+y_1)^2(x_2+y_2). \end{cases}$$

We want to show that $\Xi_1 \geq |\Xi_2|$, for which we discuss two cases. First, if $\Xi_2 \geq 0$, then

$$\begin{aligned}
& \Xi_1 - |\Xi_2| \\
= \; & (4x_1^3 + 12x_1x_2^2 - 12x_1^2x_2 - 4x_2^3) + (4y_1^3 + 12y_1y_2^2 - 12y_1^2y_2 - 4y_2^3) \\
& - \big((x_1+y_1)^3 + 3(x_1+y_1)(x_2+y_2)^2 - 3(x_1+y_1)^2(x_2+y_2) - (x_2+y_2)^3\big) \\
= \; & 4(x_1-x_2)^3 + 4(y_1-y_2)^3 - \big((x_1+y_1) - (x_2+y_2)\big)^3 \\
= \; & 4(x_1-x_2)^3 + 4(y_1-y_2)^3 - \big((x_1-x_2) + (y_1-y_2)\big)^3 \\
= \; & 3(x_1-x_2)^3 + 3(y_1-y_2)^3 - 3(x_1-x_2)^2(y_1-y_2) - 3(x_1-x_2)(y_1-y_2)^2 \\
= \; & 3\big((x_1-x_2)+(y_1-y_2)\big)\big((x_1-x_2)^2 - (x_1-x_2)(y_1-y_2) + (y_1-y_2)^2\big) \\
& -3(x_1-x_2)(y_1-y_2)\big((x_1-x_2)+(y_1-y_2)\big) \\
= \; & 3\big((x_1-x_2)+(y_1-y_2)\big)\big((x_1-x_2)-(y_1-y_2)\big)^2 \\
\geq \; & 0,
\end{aligned}$$

where the inequality is true since $x, y \in \mathcal{K}^2$. Similarly, if $\Xi_2 \leq 0$, we also have

$$\begin{aligned}
& \Xi_1 - |\Xi_2| \\
= \; & (4x_1^3 + 12x_1x_2^2 + 12x_1^2x_2 + 4x_2^3) + (4y_1^3 + 12y_1y_2^2 + 12y_1^2y_2 + 4y_2^3) \\
& - \big((x_1+y_1)^3 + 3(x_1+y_1)(x_2+y_2)^2 + 3(x_1+y_1)^2(x_2+y_2) + (x_2+y_2)^3\big) \\
= \; & 4(x_1+x_2)^3 + 4(y_1+y_2)^3 - \big((x_1+y_1) + (x_2+y_2)\big)^3 \\
= \; & 4(x_1+x_2)^3 + 4(y_1+y_2)^3 - \big((x_1+x_2) + (y_1+y_2)\big)^3 \\
= \; & 3(x_1+x_2)^3 + 3(y_1+y_2)^3 - 3(x_1+x_2)^2(y_1+y_2) - 3(x_1+x_2)(y_1+y_2)^2 \\
= \; & 3\big((x_1+x_2)+(y_1+y_2)\big)\big((x_1+x_2)^2 - (x_1+x_2)(y_1+y_2) + (y_1+y_2)^2\big) \\
& -3(x_1+x_2)(y_1+y_2)\big((x_1+x_2)+(y_1+y_2)\big) \\
= \; & 3\big((x_1+x_2)+(y_1+y_2)\big)\big((x_1+x_2)-(y_1+y_2)\big)^2 \\
\geq \; & 0,
\end{aligned}$$

where the inequality is true since $x, y \in \mathcal{K}^2$. Thus, we have verified that $f(t) = t^3$ is SOC-convex on $[0, \infty)$ for $n = 2$.

(c) Again, by Remark 2.1, we only need to provide a counterexample for case of $n = 3$. To see this, we take $x = (2, 1, -1), y = (1, 1, 0) \succeq_{\mathcal{K}^3} 0$. Then, we have

$$\frac{1}{2}\big(f^{\text{soc}}(x) + f^{\text{soc}}(y)\big) - f^{\text{soc}}\left(\frac{x+y}{2}\right) = (3, 1, -3) \not\succeq_{\mathcal{K}^3} 0,$$

which implies $f(t) = t^3$ is not even SOC-convex on the interval $[0, \infty)$. ■

Example 2.4. (a) *The function $f(t) = t^3$ is not SOC-monotone on $\mathbb{R}$.*

(b) *However, $f(t) = t^3$ is SOC-monotone of order 2 on $[0, \infty)$.*

(c) *Moreover, $f(t) = t^3$ is not SOC-monotone of order $n \geq 3$ on $[0, \infty)$.*

Solution. To see (a) and (c), let $x = (2, 1, -1) \succeq_{\mathcal{K}^3} 0$ and $y = (1, 1, 0) \succeq_{\mathcal{K}^3} 0$. It is clear that $x \succeq_{\mathcal{K}^3} y$. But, we have $f^{\mathrm{soc}}(x) = x^3 = (20, 14, -14)$ and $f^{\mathrm{soc}}(y) = y^3 = (4, 4, 0)$, which gives $f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y) = (16, 10, -14) \not\succeq_{\mathcal{K}^3} 0$. Thus, we show that $f(t) = t^3$ is not even SOC-monotone on the interval $[0, \infty)$.

To see (b), let $x = (x_1, x_2) \succeq_{\mathcal{K}^2}$ and $y = (y_1, y_2) \succeq_{\mathcal{K}^2} 0$, which means

$$|x_2| \leq x_1, \quad |y_2| \leq y_1, \quad |x_2 - y_2| \leq x_1 - y_1.$$

Then, it leads to the inequalities (2.4) again. On the other hand, we know

$$\begin{aligned} f^{\mathrm{soc}}(x) &= x^3 = \left(x_1^3 + 3x_1x_2^2, 3x_1^2x_2 + x_2^3\right), \\ f^{\mathrm{soc}}(y) &= y^3 = \left(y_1^3 + 3y_1y_2^2, 3y_1^2y_2 + y_2^3\right). \end{aligned}$$

For convenience, we denote $f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y) := (\Xi_1, \Xi_2)$, where

$$\left\{ \begin{aligned} \Xi_1 &= x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2, \\ \Xi_2 &= x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2. \end{aligned} \right.$$

We wish to prove that $f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y) = x^3 - y^3 \succeq_{\mathcal{K}^2} 0$, which suffices to show $\Xi_1 \geq |\Xi_2|$. This is true because

$$\begin{aligned} & x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - \left|x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2\right| \\ = & \left\{ \begin{array}{ll} x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2) & \text{if} \quad \Xi_2 \geq 0, \\ x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 + (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2) & \text{if} \quad \Xi_2 \leq 0, \end{array} \right. \\ = & \left\{ \begin{array}{ll} (x_1 - x_2)^3 - (y_1 - y_2)^3 & \text{if} \quad \Xi_2 \geq 0, \\ (x_1 + x_2)^3 - (y_1 + y_2)^3 & \text{if} \quad \Xi_2 \leq 0, \end{array} \right. \\ \geq & \ 0, \end{aligned}$$

where the inequalities are due to the inequalities (2.4).
Hence, we complete the verification. ■

Now, we move to another simple function $f(t) = 1/t$. We will prove that $-\frac{1}{t}$ is SOC-monotone on the interval $(0, \infty)$ and $\frac{1}{t}$ is SOC-convex on the interval $(0, \infty)$ as well. For the proof, we need the following technical lemmas.

Lemma 2.1. *Suppose that $a, b, c, d \in \mathbb{R}$. For any $a \geq b > 0$ and $c \geq d > 0$, there holds*

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) \geq \frac{a + c}{b + d}$$

Proof. The proof follows from $ac(b + d) - bd(a + c) = ab(c - d) + cd(a - b) \geq 0$. □

Lemma 2.2. *For any $x = (x_1, x_2) \in \mathcal{K}^n$ and $y = (y_1, y_2) \in \mathcal{K}^n$, we have*

(a) $(x_1+y_1)^2 - \|y_2\|^2 \geq 4x_1\sqrt{y_1^2 - \|y_2\|^2}$.

(b) $\big(x_1+y_1-\|y_2\|\big)^2 \geq 4x_1\big(y_1-\|y_2\|\big)$.

(c) $\big(x_1+y_1+\|y_2\|\big)^2 \geq 4x_1\big(y_1+\|y_2\|\big)$.

(d) $x_1y_1 - \langle x_2, y_2\rangle \geq \sqrt{x_1^2-\|x_2\|^2}\sqrt{y_1^2-\|y_2\|^2}$.

(e) $(x_1+y_1)^2 - \|x_2+y_2\|^2 \geq 4\sqrt{x_1^2-\|x_2\|^2}\sqrt{y_1^2-\|y_2\|^2}$.

Proof. (a) The proof follows from

$$\begin{aligned}(x_1+y_1)^2 - \|y_2\|^2 &= x_1^2 + (y_1^2 - \|y_2\|^2) + 2x_1y_1 \\ &\geq 2x_1\sqrt{y_1^2-\|y_2\|^2} + 2x_1y_1 \\ &\geq 2x_1\sqrt{y_1^2-\|y_2\|^2} + 2x_1\sqrt{y_1^2-\|y_2\|^2} \\ &= 4x_1\sqrt{y_1^2-\|y_2\|^2},\end{aligned}$$

where the first inequality is true due to the fact that $a+b \geq 2\sqrt{ab}$ for any positive numbers a and b.

(b) The proof follows from

$$\begin{aligned}&\big(x_1+y_1-\|y_2\|\big)^2 - 4x_1\left(y_1-\|y_2\|\right) \\ =\;& x_1^2+y_1^2+\|y_2\|^2 - 2x_1y_1 - 2y_1\|y_2\| + 2x_1\|y_2\| \\ =\;& \big(x_1-y_1+\|y_2\|\big)^2 \geq 0.\end{aligned}$$

(c) Similarly, the proof follows from

$$\begin{aligned}&\big(x_1+y_1+\|y_2\|\big)^2 - 4x_1\left(y_1+\|y_2\|\right) \\ =\;& x_1^2+y_1^2+\|y_2\|^2 - 2x_1y_1 + 2y_1\|y_2\| - 2x_1\|y_2\| \\ =\;& \big(x_1-y_1-\|y_2\|\big)^2 \geq 0.\end{aligned}$$

(d) From (1.7), we know that $x_1y_1 - \langle x_2, y_2\rangle \geq x_1y_1 - \|x_2\|\,\|y_2\| \geq 0$, and

$$\begin{aligned}&\big(x_1y_1 - \|x_2\|\,\|y_2\|\big)^2 - \big(x_1^2-\|x_2\|^2\big)\left(y_1^2-\|y_2\|^2\right) \\ =\;& x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 - 2x_1y_1\|x_2\|\,\|y_2\| \\ =\;& \big(x_1\|y_2\| - y_1\|x_2\|\big)^2 \geq 0.\end{aligned}$$

Hence, we obtain $x_1y_1 - \langle x_2, y_2\rangle \geq x_1y_1 - \|x_2\|\,\|y_2\| \geq \sqrt{x_1^2-\|x_2\|^2}\sqrt{y_1^2-\|y_2\|^2}$,

(e) The proof follows from

$$\begin{aligned}
& (x_1+y_1)^2 - \|x_2+y_2\|^2 \\
= \;& \left(x_1^2 - \|x_2\|^2\right) + \left(y_1^2 - \|y_2\|^2\right) + 2\left(x_1y_1 - \langle x_2, y_2\rangle\right) \\
\geq \;& 2\sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} + 2\left(x_1y_1 - \langle x_2, y_2\rangle\right) \\
\geq \;& 2\sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} + 2\sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)} \\
= \;& 4\sqrt{(x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2)},
\end{aligned}$$

where the first inequality is true since $a + b \geq 2\sqrt{ab}$ for all positive a, b and the second inequality is from part(d). $\square$

The inequalities in Lemma 2.2(d)-(e) can be achieved by applying Proposition 1.8(b). Next proposition is an important feature of the SOC-function corresponding to $f(t) = \frac{1}{t}$ which is very useful in the subsequent analysis and also similar to the operator setting.

Proposition 2.3. *Let $f : (0, \infty) \to (0, \infty)$ be $f(t) = \frac{1}{t}$. Then,*

(a) *$-f$ is SOC-monotone on $(0, \infty)$;*

(b) *f is SOC-convex on $(0, \infty)$.*

Proof. (a) It suffices to show that $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ implies $f^{\mathrm{soc}}(x) = x^{-1} \preceq_{\mathcal{K}^n} y^{-1} = f^{\mathrm{soc}}(y)$. For any $x = (x_1, x_2) \in \mathcal{K}^n$ and $y = (y_1, y_2) \in \mathcal{K}^n$, we know that $y^{-1} = \frac{1}{\det(y)}(y_1, -y_2)$ and $x^{-1} = \frac{1}{\det(x)}(x_1, -x_2)$, which imply

$$\begin{aligned}
f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(x) &= y^{-1} - x^{-1} \\
&= \left(\frac{y_1}{\det(y)} - \frac{x_1}{\det(x)}, \frac{x_2}{\det(x)} - \frac{y_2}{\det(y)}\right) \\
&= \frac{1}{\det(x)\det(y)}\big(\det(x)y_1 - \det(y)x_1,\, \det(y)x_2 - \det(x)y_2\big).
\end{aligned}$$

To complete the proof, we need to verify two things.
(1) First, we have to show that $\det(x)y_1 - \det(y)x_1 \geq 0$. Applying Lemma 2.1 yields

$$\frac{\det(x)}{\det(y)} = \frac{x_1^2 - \|x_2\|^2}{y_1^2 - \|y_2\|^2} = \left(\frac{x_1 + \|x_2\|}{y_1 + \|y_2\|}\right)\left(\frac{x_1 - \|x_2\|}{y_1 - \|y_2\|}\right) \geq \frac{2x_1}{2y_1} = \frac{x_1}{y_1}.$$

Then, cross multiplying gives $\det(x)y_1 \geq \det(y)x_1$, which says $\det(x)y_1 - \det(y)x_1 \geq 0$.

(2) Secondly, we need to argue that $\|\det(y)x_2 - \det(x)y_2\| \leq \det(x)y_1 - \det(y)x_1$. This

is true by

$$\begin{aligned}
& (\det(x)y_1 - \det(y)x_1)^2 - \|\det(y)x_2 - \det(x)y_2\|^2 \\
= & (\det(x))^2 y_1^2 - 2\det(x)\det(y)x_1y_1 + (\det(y))^2 x_1^2 \\
& -\big((\det(y))^2\|x_2\|^2 - 2\det(x)\det(y)\langle x_2, y_2\rangle + (\det(x))^2\|y_2\|^2\big) \\
= & (\det(x))^2(y_1^2 - \|y_2\|^2) + (\det(y))^2(x_1^2 - \|x_2\|^2) \\
& -2\det(x)\det(y)(x_1y_1 - \langle x_2, y_2\rangle) \\
= & (\det(x))^2\det(y) + (\det(y))^2\det(x) - 2\det(x)\det(y)(x_1y_1 - \langle x_2, y_2\rangle) \\
= & \det(x)\det(y)\big(\det(x) + \det(y) - 2x_1y_1 + 2\langle x_2, y_2\rangle\big) \\
= & \det(x)\det(y)\big((x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) - 2x_1y_1 + 2\langle x_2, y_2\rangle\big) \\
= & \det(x)\det(y)\big((x_1 - y_1)^2 - (\|x_2\|^2 + \|y_2\|^2 - 2\langle x_2, y_2\rangle)\big) \\
= & \det(x)\det(y)\big((x_1 - y_1)^2 - (\|x_2 - y_2\|^2)\big) \\
\geq & 0,
\end{aligned}$$

where the last step holds by the inequality (1.7).
Thus, from all the above, we prove $y^{-1} - x^{-1} \in \mathcal{K}^n$, that is, $y^{-1} \succeq_{\mathcal{K}^n} x^{-1}$.

(b) For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, using (1.7) again, there hold

$$x_1 - \|x_2\| > 0, \quad y_1 - \|y_2\| > 0, \quad |\langle x_2, y_2\rangle| \leq \|x_2\| \cdot \|y_2\| \leq x_1y_1.$$

From $x^{-1} = \frac{1}{\det(x)}(x_1, -x_2)$ and $y^{-1} = \frac{1}{\det(y)}(y_1, -y_2)$, we also have

$$\frac{1}{2}\big(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\big) = \frac{1}{2}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}, -\frac{x_2}{\det(x)} - \frac{y_2}{\det(y)}\right),$$

and

$$f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) = \left(\frac{x+y}{2}\right)^{-1} = \frac{2}{\det(x+y)}\big(x_1 + y_1, -(x_2 + y_2)\big).$$

For convenience, we denote $\frac{1}{2}\big(f^{\mathrm{soc}}(x) + f^{\mathrm{soc}}(y)\big) - f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) := \frac{1}{2}(\Xi_1, \Xi_2)$, where $\Xi_1 \in \mathbb{R}$ and $\Xi_2 \in \mathbb{R}^{n-1}$ are given by

$$\left\{\begin{aligned}
\Xi_1 &= \left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}\right) - \frac{4(x_1 + y_1)}{\det(x+y)}, \\
\Xi_2 &= \frac{4(x_2 + y_2)}{\det(x+y)} - \left(\frac{x_2}{\det(x)} + \frac{y_2}{\det(y)}\right).
\end{aligned}\right.$$

Again, in order to prove f is SOC-convex, it suffices to verify two things: $\Xi_1 \geq 0$ and $\|\Xi_2\| \leq \Xi_1$.

(1) First, we verify that $\Xi_1 \geq 0$. In fact, if we define the function

$$g(x) := \frac{x_1}{x_1^2 - \|x_2\|^2} = \frac{x_1}{\det(x)},$$

then we observe that

$$g\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(g(x)+g(y)\Big) \iff \Xi_1 \ge 0.$$

Hence, to prove $\Xi_1 \ge 0$, it is equivalent to verifying g is convex on $\mathrm{int}(\mathcal{K}^n)$. Since $\mathrm{int}(\mathcal{K}^n)$ is a convex set, it is sufficient to argue that $\nabla^2 g(x)$ is a positive semidefinite matrix. From direct computations, we have

$$\nabla^2 g(x) = \frac{1}{(x_1^2-\|x_2\|^2)^3}\left[\begin{array}{cc} 2x_1^3+6x_1\|x_2\|^2 & -(6x_1^2+2\|x_2\|^2)x_2^T \\ -(6x_1^2+2\|x_2\|^2)x_2 & 2x_1\big((x_1^2-\|x_2\|^2)I+4x_2x_2^T\big) \end{array}\right].$$

Let $\nabla^2 g(x)$ be viewed as the matrix $\left[\begin{array}{cc} A & B \\ B^T & C \end{array}\right]$ given as in Lemma 1.1 (here A is a scalar). Then, we have

$$\begin{aligned}
& AC - B^TB \\
= \;& 2x_1\left(2x_1^3+6x_1\|x_2\|^2\right)\left((x_1^2-\|x_2\|^2)I+4x_2x_2^T\right) - \left(6x_1^2+2\|x_2\|^2\right)^2 x_2x_2^T \\
= \;& \left(4x_1^4+12x_1^2\|x_2\|^2\right)\left(x_1^2-\|x_2\|^2\right)I - \left(20x_1^4-24x_1^2\|x_2\|^2+4\|x_2\|^4\right)x_2x_2^T \\
= \;& \left(4x_1^4+12x_1^2\|x_2\|^2\right)\left(x_1^2-\|x_2\|^2\right)I - 4\left(5x_1^2-\|x_2\|^2\right)\left(x_1^2-\|x_2\|^2\right)x_2x_2^T \\
= \;& \left(x_1^2-\|x_2\|^2\right)\left[\left(4x_1^4+12x_1^2\|x_2\|^2\right)I - 4\left(5x_1^2-\|x_2\|^2\right)x_2x_2^T\right] \\
= \;& \left(x_1^2-\|x_2\|^2\right)M,
\end{aligned}$$

where we denote the whole matrix in the big parenthesis of the last second equality by M. It can be verified that $x_2x_2^T$ is positive semidefinite with only one nonzero eigenvalue $\|x_2\|^2$. Hence, all the eigenvalues of the matrix M are $(4x_1^4+12x_1^2\|x_2\|^2-20x_1^2\|x_2\|^2+4\|x_2\|^4)$ and $4x_1^4+12x_1^2\|x_2\|^2$ with multiplicity of $n-2$, which are all positive since

$$\begin{aligned}
& 4x_1^4+12x_1^2\|x_2\|^2-20x_1^2\|x_2\|^2+4\|x_2\|^4 \\
= \;& 4x_1^4-8x_1^2\|x_2\|^2+4\|x_2\|^4 \\
= \;& 4\left(x_1^2-\|x_2\|^2\right) \\
> \;& 0.
\end{aligned}$$

Thus, by Lemma 1.1, we see that $\nabla^2 g(x)$ is positive definite and hence is positive semidefinite. This means g is convex on $\mathrm{int}(\mathcal{K}^n)$, which says $\Xi_1 \ge 0$.

(2) It remains to show that $\Xi_1^2 - \|\Xi_2\|^2 \geq 0$:

$$\begin{aligned}
& \Xi_1^2 - \|\Xi_2\|^2 \\
= & \left[\left(\frac{x_1^2}{\det(x)^2} + \frac{2x_1y_1}{\det(x)\det(y)} + \frac{y_1^2}{\det(y)^2}\right) - \frac{8(x_1+y_1)}{\det(x+y)}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}\right)\right. \\
& \left. + \frac{16}{\det(x+y)^2}\left(x_1^2 + 2x_1y_1 + y_1^2\right)\right] - \left\| \frac{4(x_2+y_2)}{det(x+y)} - \left(\frac{x_2}{\det(x)} + \frac{y_2}{\det(y)}\right)\right\|^2 \\
= & \left[\left(\frac{x_1^2}{\det(x)^2} + \frac{2x_1y_1}{\det(x)\det(y)} + \frac{y_1^2}{\det(y)^2}\right) - \frac{8(x_1+y_1)}{\det(x+y)}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}\right)\right. \\
& \left. + \frac{16}{\det(x+y)^2}\left(x_1^2 + 2x_1y_1 + y_1^2\right)\right] - \left[\frac{16}{\det(x+y)^2}\left(\|x_2\|^2 + 2\langle x_2, y_2\rangle + \|y_2\|^2\right)\right. \\
& \left. -8\left\langle \frac{x_2+y_2}{\det(x+y)}, \frac{x_2}{\det(x)} + \frac{y_2}{\det(y)}\right\rangle + \left(\frac{\|x_2\|^2}{\det(x)^2} + \frac{2\langle x_2, y_2\rangle}{\det(x)\det(y)} + \frac{\|y_2\|^2}{\det(y)^2}\right)\right] \\
= & \left[\frac{x_1^2 - \|x_2\|^2}{\det(x)^2} + \frac{2(x_1y_1 - \langle x_2, y_2\rangle)}{\det(x)\det(y)} + \frac{y_1^2 - \|y_2\|^2}{\det(y)^2}\right] \\
& + \frac{16}{\det(x+y)^2}\left[\left(x_1^2 - \|x_2\|^2\right) + 2\big(x_1y_1 - \langle x_2, y_2\rangle\big) + \left(y_1^2 - \|y_2\|^2\right)\right] \\
& -8\left[\frac{x_1^2 - \|x_2\|^2}{\det(x+y)\det(x)} + \frac{x_1y_1 - \langle x_2, y_2\rangle}{\det(x+y)\det(x)} + \frac{x_1y_1 - \langle x_2, y_2\rangle}{\det(x+y)\det(y)} + \frac{y_1^2 - \|y_2\|^2}{\det(x+y)\det(y)}\right] \\
= & \left(x_1^2 - \|x_2\|^2\right)\left(\frac{1}{\det(x)^2} + \frac{16}{\det(x+y)^2} - \frac{8}{\det(x+y)\det(x)}\right) \\
& + \left(y_1^2 - \|y_2\|^2\right)\left(\frac{1}{\det(y)^2} + \frac{16}{\det(x+y)^2} - \frac{8}{\det(x+y)\det(y)}\right) \\
& + 2\big(x_1y_1 - \langle x_2, y_2\rangle\big)\left(\frac{1}{\det(x)\det(y)} + \frac{16}{\det(x+y)^2} - \frac{4}{\det(x+y)det(x)} - \frac{4}{\det(x+y)\det(y)}\right) \\
= & \left(x_1^2 - \|x_2\|^2\right)\left(\frac{\det(x+y) - 4\det(x)}{\det(x)\det(x+y)}\right)^2 + \left(y_1^2 - \|y_2\|^2\right)\left(\frac{\det(x+y) - 4\det(y)}{\det(y)\det(x+y)}\right)^2 \\
& + 2\big(x_1y_1 - \langle x_2, y_2\rangle\big)\left(\frac{(\det(x+y) - 4\det(x))(\det(x+y) - 4\det(y))}{\det(x)\det(y)\det(x+y)^2}\right).
\end{aligned}$$

Now applying the facts that $\det(x) = x_1^2 - \|x_2\|^2$, $\det(y) = y_1^2 - \|y_2\|^2$, and $\det(x+y) - \det(x) - \det(y) = 2(x_1y_1 - \langle x_2, y_2\rangle)$, we can simplify the last equality (after a lot of algebra simplifications) and obtain

$$\Xi_1^2 - \|\Xi_2\|^2 = \frac{\big[\det(x+y) - 2\det(x) - 2\det(y)\big]^2}{\det(x)\det(y)\det(x+y)} \geq 0.$$

Hence, we prove that $f^{\text{soc}}\left(\frac{x+y}{2}\right) \preceq_{\mathcal{K}^n} \frac{1}{2}\left(f^{\text{soc}}(x) + f^{\text{soc}}(y)\right)$, which says the function $f(t) = \dfrac{1}{t}$ is SOC-convex on the interval $(0, \infty)$. $\square$

Proposition 2.4. **(a)** *The function* $f(t) = \frac{t}{1+t}$ *is SOC-monotone on* $(0, \infty)$.

(b) *For any* $\lambda > 0$, *the function* $f(t) = \frac{t}{\lambda+t}$ *is SOC-monotone on* $(0, \infty)$.

Proof. (a) Let $g(t) = -\frac{1}{t}$ and $h(t) = 1 + t$. Then, we see that g is SOC-monotone on $(0, \infty)$ by Proposition 2.3, while h is SOC-monotone on $\mathbb{R}$ by Proposition 2.2. Since $f(t) = 1 - \frac{1}{1+t} = h(g(1+t))$, the result follows from the fact that the composition of two SOC-monotone functions is also SOC-monotone, see Proposition 2.9.

(b) Similarly, let $g(t) = \frac{t}{1+t}$ and $h(t) = \frac{t}{\lambda}$, then both functions are SOC-monotone by part(a). Since $f(t) = g(h(t))$, the result is true by the same reason as in part(a). $\square$

Proposition 2.5. *Let* L_x *be defined as in (1.20). For any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *we have*

$$L_x \succeq L_y \iff L_y^{-1} \succeq L_x^{-1} \iff L_{y^{-1}} \succeq L_{x^{-1}}.$$

Proof. By the property of L_x that $x \succeq_{\mathcal{K}^n} y \iff L_x \succeq L_y$, and Proposition 2.3(a), then proof follows. $\square$

Next, we examine another simple function $f(t) = \sqrt{t}$. We will see that it is SOC-monotone on the interval $[0, \infty)$, and $-\sqrt{t}$ is SOC-convex on $[0, \infty)$.

Proposition 2.6. *Let* $f : [0, \infty) \to [0, \infty)$ *be* $f(t) = \sqrt{t}$. *Then,*

(a) f *is SOC-monotone on* $[0, \infty)$;

(b) $-f$ *is SOC-convex on* $[0, \infty)$.

Proof. (a) This is a consequence of Property 1.3(b).

(b) To show $-f$ is SOC-convex, it is enough to prove that $f^{\mathrm{soc}}\left(\frac{x+y}{2}\right) \succeq_{\mathcal{K}^n} \frac{f^{\mathrm{soc}}(x)+f^{\mathrm{soc}}(y)}{2}$, which is equivalent to verifying that $\left(\frac{x+y}{2}\right)^{1/2} \succeq_{\mathcal{K}^n} \frac{\sqrt{x}+\sqrt{y}}{2}$, for all $x, y \in \mathcal{K}^n$. Since $x + y \succeq_{\mathcal{K}^n} 0$, by Property 1.3(e), it is sufficient to show that $\left(\frac{x+y}{2}\right) \succeq_{\mathcal{K}^n} \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2$. This can be seen by $\left(\frac{x+y}{2}\right) - \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = \frac{(\sqrt{x}-\sqrt{y})^2}{4} \succeq_{\mathcal{K}^n} 0$. Thus, we complete the proof. $\square$

Proposition 2.7. *Let* $f : [0, \infty) \to [0, \infty)$ *be* $f(t) = t^r$ *where* $0 \le r \le 1$. *Then,*

(a) f *is SOC-monotone on* $[0, \infty)$;

(b) $-f$ *is SOC-convex on* $[0, \infty)$.

Proof. (a) Let r be a dyadic rational, i.e., a number of the form $r = \frac{m}{2^n}$, where n is any positive integer and $1 \leq m \leq 2^n$. It is enough to prove the assertion is true for such r since the dyadic rational numbers are dense in $[0,1]$. We will claim this by induction on n. Let $x, y \in \mathcal{K}^n$ with $x \succeq_{\mathcal{K}^n} y$, then by Property 1.3(b) we have $x^{1/2} \succeq_{\mathcal{K}^n} y^{1/2}$. Therefore, part(a) is true when $n = 1$. Suppose it is also true for all dyadic rational $\dfrac{m}{2^j}$, in which $1 \leq j \leq n-1$. Now let $r = \frac{m}{2^n}$ with $m \leq 2^n$. By induction hypothesis, we know $x^{\frac{m}{2^{n-1}}} \succeq_{\mathcal{K}^n} y^{\frac{m}{2^{n-1}}}$. Then, by applying Property 1.3(b), we obtain $\left(x^{\frac{m}{2^{n-1}}}\right)^{1/2} \succeq_{\mathcal{K}^n} \left(y^{\frac{m}{2^{n-1}}}\right)^{1/2}$, which says $x^{\frac{m}{2^n}} \succeq_{\mathcal{K}^n} y^{\frac{m}{2^n}}$. Thus, we have shown that $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ implies $x^r \succeq_{\mathcal{K}^n} y^r$, for all dyadic rational r in $[0,1]$. Then, the desired result follows.

(b) The proof is similar to the above arguments. First, we observe that

$$\left(\frac{x+y}{2}\right) - \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = \left(\frac{\sqrt{x}-\sqrt{y}}{2}\right)^2 \succeq_{\mathcal{K}^n} 0,$$

which implies $\left(\frac{x+y}{2}\right)^{1/2} \succeq_{\mathcal{K}^n} \frac{1}{2}\left(\sqrt{x}+\sqrt{y}\right)$ by Property 1.3(b). Hence, we show that the assertion is true when $n = 1$. By induction hypothesis, suppose $\left(\frac{x+y}{2}\right)^{\frac{m}{2^{n-1}}} \succeq_{\mathcal{K}^n} \left(\frac{x^{\frac{m}{2^{n-1}}}+y^{\frac{m}{2^{n-1}}}}{2}\right)$. Then, we have

$$\begin{aligned} \left(\frac{x+y}{2}\right)^{\frac{m}{2^{n-1}}} - \left(\frac{x^{\frac{m}{2^n}}+y^{\frac{m}{2^n}}}{2}\right)^2 \;&\succeq_{\mathcal{K}^n}\; \left(\frac{x^{\frac{m}{2^{n-1}}}+y^{\frac{m}{2^{n-1}}}}{2}\right) - \left(\frac{x^{\frac{m}{2^n}}+y^{\frac{m}{2^n}}}{2}\right)^2 \\ &= \;\left(\frac{x^{\frac{m}{2^n}}-y^{\frac{m}{2^n}}}{2}\right)^2 \succeq_{\mathcal{K}^n} 0, \end{aligned}$$

which implies $\left(\frac{x+y}{2}\right)^{\frac{m}{2^n}} \succeq_{\mathcal{K}^n} \left(\frac{x^{\frac{m}{2^n}}+y^{\frac{m}{2^n}}}{2}\right)$ by Property 1.3(b). Following the same arguments about dyadic rationals in part(a) yields the desired result. $\square$

From all the above examples, we observe that f being monotone does not imply f is SOC-monotone. Likewise, f being convex does not guarantee that f is SOC-convex. Now, we move onto some famous functions which are used very often for NCP (nonlinear complementarity problem), SDCP, and SOCCP. It would be interesting to know about the SOC-convexity and SOC-monotonicity of these functions. First, we will look at the Fischer-Burmeister function, $\phi_{\rm FB} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\phi_{\rm FB}(x,y) = (x^2+y^2)^{1/2} - (x+y), \tag{2.5}$$

which is a well-known merit function for complementarity problem, see [88, 139]. Here, $(\cdot)^2$ and $(\cdot)^{1/2}$ are defined through Jordan product introduced as in (1.5) in Chapter 1. For SOCCP, it has been shown that squared norm of $\phi_{\rm FB}$, i.e.,

$$\psi_{\rm FB}(x,y) = \|\phi(x,y)\|^2, \tag{2.6}$$

is continuously differentiable (see [49]) whereas $\psi_{\rm FB}$ is only shown differentiable for SDCP (see [145]). In addition, $\phi_{\rm FB}$ is proved to have semismoothness and Lipschitz continuity in the recent paper [142] for both cases of SOCCP and SDCP. For more details regarding further properties of these functions associated with SOC and the roles they play in the solutions methods, please refer to [48, 120–122]. In NCP setting, $\phi_{\rm FB}$ is a convex function, so we may wish to have an analogy for SOCCP. Unfortunately, as shown below, it is not an SOC-convex function.

Example 2.5. *Let $\phi_{\rm FB}$ be defined as in (2.5) and $\psi_{\rm FB}$ defined as in (2.6).*

(a) *The function $\rho(x,y) = (x^2+y^2)^{1/2}$ does not satisfy (2.2).*

(b) *The Fischer-Burmeister function $\phi_{\rm FB}$ does not satisfy (2.2).*

(c) *The function $\psi_{\rm FB} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is not convex.*

Solution. (a) A counterexample occurs when taking $x = (1,1)$ and $y = (1,0)$.

(b) Suppose that it satisfies (2.2). Then, we will have ρ satisfies (2.2) by $\rho(x,y) = \phi_{\rm FB}(x,y) + (x+y)$, which is a contradiction to part(a). Thus, $\phi_{\rm FB}$ does not satisfy (2.2).

(c) Let $x = (1,-2)$, $y = (1,-1)$ and $u = (0,-1)$, $v = (1,-1)$. Then, we have

$$\phi_{\rm FB}(x,y) = \left(\frac{-3+\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2} \right) \implies \psi_{\rm FB}(x,y) = \|\phi_{\rm FB}(x,y)\|^2 = 21 - 5\sqrt{13}.$$

$$\phi_{\rm FB}(u,v) = \left(\frac{-1+\sqrt{5}}{2}, \frac{5-\sqrt{5}}{2} \right) \implies \psi_{\rm FB}(u,v) = \|\phi_{\rm FB}(u,v)\|^2 = 9 - 3\sqrt{5}.$$

Thus, $\frac{1}{2}\big(\psi_{\rm FB}(x,y) + \psi(u,v)\big) = \frac{1}{2}(30 - 5\sqrt{13} - 3\sqrt{5}) \approx 2.632$.

On the other hand, let $(\hat{x},\hat{y}) := \frac{1}{2}(x,y) + \frac{1}{2}(u,v)$, that is, $\hat{x} = (\frac{1}{2}, -\frac{3}{2})$ and $\hat{y} = (1,-1)$. Indeed, we have $\hat{x}^2 + \hat{y}^2 = \big(\frac{9}{2}, -\frac{7}{2}\big)$ and hence $(\hat{x}^2+\hat{y}^2)^{1/2} = \big(\frac{1+2\sqrt{2}}{2}, \frac{1-2\sqrt{2}}{2}\big)$, which implies $\psi_{\rm FB}(\hat{x},\hat{y}) = \|\phi_{\rm FB}(\hat{x},\hat{y})\|^2 = 14 - 8\sqrt{2} \approx 2.686$. Therefore, we obtain

$$\psi_{\rm FB}\left(\frac{1}{2}(x,y) + \frac{1}{2}(u,v)\right) > \frac{1}{2}\psi_{\rm FB}(x,y) + \frac{1}{2}\psi_{\rm FB}(u,v),$$

which shows $\psi_{\rm FB}$ is not convex. ■

Another function based on the Fischer-Burmeister function is $\psi_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, given by

$$\psi_1(x,y) := \|[\phi_{\rm FB}(x,y)]_+\|^2, \tag{2.7}$$

where $\phi_{\rm FB}$ is the Fischer-Burmeister function given as in (2.5). In the NCP case, it is known that ψ_1 is convex. It has been an open question whether this is still true for SDCP and SOCCP (see Question 3 on page 182 of [145]). In fact, Qi and Chen [128] gave the negative answer for the SDCP case. Here we provide an answer to the question for SOCCP: ψ_1 is not convex in the SOCCP case.

Example 2.6. *Let $\phi_{\rm FB}$ be defined as in (2.5) and ψ_1 defined as in (2.7).*

(a) *The function $[\phi_{\rm FB}(x,y)]_+ = [(x^2+y^2)^{1/2} - (x+y)]_+$ does not satisfy (2.2).*

(b) *The function ψ_1 is not convex.*

Solution. (a) Let $x=(2,1,-1)$, $y=(1,1,0)$ and $u=(1,-2,5)$, $v=(-1,5,0)$. For simplicity, we denote $\phi_1(x,y) := [\phi_{\rm FB}(x,y)]_+$. Then, by direct computations, we obtain

$$\frac{1}{2}\phi_1(x,y) + \frac{1}{2}\phi_1(u,v) - \phi_1\left(\frac{1}{2}(x,y) + \frac{1}{2}(u,v)\right) = \big(1.0794, 0.4071, -1.0563\big) \not\succeq_{\mathcal{K}^3} 0,$$

which says ϕ_1 does not satisfy (2.2).

(b) Let $x=(17,5,16)$, $y=(20,-3,15)$ and $u=(2,3,3)$, $v=(9,-7,2)$. Then, it can be easily verified that $\frac{1}{2}\psi_1(x,y) + \frac{1}{2}\psi_1(u,v) - \psi_1\big(\frac{1}{2}(x,y)+\frac{1}{2}(u,v)\big) < 0$, which implies ψ_1 is not convex. ■

Example 2.7. **(a)** *The function $f(t)=|t|$ is not SOC-monotone on $\mathbb{R}$.*

(b) *The function $f(t)=|t|$ is not SOC-convex on $\mathbb{R}$.*

(c) *The function $f(t)=[t]_+$ is not SOC-monotone on $\mathbb{R}$.*

(d) *The function $f(t)=[t]_+$ is not SOC-convex on $\mathbb{R}$.*

Solution. To see (a), let $x=(1,0), y=(-2,0)$. It is clear that $x \succeq_{\mathcal{K}^2} y$. Besides, we have $x^2=(1,0)$, $y^2=(4,0)$ which yields $|x|=(1,0)$ and $|y|=(2,0)$. But, $|x|-|y| = (-1,0) \not\succeq_{\mathcal{K}^2} 0$.

To see (b), let $x=(1,1,1)$, $y=(-1,1,0)$. In fact, we have $|x| = \left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $|y| = (1,-1,0)$, and $|x+y| = (\sqrt{5},0,0)$. Therefore,

$$|x|+|y|-|x+y| = \left(\sqrt{2}+1-\sqrt{5}, -1+\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \not\succ_{\mathcal{K}^3} 0,$$

which says $f^{\rm soc}\left(\frac{x+y}{2}\right) \not\preceq_{\mathcal{K}^3} \frac{1}{2}\left(f^{\rm soc}(x)+f^{\rm soc}(y)\right)$. Thus, $f(t)=|t|$ is not SOC-convex on $\mathbb{R}$.

To see (c) and (d), just follows (a) and (b) and the facts that $[t]_+ = \frac{1}{2}(t+|t|)$ where $t\in\mathbb{R}$, and Property 1.2(f): $[x]_+ = \frac{1}{2}(x+|x|)$ where $x\in\mathbb{R}^n$. ■

To close this section, we check with one popular smoothing function,

$$f(t) = \frac{1}{2}\left(\sqrt{t^2+4}+t\right),$$

which was proposed by Chen and Harker [39], Kanzow [85], and Smale [138]; and is called the CHKS function. Its corresponding SOC-function is defined by

$$f^{\text{soc}}(x) = \frac{1}{2}\left((x^2+4e)^{\frac{1}{2}} + x\right),$$

where $e = (1, 0, \cdots, 0)$. The function $f(t)$ is convex and monotone, so we may also wish to know whether it is SOC-convex or SOC-monotone or not. Unfortunately, it is neither SOC-convex nor SOC-monotone for $n \geq 3$, though it is both SOC-convex and SOC-monotone for $n = 2$. The following example demonstrates what we have just said.

Example 2.8. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be* $f(t) = \dfrac{\sqrt{t^2+4}+t}{2}$. *Then,*

(a) *f is not SOC-monotone of order $n \geq 3$ on $\mathbb{R}$;*

(b) *however, f is SOC-monotone of order 2 on $\mathbb{R}$;*

(c) *f is not SOC-convex of order $n \geq 3$ on $\mathbb{R}$;*

(d) *however, f is SOC-convex of order 2 on $\mathbb{R}$.*

Solution. Again, by Remark 2.1, taking $x = (2, 1, -1)$ and $y = (1, 1, 0)$ gives a counterexample for both (a) and (c).

To see (b) and (d), it follows by direct verifications as what we have done before. ■

2.2 Characterizations of SOC-monotone and SOC-convex functions

Based on all the results in the previous section, one may expect some certain relation between SOC-convex function and SOC-monotone function. One may also like to know under what conditions a function is SOC-convex. The same question arises for SOC-monotone. In this section, we aim to answer these questions. In fact, there already have some analogous results for matrix-functions (see Chapter V of [22]). However, not much yet for this kind of vector-valued SOC-functions, so further study on these topics is necessary.

Originally, in light of all the above observations, two conjectures were proposed in [42]) as below. The answers for these two conjectures will turn clear later after Section 2.2 and Section 2.3.

Conjecture 2.1. *Let $f : (0, \infty) \to \mathbb{R}$ be continuous, convex, and nonincreasing. Then,*

(a) *f is SOC-convex;*

(b) *$-f$ is SOC-monotone.*

Conjecture 2.2. *Let $f : [0,\infty) \to [0,\infty)$ be continuous. Then,*

$$-f \text{ is SOC-convex} \iff f \text{ is SOC-monotone.}$$

Proposition 2.8. *Let $f : [0,\infty) \to [0,\infty)$ be continuous. If $-f$ is SOC-convex, then f is SOC-monotone.*

Proof. Suppose that $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$. For any $0 < \lambda < 1$, we can write

$$\lambda x = \lambda y + (1-\lambda)\frac{\lambda}{1-\lambda}(x-y).$$

Then, using the SOC-convexity of $-f$ yields that

$$f^{\text{soc}}(\lambda x) \succeq_{\mathcal{K}^n} \lambda f^{\text{soc}}(y) + (1-\lambda) f^{\text{soc}}\left(\frac{\lambda}{1-\lambda}(x-y)\right) \succeq_{\mathcal{K}^n} 0,$$

where the second inequality is true since f is from $[0,\infty)$ into itself and $x-y \succeq_{\mathcal{K}^n} 0$. This yields $f^{\text{soc}}(\lambda x) \succeq_{\mathcal{K}^n} \lambda f^{\text{soc}}(y)$. Now, letting $\lambda \to 1$, we obtain that $f^{\text{soc}}(x) \succeq_{\mathcal{K}^n} f^{\text{soc}}(y)$, which says that f is SOC-monotone. $\Box$

The converse of Proposition 2.8 is not true, in general. For counterexample, we consider

$$f(t) = -\cot\left(-\frac{\pi}{2}(1+t)^{-1} + \pi\right), \qquad t \in [0,\infty).$$

Notice that $-\cot(t)$ is SOC-monotone on $[\pi/2, \pi)$, whereas $-\frac{\pi}{2}(1+t)^{-1}$ is SOC-monotone on $[0,\infty)$. Hence, their compound function $f(t)$ is SOC-monotone on $[0,\infty)$. However, $-f(t)$ does not satisfy the inequality (2.36) for all $t \in (0,\infty)$. For example, when $t_1 = 7.7$ and $t_2 = 7.6$, the left hand side of (2.36) equals 0.0080, whereas the right hand side equals 27.8884. This shows that $f(t) = -\cot(t)$ is not SOC-concave of order $n \geq 3$. In summary, only one direction ("$\Longrightarrow$") of Conjecture 2.2 holds. Whether Conjecture 2.1 is true or not will be confirmed at the end of Section 2.3.

We notice that if f is not a function from $[0,\infty)$ into itself, then Proposition 2.8 may be false. For instance, $f(t) = -t^2$ is SOC-concave, but not SOC-monotone. In other words, the domain of function f is an important factor for such relation. From now on, we will demonstrate various characterizations regarding SOC-convex and SOC-monotone functions.

Proposition 2.9. *Let $g : J \to \mathbb{R}$ and $h : I \to J$, where $J \subseteq \mathbb{R}$ and $I \subseteq \mathbb{R}$. Then, the following hold.*

(a) *If g is SOC-concave and SOC-monotone on J and h is SOC-concave on I, then their composition $g \circ h = g(h(\cdot))$ is also SOC-concave on I.*

(b) *If g is SOC-monotone on J and h is SOC-monotone on I, then $g \circ h = g(h(\cdot))$ is SOC-monotone on I.*

Proof. (a) For the sake of notation, let $g^{\rm soc} : \widehat{S} \to \mathbb{R}^n$ and $h^{\rm soc} : S \to \widehat{S}$ be the vector-valued functions associated with g and h, respectively, where $S \subseteq \mathbb{R}^n$ and $\widehat{S} \subseteq \mathbb{R}^n$. Define $\widehat{g}(t) = g(h(t))$. Then, for any $x \in S$, it follows from (1.2) and (1.8) that

$$\begin{aligned} g^{\rm soc}(h^{\rm soc}(x)) &= g^{\rm soc}\Big[h(\lambda_1(x))u_x^{(1)} + h(\lambda_2(x))u_x^{(2)}\Big] \\ &= g\big[h(\lambda_1(x))\big]u_x^{(1)} + g\big[h(\lambda_2(x))\big]u_x^{(2)} \\ &= \widehat{g}^{\rm soc}(x). \end{aligned} \tag{2.8}$$

We next prove that $\widehat{g}(t)$ is SOC-concave on I. For any $x, y \in S$ and $0 \le \beta \le 1$, from the SOC-concavity of $h(t)$ it follows that

$$h^{\rm soc}(\beta x + (1-\beta)y) \succeq_{\mathcal{K}^n} \beta h^{\rm soc}(x) + (1-\beta)h^{\rm soc}(y).$$

Using the SOC-monotonicity and SOC-concavity of g, we then obtain that

$$\begin{aligned} g^{\rm soc}\Big[h^{\rm soc}(\beta x + (1-\beta)y)\Big] &\succeq_{\mathcal{K}^n} g^{\rm soc}\Big[\beta h^{\rm soc}(x) + (1-\beta)h^{\rm soc}(y)\Big] \\ &\succeq_{\mathcal{K}^n} \beta g^{\rm soc}[h^{\rm soc}(x)] + (1-\beta)g^{\rm soc}[h^{\rm soc}(y)]. \end{aligned}$$

This together with (2.8) implies that for any $x, y \in S$ and $0 \le \beta \le 1$,

$$(\widehat{g})^{\rm soc}\Big(\beta x + (1-\beta)y\Big) \ \succeq_{\mathcal{K}^n} \ \beta(\widehat{g})^{\rm soc}(x) + (1-\beta)(\widehat{g})^{\rm soc}(y).$$

Consequently, the function $\widehat{g}(t)$, i.e. $g(h(\cdot))$ is SOC-concave on I.

(b) It is clear that for all $x, y \in \mathbb{R}^n$, $x \succeq_{\mathcal{K}^n} y$ if and only if $\lambda_i(x) \ge \lambda_i(y)$ with $i = 1, 2$. In addition, g is increasing on J since it is SOC-monotone. From the two facts, we immediately obtain the result. $\Box$

Proposition 2.10. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $z \in \mathbb{R}^n$. Let $g_z : \mathbb{R}^n \to \mathbb{R}$ be defined by $g_z(x) := \langle f^{\rm soc}(x), z\rangle$. Then, f is SOC-convex if and only if g_z is a convex function for all $z \succeq_{\mathcal{K}^n} 0$.*

Proof. Suppose f is SOC-convex and let $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$. Then, we have

$$f^{\rm soc}\big((1-\lambda)x + \lambda y\big) \succeq_{\mathcal{K}^n} (1-\lambda)f^{\rm soc}(x) + \lambda f^{\rm soc}(y),$$

which implies

$$\begin{aligned} g_z\left((1-\lambda)x + \lambda y\right) &= \Big\langle f^{\rm soc}\big((1-\lambda)x + \lambda y\big), z\Big\rangle \\ &\le \Big\langle (1-\lambda)f^{soc}(x) + \lambda f^{\rm soc}(y), z\Big\rangle \\ &= (1-\lambda)\langle f^{\rm soc}(x)\ ,\ z\rangle + \langle f^{\rm soc}(y), z\rangle \\ &= (1-\lambda)g_z(x) + \lambda g_z(y), \end{aligned}$$

where the inequality holds by Property 1.3(d). This says that g_z is a convex function. For the other direction, from the convexity of g, we obtain

$$\left\langle f^{\mathrm{soc}}\big((1-\lambda)x+\lambda y\big), z\right\rangle \le \left\langle (1-\lambda)f^{\mathrm{soc}}(x)+\lambda f^{\mathrm{soc}}(y), z\right\rangle.$$

Since $z \succeq_{\mathcal{K}^n} 0$, by Property 1.3(d) again, the above yields

$$f^{\mathrm{soc}}\big((1-\lambda)x+\lambda y\big) \succeq_{\mathcal{K}^n} (1-\lambda)f^{\mathrm{soc}}(x)+\lambda f^{\mathrm{soc}}(y),$$

which says f is SOC-convex. $\square$

Proposition 2.11. *A differentiable function $f:\mathbb{R}\to\mathbb{R}$ is SOC-convex if and only if $f^{\mathrm{soc}}(y) \succeq_{\mathcal{K}^n} f^{\mathrm{soc}}(x)+\nabla f^{\mathrm{soc}}(x)(y-x)$ for all $x,y\in\mathbb{R}^n$.*

Proof. From Proposition 1.13, we know that f is differentiable if and only if f^{soc} is differentiable. Using the gradient formula given therein and following the arguments as in [21, Proposition B.3] or [30, Theorem 2.3.5], the proof can be done easily. We omit the details. $\square$

To discover more characterizations and try to answer the aforementioned conjectures, we develop the second-order Taylor's expansion for the vector-valued SOC-function f^{soc} defined as in (1.8), which is crucial to our subsequent analysis. To the end, we assume that $f\in C^{(2)}(J)$ with J being an open interval in $\mathbb{R}$ and $\mathrm{dom}(f^{\mathrm{soc}})$ is open in $\mathbb{R}^n$ (this is true by Proposition 1.4(a)). Given any $x\in\mathrm{dom}(f^{\mathrm{soc}})$ and $h=(h_1,h_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, we have $x+th\in\mathrm{dom}(f^{\mathrm{soc}})$ for any sufficiently small $t>0$. We wish to calculate the Taylor's expansion of the function $f^{\mathrm{soc}}(x+th)$ at x for any sufficiently small $t>0$. In particular, we are interested in finding matrices $\nabla f^{\mathrm{soc}}(x)$ and $A_i(x)$ for $i=1,2,\ldots,n$ such that

$$f^{\mathrm{soc}}(x+th) = f^{\mathrm{soc}}(x)+t\nabla f^{\mathrm{soc}}(x)h+\frac{1}{2}t^2\begin{bmatrix} h^TA_1(x)h \\ h^TA_2(x)h \\ \vdots \\ h^TA_n(x)h\end{bmatrix}+o(t^2). \tag{2.9}$$

Again, for convenience, we omit the variable notion x in $\lambda_i(x)$ and $u_x^{(i)}$ for $i=1,2$ in the subsequent discussions.

It is known that f^{soc} is differentiable (respectively, smooth) if and only if f is differentiable (respectively, smooth), see Proposition 1.13. Moreover, there holds that

$$\nabla f^{\mathrm{soc}}(x) = \begin{bmatrix} b^{(1)} & c^{(1)}\dfrac{x_2^T}{\|x_2\|} \\ c^{(1)}\dfrac{x_2}{\|x_2\|} & a^{(0)}I+(b^{(1)}-a^{(0)})\dfrac{x_2x_2^T}{\|x_2\|^2}\end{bmatrix} \tag{2.10}$$

if $x_2 \neq 0$; and otherwise

$$\nabla f^{\mathrm{soc}}(x) = f'(x_1) I, \tag{2.11}$$

where

$$a^{(0)} = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}, \quad b^{(1)} = \frac{f'(\lambda_2) + f'(\lambda_1)}{2}, \quad c^{(1)} = \frac{f'(\lambda_2) - f'(\lambda_1)}{2}.$$

Therefore, we only need to derive the formula of $A_i(x)$ for $i = 1, 2, \cdots, n$ in (2.9).

We first consider the case where $x_2 \neq 0$ and $x_2 + th_2 \neq 0$. By the definition (1.8), we see that

$$\begin{aligned}
f^{\mathrm{soc}}(x+th) &= \frac{1}{2} f(x_1 + th_1 - \|x_2 + th_2\|) \begin{bmatrix} 1 \\ -\frac{x_2 + th_2}{\|x_2 + th_2\|} \end{bmatrix} \\
&\quad + \frac{1}{2} f(x_1 + th_1 + \|x_2 + th_2\|) \begin{bmatrix} 1 \\ \frac{x_2 + th_2}{\|x_2 + th_2\|} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{f(x_1 + th_1 - \|x_2 + th_2\|) + f(x_1 + th_1 + \|x_2 + th_2\|)}{2} \\ \dfrac{f(x_1 + th_1 + \|x_2 + th_2\|) - f(x_1 + th_1 - \|x_2 + th_2\|)}{2} \dfrac{x_2 + th_2}{\|x_2 + th_2\|} \end{bmatrix} \\
&:= \begin{bmatrix} \Xi_1 \\ \Xi_2 \end{bmatrix}.
\end{aligned}$$

To derive the Taylor's expansion of $f^{\mathrm{soc}}(x+th)$ at x with $x_2 \neq 0$, we first write out and expand $\|x_2 + th_2\|$. Notice that

$$\|x_2 + th_2\| = \sqrt{\|x_2\|^2 + 2t x_2^T h_2 + t^2 \|h_2\|^2} = \|x_2\| \sqrt{1 + 2t \frac{x_2^T h_2}{\|x_2\|^2} + t^2 \frac{\|h_2\|^2}{\|x_2\|^2}}.$$

Therefore, using the fact that

$$\sqrt{1+\varepsilon} = 1 + \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2 + o(\varepsilon^2),$$

we may obtain

$$\|x_2 + th_2\| = \|x_2\| \left(1 + t \frac{\alpha}{\|x_2\|} + \frac{1}{2} t^2 \frac{\beta}{\|x_2\|^2} \right) + o(t^2), \tag{2.12}$$

where

$$\alpha = \frac{x_2^T h_2}{\|x_2\|}, \quad \beta = \|h_2\|^2 - \frac{(x_2^T h_2)^2}{\|x_2\|^2} = \|h_2\|^2 - \alpha^2 = h_2^T M_{x_2} h_2,$$

with

$$M_{x_2} = I - \frac{x_2 x_2^T}{\|x_2\|^2}.$$

Furthermore, from (2.12) and the fact that $(1+\varepsilon)^{-1} = 1-\varepsilon+\varepsilon^2+o(\varepsilon^2)$, it follows that

$$\|x_2+th_2\|^{-1} = \|x_2\|^{-1}\left(1 - t\frac{\alpha}{\|x_2\|} + \frac{1}{2}t^2\left(2\frac{\alpha^2}{\|x_2\|^2} - \frac{\beta}{\|x_2\|^2}\right) + o(t^2)\right). \tag{2.13}$$

Combining equations (2.12) and (2.13) then yields that

$$\begin{aligned}
\frac{x_2+th_2}{\|x_2+th_2\|} &= \frac{x_2}{\|x_2\|} + t\left(\frac{h_2}{\|x_2\|} - \frac{\alpha}{\|x_2\|}\frac{x_2}{\|x_2\|}\right) \\
&\quad + \frac{1}{2}t^2\left(\left(2\frac{\alpha^2}{\|x_2\|^2} - \frac{\beta}{\|x_2\|^2}\right)\frac{x_2}{\|x_2\|} - 2\frac{h_2}{\|x_2\|}\frac{\alpha}{\|x_2\|}\right) + o(t^2) \\
&= \frac{x_2}{\|x_2\|} + tM_{x_2}\frac{h_2}{\|x_2\|} \qquad (2.14)\\
&\quad + \frac{1}{2}t^2\left(3\frac{h_2^Tx_2x_2^Th_2}{\|x_2\|^4}\frac{x_2}{\|x_2\|} - \frac{\|h_2\|^2}{\|x_2\|^2}\frac{x_2}{\|x_2\|} - 2\frac{h_2h_2^T}{\|x_2\|^2}\frac{x_2}{\|x_2\|}\right) + o(t^2).
\end{aligned}$$

In addition, from (2.12), we have the following equalities

$$\begin{aligned}
& f(x_1+th_1-\|x_2+th_2\|) \\
= \;& f\left(x_1+th_1 - \left(\|x_2\|\left(1+t\frac{\alpha}{\|x_2\|} + \frac{1}{2}t^2\frac{\beta}{\|x_2\|^2}\right) + o(t^2)\right)\right) \\
= \;& f\left(\lambda_1 + t(h_1-\alpha) - \frac{1}{2}t^2\frac{\beta}{\|x_2\|} + o(t^2)\right) \qquad (2.15)\\
= \;& f(\lambda_1) + tf'(\lambda_1)(h_1-\alpha) + \frac{1}{2}t^2\left(-f'(\lambda_1)\frac{\beta}{\|x_2\|} + f''(\lambda_1)(h_1-\alpha)^2\right) + o(t^2)
\end{aligned}$$

and

$$\begin{aligned}
& f(x_1+th_1+\|x_2+th_2\|) \\
= \;& f\left(\lambda_2 + t(h_1+\alpha) + \frac{1}{2}t^2\frac{\beta}{\|x_2\|} + o(t^2)\right) \qquad (2.16)\\
= \;& f(\lambda_2) + tf'(\lambda_2)(h_1+\alpha) + \frac{1}{2}t^2\left(f'(\lambda_2)\frac{\beta}{\|x_2\|} + f''(\lambda_2)(h_1+\alpha)^2\right) + o(t^2).
\end{aligned}$$

For $i = 0, 1, 2$, we define

$$a^{(i)} = \frac{f^{(i)}(\lambda_2) - f^{(i)}(\lambda_1)}{\lambda_2-\lambda_1}, \quad b^{(i)} = \frac{f^{(i)}(\lambda_2) + f^{(i)}(\lambda_1)}{2}, \quad c^{(i)} = \frac{f^{(i)}(\lambda_2) - f^{(i)}(\lambda_1)}{2}, \tag{2.17}$$

where $f^{(i)}$ means the i-th derivative of f and $f^{(0)}$ is the same as the original f. Then, by the equations (2.15)–(2.17), it can be verified that

$$\begin{aligned}
\Xi_1 &= \frac{1}{2}\Big(f(x_1+th_1+\|x_2+th_2\|) + f(x_1+th_1-\|x_2+th_2\|)\Big) \\
&= b^{(0)} + t\left(b^{(1)}h_1 + c^{(1)}\alpha\right) + \frac{1}{2}t^2\left(a^{(1)}\beta + b^{(2)}(h_1^2+\alpha^2) + 2c^{(2)}h_1\alpha\right) + o(t^2) \\
&= b^{(0)} + t\left(b^{(1)}h_1 + c^{(1)}h_2^T\frac{x_2}{\|x_2\|}\right) + \frac{1}{2}t^2h^TA_1(x)h + o(t^2),
\end{aligned}$$

where

$$A_1(x) = \begin{bmatrix} b^{(2)} & c^{(2)}\dfrac{x_2^T}{\|x_2\|} \\ c^{(2)}\dfrac{x_2}{\|x_2\|} & a^{(1)}I + \left(b^{(2)} - a^{(1)}\right)\dfrac{x_2x_2^T}{\|x_2\|^2} \end{bmatrix}. \tag{2.18}$$

Note that in the above expression for Ξ_1, $b^{(0)}$ is exactly the first component of $f^{\mathrm{soc}}(x)$ and $(b^{(1)}h_1 + c^{(1)}h_2^T\frac{x_2}{\|x_2\|})$ is the first component of $\nabla f^{\mathrm{soc}}(x)h$. Using the same techniques again,

$$\begin{aligned} & \frac{1}{2}\big(f(x_1+th_1+\|x_2+th_2\|) - f(x_1+th_1-\|x_2+th_2\|)\big) \\ = \ & c^{(0)} + t\left(c^{(1)}h_1 + b^{(1)}\alpha\right) + \frac{1}{2}t^2\left(b^{(1)}\frac{\beta}{\|x_2\|} + c^{(2)}(h_1^2+\alpha^2) + 2b^{(2)}h_1\alpha\right) + o(t^2) \\ = \ & c^{(0)} + t\left(c^{(1)}h_1 + b^{(1)}\alpha\right) + \frac{1}{2}t^2h^TB(x)h + o(t^2), \end{aligned} \tag{2.19}$$

where

$$B(x) = \begin{bmatrix} c^{(2)} & b^{(2)}\dfrac{x_2^T}{\|x_2\|} \\ b^{(2)}\dfrac{x_2}{\|x_2\|} & c^{(2)}I + \left(\dfrac{b^{(1)}}{\|x_2\|} - c^{(2)}\right)M_{x_2} \end{bmatrix}. \tag{2.20}$$

Using equations (2.19) and (2.14), we obtain that

$$\begin{aligned} \Xi_2 &= \frac{1}{2}\Big(f(x_1+th_1+\|x_2+th_2\|) - f(x_1+th_1-\|x_2+th_2\|)\Big)\frac{x_2+th_2}{\|x_2+th_2\|} \\ &= c^{(0)}\frac{x_2}{\|x_2\|} + t\left(\frac{x_2}{\|x_2\|}(c^{(1)}h_1+b^{(1)}\alpha) + c^{(0)}M_{x_2}\frac{h_2}{\|x_2\|}\right) + \frac{1}{2}t^2W + o(t^2), \end{aligned}$$

where

$$\begin{aligned} W &= \frac{x_2}{\|x_2\|}h^TB(x)h + 2M_{x_2}\frac{h_2}{\|x_2\|}\left(c^{(1)}h_1 + b^{(1)}\alpha\right) \\ &\quad + c^{(0)}\left(3\frac{h_2^Tx_2x_2^Th_2}{\|x_2\|^4}\frac{x_2}{\|x_2\|} - \frac{\|h_2\|^2}{\|x_2\|^2}\frac{x_2}{\|x_2\|} - 2\frac{h_2h_2^T}{\|x_2\|^2}\frac{x_2}{\|x_2\|}\right). \end{aligned}$$

Now we denote

$$\begin{aligned} d &:= \frac{b^{(1)} - a^{(0)}}{\|x_2\|} = \frac{2(b^{(1)} - a^{(0)})}{\lambda_2 - \lambda_1}, \qquad U := h^TC(x)h \\ V &:= 2\frac{c^{(1)}h_1 + b^{(1)}\alpha}{\|x_2\|} - c^{(0)}2\frac{x_2^Th_2}{\|x_2\|^3} = 2a^{(1)}h_1 + 2d\frac{x_2^Th_2}{\|x_2\|}, \end{aligned}$$

where

$$C(x) := \begin{bmatrix} c^{(2)} & (b^{(2)} - a^{(1)})\dfrac{x_2^T}{\|x_2\|} \\ (b^{(2)} - a^{(1)})\dfrac{x_2}{\|x_2\|} & dI + \left(c^{(2)} - 3d\right)\dfrac{x_2x_2^T}{\|x_2\|^2} \end{bmatrix}. \tag{2.21}$$

Then U can be further recast as

$$U = h^T B(x)h + c^{(0)} 3\frac{h_2^T x_2 x_2^T h_2}{\|x_2\|^4} - c^{(0)}\frac{\|h_2\|^2}{\|x_2\|^2} - 2\frac{x_2^T h_2}{\|x_2\|^2}(c^{(1)}h_1 + b^{(1)}\alpha).$$

Consequently,

$$W = \frac{x_2}{\|x_2\|}U + h_2 V.$$

We next consider the case where $x_2 = 0$ and $x_2 + th_2 \neq 0$. By definition (1.8),

$$\begin{aligned} f^{\mathrm{soc}}(x+th) &= \frac{f(x_1 + t(h_1 - \|h_2\|))}{2}\begin{bmatrix} 1 \\ -\dfrac{h_2}{\|h_2\|}\end{bmatrix} + \frac{f(x_1 + t(h_1 + \|h_2\|))}{2}\begin{bmatrix} 1 \\ \dfrac{h_2}{\|h_2\|}\end{bmatrix} \\ &= \begin{bmatrix} \dfrac{f(x_1 + t(h_1 - \|h_2\|)) + f(x_1 + t(h_1 + \|h_2\|))}{2} \\ \dfrac{f(x_1 + t(h_1 + \|h_2\|)) - f(x_1 + t(h_1 - \|h_2\|))}{2}\dfrac{h_2}{\|h_2\|}\end{bmatrix}. \end{aligned}$$

Using the Taylor expansion of f at x_1, we can obtain that

$$\begin{aligned} & \frac{1}{2}\Big[f(x_1 + t(h_1 - \|h_2\|)) + f(x_1 + t(h_1 + \|h_2\|))\Big] \\ = \ & f(x_1) + tf^{(1)}(x_1)h_1 + \frac{1}{2}t^2 f^{(2)}(x_1)h^T h + o(t^2), \\ & \frac{1}{2}\Big[f(x_1 + t(h_1 - \|h_2\|)) - f(x_1 + t(h_1 + \|h_2\|))\Big] \\ = \ & tf^{(1)}(x_1)h_2 + \frac{1}{2}t^2 f^{(2)}(x_1)2h_1h_2 + o(t^2). \end{aligned}$$

Therefore,

$$f^{\mathrm{soc}}(x+th) = f^{\mathrm{soc}}(x) + tf^{(1)}(x_1)h + \frac{1}{2}t^2 f^{(2)}(x_1)\begin{bmatrix} h^T h \\ 2h_1h_2 \end{bmatrix}.$$

Thus, under this case, we have that

$$A_1(x) = f^{(2)}(x_1)I, \quad A_i(x) = f^{(2)}(x_1)\begin{bmatrix} 0 & \bar{e}_{i-1}^T \\ \bar{e}_{i-1} & \mathbf{O}\end{bmatrix} \quad i = 2, \cdots, n, \tag{2.22}$$

where $\bar{e}_j \in \mathbb{R}^{n-1}$ is the vector whose j-th component is 1 and the others are 0.

Summing up the above discussions gives the following conclusion.

Proposition 2.12. *Let $f \in C^{(2)}(J)$ with J being an open interval in $\mathbb{R}$ and $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. Then, for any $x \in \mathrm{dom}(f^{\mathrm{soc}})$, $h \in \mathbb{R}^n$ and any sufficiently small $t > 0$, there holds*

$$f^{\mathrm{soc}}(x+th) = f^{\mathrm{soc}}(x) + t\nabla f^{\mathrm{soc}}(x)h + \frac{1}{2}t^2\begin{bmatrix} h^T A_1(x)h \\ h^T A_2(x)h \\ \vdots \\ h^T A_n(x)h \end{bmatrix} + o(t^2),$$

where $\nabla f^{\mathrm{soc}}(x)$ *and* $A_i(x)$ *for* $i = 1, 2, \cdots, n$ *are given by (2.11) and (2.22) if* $x_2 = 0$*; and otherwise* $\nabla f^{\mathrm{soc}}(x)$ *and* $A_1(x)$ *are given by (2.10) and (2.18), respectively, and for* $i \geq 2$,

$$A_i(x) = C(x)\frac{x_{2i}}{\|x_2\|} + B_i(x)$$

where

$$B_i(x) = ve_i^T + e_i v^T, \quad v = \left[a^{(1)} \quad d\frac{x_2^T}{\|x_2\|}\right]^T = \left(a^{(1)}, \ \frac{d}{\|x_2\|}x_2\right).$$

From Proposition 2.11 and Proposition 2.12, the following consequence is obtained.

Proposition 2.13. *Let* $f \in C^{(2)}(J)$ *with* J *being an open interval in* $\mathbb{R}$ *and* $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$*. Then,* f *is SOC-convex if and only if for any* $x \in \mathrm{dom}(f^{\mathrm{soc}})$ *and* $h \in \mathbb{R}^n$*, the vector*

$$\begin{bmatrix} h^T A_1(x)h \\ h^T A_2(x)h \\ \vdots \\ h^T A_n(x)h \end{bmatrix} \in \mathcal{K}^n,$$

where $A_i(x)$ *is given as in (2.22).*

Now we are ready to show our another main result about the characterization of SOC-monotone functions. Two technical lemmas are needed for the proof. The first one is so-called S-Lemma whose proof can be found in [125].

Lemma 2.3. *Let* A, B *be symmetric matrices and* $y^T Ay > 0$ *for some* y*. Then, the implication* $\left[z^T Az \geq 0 \Rightarrow z^T Bz \geq 0\right]$ *is valid if and only if* $B \succeq \lambda A$ *for some* $\lambda \geq 0$.

Lemma 2.4. *Given* $\theta \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^{n-1}$*, and a symmetric matrix* $A \in \mathbb{R}^{n\times n}$*. Let* $\mathcal{B}^{n-1} := \{z \in \mathbb{R}^{n-1} | \ \|z\| \leq 1\}$*. Then, the following hold.*

(a) *For any* $h \in \mathcal{K}^n$, $Ah \in \mathcal{K}^n$ *is equivalent to* $A\begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathcal{K}^n$ *for any* $z \in \mathcal{B}^{n-1}$.

(b) *For any* $z \in \mathcal{B}^{n-1}$, $\theta + \mathbf{a}^T z \geq 0$ *is equivalent to* $\theta \geq \|\mathbf{a}\|$.

(c) *If* $A = \begin{bmatrix} \theta & \mathbf{a}^T \\ \mathbf{a} & H \end{bmatrix}$ *with* H *being an* $(n-1)\times(n-1)$ *symmetric matrix, then for any* $h \in \mathcal{K}^n$, $Ah \in \mathcal{K}^n$ *is equivalent to* $\theta \geq \|\mathbf{a}\|$ *and there exists* $\lambda \geq 0$ *such that the matrix*

$$\begin{bmatrix} \theta^2 - \|\mathbf{a}\|^2 - \lambda & \theta\mathbf{a}^T - \mathbf{a}^T H \\ \theta\mathbf{a} - H^T\mathbf{a} & \mathbf{a}\mathbf{a}^T - H^T H + \lambda I \end{bmatrix} \succeq O.$$

Proof. (a) For any $h \in \mathcal{K}^n$, suppose that $Ah \in \mathcal{K}^n$. Let $h = \begin{bmatrix} 1 \\ z \end{bmatrix}$ where $z \in \mathcal{B}^{n-1}$. Then $h \in \mathcal{K}^n$ and the desired result follows. For the other direction, if $h = 0$, the conclusion is obvious. Now let $h := (h_1, h_2)$ be any nonzero vector in $\mathcal{K}^n$. Then, $h_1 > 0$ and $\|h_2\| \le h_1$. Consequently, $\frac{h_2}{h_1} \in \mathcal{B}^{n-1}$ and $A \begin{bmatrix} 1 \\ \frac{h_2}{h_1} \end{bmatrix} \in \mathcal{K}^n$. Since $\mathcal{K}^n$ is a cone, we have

$$h_1 A \begin{bmatrix} 1 \\ \frac{h_2}{h_1} \end{bmatrix} = Ah \in \mathcal{K}^n.$$

(b) For $z \in \mathcal{B}^{n-1}$, suppose $\theta + \mathbf{a}^T z \ge 0$. If $\mathbf{a} = 0$, then the result is clear since $\theta \ge 0$. If $\mathbf{a} \neq 0$, let $z := -\frac{\mathbf{a}}{\|\mathbf{a}\|}$. Clearly, $z \in \mathcal{B}^{n-1}$ and hence $\theta + \frac{-\mathbf{a}^T \mathbf{a}}{\|\mathbf{a}\|} \ge 0$ which gives $\theta - \|\mathbf{a}\| \ge 0$. For the other direction, the result follows from the Cauchy Schwarz Inequality:

$$\theta + \mathbf{a}^T z \ge \theta - \|\mathbf{a}\| \cdot \|z\| \ge \theta - \|\mathbf{a}\| \ge 0.$$

(c) From part(a), $Ah \in \mathcal{K}^n$ for any $h \in \mathcal{K}^n$ is equivalent to $A \begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathcal{K}^n$ for any $z \in \mathcal{B}^{n-1}$. Notice that

$$A \begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} \theta & \mathbf{a}^T \\ \mathbf{a} & H \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} \theta + \mathbf{a}^T z \\ \mathbf{a} + Hz \end{bmatrix}.$$

Then, $Ah \in \mathcal{K}^n$ for any $h \in \mathcal{K}^n$ is equivalent to the following two things:

$$\theta + \mathbf{a}^T z \ge 0 \quad \text{for any} \quad z \in \mathcal{B}^{n-1} \tag{2.23}$$

and

$$(\mathbf{a} + Hz)^T (\mathbf{a} + Hz) \le (\theta + \mathbf{a}^T z)^2, \quad \text{for any} \quad z \in \mathcal{B}^{n-1}. \tag{2.24}$$

By part(b), (2.23) is equivalent to $\theta \ge \|\mathbf{a}\|$. Now, we write the expression of (2.24) as below:

$$z^T \left(\mathbf{a}\mathbf{a}^T - H^T H \right) z + 2 \left(\theta \mathbf{a}^T - \mathbf{a}^T H \right) z + \theta^2 - \mathbf{a}^T \mathbf{a} \ge 0, \quad \text{for any} \quad z \in \mathcal{B}^{n-1},$$

which can be further simplified as

$$\begin{bmatrix} 1 & z^T \end{bmatrix} \begin{bmatrix} \theta^2 - \|\mathbf{a}\|^2 & \theta \mathbf{a}^T - \mathbf{a}^T H \\ \theta \mathbf{a} - H^T \mathbf{a} & \mathbf{a}\mathbf{a}^T - H^T H \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} \ge 0, \quad \text{for any} \quad z \in \mathcal{B}^{n-1}.$$

Observe that $z \in \mathcal{B}^{n-1}$ is the same as

$$\begin{bmatrix} 1 & z^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} \ge 0.$$

Thus, by applying the S-Lemma (Lemma 2.3), there exists $\lambda \ge 0$ such that

$$\begin{bmatrix} \theta^2 - \|\mathbf{a}\|^2 & \theta \mathbf{a}^T - \mathbf{a}^T H \\ \theta \mathbf{a} - H^T \mathbf{a} & \mathbf{a}\mathbf{a}^T - H^T H \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \succeq O.$$

This completes the proof of part(c). $\square$

Proposition 2.14. *Let $f \in C^{(1)}(J)$ with J being an open interval and $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. Then, the following hold.*

(a) *f is SOC-monotone of order 2 if and only if $f'(\tau) \ge 0$ for any $\tau \in J$.*

(b) *f is SOC-monotone of order $n \ge 3$ if and only if the 2×2 matrix*

$$\begin{bmatrix} f^{(1)}(t_1) & \dfrac{f(t_2)-f(t_1)}{t_2-t_1} \\ \dfrac{f(t_2)-f(t_1)}{t_2-t_1} & f^{(1)}(t_2) \end{bmatrix} \succeq O, \quad \forall\, t_1, t_2 \in J.$$

Proof. By the definition of SOC-monotonicity, f is SOC-monotone if and only if

$$f^{\mathrm{soc}}(x+h) - f^{\mathrm{soc}}(x) \in \mathcal{K}^n \tag{2.25}$$

for any $x \in \mathrm{dom}(f^{\mathrm{soc}})$ and $h \in \mathcal{K}^n$ such that $x+h \in \mathrm{dom}(f^{\mathrm{soc}})$. By the first-order Taylor expansion of f^{soc}, i.e.,

$$f^{\mathrm{soc}}(x+h) = f^{\mathrm{soc}}(x) + \nabla f^{\mathrm{soc}}(x+th)h \ \text{ for some } \ t \in (0,1),$$

it is clear that (2.25) is equivalent to $\nabla f^{\mathrm{soc}}(x+th)h \in \mathcal{K}^n$ for any $x \in \mathrm{dom}(f^{\mathrm{soc}})$ and $h \in \mathcal{K}^n$ such that $x+h \in \mathrm{dom}(f^{\mathrm{soc}})$, and some $t \in (0,1)$. Let $y := x+th = \mu_1 v^{(1)} + \mu_2 v^{(2)}$ for such x, h and t. We next proceed the arguments by the two cases of $y_2 \neq 0$ and $y_2 = 0$.

Case (1): $y_2 \neq 0$. Under this case, we notice that

$$\nabla f^{\mathrm{soc}}(y) = \begin{bmatrix} \theta & \mathbf{a}^T \\ \mathbf{a} & H \end{bmatrix},$$

where

$$\theta = \tilde{b}^{(1)}, \quad \mathbf{a} = \tilde{c}^{(1)} \frac{y_2}{\|y_2\|}, \quad \text{and } H = \tilde{a}^{(0)} I + (\tilde{b}^{(1)} - \tilde{a}^{(0)}) \frac{y_2 y_2^T}{\|y_2\|^2},$$

with

$$\tilde{a}^{(0)} = \frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1}, \quad \tilde{b}^{(1)} = \frac{f'(\mu_2)+f'(\mu_1)}{2}, \quad \tilde{c}^{(1)} = \frac{f'(\mu_2)-f'(\mu_1)}{2}.$$

In addition, we also observe that

$$\theta^2 - \|\mathbf{a}\|^2 = (\tilde{b}^{(1)})^2 - (\tilde{c}^{(1)})^2, \qquad \theta \mathbf{a}^T - \mathbf{a}^T H = 0$$

and

$$\mathbf{a}\mathbf{a}^T - H^T H = -(\tilde{a}^{(0)})^2 I + \left((\tilde{c}^{(1)})^2 - (\tilde{b}^{(1)})^2 + (\tilde{a}^{(0)})^2 \right) \frac{y_2 y_2^T}{\|y_2\|^2}.$$

Thus, by Lemma 2.4, f is SOC-monotone if and only if

(i) $\tilde{b}^{(1)} \ge |\tilde{c}^{(1)}|$;

(ii) and there exists $\lambda \geq 0$ such that the matrix

$$\begin{bmatrix} (\tilde{b}^{(1)})^2 - (\tilde{c}^{(1)})^2 - \lambda & 0 \\ 0 & (\lambda - (\tilde{a}^{(0)})^2)I + \left((\tilde{c}^{(1)})^2 - (\tilde{b}^{(1)})^2 + (\tilde{a}^{(0)})^2\right)\dfrac{y_2y_2^T}{\|y_2\|^2} \end{bmatrix} \succeq O.$$

When $n = 2$, (i) together with (ii) is equivalent to saying that $f'(\mu_1) \geq 0$ and $f'(\mu_2) \geq 0$. Then we conclude that f is SOC-monotone if and only if $f'(\tau) \geq 0$ for any $\tau \in J$.

When $n \geq 3$, (ii) is equivalent to saying that $(\tilde{b}^{(1)})^2 - (\tilde{c}^{(1)})^2 - \lambda \geq 0$ and $\lambda - (\tilde{a}^{(0)})^2 \geq 0$, i.e., $(\tilde{b}^{(1)})^2 - (\tilde{c}^{(1)})^2 \geq (\tilde{a}^{(0)})^2$. Therefore, (i) together with (ii) is equivalent to

$$\begin{bmatrix} f^{(1)}(\mu_1) & \dfrac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} \\ \dfrac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} & f^{(1)}(\mu_2) \end{bmatrix} \succeq O$$

for any $x \in \mathbb{R}^n, h \in \mathcal{K}^n$ such that $x + h \in \text{dom}f^{\text{soc}}$, and some $t \in (0, 1)$. Thus, we conclude that f is SOC-monotone if and only if

$$\begin{bmatrix} f^{(1)}(t_1) & \dfrac{f(t_2) - f(t_1)}{t_2 - t_1} \\ \dfrac{f(t_2) - f(t_1)}{t_2 - t_1} & f^{(1)}(t_2) \end{bmatrix} \succeq O \quad \text{for all } t_1, t_2 \in J.$$

Case (2): $y_2 = 0$. Now we have $\mu_1 = \mu_2$ and $\nabla f^{\text{soc}}(y) = f^{(1)}(\mu_1)I = f^{(1)}(\mu_2)I$. Hence, f is SOC-monotone is equivalent to $f^{(1)}(\mu_1) \geq 0$, which is also equivalent to

$$\begin{bmatrix} f^{(1)}(\mu_1) & \dfrac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} \\ \dfrac{f(\mu_2) - f(\mu_1)}{\mu_2 - \mu_1} & f^{(1)}(\mu_2) \end{bmatrix} \succeq O$$

since $f^{(1)}(\mu_1) = f^{(1)}(\mu_2)$ and $\frac{f(\mu_2)-f(\mu_1)}{\mu_2-\mu_1} = f^{(1)}(\mu_1) = f^{(1)}(\mu_2)$ by the Taylor formula and $\mu_1 = \mu_2$. Thus, similar to Case (1), the conclusion also holds under this case. $\square$

The SOC-convexity and SOC-monotonicity are also connected to their counterparts, matrix-convexity and matrix-monotonicity. Before illustrating their relations, we briefly recall definitions of matrix-convexity and matrix-monotonicity.

Definition 2.2. *Let $\mathbb{M}_n^{sa}$ denote $n \times n$ self-adjoint complex matrices, $\sigma(A)$ be the spectrum of a matrix A, and $J \subseteq \mathbb{R}$ be an interval.*

(a) *A function $f : J \to \mathbb{R}$ is called matrix monotone of degree n or n-matrix monotone if, for every $A, B \in \mathbb{M}_n^{sa}$ with $\sigma(A) \subseteq J$ and $\sigma(B) \subseteq J$, it holds that*

$$A \preceq B \Longrightarrow f(A) \preceq f(B).$$

(b) *A function $f : J \to \mathbb{R}$ is called operator monotone or matrix monotone if it is n-matrix monotone for all $n \in \mathbb{N}$.*

(c) *A function $f : J \to \mathbb{R}$ is called matrix convex of degree n or n-matrix convex if, for every $A, B \in \mathbb{M}_n^{sa}$ with $\sigma(A) \subseteq J$ and $\sigma(B) \subseteq J$, it holds that*

$$f((1-\lambda)A + \lambda B) \preceq (1-\lambda)f(A) + \lambda f(B).$$

(d) *A function $f : J \to \mathbb{R}$ is called operator convex or matrix convex if it is n-matrix convex for all $n \in \mathbb{N}$.*

(e) *A function $f : J \to \mathbb{R}$ is called matrix concave of degree n or n-matrix concave if $-f$ is n-matrix convex.*

(f) *A function $f : J \to \mathbb{R}$ is called operator concave or matrix concave if it is n-matrix concave for all $n \in \mathbb{N}$.*

In fact, from Proposition 2.14 and [76, Theorem 6.6.36], we immediately have the following consequences.

Proposition 2.15. *Let $f \in C^{(1)}(J)$ with J being an open interval in $\mathbb{R}$. Then, the following hold.*

(a) *f is SOC-monotone of order $n \geq 3$ if and only if it is 2-matrix monotone, and f is SOC-monotone of order $n \leq 2$ if it is 2-matrix monotone.*

(b) *Suppose that $n \geq 3$ and f is SOC-monotone of order n. Then, $f'(t_0) = 0$ for some $t_0 \in J$ if and only if $f(\cdot)$ is a constant function on J.*

We illustrate a few examples by using either Proposition 2.14 or Proposition 2.15.

Example 2.9. *Let $f : (0,\infty) \to \mathbb{R}$ be $f(t) = \ln t$. Then, $f(t)$ is SOC-monotone on $(0,\infty)$.*

Solution. To see this, it needs to verify that the 2×2 matrix

$$\begin{bmatrix} f^{(1)}(t_1) & \dfrac{f(t_2)-f(t_1)}{t_2-t_1} \\ \dfrac{f(t_2)-f(t_1)}{t_2-t_1} & f^{(1)}(t_2) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{t_1} & \dfrac{\ln(t_2)-\ln(t_1)}{t_2-t_1} \\ \dfrac{\ln(t_2)-\ln(t_1)}{t_2-t_1} & \dfrac{1}{t_2} \end{bmatrix}$$

is positive semidefinite for all $t_1, t_2 \in (0,\infty)$. ■

Example 2.10. **(a)** *For any fixed $\sigma \in \mathbb{R}$, the function $f(t) = \frac{1}{\sigma - t}$ is SOC-monotone on (σ, ∞).*

(b) *For any fixed* $\sigma \in \mathbb{R}$*, the function* $f(t) = \sqrt{t-\sigma}$ *is SOC-monotone on* $[\sigma, \infty)$*.*

(c) *For any fixed* $\sigma \in \mathbb{R}$*, the function* $f(t) = \ln(t-\sigma)$ *is SOC-monotone on* (σ, ∞)*.*

(d) *For any fixed* $\sigma \geq 0$*, the function* $f(t) = \frac{t}{t+\sigma}$ *is SOC-monotone on* $(-\sigma, \infty)$*.*

Solution. (a) For any $t_1, t_2 \in (\sigma, \infty)$, it is clear to see that

$$\begin{bmatrix} \dfrac{1}{(\sigma-t_1)^2} & \dfrac{1}{(\sigma-t_2)(\sigma-t_1)} \\ \dfrac{1}{(\sigma-t_2)(\sigma-t_1)} & \dfrac{1}{(\sigma-t_2)^2} \end{bmatrix} \succeq O.$$

Then, applying Proposition 2.14 yields the desired result.

(b) If $x \succeq_{\mathcal{K}^n} \sigma e$, then $(x-\sigma e)^{1/2} \succeq_{\mathcal{K}^n} 0$. Thus, by Proposition 2.14, it suffices to show

$$\begin{bmatrix} \dfrac{1}{2\sqrt{t_1-\sigma}} & \dfrac{\sqrt{t_2-\sigma}-\sqrt{t_1-\sigma}}{t_2-t_1} \\ \dfrac{\sqrt{t_2-\sigma}-\sqrt{t_1-\sigma}}{t_2-t_1} & \dfrac{1}{2\sqrt{t_2-\sigma}} \end{bmatrix} \succeq O \quad \text{for any} \quad t_1, t_2 > 0,$$

which is equivalent to proving that

$$\frac{1}{4\sqrt{t_1-\sigma}\sqrt{t_2-\sigma}} - \frac{1}{(\sqrt{t_2-\sigma}+\sqrt{t_1-\sigma})^2} \geq 0.$$

This inequality holds by $4\sqrt{t_1-\sigma}\sqrt{t_2-\sigma} \leq (\sqrt{t_2-\sigma}+\sqrt{t_1-\sigma})^2$ for any $t_1, t_2 \in (\sigma, \infty)$.

(c) By Proposition 2.14, it suffices to prove that for any $t_1, t_2 \in (\sigma, \infty)$,

$$\begin{bmatrix} \dfrac{1}{(t_1-\sigma)} & \dfrac{1}{(t_2-t_1)} \ln\left(\dfrac{t_2-\sigma}{t_1-\sigma}\right) \\ \dfrac{1}{(t_2-t_1)} \ln\left(\dfrac{t_2-\sigma}{t_1-\sigma}\right) & \dfrac{1}{(t_2-\sigma)} \end{bmatrix} \succeq O,$$

which is equivalent to showing that

$$\frac{1}{(t_1-\sigma)(t_2-\sigma)} - \left[\frac{1}{(t_2-t_1)} \ln\left(\frac{t_2-\sigma}{t_1-\sigma}\right)\right]^2 \geq 0.$$

Notice that $\ln t \leq t-1$ $(t > 0)$, and hence it is easy to verify that

$$\left[\frac{1}{(t_2-t_1)} \ln\left(\frac{t_2-\sigma}{t_1-\sigma}\right)\right]^2 \leq \frac{1}{(t_1-\sigma)(t_2-\sigma)}.$$

Consequently, the desired result follows.

(d) Since for any fixed $\sigma \geq 0$ and any $t_1, t_2 \in (-\sigma, \infty)$, there holds that

$$\begin{bmatrix} \frac{\sigma}{(\sigma+t_1)^2} & \frac{\sigma}{(\sigma+t_2)(\sigma+t_1)} \\ \frac{\sigma}{(\sigma+t_2)(\sigma+t_1)} & \frac{\sigma}{(\sigma+t_2)^2} \end{bmatrix} \succeq O,$$

we immediately obtain the desired result from Proposition 2.14. ■

We point out that the SOC-monotonicity of order 2 does not imply the 2-matrix monotonicity. For example, $f(t) = t^2$ is SOC-monotone of order 2 on $(0, \infty)$ by Example 2.2(a), but by [76, Theorem 6.6.36] we can verify that it is not 2-matrix monotone. Proposition 2.15(a) indicates that a continuously differentiable function defined on an open interval must be SOC-monotone if it is 2-matrix monotone.

Next, we exploit Peirce decomposition to derive some characterizations for SOC-convex functions. Let $f \in C^{(2)}(J)$ with J being an open interval in $\mathbb{R}$ and $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. For any $x \in \mathrm{dom}(f^{\mathrm{soc}})$ and $h \in \mathbb{R}^n$, if $x_2 = 0$, from Proposition 2.12, we have

$$\begin{bmatrix} h^T A_1(x) h \\ h^T A_2(x) h \\ \vdots \\ h^T A_n(x) h \end{bmatrix} = f^{(2)}(x_1) \begin{bmatrix} h^T h \\ 2h_1 h_2 \end{bmatrix}.$$

Since $(h^T h, 2h_1h_2) \in \mathcal{K}^n$, from Proposition 2.13, it follows that f is SOC-convex if and only if $f^{(2)}(x_1) \geq 0$. By the arbitrariness of x_1, f is SOC-convex if and only if f is convex on J.

For the case of $x_2 \neq 0$, we let $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$, where $u^{(1)}$ and $u^{(2)}$ are given by (1.4) with $\bar{x}_2 = \frac{x_2}{\|x_2\|}$. Let $u^{(i)} = (0, \upsilon_2^{(i)})$ for $i = 3, \cdots, n$, where $\upsilon_2^{(3)}, \cdots, \upsilon_2^{(n)}$ is any orthonormal set of vectors that span the subspace of $\mathbb{R}^{n-2}$ orthogonal to x_2. It is easy to verify that the vectors $u^{(1)}, u^{(2)}, u^{(3)}, \cdots, u^{(n)}$ are linearly independent. Hence, for any given $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, there exists μ_i, $i = 1, 2, \cdots, n$ such that

$$h = \mu_1 \sqrt{2}\, u^{(1)} + \mu_2 \sqrt{2}\, u^{(2)} + \sum_{i=3}^{n} \mu_i\, u^{(i)}.$$

From (2.18), we can verify that $b^{(2)} + c^{(2)}$ and $b^{(2)} - c^{(2)}$ are the eigenvalues of $A_1(x)$ with $u^{(2)}$ and $u^{(1)}$ being the corresponding eigenvectors, and $a^{(1)}$ is the eigenvalue of multiplicity $n-2$ with $u^{(i)} = (0, \upsilon_2^{(i)})$ for $i = 3, \ldots, n$ being the corresponding eigenvectors. Therefore,

$$\begin{aligned} h^T A_1(x) h &= \mu_1^2(b^{(2)} - c^{(2)}) + \mu_2^2(b^{(2)} + c^{(2)}) + a^{(1)} \sum_{i=3}^{n} \mu_i^2 \\ &= f^{(2)}(\lambda_1)\mu_1^2 + f^{(2)}(\lambda_2)\mu_2^2 + a^{(1)}\mu^2, \end{aligned} \tag{2.26}$$

where

$$\mu^2 = \sum_{i=3}^n \mu_i^2.$$

Similarly, we can verify that $c^{(2)} + b^{(2)} - a^{(1)}$ and $c^{(2)} - b^{(2)} + a^{(1)}$ are the eigenvalues of

$$\begin{bmatrix} c^{(2)} & (b^{(2)} - a^{(1)})\dfrac{x_2^T}{\|x_2\|} \\ (b^{(2)} - a^{(1)})\dfrac{x_2}{\|x_2\|} & dI + \left(c^{(2)} - d\right) \dfrac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}$$

with $u^{(2)}$ and $u^{(1)}$ being the corresponding eigenvectors, and d is the eigenvalue of multiplicity $n-2$ with $u^{(i)} = (0, v_2^{(i)})$ for $i = 3, \cdots, n$ being the corresponding eigenvectors. Notice that $C(x)$ in (2.21) can be decomposed the sum of the above matrix and

$$\begin{bmatrix} 0 & 0 \\ 0 & -2d\,\dfrac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}.$$

Consequently,

$$h^T C(x) h = \mu_1^2(c^{(2)} - b^{(2)} + a^{(1)}) + \mu_2^2(c^{(2)} + b^{(2)} - a^{(1)}) - d(\mu_2 - \mu_1)^2 + d\mu^2. \tag{2.27}$$

In addition, by the definition of $B_i(x)$, it is easy to compute that

$$h^T B_i(x) h = \sqrt{2} h_{2,i-1}(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d)), \tag{2.28}$$

where $h_{2i} = (h_{21}, \ldots, h_{2,n-1})$. From equations (2.26)-(2.28) and the definition of $A_i(x)$ in (2.22), we thus have

$$\begin{aligned}
\sum_{i=2}^n (h^T A_i(x) h)^2 &= [h^T C(x) h]^2 + 2\|h_2\|^2 (\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))^2 \\
&\quad + 2(\mu_2 - \mu_1) h^T C(x) h (\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d)) \\
&= [h^T C(x) h]^2 + 2(\frac{1}{2}(\mu_2 - \mu_1)^2 + \mu^2)(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))^2 \\
&\quad + 2(\mu_2 - \mu_1) h^T C(x) h (\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d)) \\
&= [h^T C(x) h + (\mu_2 - \mu_1)(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))]^2 \\
&\quad + 2\mu^2(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))^2 \\
&= [-f^{(2)}(\lambda_1)\mu_1^2 + f^{(2)}(\lambda_2)\mu_2^2 + d\mu^2]^2 \\
&\quad + 2\mu^2(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))^2.
\end{aligned} \tag{2.29}$$

On the other hand, by Proposition 2.13, f is SOC-convex if and only if

$$A_1(x) \succeq O \quad \text{and} \quad \sum_{i=2}^n (h^T A_i(x) h)^2 \leq (h^T A_1(x) h)^2. \tag{2.30}$$

From (2.26) and (2.29)-(2.40), we have that f is SOC-convex if and only if $A_1(x) \succeq O$ and

$$\begin{aligned}&\left[-f^{(2)}(\lambda_1)\mu_1^2 + f^{(2)}(\lambda_2)\mu_2^2 + d\mu^2\right]^2 + 2\mu^2(\mu_1(a^{(1)} - d) + \mu_2(a^{(1)} + d))^2 \\ &\leq \left[f^{(2)}(\lambda_1)\mu_1^2 + f^{(2)}(\lambda_2)\mu_2^2 + a^{(1)}\mu^2\right]^2. \end{aligned} \tag{2.31}$$

When $n = 2$, it is clear that $\mu = 0$. Then, f is SOC-convex if and only if

$$A_1(x) \succeq O \quad \text{and} \quad f^{(2)}(\lambda_1)f^{(2)}(\lambda_2) \geq 0.$$

From the previous discussions, we know that $b^{(2)} - c^{(2)} = f^{(2)}(\lambda_1)$, $b^{(2)} + c^{(2)} = f^{(2)}(\lambda_2)$ and $a^{(1)} = \frac{f^{(1)}(\lambda_2) - f^{(1)}(\lambda_1)}{\lambda_2 - \lambda_1}$ are all eigenvalues of $A_1(x)$. Thus, f is SOC-convex if and only if

$$f^{(2)}(\lambda_2) \geq 0, \quad f^{(2)}(\lambda_1) \geq 0, \quad f^{(1)}(\lambda_2) \geq f^{(1)}(\lambda_1),$$

which by the arbitrariness of x is equivalent to saying that f is convex on J.

When $n \geq 3$, if $\mu = 0$, then from the discussions above, we know that f is SOC-convex if and only if f is convex. If $\mu \neq 0$, without loss of generality, we assume that $\mu^2 = 1$. Then, the inequality (2.41) above is equivalent to

$$\begin{aligned}&4f^{(2)}(\lambda_1)f^{(2)}(\lambda_2)\mu_1^2\mu_2^2 + (a^{(1)})^2 - d^2 \\ &+2f^{(2)}(\lambda_2)\mu_2^2(a^{(1)} - d) + 2f^{(2)}(\lambda_1)\mu_1^2(a^{(1)} + d) \\ &-2\Big(\mu_1^2(a^{(1)} - d)^2 + \mu_2^2(a^{(1)} + d)^2 + 2\mu_1\mu_2((a^{(1)})^2 - d^2)\Big) \\ &\geq 0 \ \text{ for any } \mu_1, \mu_2. \end{aligned} \tag{2.32}$$

Now we show that $A_1(x) \succeq O$ and (2.32) holds if and only if f is convex on J and

$$f^{(2)}(\lambda_1)(a^{(1)} + d) \;\geq\; (a^{(1)} - d)^2, \tag{2.33}$$

$$f^{(2)}(\lambda_2)(a^{(1)} - d) \;\geq\; (a^{(1)} + d)^2. \tag{2.34}$$

Indeed, if f is convex on J, then by the discussions above $A_1(x) \succeq O$ clearly holds. If the inequalities (2.33) and (2.34) hold, then by the convexity of f we have $a^{(1)} \geq |d|$. If $\mu_1\mu_2 \leq 0$, then we readily have the inequality (2.32). If $\mu_1\mu_2 > 0$, then using $a^{(1)} \geq |d|$ yields that

$$f^{(2)}(\lambda_1)f^{(2)}(\lambda_2)\mu_1^2\mu_2^2 \geq (a^{(1)})^2 - d^2.$$

Combining with equations (2.33) and (2.34) thus leads to the inequality (2.32). On the other hand, if $A_1(x) \succeq O$, then f must be convex on J by the discussions above, whereas if the inequality (2.32) holds for any μ_1, μ_2, then by letting $\mu_1 = \mu_2 = 0$ yields that

$$a^{(1)} \geq |d|. \tag{2.35}$$

Using the inequality (2.35) and letting $\mu_1 = 0$ in (2.32) then yields (2.33), whereas using (2.35) and letting $\mu_2 = 0$ in (2.32) leads to (2.34). Thus, when $n \geq 3$, f is SOC-convex if and only if f is convex on J and (2.33) and (2.34) hold. We notice that (2.33) and (2.34) are equivalent to

$$\begin{aligned} & \frac{1}{2} f^{(2)}(\lambda_1) \frac{[f(\lambda_1) - f(\lambda_2) + f^{(1)}(\lambda_2)(\lambda_2 - \lambda_1)]}{(\lambda_2 - \lambda_1)^2} \\ \geq \; & \frac{[f(\lambda_2) - f(\lambda_1) - f^{(1)}(\lambda_1)(\lambda_2 - \lambda_1)]^2}{(\lambda_2 - \lambda_1)^4} \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} f^{(2)}(\lambda_2) \frac{[f(\lambda_2) - f(\lambda_1) - f^{(1)}(\lambda_1)(\lambda_2 - \lambda_1)]}{(\lambda_2 - \lambda_1)^2} \\ \geq \; & \frac{[f(\lambda_1) - f(\lambda_2) + f^{(1)}(\lambda_2)(\lambda_2 - \lambda_1)]^2}{(\lambda_2 - \lambda_1)^4}. \end{aligned}$$

Therefore, f is SOC-convex if and only if f is convex on J, and

$$\begin{aligned} & \frac{1}{2} f^{(2)}(t_0) \frac{[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t)]}{(t_0 - t)^2} \\ \geq \; & \frac{[f(t) - f(t_0) - f^{(1)}(t_0)(t - t_0)]^2}{(t_0 - t)^4}, \quad \forall\, t_0, t \in J. \end{aligned} \tag{2.36}$$

Summing up the above analysis, we can characterize the SOC-convexity as follows.

Proposition 2.16. *Let $f \in C^{(2)}(J)$ with J being an open interval in* $\mathbb{R}$ *and* $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. *Then, the following hold.*

(a) *f is SOC-convex of order 2 if and only if f is convex.*

(b) *f is SOC-convex of order $n \geq 3$ if and only if f is convex and the inequality (2.36) holds for any $t_0, t \in J$.*

By the formulas of divided differences, it is not hard to verify that f is convex on J and (2.36) holds for any $t_0, t \in J$ if and only if

$$\begin{bmatrix} \triangle^2 f(t_0, t_0, t_0) & \triangle^2 f(t_0, t, t_0) \\ \triangle^2 f(t, t_0, t_0) & \triangle^2 f(t, t, t_0) \end{bmatrix} \succeq O. \tag{2.37}$$

This, together with Proposition 2.16 and [76, Theorem 6.6.52], leads to the following results.

Proposition 2.17. *Let $f \in C^{(2)}(J)$ with J being an open interval in* $\mathbb{R}$ *and* $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. *Then, the following hold.*

(a) *f is SOC-convex of order $n \geq 3$ if and only if it is 2-matrix convex.*

(b) *f is SOC-convex of order $n \le 2$ if it is 2-matrix convex.*

Proposition 2.17 implies that, if f is a twice continuously differentiable function defined on an open interval J and 2-matrix convex, then it must be SOC-convex. Similar to Proposition 2.15(a), when f is SOC-convex of order 2, it may not be 2-matrix convex. For example, $f(t) = t^3$ is SOC-convex of order 2 on $(0, +\infty)$ by Example 2.3(c), but it is easy to verify that (2.37) does not hold for this function, and consequently, f is not 2-matrix convex. Using Proposition 2.17, we may prove that the direction "$\Leftarrow$" of Conjecture 2.2 does not hold in general, although the other direction is true due to Proposition 2.8. Particularly, from Proposition 2.17 and [69, Theorem 2.3], we can establish the following characterizations for SOC-convex functions.

Proposition 2.18. *Let $f \in C^{(4)}(J)$ with J being an open interval in $\mathbb{R}$ and $\mathrm{dom}(f^{\mathrm{soc}}) \subseteq \mathbb{R}^n$. If $f^{(2)}(t) > 0$ for every $t \in J$, then f is SOC-convex of order n with $n \ge 3$ if and only if one of the following conditions holds.*

(a) *For every $t \in J$, the 2×2 matrix*

$$\begin{bmatrix} \frac{f^{(2)}(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} \succeq O.$$

(b) *There is a positive concave function $c(\cdot)$ on I such that $f^{(2)}(t) = c(t)^{-3}$ for every $t \in J$.*

(c) *There holds that*

$$\begin{aligned} & \left(\frac{\left[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t) \right]}{(t_0 - t)^2} \right) \left(\frac{\left[f(t) - f(t_0) - f^{(1)}(t_0)(t - t_0) \right]}{(t_0 - t)^2} \right) \\ \le\ & \frac{1}{4} f^{(2)}(t_0) f^{(2)}(t). \end{aligned} \tag{2.38}$$

Moreover, f is also SOC-convex of order 2 under one of the above conditions.

Proof. We note that f is convex on J. Therefore, by Proposition 2.17, it suffices to prove the following equivalence:

$$(2.36) \iff \text{assertion (a)} \iff \text{assertion (b)} \iff \text{assertion (c)}.$$

Case (1). (2.36) $\Rightarrow$ assertion (a): From the previous discussions, we know that (2.36) is equivalent to (2.33) and (2.34). We expand (2.33) using Taylor's expansion at λ_1 to the forth order and get

$$\frac{3}{4} f^{(2)}(\lambda_1) f^{(4)}(\lambda_1) \ge (f^{(3)}(\lambda_1))^2.$$

We do the same for the inequality (2.34) at λ_2 and get the inequality

$$\frac{3}{4} f^{(2)}(\lambda_2) f^{(4)}(\lambda_2) \geq (f^{(3)}(\lambda_2))^2.$$

The above two inequalities are precisely

$$\frac{3}{4} f^{(2)}(t) f^{(4)}(t) \geq (f^{(3)}(t))^2, \quad \forall t \in J, \tag{2.39}$$

which is clearly equivalent to saying that the 2×2 matrix in (a) is positive semidefinite.

Case (2). assertion (a) $\Rightarrow$ assertion (b): Take $c(t) = [f^{(2)}(t)]^{-1/3}$ for $t \in J$. Then c is a positive function and $f^{(2)}(t) = c(t)^{-3}$. By twice differentiation, we obtain

$$f^{(4)}(t) = 12c(t)^{-5}[c'(t)(t)]^2 - 3c(t)^{-4}c''(t).$$

Substituting the last equality into the matrix in (a) then yields that

$$-\frac{1}{16} c(t)^{-7} c''(t) \geq 0,$$

which, together with $c(t) > 0$ for every $t \in J$, implies that c is concave.

Case (3). assertion (b) $\Rightarrow$ assertion (c): We first prove the following fact: if $f^{(2)}(t)$ is strictly positive for every $t \in J$ and the function $c(t) = \left[f^{(2)}(t)\right]^{-1/3}$ is concave on J, then

$$\frac{[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t)]}{(t_0 - t)^2} \leq \frac{1}{2} f^{(2)}(t_0)^{1/3} f^{(2)}(t)^{2/3}, \quad \forall\, t_0, t \in J. \tag{2.40}$$

Indeed, using the concavity of the function c, it follows that

$$\begin{aligned}
\frac{[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t)]}{(t_0 - t)^2} &= \int_0^1 \int_0^{u_1} f^{(2)}\left[t + u_2(t_0 - t)\right] du_2 du_1 \\
&= \int_0^1 \int_0^{u_1} c\left((1 - u_2)t + u_2 t_0)\right)^{-3} du_2 du_1 \\
&\leq \int_0^1 \int_0^{u_1} \left((1 - u_2)c(t) + u_2 c(t_0)\right)^{-3} du_2 du_1.
\end{aligned}$$

Notice that $g(t) = 1/t$ $(t > 0)$ has the second-order derivative $g^{(2)}(t) = 2/t^3$. Hence,

$$\begin{aligned}
\frac{[f(t_0) - f(t) - f^{(1)}(t)(t_0 - t)]}{(t_0 - t)^2} &\leq \frac{1}{2} \int_0^1 \int_0^{u_1} g^{(2)}\left((1 - u_2)c(t) + u_2 c(t_0)\right) du_2 du_1 \\
&= \frac{1}{2} \left(\frac{g(c(t_0)) - g(c(t))}{(c(t_0) - c(t))^2} - \frac{g^{(1)}(c(t))}{c(t_0) - c(t)} \right) \\
&= \frac{1}{2c(t_0)c(t)c(t)} \\
&= \frac{1}{2} f^{(2)}(t_0)^{1/3} f^{(2)}(t)^{2/3},
\end{aligned}$$

which implies the inequality (2.40). Now exchanging t_0 with t in (2.40), we obtain

$$\frac{[f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)]}{(t_0-t)^2} \le \frac{1}{2} f^{(2)}(t)^{1/3} f^{(2)}(t_0)^{2/3}, \quad \forall\, t, t_0 \in J. \tag{2.41}$$

Since f is convex on J by the given assumption, the left hand sides of the inequalities (2.40) and (2.41) are nonnegative, and their product satisfies the inequality of (2.38).

Case (4). assertion (c) $\Rightarrow$ (2.36): We introduce a function $F : J \to \mathbb{R}$ defined by

$$F(t) = \frac{1}{2} f^{(2)}(t_0)[f(t_0)-f(t)-f^{(1)}(t)(t_0-t)] - \frac{[f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)]^2}{(t_0-t)^2}$$

if $t \neq t_0$, and otherwise $F(t_0)=0$. We next prove that F is nonnegative on J. It is easy to verify that such $F(t)$ is differentiable on J, and moreover,

$$\begin{aligned}
F'(t) &= \frac{1}{2} f^{(2)}(t_0) f^{(2)}(t)(t-t_0) \\
&\quad -2(t-t_0)^{-2}[f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)](f^{(1)}(t)-f^{(1)}(t_0)) \\
&\quad +2(t-t_0)^{-3}[f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)]^2 \\
&= \frac{1}{2} f^{(2)}(t_0) f^{(2)}(t)(t-t_0) \\
&\quad -2(t-t_0)^{-3}[f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)][f(t_0)-f(t)-f^{(1)}(t)(t_0-t)] \\
&= 2(t-t_0)\left[\frac{1}{4} f^{(2)}(t_0) f^{(2)}(t) - (t-t_0)^{-4}\left(f(t)-f(t_0)-f^{(1)}(t_0)(t-t_0)\right)\right. \\
&\quad \left.\left(f(t_0)-f(t)-f^{(1)}(t)(t_0-t)\right)\right].
\end{aligned}$$

Using the inequality in part(c), we can verify that $F(t)$ has a minimum value 0 at $t=t_0$, and therefore, $F(t)$ is nonnegative on J. This implies the inequality (2.36). □

We demonstrate a few examples by using either Proposition 2.16, Proposition 2.17, or Proposition 2.18.

Example 2.11. *Let $f : \mathbb{R} \to [0,\infty)$ be $f(t)=e^t$. Then,*

(a) *f is SOC-convex of order 2 on $\mathbb{R}$;*

(b) *f is not SOC-convex of order $n \ge 3$ on $\mathbb{R}$.*

Solution. (a) By applying Proposition 2.16(a), it is clear that f is SOC-convex because exponential function is a convex function on $\mathbb{R}$.

(b) As below, it is a counterexample which shows $f(t)=e^t$ is not SOC-convex of order $n \ge 3$. To see this, we compute that

$$\begin{aligned}
e^{[(2,0,-1)+(6,-4,-3)]/2} &= e^{(4,-2,-2)} \\
&= e^4\left(\cosh(2\sqrt{2})\,,\, \sinh(2\sqrt{2})\cdot(-2,-2)/(2\sqrt{2})\right) \\
&\approx (463.48, -325.45, -325.45)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2}\left(e^{(2,0,-1)}+e^{(6,-4,-3)}\right) \\
= & \frac{1}{2}\left[e^2(\cosh(1),0,-\sinh(1))+e^6(\cosh(5),\sinh(5)\cdot(-4,-3)/5)\right] \\
= & (14975,-11974,-8985).
\end{aligned}$$

We see that $14975-463.48=14511.52$, but

$$\left\|(-11974,-8985)-(-325.4493,-325.4493)\right\| = 14515 > 14511.52$$

which is a contradiction. ■

Example 2.12. **(a)** *For any fixed $\sigma \in \mathbb{R}$, the function $f(t)=(t-\sigma)^{-r}$ with $r \geq 0$ is SOC-convex on (σ,∞) if and only if $0 \leq r \leq 1$.*

(a) *For any fixed $\sigma \in \mathbb{R}$, the function $f(t)=(t-\sigma)^{r}$ with $r \geq 0$ is SOC-convex on $[\sigma,\infty)$ if and only if $1 \leq r \leq 2$, and f is SOC-concave on $[\sigma,\infty)$ if and only if $0 \leq r \leq 1$.*

(c) *For any fixed $\sigma \in \mathbb{R}$, the function $f(t)=\ln(t-\sigma)$ is SOC-concave on (σ,∞).*

(d) *For any fixed $\sigma \geq 0$, the function $f(t)=\frac{t}{t+\sigma}$ is SOC-concave on $(-\sigma,\infty)$.*

Solution. (a) For any fixed $\sigma \in \mathbb{R}$, by a simple computation, we have that

$$\begin{bmatrix} \frac{f^{(2)}(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} = \begin{bmatrix} \frac{r(r+1)(t-\sigma)^{-r-2}}{2} & \frac{r(r+1)(-r-2)(t-\sigma)^{-r-3}}{6} \\ \frac{r(r+1)(-r-2)(t-\sigma)^{-r-3}}{6} & \frac{r(r+1)(r+2)(r+3)(t-\sigma)^{-r-4}}{24} \end{bmatrix}.$$

The sufficient and necessary condition for the above matrix being positive semidefinite is

$$\frac{r^2(r+1)^2(r+2)(r+3)(t-\sigma)^{-2r-6}}{24} - \frac{r^2(r+1)^2(r+2)^2(t-\sigma)^{-2r-6}}{18} \geq 0, \qquad (2.42)$$

which is equivalent to requiring $0 \leq r \leq 1$. By Proposition 2.18, it then follows that f is SOC-convex on $(\sigma,+\infty)$ if and only if $0 \leq r \leq 1$.

(b) For any fixed $\sigma \in \mathbb{R}$, by a simple computation, we have that

$$\begin{bmatrix} \frac{f^{(2)}(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} = \begin{bmatrix} \frac{r(r-1)(t-\sigma)^{r-2}}{2} & \frac{r(r-1)(r-2)(t-\sigma)^{r-3}}{6} \\ \frac{r(r-1)(r-2)(t-\sigma)^{r-3}}{6} & \frac{r(r-1)(r-2)(r-3)(t-\sigma)^{r-4}}{24} \end{bmatrix}.$$

The sufficient and necessary condition for the above matrix being positive semidefinite is

$$r \geq 1 \quad \text{and} \quad \frac{r^2(r-1)^2(r-2)(r-3)(t-\sigma)^{2r-6}}{24} - \frac{r^2(r-1)^2(r-2)^2(t-\sigma)^{2r-6}}{18} \geq 0, \qquad (2.43)$$

whereas the sufficient and necessary condition for it being negative semidefinite is

$$0 \leq r \leq 1 \ \text{ and } \ \frac{r^2(r-1)^2(r-2)(r-3)(t-\sigma)t^{2r-6}}{24} - \frac{r^2(r-1)^2(r-2)^2(t-\sigma)^{2r-6}}{18} \geq 0. \tag{2.44}$$

It is easily shown that (2.43) holds if and only if $1 \leq r \leq 2$, and (2.44) holds if and only if $0 \leq r \leq 1$. By Proposition 2.18, this shows that f is SOC-convex on (σ, ∞) if and only if $1 \leq r \leq 2$, and f is SOC-concave on (σ, ∞) if and only if $0 \leq r \leq 1$. This together with the definition of SOC-convexity yields the desired result.

(c) Notice that for any $t > \sigma$, there always holds that

$$-\begin{bmatrix} \frac{f^{(2)}(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} = \begin{bmatrix} \frac{1}{2(t-\sigma)^2} & -\frac{1}{3(t-\sigma)^3} \\ -\frac{1}{3(t-\sigma)^3} & \frac{1}{4(t-\sigma)^4} \end{bmatrix} \succeq O.$$

Consequently, from Proposition 2.18(a), we conclude that f is SOC-concave on (σ, ∞).

(d) For any $t > -\sigma$, it is easy to compute that

$$-\begin{bmatrix} \frac{f^{(2)}(t)}{2} & \frac{f^{(3)}(t)}{6} \\ \frac{f^{(3)}(t)}{6} & \frac{f^{(4)}(t)}{24} \end{bmatrix} = \begin{bmatrix} \frac{1}{(t+\sigma)^3} & -\frac{1}{(t+\sigma)^4} \\ -\frac{1}{(t+\sigma)^4} & \frac{1}{(t+\sigma)^5} \end{bmatrix} \succeq O.$$

By Proposition 2.18 again, we then have that the function f is SOC-concave on $(-\sigma, \infty)$. ■

2.3 Further characterizations in Hilbert space

In this section, we establish further characterizations in the setting of Hilbert space. The main idea is similar, nonetheless, the approach is slightly different. Let $\mathbb{H}$ be a real Hilbert space of dimension $\dim(\mathbb{H}) \geq 3$ endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Fix a unit vector $e \in \mathbb{H}$ and denote by $\langle e \rangle^\perp$ the orthogonal complementary space of e, i.e., $\langle e \rangle^\perp = \{x \in \mathbb{H} \,|\, \langle x, e \rangle = 0\}$. Then each x can be written as

$$x = x_e + x_0 e \ \text{ for some } x_e \in \langle e \rangle^\perp \text{ and } x_0 \in \mathbb{R}.$$

The second-order cone (SOC) in $\mathbb{H}$, also called the Lorentz cone, is a set defined by

$$K := \left\{ x \in \mathbb{H} \,\middle|\, \langle x, e \rangle \geq \frac{1}{\sqrt{2}} \|x\| \right\} = \{x_e + x_0 e \in \mathbb{H} \,|\, x_0 \geq \|x_e\|\}.$$

We also call this K the second-order cone because it reduces to SOC when $\mathbb{H}$ equals the space $\mathbb{R}^n$. From [53, Section 2], we know that K is a pointed closed convex self-dual cone. Hence, $\mathbb{H}$ becomes a partially ordered space via the relation $\succeq_{\mathcal{K}^n}$. In the sequel, for any $x, y \in \mathbb{H}$, we always write $x \succeq_{\mathcal{K}^n} y$ (respectively, $x \succ_{\mathcal{K}^n} y$) when $x - y \in K$ (respectively, $x - y \in \text{int}K$); and denote $\overline{x}_e$ by the vector $\frac{x_e}{\|x_e\|}$ if $x_e \neq 0$, and otherwise by any unit vector from $\langle e \rangle^{\perp}$.

Likewise, associated with the second-order cone K, each $x = x_e + x_0 e \in \mathbb{H}$ can be decomposed as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)},$$

where $\lambda_i(x) \in \mathbb{R}$ and $u_i(x) \in \mathbb{H}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of x, defined by

$$\lambda_i(x) = x_0 + (-1)^i \|x_e\|, \quad u_x^{(i)} = \frac{1}{2}\big(e + (-1)^i \overline{x}_e\big).$$

Clearly, when $x_e \neq 0$, the spectral factorization of x is unique by definition. In addition, the SOC function is given by

$$f^{\text{soc}}(x) := f(\lambda_1(x)) u_x^{(1)} + f(\lambda_2(x)) u_x^{(2)}, \qquad \forall x \in S.$$

We will not distinguish this decomposition from the earlier spectral decomposition (1.2) given in Chapter 1 since they possess the same properties. Analogous to Property 1.4, there also holds

$$\begin{aligned}
& (\lambda_1(x) - \lambda_1(y))^2 + (\lambda_2(x) - \lambda_2(y))^2 \\
= \;& 2(\|x\|^2 + \|y\|^2 - 2x_0 y_0 - 2\|x_e\|\|y_e\|) \\
\leq \;& 2\left(\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle\right) \\
= \;& 2\|x - y\|^2.
\end{aligned}$$

We may verify that the domain S of f^{soc} is open in $\mathbb{H}$ if and only if J is open in $\mathbb{R}$. Also, S is always convex since, for any $x = x_e + x_0 e$, $y = y_e + y_0 e \in S$ and $\beta \in [0, 1]$,

$$\begin{aligned}
\lambda_1\left[\beta x + (1-\beta)y\right] &= \big(\beta x_0 + (1-\beta)y_0\big) - \|\beta x_e + (1-\beta)y_e\| \geq \min\{\lambda_1(x), \lambda_1(y)\}, \\
\lambda_2\left[\beta x + (1-\beta)y\right] &= \big(\beta x_0 + (1-\beta)y_0\big) + \|\beta x_e + (1-\beta)y_e\| \leq \max\{\lambda_2(x), \lambda_2(y)\},
\end{aligned}$$

which implies that $\beta x + (1-\beta)y \in S$. Thus, $f^{\text{soc}}(\beta x + (1-\beta)y)$ is well defined.

Throughout this section, all differentiability means Fréchet differentiability. If $F : \mathbb{H} \to \mathbb{H}$ is (twice) differentiable at $x \in \mathbb{H}$, we denote by $F'(x)$ ($F''(x)$) the first-order F-derivative (the second-order F-derivative) of F at x. In addition, we use $C^n(J)$ and $C^\infty(J)$ to denote the set of n times and infinite times continuously differentiable real functions on J, respectively. When $f \in C^1(J)$, we denote by $f^{[1]}$ the function on $J \times J$ defined by

$$f^{[1]}(\lambda, \mu) := \begin{cases} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} & \text{if } \lambda \neq \mu, \\ f'(\lambda) & \text{if } \lambda = \mu, \end{cases} \tag{2.45}$$

and when $f \in C^2(J)$, denote by $f^{[2]}$ the function on $J \times J \times J$ defined by

$$f^{[2]}(\tau_1, \tau_2, \tau_3) := \frac{f^{[1]}(\tau_1, \tau_2) - f^{[1]}(\tau_1, \tau_3)}{\tau_2 - \tau_3} \tag{2.46}$$

if τ_1, τ_2, τ_3 are distinct, and for other values of τ_1, τ_2, τ_3, $f^{[2]}$ is defined by continuity; e.g.,

$$f^{[2]}(\tau_1, \tau_1, \tau_3) = \frac{f(\tau_3) - f(\tau_1) - f'(\tau_1)(\tau_3 - \tau_1)}{(\tau_3 - \tau_1)^2}, \quad f^{[2]}(\tau_1, \tau_1, \tau_1) = \frac{1}{2} f''(\tau_1).$$

For a linear operator $\mathcal{L}$ from $\mathbb{H}$ into $\mathbb{H}$, we write $\mathcal{L} \geq 0$ (respectively, $\mathcal{L} > 0$) to mean that $\mathcal{L}$ is positive semidefinite (respectively, positive definite), i.e., $\langle h, \mathcal{L}h \rangle \geq 0$ for any $h \in \mathbb{H}$ (respectively, $\langle h, \mathcal{L}h \rangle > 0$ for any $0 \neq h \in \mathbb{H}$).

Lemma 2.5. *Let $\mathbb{B} := \{z \in \langle e \rangle^{\perp} \mid \|z\| \leq 1\}$. Then, for any given $u \in \langle e \rangle^{\perp}$ with $\|u\| = 1$ and $\theta, \lambda \in \mathbb{R}$, the following results hold.*

(a) *$\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$ if and only if $\theta \geq |\lambda|$.*

(b) *$\theta - \|\lambda z\|^2 \geq (\theta - \lambda^2)\langle u, z \rangle^2$ for any $z \in \mathbb{B}$ if and only if $\theta - \lambda^2 \geq 0$.*

Proof. (a) Suppose that $\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$. If $\lambda = 0$, then $\theta \geq |\lambda|$ clearly holds. If $\lambda \neq 0$, take $z = -\text{sign}(\lambda)u$. Since $\|u\| = 1$, we have $z \in \mathbb{B}$, and consequently, $\theta + \lambda \langle u, z \rangle \geq 0$ reduces to $\theta - |\lambda| \geq 0$. Conversely, if $\theta \geq |\lambda|$, then using the Cauchy-Schwartz Inequality yields $\theta + \lambda \langle u, z \rangle \geq 0$ for any $z \in \mathbb{B}$.

(b) Suppose that $\theta - \|\lambda z\|^2 \geq (\theta - \lambda^2)\langle u, z \rangle^2$ for any $z \in \mathbb{B}$. Then, we must have $\theta - \lambda^2 \geq 0$. If not, for those $z \in \mathbb{B}$ with $\|z\| = 1$ but $\langle u, z \rangle \neq \|u\|\|z\|$, it holds that

$$(\theta - \lambda^2)\langle u, z \rangle^2 > (\theta - \lambda^2)\|u\|^2\|z\|^2 = \theta - \|\lambda z\|^2,$$

which contradicts the given assumption. Conversely, if $\theta - \lambda^2 \geq 0$, the Cauchy-Schwartz inequality implies that $(\theta - \lambda^2)\langle u, z \rangle^2 \leq \theta - \|\lambda z\|^2$ for any $z \in \mathbb{B}$. $\square$

Lemma 2.6. *For any given $a, b, c \in \mathbb{R}$ and $x = x_e + x_0 e$ with $x_e \neq 0$, the inequality*

$$a\left[\|h_e\|^2 - \langle h_e, \overline{x}_e \rangle^2\right] + b\big[h_0 + \langle \overline{x}_e, h_e \rangle\big]^2 + c\big[h_0 - \langle \overline{x}_e, h_e \rangle\big]^2 \geq 0 \tag{2.47}$$

holds for all $h = h_e + h_0 e \in \mathbb{H}$ if and only if $a \geq 0$, $b \geq 0$ and $c \geq 0$.

Proof. Suppose that (2.47) holds for all $h = h_e + h_0 e \in \mathbb{H}$. By letting $h_e = \overline{x}_e$, $h_0 = 1$ and $h_e = -\overline{x}_e$, $h_0 = 1$, respectively, we get $b \geq q0$ and $c \geq 0$ from (2.47). If $a \geq 0$ does not hold, then by taking $h_e = \sqrt{\frac{b+c+1}{|a|}} \frac{z_e}{\|z_e\|}$ with $\langle z_e, x_e \rangle = 0$ and $h_0 = 1$, (2.47) gives a contradiction $-1 \geq 0$. Conversely, if $a \geq 0$, $b \geq 0$ and $c \geq 0$, then (2.47) clearly holds for all $h \in \mathbb{H}$. $\square$

Lemma 2.7. *Let $f \in C^2(J)$ and $u_e \in \langle e\rangle^\perp$ with $\|u_e\| = 1$. For any $h = h_e + h_0 e \in \mathbb{H}$, define*

$$\mu_1(h) := \frac{h_0 - \langle u_e, h_e\rangle}{\sqrt{2}},\ \mu_2(h) := \frac{h_0 + \langle u_e, h_e\rangle}{\sqrt{2}},\ \mu(h) := \sqrt{\|h_e\|^2 - \langle u_e, h_e\rangle^2}.$$

Then, for any given $a, d \in \mathbb{R}$ and $\lambda_1, \lambda_2 \in J$, the following inequality

$$\begin{aligned} &4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 + 2(a-d)f''(\lambda_2)\mu_2(h)^2\mu(h)^2 \\ &+2\,(a+d)\,f''(\lambda_1)\mu_1(h)^2\mu(h)^2 + \left(a^2-d^2\right)\mu(h)^4 \\ &-2\left[(a-d)\,\mu_1(h) + (a+d)\,\mu_2(h)\right]^2\mu(h)^2 \geq 0 \end{aligned} \tag{2.48}$$

holds for all $h = h_e + h_0 e \in \mathbb{H}$ if and only if

$$a^2 - d^2 \geq 0,\ f''(\lambda_2)(a-d) \geq (a+d)^2 \text{ and } f''(\lambda_1)(a+d) \geq (a-d)^2. \tag{2.49}$$

Proof. Suppose that (2.48) holds for all $h = h_e + h_0 e \in \mathbb{H}$. Taking $h_0 = 0$ and $h_e \neq 0$ with $\langle h_e, u_e\rangle = 0$, we have $\mu_1(h) = 0, \mu_2(h) = 0$ and $\mu(h) = \|h_e\| > 0$, and then (2.48) gives $a^2 - d^2 \geq 0$. Taking $h_e \neq 0$ such that $|\langle u_e, h_e\rangle| < \|h_e\|$ and $h_0 = \langle u_e, h_e\rangle \neq 0$, we have $\mu_1(h) = 0, \mu_2(h) = \sqrt{2}h_0$ and $\mu(h) > 0$, and then (2.48) reduces to the following inequality

$$4\left[(a-d)f''(\lambda_2) - (a+d)^2\right]h_0^2 + (a^2-d^2)(\|h_e\|^2 - h_0^2) \geq 0.$$

This implies that $(a-d)f''(\lambda_2) - (a+d)^2 \geq 0$. If not, by letting h_0 be sufficiently close to $\|h_e\|$, the last inequality yields a contradiction. Similarly, taking h with $h_e \neq 0$ satisfying $|\langle u_e, h_e\rangle| < \|h_e\|$ and $h_0 = -\langle u_e, h_e\rangle$, we get $f''(\lambda_1)(a+d) \geq (a-d)^2$ from (2.48).

Next, suppose that (2.49) holds. Then, the inequalities $f''(\lambda_2)(a-d) \geq (a+d)^2$ and $f''(\lambda_1)(a+d) \geq (a-d)^2$ imply that the left-hand side of (2.48) is greater than

$$4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2-d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + \left(a^2-d^2\right)\mu(h)^4,$$

which is obviously nonnegative if $\mu_1(h)\mu_2(h) \leq 0$. Now assume that $\mu_1(h)\mu_2(h) > 0$. If $a^2 - d^2 = 0$, then the last expression is clearly nonnegative, and if $a^2 - d^2 > 0$, then the last two inequalities in (2.49) imply that $f''(\lambda_1)f''(\lambda_2) \geq (a^2-d^2) > 0$, and therefore,

$$\begin{aligned} & 4f''(\lambda_1)f''(\lambda_2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2-d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + \left(a^2-d^2\right)\mu(h)^4 \\ \geq\ & 4(a^2-d^2)\mu_1(h)^2\mu_2(h)^2 - 4(a^2-d^2)\mu_1(h)\mu_2(h)\mu(h)^2 + \left(a^2-d^2\right)\mu(h)^4 \\ =\ & (a^2-d^2)\left[2\mu_1(h)\mu_2(h) - \mu(h)^2\right]^2 \geq 0. \end{aligned}$$

Thus, we prove that inequality (2.48) holds. The proof is complete. □

To proceed, we introduce the regularization of a locally integrable real function. Let φ be a real function of class C^∞ with the following properties: $\varphi \geq 0$, φ is even, the support supp $\varphi = [-1,1]$, and $\int_{\mathbb{R}} \varphi(t)dt = 1$. For each $\varepsilon > 0$, let $\varphi_\varepsilon(t) = \frac{1}{\varepsilon}\varphi(\frac{t}{\varepsilon})$. Then, supp $\varphi_\varepsilon = [-\varepsilon, \varepsilon]$ and φ_ε has all the properties of φ listed above. If f is a locally integrable real function, we define its regularization of order ε as the function

$$f_\varepsilon(s) := \int f(s-t)\varphi_\varepsilon(t)dt = \int f(s-\varepsilon t)\varphi(t)dt. \tag{2.50}$$

Note that f_ε is a C^∞ function for each $\varepsilon > 0$, and $\lim_{\varepsilon\to 0} f_\varepsilon(x) = f(x)$ if f is continuous.

Lemma 2.8. *For any given $f : J \to \mathbb{R}$ with J open, let $f^{\text{soc}} : S \to \mathbb{H}$ be defined by (1.8).*

(a) *f^{soc} is continuous on S if and only if f is continuous on J.*

(b) *f^{soc} is (continuously) differentiable on S if and only if f is (continuously) differentiable on J. Also, when f is differentiable on J, for any $x = x_e + x_0 e \in S$ and $v = v_e + v_0 e \in \mathbb{H}$,*

$$(f^{\text{soc}})'(x)v = \begin{cases} f'(x_0)v & \text{if } x_e = 0; \\ (b_1(x) - a_0(x))\langle \overline{x}_e, v_e\rangle \overline{x}_e + c_1(x)v_0\overline{x}_e & \\ +a_0(x)v_e + b_1(x)v_0 e + c_1(x)\langle \overline{x}_e, v_e\rangle e & \text{if } x_e \neq 0, \end{cases} \tag{2.51}$$

where

$$\begin{aligned} a_0(x) &= \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, \\ b_1(x) &= \frac{f'(\lambda_2(x)) + f'(\lambda_1(x))}{2}, \\ c_1(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{2}. \end{aligned}$$

(c) *If f is differentiable on J, then for any given $x \in S$ and all $v \in \mathbb{H}$,*

$$(f^{\text{soc}})'(x)e = (f')^{\text{soc}}(x) \quad \text{and} \quad \langle e, (f^{\text{soc}})'(x)v\rangle = \langle v, (f')^{\text{soc}}(x)\rangle .$$

(d) *If f' is nonnegative (respectively, positive) on J, then for each $x \in S$,*

$$(f^{\text{soc}})'(x) \geq 0 \quad (\text{respectively}, (f^{\text{soc}})'(x) > 0).$$

Proof. (a) Suppose that f^{soc} is continuous. Let Ω be the set composed of those $x = te$ with $t \in J$. Clearly, $\Omega \subseteq S$, and f^{soc} is continuous on Ω. Noting that $f^{\text{soc}}(x) = f(t)e$ for any $x \in \Omega$, it follows that f is continuous on J. Conversely, if f is continuous on J, then f^{soc} is continuous at any $x = x_e + x_0 e \in S$ with $x_e \neq 0$ since $\lambda_i(x)$ and $u_i(x)$ for $i = 1, 2$ are continuous at such points. Next, let $x = x_e + x_0 e$ be an arbitrary element from S

with $x_e = 0$, and we prove that $f^{\rm soc}$ is continuous at x. Indeed, for any $z = z_e + z_0 e \in S$ sufficiently close to x, it is not hard to verify that

$$\|f^{\rm soc}(z) - f^{\rm soc}(x)\| \leq \frac{|f(\lambda_2(z)) - f(x_0)|}{2} + \frac{|f(\lambda_1(z)) - f(x_0)|}{2} + \frac{|f(\lambda_2(z)) - f(\lambda_1(z))|}{2}.$$

Since f is continuous on J, and $\lambda_1(z), \lambda_2(z) \to x_0$ as $z \to x$, it follows that

$$f(\lambda_1(z)) \to f(x_0) \ \text{ and } \ f(\lambda_2(z)) \to f(x_0) \ \text{ as } \ z \to x.$$

The last two equations imply that $f^{\rm soc}$ is continuous at x.

(b) When $f^{\rm soc}$ is (continuously) differentiable, using the similar arguments as in part(a) can show that f is (continuously) differentiable. Next, assume that f is differentiable. Fix any $x = x_e + x_0 e \in S$. We first consider the case where $x_e \neq 0$. Since $\lambda_i(x)$ for $i = 1, 2$ and $\frac{x_e}{\|x_e\|}$ are continuously differentiable at such x, it follows that $f(\lambda_i(x))$ and $u_i(x)$ are differentiable and continuously differentiable, respectively, at x. Then, $f^{\rm soc}$ is differentiable at such x by the definition of $f^{\rm soc}$. Also, an elementary computation shows that

$$[\lambda_i(x)]'v = \langle v, e\rangle + (-1)^i \frac{\langle x_e, v - \langle v, e\rangle e\rangle}{\|x_e\|} = v_0 + (-1)^i \frac{\langle x_e, v_e\rangle}{\|x_e\|}, \tag{2.52}$$

$$\left(\frac{x_e}{\|x_e\|}\right)' v = \frac{v - \langle v, e\rangle e}{\|x_e\|} - \frac{\langle x_e, v - \langle v, e\rangle e\rangle x_e}{\|x_e\|^3} = \frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e\rangle x_e}{\|x_e\|^3} \tag{2.53}$$

for any $v = v_e + v_0 e \in \mathbb{H}$, and consequently,

$$\begin{aligned} [f(\lambda_i(x))]' v &= f'(\lambda_i(x)) \left[v_0 + (-1)^i \frac{\langle x_e, v_e\rangle}{\|x_e\|}\right], \\ [u_i(x)]' v &= \frac{1}{2}(-1)^i \left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e\rangle x_e}{\|x_e\|^3}\right]. \end{aligned}$$

Together with the definition of $f^{\rm soc}$, we calculate that $(f^{\rm soc})'(x)v$ is equal to

$$\begin{aligned} & \frac{f'(\lambda_1(x))}{2}\left[v_0 - \frac{\langle x_e, v_e\rangle}{\|x_e\|}\right]\left(e - \frac{x_e}{\|x_e\|}\right) - \frac{f(\lambda_1(x))}{2}\left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e\rangle x_e}{\|x_e\|^3}\right] \\ & + \frac{f'(\lambda_2(x))}{2}\left[v_0 + \frac{\langle x_e, v_e\rangle}{\|x_e\|}\right]\left(e + \frac{x_e}{\|x_e\|}\right) + \frac{f(\lambda_2(x))}{2}\left[\frac{v_e}{\|x_e\|} - \frac{\langle x_e, v_e\rangle x_e}{\|x_e\|^3}\right] \\ = \ & b_1(x) v_0 e + c_1(x)\langle \overline{x}_e, v_e\rangle e + c_1(x) v_0 \overline{x}_e + b_1(x)\langle \overline{x}_e, v_e\rangle \overline{x}_e \\ & + a_0(x) v_e - a_0(x)\langle \overline{x}_e, v_e\rangle \overline{x}_e, \end{aligned}$$

where $\lambda_2(x) - \lambda_1(x) = 2\|x_e\|$ is used for the last equality. Thus, we obtain (2.51) for

$x_e \neq 0$. We next consider the case where $x_e = 0$. Under this case, for any $v = v_e + v_0 e \in \mathbb{H}$,

$$\begin{aligned}
& f^{\rm soc}(x+v) - f^{\rm soc}(x) \\
= \ & \frac{f(x_0+v_0-\|v_e\|)}{2}(e-\overline{v}_e) + \frac{f(x_0+v_0+\|v_e\|)}{2}(e+\overline{v}_e) - f(x_0)e \\
= \ & \frac{f'(x_0)(v_0-\|v_e\|)}{2}e + \frac{f'(x_0)(v_0+\|v_e\|)}{2}e \\
& + \frac{f'(x_0)(v_0+\|v_e\|)}{2}\overline{v}_e - \frac{f'(x_0)(v_0-\|v_e\|)}{2}\overline{v}_e + o(\|v\|) \\
= \ & f'(x_0)(v_0 e + \|v_e\|\overline{v}_e) + o(\|v\|),
\end{aligned}$$

where $\overline{v}_e = \frac{v_e}{\|v_e\|}$ if $v_e \neq 0$, and otherwise $\overline{v}_e$ is an arbitrary unit vector from $\langle e \rangle^\perp$. Hence,

$$\left\| f^{\rm soc}(x+v) - f^{\rm soc}(x) - f'(x_0)v \right\| = o(\|v\|).$$

This shows that $f^{\rm soc}$ is differentiable at such x with $(f^{\rm soc})'(x)v = f'(x_0)v$.

Assume that f is continuously differentiable. From (2.51), it is easy to see that $(f^{\rm soc})'(x)$ is continuous at every x with $x_e \neq 0$. We next argue that $(f^{\rm soc})'(x)$ is continuous at every x with $x_e = 0$. Fix any $x = x_0 e$ with $x_0 \in J$. For any $z = z_e + z_0 e$ with $z_e \neq 0$, we have

$$\begin{aligned}
& \|(f^{\rm soc})'(z)v - (f^{\rm soc})'(x)v\| \\
\leq \ & |b_1(z) - a_0(z)|\|v_e\| + |b_1(z) - f'(x_0)||v_0| \\
& + |a_0(z) - f'(x_0)|\|v_e\| + |c_1(z)|(|v_0| + \|v_e\|).
\end{aligned} \tag{2.54}$$

Since f is continuously differentiable on J and $\lambda_2(z) \to x_0$, $\lambda_1(z) \to x_0$ as $z \to x$, we have

$$a_0(z) \to f'(x_0), \quad b_1(z) \to f'(x_0) \quad \text{and} \quad c_1(z) \to 0.$$

Together with equation (2.54), we obtain that $(f^{\rm soc})'(z) \to (f^{\rm soc})'(x)$ as $z \to x$.

(c) The result is direct by the definition of $f^{\rm soc}$ and a simple computation from (2.51).

(d) Suppose that $f'(t) \geq 0$ for all $t \in J$. Fix any $x = x_e + x_0 e \in S$. If $x_e = 0$, the result is direct. It remains to consider the case $x_e \neq 0$. Since $f'(t) \geq 0$ for all $t \in J$, we have $b_1(x) \geq 0$, $b_1(x) - c_1(x) = f'(\lambda_1(x)) \geq 0$, $b_1(x) + c_1(x) = f'(\lambda_2(x)) \geq 0$ and $a_0(x) \geq 0$. From part(b) and the definitions of $b_1(x)$ and $c_1(x)$, it follows that for any $h = h_e + h_0 e \in \mathbb{H}$,

$$\begin{aligned}
\langle h, (f^{\rm soc})'(x)h \rangle &= (b_1(x) - a_0(x))\langle \overline{x}_e, h_e \rangle^2 + 2c_1(x)h_0\langle \overline{x}_e, h_e \rangle + b_1(x)h_0^2 + a_0(x)\|h_e\|^2 \\
&= a_0(x)\left[\|h_e\|^2 - \langle \overline{x}_e, h_e \rangle^2\right] + \frac{1}{2}(b_1(x) - c_1(x))\left[h_0 - \langle \overline{x}_e, h_e \rangle\right]^2 \\
&\quad + \frac{1}{2}(b_1(x) + c_1(x))\left[h_0 + \langle \overline{x}_e, h_e \rangle\right]^2 \geq 0.
\end{aligned}$$

This implies that the operator $(f^{\rm soc})'(x)$ is positive semidefinite. Particularly, if $f'(t) > 0$ for all $t \in J$, we have that $\langle h, (f^{\rm soc})'(x)h \rangle > 0$ for all $h \neq 0$. The proof is complete. □

Lemma 2.8(d) shows that the differential operator $(f^{\mathrm{soc}})'(x)$ corresponding to a differentiable nondecreasing f is positive semidefinite. Therefore, the differential operator $(f^{\mathrm{soc}})'(x)$ associated with a differentiable SOC-monotone function is also positive semidefinite.

Proposition 2.19. *Assume that $f \in C^1(J)$ with J being an open interval in $\mathbb{R}$. Then, f is SOC-monotone if and only if $(f^{\mathrm{soc}})'(x)h \in K$ for any $x \in S$ and $h \in K$.*

Proof. If f is SOC-monotone, then for any $x \in S$, $h \in K$ and $t > 0$, we have

$$f^{\mathrm{soc}}(x+th) - f^{\mathrm{soc}}(x) \succeq_{\mathcal{K}^n} 0,$$

which, by the continuous differentiability of f^{soc} and the closedness of K, implies that

$$(f^{\mathrm{soc}})'(x)h \succeq_{\mathcal{K}^n} 0.$$

Conversely, for any $x, y \in S$ with $x \succeq_{\mathcal{K}^n} y$, from the given assumption we have that

$$f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y) = \int_0^1 (f^{\mathrm{soc}})'(x + t(x-y))(x-y)dt \in K.$$

This shows that $f^{\mathrm{soc}}(x) \succeq_{\mathcal{K}^n} f^{\mathrm{soc}}(y)$, i.e., f is SOC-monotone. The proof is complete. □

Proposition 2.19 shows that the differential operator $(f^{\mathrm{soc}})'(x)$ associated with a differentiable SOC-monotone function f leaves K invariant. If, in addition, the linear operator $(f^{\mathrm{soc}})'(x)$ is bijective, then $(f^{\mathrm{soc}})'(x)$ belongs to the automorphism group of K. Such linear operators are important to study the structure of the cone K (see [62]).

Proposition 2.20. *Assume that $f \in C^1(J)$ with J being an open interval in $\mathbb{R}$. If f is SOC-monotone, then*

(a) *$f^{\mathrm{soc}}(x) \in K$ for any $x \in S$;*

(b) *f^{soc} is a monotone function, that is, $\langle f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y), x-y\rangle \geq 0$ for any $x, y \in S$.*

Proof. Part(a) is direct by using Proposition 2.19 with $h = e$ and Lemma 2.8(c). By part(a), $f'(\tau) \geq 0$ for all $\tau \in J$. Together with Lemma 2.8(d), $(f^{\mathrm{soc}})'(x) \geq 0$ for any $x \in S$. Applying the integral mean-value theorem, it then follows that

$$\langle f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y), x-y\rangle = \int_0^1 \langle x-y, (f^{\mathrm{soc}})'(y + t(x-y))(x-y)\rangle dt \geq 0.$$

This proves the desired result of part (b). The proof is complete. □

Note that the converse of Proposition 2.20(a) is not correct. For example, for the function $f(t) = -t^{-2}$ $(t > 0)$, it is clear that $f^{\mathrm{soc}}(x) \in K$ for any $x \in \mathrm{int}K$, but it is not SOC-monotone by Example 2.13(b). The following proposition provides another sufficient and necessary characterization for differentiable SOC-monotone functions.

Proposition 2.21. *Let $f \in C^1(J)$ with J being an open interval in $\mathbb{R}$. Then, f is SOC-monotone if and only if*

$$\begin{bmatrix} f^{[1]}(\tau_1,\tau_1) & f^{[1]}(\tau_1,\tau_2) \\ f^{[1]}(\tau_2,\tau_1) & f^{[1]}(\tau_2,\tau_2) \end{bmatrix} = \begin{bmatrix} f'(\tau_1) & \frac{f(\tau_2)-f(\tau_1)}{\tau_2-\tau_1} \\ \frac{f(\tau_1)-f(\tau_2)}{\tau_1-\tau_2} & f'(\tau_2) \end{bmatrix} \succeq O, \quad \forall \tau_1,\tau_2 \in J. \tag{2.55}$$

Proof. The equality is direct by the definition of $f^{[1]}$ given as in (2.45). It remains to prove that f is SOC-monotone if and only if the inequality in (2.55) holds for any $\tau_1, \tau_2 \in J$. Assume that f is SOC-monotone. By Proposition 2.19, $(f^{\mathrm{soc}})'(x)h \in K$ for any $x \in S$ and $h \in K$. Fix any $x = x_e + x_0 e \in S$. It suffices to consider the case where $x_e \neq 0$. Since $(f^{\mathrm{soc}})'(x)h \in K$ for any $h \in K$, we particularly have $(f^{\mathrm{soc}})'(x)(z+e) \in K$ for any $z \in \mathbb{B}$, where $\mathbb{B}$ is the set defined in Lemma 2.5. From Lemma 2.8(b), it follows that

$$(f^{\mathrm{soc}})'(x)(z+e) = [(b_1(x)-a_0(x))\langle \overline{x}_e, z\rangle + c_1(x)]\,\overline{x}_e + a_0(x)z + [b_1(x) + c_1(x)\langle \overline{x}_e, z\rangle]\,e.$$

This means that $(f^{\mathrm{soc}})'(x)(z+e) \in K$ for any $z \in \mathbb{B}$ if and only if

$$b_1(x) + c_1(x)\langle \overline{x}_e, z\rangle \geq 0, \tag{2.56}$$

$$[b_1(x) + c_1(x)\langle \overline{x}_e, z\rangle]^2 \geq \left\| \left[(b_1(x)-a_0(x))\langle \overline{x}_e, z\rangle + c_1(x)\right]\overline{x}_e + a_0(x)z \right\|^2. \tag{2.57}$$

By Lemma 2.5(a), we know that (2.56) holds for any $z \in \mathbb{B}$ if and only if $b_1(x) \geq |c_1(x)|$. Since by a simple computation the inequality in (2.57) can be simplified as

$$b_1(x)^2 - c_1(x)^2 - a_0(x)^2\,\|z\|^2 \geq \left[b_1(x)^2 - c_1(x)^2 - a_0(x)^2\right]\langle z, \overline{x}_e\rangle^2,$$

applying Lemma 2.5(b) yields that (2.57) holds for any $z \in \mathbb{B}$ if and only if

$$b_1(x)^2 - c_1(x)^2 - a_0(x)^2 \geq 0.$$

This shows that $(f^{\mathrm{soc}})'(x)(z+e) \in K$ for any $z \in \mathbb{B}$ if and only if

$$b_1(x) \geq |c_1(x)| \;\text{ and }\; b_1(x)^2 - c_1(x)^2 - a_0(x)^2 \geq 0. \tag{2.58}$$

The first condition in (2.58) is equivalent to $b_1(x) \geq 0$, $b_1(x) - c_1(x) \geq 0$ and $b_1(x) + c_1(x) \geq 0$, which, by the expressions of $b_1(x)$ and $c_1(x)$ and the arbitrariness of x, is equivalent to $f'(\tau) \geq 0$ for all $\tau \in J$; whereas the second condition in (2.58) is equivalent to

$$f'(\tau_1)f'(\tau_2) - \left[\frac{f(\tau_2)-f(\tau_1)}{\tau_2-\tau_1}\right]^2 \geq 0, \quad \forall \tau_1,\tau_2 \in J.$$

The two sides show that the inequality in (2.55) holds for all $\tau_1, \tau_2 \in J$.

Conversely, if the inequality in (2.55) holds for all τ_1, $\tau_2 \in J$, then from the arguments above we have $(f^{\mathrm{soc}})'(x)(z+e) \in K$ for any $x = x_e + x_0 e \in S$ and $z \in \mathbb{B}$. This implies that $(f^{\rm soc})'(x)h \in K$ for any $x \in S$ and $h \in K$. By Proposition 2.19, f is SOC-monotone. $\square$

Propositions 2.19 and 2.21 provide the characterizations for continuously differentiable SOC-monotone functions. When f does not belong to $C^1(J)$, one may check the SOC-monotonicity of f by combining the following proposition with Propositions 2.19 and 2.21.

Proposition 2.22. *Let $f : J \to \mathbb{R}$ be a continuous function on the open interval J, and f_ε be its regularization defined by (2.50). Then, f is SOC-monotone if and only if f_ε is SOC-monotone on J_ε for every sufficiently small $\varepsilon > 0$, where $J_\varepsilon := (a+\varepsilon, b-\varepsilon)$ for $J = (a, b)$.*

Proof. Throughout the proof, for every sufficiently small $\varepsilon > 0$, we let S_ε be the set of all $x \in \mathbb{H}$ whose spectral values $\lambda_1(x)$, $\lambda_2(x)$ belong to J_ε. Assume that f_ε is SOC-monotone on J_ε for every sufficiently small $\varepsilon > 0$. Let x, y be arbitrary vectors from S with $x \succeq_{\mathcal{K}^n} y$. Then, for any sufficiently small $\varepsilon > 0$, we have $x + \varepsilon e$, $y + \varepsilon e \in S_\varepsilon$ and $x + \varepsilon e \succeq_{\mathcal{K}^n} y + \varepsilon e$.

Using the SOC-monotonicity of f_ε on J_ε yields that $f_\varepsilon^{\rm soc}(x+\varepsilon e) \succeq_{\mathcal{K}^n} f_\varepsilon^{\rm soc}(y+\varepsilon e)$. Taking the limit $\varepsilon \to 0$ and using the convergence of $f_\varepsilon^{\rm soc}(x) \to f^{\rm soc}(x)$ and the continuity of $f^{\rm soc}$ on S implied by Lemma 2.8(a), we readily obtain that $f^{\rm soc}(x) \succeq_{\mathcal{K}^n} f^{\rm soc}(y)$. This shows that f is SOC-monotone.

Now assume that f is SOC-monotone. Let $\varepsilon > 0$ be an arbitrary sufficiently small real number. Fix any $x, y \in S_\varepsilon$ with $x \succeq_{\mathcal{K}^n} y$. Then, for all $t \in [-1, 1]$, we have $x - t\varepsilon e, y - t\varepsilon e \in S$ and $x - t\varepsilon e \succeq_{\mathcal{K}^n} y - t\varepsilon e$. Therefore, $f^{\rm soc}(x - t\varepsilon e) \succeq_{\mathcal{K}^n} f^{\rm soc}(y - t\varepsilon e)$, which is equivalent to

$$\begin{aligned}
& \frac{f(\lambda_1 - t\varepsilon) + f(\lambda_2 - t\varepsilon)}{2} - \frac{f(\mu_1 - t\varepsilon) + f(\mu_2 - t\varepsilon)}{2} \\
\geq \ & \left\| \frac{f(\lambda_1 - t\varepsilon) - f(\lambda_2 - t\varepsilon)}{2}\overline{x}_e - \frac{f(\mu_1 - t\varepsilon) - f(\mu_2 - t\varepsilon)}{2}\overline{y}_e \right\|.
\end{aligned}$$

Together with the definition of f_ε, it then follows that

$$\begin{aligned}
& \frac{f_\varepsilon(\lambda_1) + f_\varepsilon(\lambda_2)}{2} - \frac{f_\varepsilon(\mu_1) + f_\varepsilon(\mu_2)}{2} \\
= \ & \int \left[\frac{f(\lambda_1 - t\varepsilon) + f(\lambda_2 - t\varepsilon)}{2} - \frac{f(\mu_1 - t\varepsilon) + f(\mu_2 - t\varepsilon)}{2} \right] \varphi(t)dt \\
\geq \ & \int \left\| \frac{f(\lambda_1 - \varepsilon) - f(\lambda_2 - \varepsilon)}{2}\overline{x}_e - \frac{f(\mu_1 - \varepsilon) - f(\mu_2 - \varepsilon)}{2}\overline{y}_e \right\| \varphi(t)dt \\
\geq \ & \left\| \int \left[\frac{f(\lambda_1 - \varepsilon) - f(\lambda_2 - \varepsilon)}{2}\overline{x}_e - \frac{f(\mu_1 - \varepsilon) - f(\mu_2 - \varepsilon)}{2}\overline{y}_e \right] \varphi(t)dt \right\| \\
= \ & \left\| \frac{f_\varepsilon(\lambda_1) - f_\varepsilon(\lambda_2)}{2}\overline{x}_e - \frac{f_\varepsilon(\mu_1) - f_\varepsilon(\mu_2)}{2}\overline{y}_e \right\|.
\end{aligned}$$

By the definition of $f_\varepsilon^{\rm soc}$, this shows that $f_\varepsilon^{\rm soc}(x) \succeq_{\mathcal{K}^n} f_\varepsilon^{\rm soc}(y)$, i.e., f_ε is SOC-monotone. □

From Proposition 2.21 and [22, Theorem V. 3.4], $f \in C^1(J)$ is SOC-monotone if and only if it is matrix monotone of order 2. When the continuous f is not in the class $C^1(J)$, the result also holds due to Proposition 2.22 and the fact that f is matrix monotone of order n if and only if f_ε is matrix monotone of order n. Thus, we have the following main result.

Proposition 2.23. *The set of continuous SOC-monotone functions on the open interval J coincides with that of continuous matrix monotone functions of order 2 on J.*

Remark 2.2. *Combining Proposition 2.23 with Löwner's Theorem [104] shows that if $f : J \to \mathbb{R}$ is a continuous SOC-monotone function on the open interval J, then $f \in C^1(J)$.*

We now move to the characterizations of SOC-convex functions, and shows that the continuous f is SOC-convex if and only if it is matrix convex of order 2. First, for the first-order differentiable SOC-convex functions, we have the following characterizations.

Proposition 2.24. *Assume that $f \in C^1(J)$ with J being an open interval in $\mathbb{R}$. Then, the following hold.*

(a) *f is SOC-convex if and only if for any $x, y \in S$,*

$$f^{\rm soc}(y) - f^{\rm soc}(x) - (f^{\rm soc})'(x)(y-x) \succeq_{\mathcal{K}^n} 0.$$

(b) *If f is SOC-convex, then $(f')^{\rm soc}$ is a monotone function on S.*

Proof. (a) By following the arguments as in [21, Proposition B.3(a)], the proof can be done easily. We omit the details.

(b) From part(a), it follows that for any $x, y \in S$,

$$\begin{aligned} f^{\rm soc}(x) - f^{\rm soc}(y) - (f^{\rm soc})'(y)(x-y) &\succeq_{\mathcal{K}^n} 0, \\ f^{\rm soc}(y) - f^{\rm soc}(x) - (f^{\rm soc})'(x)(y-x) &\succeq_{\mathcal{K}^n} 0. \end{aligned}$$

Adding the last two inequalities, we immediately obtain that

$$\left[(f^{\rm soc})'(y) - (f^{\rm soc})'(x)\right](y-x) \succeq_{\mathcal{K}^n} 0.$$

Using the self-duality of K and Lemma 2.8(c) then yields

$$0 \le \left\langle e, \left[(f^{\rm soc})'(y) - (f^{\rm soc})'(x)\right](y-x)\right\rangle = \left\langle y-x, (f')^{\rm soc}(y) - (f')^{\rm soc}(x)\right\rangle.$$

This shows that $(f')^{\rm soc}$ is monotone. The proof is complete. □

To provide sufficient and necessary characterizations for twice differentiable SOC-convex functions, we need the following lemma that offers the second-order differential of $f^{\rm soc}$.

Lemma 2.9. *For any given $f : J \to \mathbb{R}$ with J open, let $f^{\mathrm{soc}} : S \to \mathbb{H}$ be defined by (1.8).*

(a) *f^{soc} is twice (continuously) differentiable on S if and only if f is twice (continuously) differentiable on J. Furthermore, when f is twice differentiable on J, for any given $x = x_e + x_0 e \in S$ and $u = u_e + u_0 e, v = v_e + v_0 e \in \mathbb{H}$, we have that*

$$(f^{\mathrm{soc}})''(x)(u,v) = f''(x_0)u_0v_0e + f''(x_0)(u_0v_e + v_0u_e) + f''(x_0)\langle u_e, v_e\rangle e$$

if $x_e = 0$; and otherwise

$$\begin{aligned}(f^{\mathrm{soc}})''(x)(u,v) = {} & (b_2(x) - a_1(x))u_0\langle \overline{x}_e, v_e\rangle \overline{x}_e + (c_2(x) - 3d(x))\langle \overline{x}_e, u_e\rangle\langle \overline{x}_e, v_e\rangle \overline{x}_e \\ & + d(x)\big[\langle u_e, v_e\rangle \overline{x}_e + \langle \overline{x}_e, v_e\rangle u_e + \langle \overline{x}_e, u_e\rangle v_e\big] + c_2(x)u_0v_0\overline{x}_e \\ & + \big(b_2(x) - a_1(x)\big)\langle \overline{x}_e, u_e\rangle v_0\overline{x}_e + a_1(x)\big(v_0u_e + u_0v_e\big) \\ & + b_2(x)u_0v_0e + c_2(x)\big[v_0\langle \overline{x}_e, u_e\rangle + u_0\langle \overline{x}_e, v_e\rangle\big]e \\ & + a_1(x)\langle u_e, v_e\rangle e + (b_2(x) - a_1(x))\langle \overline{x}_e, u_e\rangle\langle \overline{x}_e, v_e\rangle e, \end{aligned} \tag{2.59}$$

where

$$c_2(x) = \frac{f''(\lambda_2(x)) - f''(\lambda_1(x))}{2}, \qquad b_2(x) = \frac{f''(\lambda_2(x)) + f''(\lambda_1(x))}{2},$$
$$a_1(x) = \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, \qquad d(x) = \frac{b_1(x) - a_0(x)}{\|x_e\|}.$$

(b) *If f is twice differentiable on J, then for any given $x \in S$ and $u, v \in \mathbb{H}$,*

$$\begin{aligned}(f^{\mathrm{soc}})''(x)(u,v) &= (f^{\mathrm{soc}})''(x)(v,u), \\ \langle u, (f^{\mathrm{soc}})''(x)(u,v)\rangle &= \langle v, (f^{\mathrm{soc}})''(x)(u,u)\rangle.\end{aligned}$$

Proof. (a) The first part is direct by the given conditions and Lemma 2.8(b), and we only need to derive the differential formula. Fix any $u = u_e + u_0e, v = v_e + v_0e \in \mathbb{H}$. We first consider the case where $x_e = 0$. Without loss of generality, assume that $u_e \neq 0$. For any sufficiently small $t > 0$, using Lemma 2.8(b) and $x + tu = (x_0 + tu_0) + tu_e$, we have that

$$\begin{aligned}(f^{\mathrm{soc}})'(x+tu)v = {} & [b_1(x+tu) - a_0(x+tu)]\,\langle \overline{u}_e, v_e\rangle \overline{u}_e + c_1(x+tu)v_0\overline{u}_e \\ & + a_0(x+tu)v_e + b_1(x+tu)v_0e + c_1(x+tu)\langle \overline{u}_e, v_e\rangle e.\end{aligned}$$

In addition, from Lemma 2.8(b), we also have that $(f^{\mathrm{soc}})'(x)v = f'(x_0)v_0e + f'(x_0)v_e$. Using the definition of $b_1(x)$ and $a_0(x)$, and the differentiability of f' on J, it follows that

$$\begin{aligned}\lim_{t\to 0}\frac{b_1(x+tu)v_0e - f'(x_0)v_0e}{t} &= f''(x_0)u_0v_0e, \\ \lim_{t\to 0}\frac{a_0(x+tu)v_e - f'(x_0)v_e}{t} &= f''(x_0)u_0v_e, \\ \lim_{t\to 0}\frac{b_1(x+tu) - a_0(x+tu)}{t} &= 0, \\ \lim_{t\to 0}\frac{c_1(x+tu)}{t} &= f''(x_0)\|u_e\|.\end{aligned}$$

Using the above four limits, it is not hard to obtain that

$$\begin{aligned}(f^{\mathrm{soc}})''(x)(u,v) &= \lim_{t\to 0}\frac{(f^{\mathrm{soc}})'(x+tu)v-(f^{\mathrm{soc}})'(x)v}{t}\\ &= f''(x_0)u_0v_0e+f''(x_0)(u_0v_e+v_0u_e)+f''(x_0)\langle u_e,v_e\rangle e.\end{aligned}$$

We next consider the case where $x_e\neq 0$. From Lemma 2.8(b), it follows that

$$\begin{aligned}(f^{\mathrm{soc}})'(x)v &= (b_1(x)-a_0(x))\langle \overline{x}_e,v_e\rangle\,\overline{x}_e+c_1(x)v_0\overline{x}_e\\ &\quad +a_0(x)v_e+b_1(x)v_0e+c_1(x)\langle \overline{x}_e,v_e\rangle\, e,\end{aligned}$$

which in turn implies that

$$\begin{aligned}(f^{\mathrm{soc}})''(x)(u,v) &= \left[(b_1(x)-a_0(x))\langle \overline{x}_e,v_e\rangle\,\overline{x}_e\right]' u+\left[c_1(x)v_0\overline{x}_e\right]' u\\ &\quad +\left[a_0(x)v_e+b_1(x)v_0e\right]' u+\left[c_1(x)\langle \overline{x}_e,v_e\rangle\, e\right]' u. \end{aligned}\tag{2.60}$$

By the expressions of $a_0(x), b_1(x)$ and $c_1(x)$ and equations (2.52)-(2.53), we calculate that

$$\begin{aligned}(b_1(x))'u &= \frac{f''(\lambda_2(x))\left[u_0+\langle \overline{x}_e,u_e\rangle\right]}{2}+\frac{f''(\lambda_1(x))\left[u_0-\langle \overline{x}_e,u_e\rangle\right]}{2}\\ &= b_2(x)u_0+c_2(x)\langle \overline{x}_e,u_e\rangle,\\ (c_1(x))'u &= c_2(x)u_0+b_2(x)\langle \overline{x}_e,u_e\rangle,\\ (a_0(x))'u &= \frac{f'(\lambda_2(x))-f'(\lambda_1(x))}{\lambda_2(x)-\lambda_1(x)}u_0+\frac{b_1(x)-a_0(x)}{\|x_e\|}\langle \overline{x}_e,u_e\rangle\\ &= a_1(x)u_0+d(x)\langle \overline{x}_e,u_e\rangle,\\ (\langle \overline{x}_e,v_e\rangle)'u &= \left\langle \frac{1}{\|x_e\|}u_e-\frac{\langle \overline{x}_e,u_e\rangle}{\|x_e\|}\overline{x}_e,v_e\right\rangle.\end{aligned}$$

Using these equalities and noting that $a_1(x)=c_1(x)/\|x_e\|$, we obtain that

$$\begin{aligned}\Big[\big(b_1(x)-a_0(x)\big)\langle \overline{x}_e,v_e\rangle\overline{x}_e\Big]'u &= \Big[\big(b_2(x)-a_1(x)\big)u_0+(c_2(x)-d(x))\langle \overline{x}_e,u_e\rangle\Big]\langle \overline{x}_e,v_e\rangle\overline{x}_e\\ &\quad +(b_1(x)-a_0(x))\left\langle \frac{1}{\|x_e\|}u_e-\frac{\langle \overline{x}_e,u_e\rangle}{\|x_e\|}\overline{x}_e,v_e\right\rangle\overline{x}_e\\ &\quad +(b_1(x)-a_0(x))\langle \overline{x}_e,v_e\rangle\left[\frac{1}{\|x_e\|}u_e-\frac{\langle \overline{x}_e,u_e\rangle}{\|x_e\|}\overline{x}_e\right]\\ &= \Big[(b_2(x)-a_1(x))u_0+(c_2(x)-d(x))\langle \overline{x}_e,u_e\rangle\Big]\langle \overline{x}_e,v_e\rangle\overline{x}_e\\ &\quad +d(x)\langle u_e,v_e\rangle\overline{x}_e-2d(x)\langle \overline{x}_e,v_e\rangle\langle \overline{x}_e,u_e\rangle\overline{x}_e+d(x)\langle \overline{x}_e,v_e\rangle u_e;\\ \Big[a_0(x)v_e+b_1(x)v_0e\Big]'u &= \Big[a_1(x)u_0+d(x)\langle \overline{x}_e,u_e\rangle\Big]v_e+\Big[b_2(x)u_0+c_2(x)\langle \overline{x}_e,u_e\rangle\Big]v_0e;\\ \Big[c_1(x)v_0\overline{x}_e\Big]'u &= \Big[c_2(x)u_0+b_2(x)\langle \overline{x}_e,u_e\rangle\Big]v_0\overline{x}_e+c_1(x)v_0\frac{u_e-\langle \overline{x}_e,u_e\rangle\overline{x}_e}{\|x_e\|}\\ &= \Big[c_2(x)u_0+b_2(x)\langle \overline{x}_e,u_e\rangle\Big]v_0\overline{x}_e+a_1(x)v_0\Big[u_e-\langle \overline{x}_e,u_e\rangle\overline{x}_e\Big];\end{aligned}$$

and

$$\begin{aligned}\Big[c_1(x)\langle\overline{x}_e, v_e\rangle e\Big]' u &= \Big[c_2(x)u_0 + b_2(x)\langle\overline{x}_e, u_e\rangle\Big]\langle\overline{x}_e, v_e\rangle e + c_1(x)\left\langle \frac{u_e - \langle\overline{x}_e, u_e\rangle\overline{x}_e}{\|x_e\|}, v_e\right\rangle e \\ &= c_2(x)u_0\langle\overline{x}_e, v_e\rangle e + \big(b_2(x) - a_1(x)\big)\langle\overline{x}_e, u_e\rangle\,\langle\overline{x}_e, v_e\rangle\, e + a_1(x)\langle u_e, v_e\rangle e.\end{aligned}$$

Adding the equalities above and using equation (2.60) yields the formula in (2.59).

(b) By the formula in part (a), a simple computation yields the desired result. □

Proposition 2.25. *Assume that $f \in C^2(J)$ with J being an open interval in $\mathbb{R}$. Then, the following hold.*

(a) *f is SOC-convex if and only if for any $x \in S$ and $h \in \mathbb{H}$, $(f^{\rm soc})''(x)(h,h) \in K$.*

(b) *f is SOC-convex if and only if f is convex and for any $\tau_1, \tau_2 \in J$,*

$$\begin{aligned}&\left(\frac{f''(\tau_2)}{2}\right)\left(\frac{f(\tau_2) - f(\tau_1) - f'(\tau_1)(\tau_2 - \tau_1)}{(\tau_2 - \tau_1)^2}\right) \\ \geq\ & \left[\frac{f(\tau_1) - f(\tau_2) - f'(\tau_2)(\tau_1 - \tau_2)}{(\tau_2 - \tau_1)^2}\right]^2.\end{aligned} \tag{2.61}$$

(c) *f is SOC-convex if and only if f is convex and for any $\tau_1, \tau_2 \in J$,*

$$\begin{aligned}&\frac{1}{4}f''(\tau_1)f''(\tau_2) \\ \geq\ & \left(\frac{f(\tau_2) - f(\tau_1) - f'(\tau_1)(\tau_2 - \tau_1)}{(\tau_2 - \tau_1)^2}\right)\left(\frac{f(\tau_1) - f(\tau_2) - f'(\tau_2)(\tau_1 - \tau_2)}{(\tau_2 - \tau_1)^2}\right).\end{aligned} \tag{2.62}$$

(d) *f is SOC-convex if and only if for any $\tau_1, \tau_2 \in J$ and $s = \tau_1, \tau_2$,*

$$\begin{bmatrix} f^{[2]}(\tau_2, s, \tau_2) & f^{[2]}(\tau_2, s, \tau_1) \\ f^{[2]}(\tau_1, s, \tau_2) & f^{[2]}(\tau_1, s, \tau_1)\end{bmatrix} \succeq O.$$

Proof. (a) Suppose that f is SOC-convex. Since $f^{\rm soc}$ is twice continuously differentiable by Lemma 2.9(a), we have for any given $x \in S, h \in \mathbb{H}$ and sufficiently small $t > 0$,

$$f^{\rm soc}(x + th) = f^{\rm soc}(x) + t(f^{\rm soc})'(x)h + \frac{1}{2}t^2(f^{\rm soc})''(x)(h,h) + o(t^2).$$

Applying Proposition 2.24(a) yields that $\frac{1}{2}(f^{\rm soc})''(x)(h,h) + o(t^2)/t^2 \succeq_{\mathcal{K}^n} 0$. Taking the limit $t \downarrow 0$, we obtain $(f^{\rm soc})''(x)(h,h) \in K$. Conversely, fix any $z \in K$ and $x, y \in S$. Applying the mean-value theorem for the twice continuously differentiable $\langle f^{\rm soc}(\cdot), z\rangle$ at x, we have

$$\begin{aligned}\langle f^{\rm soc}(y), z\rangle &= \langle f^{\rm soc}(x), z\rangle + \langle (f^{\rm soc})'(x)(y - x), z\rangle \\ &\quad + \frac{1}{2}\langle (f^{\rm soc})''(x + t_1(y - x))(y - x, y - x), z\rangle\end{aligned}$$

for some $t_1 \in (0,1)$. Since $x + t_1(y-x) \in S$, the given assumption implies that

$$\langle f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(x) - (f^{\mathrm{soc}})'(x)(y-x), z\rangle \geq 0.$$

This, by the arbitrariness of z in K, implies that $f^{\mathrm{soc}}(y) - f^{\mathrm{soc}}(x) - (f^{\mathrm{soc}})'(x)(y-x) \succeq_{\mathcal{K}^n} 0$. From Proposition 2.24(a), it then follows that f is SOC-convex.

(b) By part (a), it suffices to prove that $(f^{\mathrm{soc}})''(x)(h,h) \in K$ for any $x \in S$ and $h \in \mathbb{H}$ if and only if f is convex and (2.61) holds. Fix any $x = x_e + x_0 e \in S$. By the continuity of $(f^{\mathrm{soc}})''(x)$, we may assume that $x_e \neq 0$. From Lemma 2.9(a), for any $h = h_e + h_0 e \in \mathbb{H}$,

$$\begin{aligned}
(f^{\mathrm{soc}})''(x)(h,h) = & \Big[\big(c_2(x) - 3d(x)\big)\langle \overline{x}_e, h_e\rangle^2 + 2\big(b_2(x) - a_1(x)\big)h_0\langle \overline{x}_e, h_e\rangle\Big]\overline{x}_e \\
& + \Big[c_2(x)h_0^2 + d(x)\|h_e\|^2\Big]\overline{x}_e + \Big[2a_1(x)h_0 + 2d(x)\langle \overline{x}_e, h_e\rangle\Big]h_e \\
& + \Big[2c_2(x)h_0\,\langle \overline{x}_e, h_e\rangle + b_2(x)h_0^2 + a_1(x)\|h_e\|^2\Big]e \\
& + (b_2(x) - a_1(x))\langle \overline{x}_e, h_e\rangle^2 e.
\end{aligned}$$

Therefore, $(f^{\mathrm{soc}})''(x)(h,h) \in K$ if and only if the following two inequalities hold:

$$b_2(x)\left(h_0^2 + \langle \overline{x}_e, h_e\rangle^2\right) + 2c_2(x)h_0\langle \overline{x}_e, h_e\rangle + a_1(x)\left(\|h_e\|^2 - \langle \overline{x}_e, h_e\rangle^2\right) \geq 0 \tag{2.63}$$

and

$$\begin{aligned}
& \Big[b_2(x)\left(h_0^2 + \langle \overline{x}_e, h_e\rangle^2\right) + 2c_2(x)h_0\langle \overline{x}_e, h_e\rangle + a_1(x)\left(\|h_e\|^2 - \langle \overline{x}_e, h_e\rangle^2\right)\Big]^2 \\
& \quad \geq \Big\|\left(c_2(x)h_0^2 + d(x)\|h_e\|^2\right)\overline{x}_e + 2\left(b_2(x) - a_1(x)\right)h_0\langle \overline{x}_e, h_e\rangle\overline{x}_e \\
& \quad + \left(c_2(x) - 3d(x)\right)\langle \overline{x}_e, h_e\rangle^2\overline{x}_e + 2\left(a_1(x)h_0 + d(x)\langle \overline{x}_e, h_e\rangle\right)h_e\Big\|^2.
\end{aligned} \tag{2.64}$$

Observe that the left-hand side of (2.63) can be rewritten as

$$\frac{f''(\lambda_2(x))(h_0 + \langle \overline{x}_e, h_e\rangle)^2}{2} + \frac{f''(\lambda_1(x))(h_0 - \langle \overline{x}_e, h_e\rangle)^2}{2} + a_1(x)(\|h_e\|^2 - \langle \overline{x}_e, h_e\rangle^2).$$

From Lemma 2.6, it then follows that (2.63) holds for all $h = h_e + h_0 e \in \mathbb{H}$ if and only if

$$f''(\lambda_1(x)) \geq 0, \quad f''(\lambda_2(x)) \geq 0 \ \text{ and } \ a_1(x) \geq 0. \tag{2.65}$$

In addition, by the definition of $b_2(x), c_2(x)$ and $a_1(x)$, the left-hand side of (2.64) equals

$$\Big[f''(\lambda_2(x))\mu_2(h)^2 + f''(\lambda_1(x))\mu_1(h)^2 + a_1(x)\mu(h)^2\Big]^2, \tag{2.66}$$

where $\mu_1(h), \mu_2(h)$ and $\mu(h)$ are defined as in Lemma 2.7 with u_e replaced by $\overline{x}_e$. In the following, we use μ_1, μ_2 and μ to represent $\mu_1(h), \mu_2(h)$ and $\mu(h)$ respectively. Note that the sum of the first three terms in $\|\cdot\|^2$ on the right-hand side of (2.64) equals

$$\begin{aligned}
& \frac{1}{2}\left(c_2(x)+b_2(x)-a_1(x)\right)\left(h_0+\langle \overline{x}_e, h_e\rangle\right)^2 \overline{x}_e \\
& +\frac{1}{2}\left(c_2(x)-b_2(x)+a_1(x)\right)\left(h_0-\langle \overline{x}_e, h_e\rangle\right)^2 \overline{x}_e \\
& +d(x)\left(\|h_e\|^2-\langle \overline{x}_e, h_e\rangle^2\right)\overline{x}_e-2d(x)\langle \overline{x}_e, h_e\rangle^2\overline{x}_e \\
= \;& f''(\lambda_2(x))\mu_2^2\overline{x}_e-f''(\lambda_1(x))\mu_1^2\overline{x}_e-\big(a_1(x)+d(x)\big)\mu_2^2\overline{x}_e \\
& +\big(a_1(x)-d(x)\big)\mu_1^2\overline{x}_e+2d(x)\mu_2\mu_1\overline{x}_e+d(x)\mu^2\overline{x}_e \\
=: \;& E(x,h)\overline{x}_e,
\end{aligned}$$

where $(\mu_2-\mu_1)^2=2\langle \overline{x}_e, h_e\rangle^2$ is used for the equality, while the last term is

$$\begin{aligned}
& \left(a_1(x)-d(x)\right)\left(h_0-\langle \overline{x}_e, h_e\rangle\right)h_e+\left(a_1(x)+d(x)\right)\left(h_0+\langle \overline{x}_e, h_e\rangle\right)h_e \\
= \;& \sqrt{2}\left(a_1(x)-d(x)\right)\mu_1 h_e+\sqrt{2}\left(a_1(x)+d(x)\right)\mu_2 h_e.
\end{aligned}$$

Thus, we calculate that the right-hand side of (2.64) equals

$$\begin{aligned}
& E(x,h)^2+2\Big[\big(a_1(x)-d(x)\big)\mu_1+\big(a_1(x)+d(x)\big)\mu_2\Big]^2\|h_e\|^2 \\
& +2\sqrt{2}E(x,h)\big[a_1(x)-d(x)\big]\mu_1\langle \overline{x}_e, h_e\rangle+2\sqrt{2}E(x,h)\big[a_1(x)+d(x)\big]\mu_2\langle \overline{x}_e, h_e\rangle \\
= \;& E(x,h)^2+2\Big[\big(a_1(x)-d(x)\big)\mu_1+\big(a_1(x)+d(x)\big)\mu_2\Big]^2\left[\mu^2+\frac{(\mu_2-\mu_1)^2}{2}\right] \\
& +2E(x,h)(\mu_2-\mu_1)\Big[(a_1(x)-d(x))\mu_1+(a_1(x)+d(x))\mu_2\Big] \\
= \;& \Big[E(x,h)+(\mu_2-\mu_1)\left[(a_1(x)-d(x))\mu_1+(a_1(x)+d(x))\mu_2\right]\Big]^2 \\
& +2\Big[\left(a_1(x)-d(x)\right)\mu_1+\left(a_1(x)+d(x)\right)\mu_2\Big]^2\mu^2, \qquad (2.67)
\end{aligned}$$

where the expressions of μ_1, μ_2 and μ are used for the first equality. Now substituting the expression of $E(x,h)$ into (2.67) yields that the right-hand side of (2.67) equals

$$\Big[f''(\lambda_2(x))\mu_2^2-f''(\lambda_1(x))\mu_1^2+d(x)\mu^2\Big]^2+2\Big[\left(a_1(x)-d(x)\right)\mu_1+\left(a_1(x)+d(x)\right)\mu_2\Big]^2\mu^2.$$

Together with equation (2.66), it follows that (2.64) is equivalent to

$$\begin{aligned}
& 4f''(\lambda_1(x))f''(\lambda_2(x))\mu_1^2\mu_2^2+2\left(a_1(x)-d(x)\right)f''(\lambda_2(x))\mu_2^2\mu^2 \\
& +2\left(a_1(x)+d(x)\right)f''(\lambda_1(x))\mu_1^2\mu^2+\left(a_1(x)^2-d(x)^2\right)\mu^4 \\
& -2\left[\left(a_1(x)-d(x)\right)\mu_1+\left(a_1(x)+d(x)\right)\mu_2\right]^2\mu^2\geq 0.
\end{aligned}$$

By Lemma 2.7, this inequality holds for any $h=h_e+h_0e\in\mathbb{H}$ if and only if

$$\begin{aligned}
a_1(x)^2-d(x)^2\geq 0, \quad & f''(\lambda_2(x))\big(a_1(x)-d(x)\big)\geq\big(a_1(x)+d(x)\big)^2, \\
& f''(\lambda_1(x))\big(a_1(x)+d(x)\big)\geq\big(a_1(x)-d(x)\big)^2,
\end{aligned}$$

which, by the expression of $a_1(x)$ and $d(x)$, are respectively equivalent to

$$\begin{aligned}
\frac{f(\lambda_2)-f(\lambda_1)-f'(\lambda_1)(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)^2}\cdot\frac{f(\lambda_1)-f(\lambda_2)-f'(\lambda_2)(\lambda_1-\lambda_2)}{(\lambda_2-\lambda_1)^2} &\geq 0,\\
\frac{f''(\lambda_2)}{2}\frac{f(\lambda_2)-f(\lambda_1)-f'(\lambda_1)(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)^2} &\geq \left[\frac{f(\lambda_1)-f(\lambda_2)-f'(\lambda_2)(\lambda_1-\lambda_2)}{(\lambda_2-\lambda_1)^2}\right]^2, \qquad (2.68)\\
\frac{f''(\lambda_1)}{2}\frac{f(\lambda_1)-f(\lambda_2)-f'(\lambda_2)(\lambda_1-\lambda_2)}{(\lambda_2-\lambda_1)^2} &\geq \left[\frac{f(\lambda_2)-f(\lambda_1)-f'(\lambda_1)(\lambda_2-\lambda_1)}{(\lambda_2-\lambda_1)^2}\right]^2,
\end{aligned}$$

where $\lambda_1=\lambda_1(x)$ and $\lambda_2=\lambda_2(x)$. Summing up the discussions above, f is SOC-convex if and only if (2.65) and (2.68) hold. In view of the arbitrariness of x, we have that f is SOC-convex if and only if f is convex and (2.61) holds.

(c) It suffices to prove that (2.61) is equivalent to (2.62). Clearly, (2.61) implies (2.62). We next prove that (2.62) implies (2.61). Fixing any $\tau_2\in J$, we consider $g(t):J\to\mathbb{R}$ defined by

$$g(t)=\frac{f''(\tau_2)}{2}\left[f(\tau_2)-f(t)-f'(t)(\tau_2-t)\right]-\frac{[f(t)-f(\tau_2)-f'(\tau_2)(t-\tau_2)]^2}{(t-\tau_2)^2}$$

if $t\neq\tau_2$, and otherwise $g(\tau_2)=0$. From the proof of [69, Theorem 2.3], we know that (2.61) implies that $g(t)$ attains its global minimum at $t=\tau_2$. Consequently, (2.61) follows.

(d) The result is immediate by part(b) and the definition of $f^{[2]}$ given as in (2.46). □

Propositions 2.24 and 2.25 provide the characterizations for continuously differentiable SOC-convex functions, which extend the corresponding results of [45, Section 4]. When f is not continuously differentiable, the following proposition shows that one may check the SOC-convexity of f by checking that of its regularization f_ε. Since the proof can be done easily by following that of Proposition 2.22, we omit the details.

Proposition 2.26. *Let $f:J\to\mathbb{R}$ be a continuous function on the open interval J, and f_ε be its regularization defined by (2.50). Then, f is SOC-convex if and only if f_ε is SOC-convex on J_ε for every sufficiently small $\varepsilon>0$, where $J_\varepsilon:=(a+\varepsilon,b-\varepsilon)$ for $J=(a,b)$.*

By [69, Theorem 2.3] and Proposition 2.26, we can obtain the below consequence immediately.

Proposition 2.27. *The set of continuous SOC-convex functions on the open interval J coincides with that of continuous matrix convex functions of order 2 on J.*

Remark 2.3. *Combining Proposition 2.27 with Kraus' theorem [93] shows that if $f : J \to \mathbb{R}$ is a continuous SOC-convex function, then $f \in C^2(J)$.*

We establish another sufficient and necessary characterization for twice continuously differentiable SOC-convex functions f by the differential operator $(f^{\mathrm{soc}})'$.

Proposition 2.28. *Let $f \in C^2(J)$ with J being an open interval in $\mathbb{R}$. Then, f is SOC-convex if and only if*

$$x \succeq_{\mathcal{K}^n} y \implies (f^{\mathrm{soc}})'(x) - (f^{\mathrm{soc}})'(y) \geq 0, \qquad \forall x, y \in S. \tag{2.69}$$

Proof. Suppose that f is SOC-convex. Fix any $x, y \in S$ with $x \succeq_{\mathcal{K}^n} y$, and $h \in \mathbb{H}$. Since f^{soc} is twice continuously differentiable by Lemma 2.9(a), applying the mean-value theorem for the twice continuously differentiable $\langle h, (f^{\mathrm{soc}})'(\cdot)h\rangle$ at y, we have

$$\begin{aligned} \left\langle h, \left[(f^{\mathrm{soc}})'(x) - (f^{\mathrm{soc}})'(y)\right] h\right\rangle &= \left\langle h, (f^{\mathrm{soc}})''(y + t_1(x-y))(x-y, h)\right\rangle \\ &= \left\langle x - y, (f^{\mathrm{soc}})''(y + t_1(x-y))(h, h)\right\rangle \end{aligned} \tag{2.70}$$

for some $t_1 \in (0, 1)$, where Lemma 2.9(b) is used for the second equality. Noting that $y + t_1(x - y) \in S$ and f is SOC-convex, from Proposition 2.25(a) we have

$$(f^{\mathrm{soc}})''(y + t_1(x-y))(h, h) \in K.$$

This, together with $x - y \in K$, yields that $\left\langle x - y, (f^{\mathrm{soc}})''(x + t_1(x-y))(h, h)\right\rangle \geq 0$. Then, from (2.70) and the arbitrariness of h, we have $(f^{\mathrm{soc}})'(x) - (f^{\mathrm{soc}})'(y) \geq 0$.

Conversely, assume that the implication in (2.69) holds for any $x, y \in S$. For any fixed $u \in K$, clearly, $x + tu \succeq_{\mathcal{K}^n} x$ for all $t > 0$. Consequently, for any $h \in \mathbb{H}$, we have

$$\left\langle h, \left[(f^{\mathrm{soc}})'(x + tu) - (f^{\mathrm{soc}})'(x)\right] h\right\rangle \geq 0.$$

Note that $(f^{\mathrm{soc}})'(x)$ is continuously differentiable. The last inequality implies that

$$0 \leq \left\langle h, (f^{\mathrm{soc}})''(x)(u, h)\right\rangle = \langle u, (f^{\mathrm{soc}})''(x)(h, h)\rangle.$$

By the self-duality of K and the arbitrariness of u in K, this means that $(f^{\mathrm{soc}})''(x)(h, h) \in K$. Together with Proposition 2.25(a), it follows that f is SOC-convex. $\square$

Example 2.13. *The following functions are SOC-monotone.*

(a) *The function $f(t) = t^r$ is SOC-monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$.*

(b) *The function $f(t) = -t^{-r}$ is SOC-monotone on $(0, \infty)$ if and only if $0 \leq r \leq 1$.*

(c) *The function $f(t) = \ln(t)$ is SOC-monotone on $(0, \infty)$.*

(d) *The function* $f(t) = -\cot(t)$ *is SOC-monotone on* $(0, \pi)$.

(e) *The function* $f(t) = \frac{t}{c+t}$ *with* $c \geq 0$ *is SOC-monotone on* $(-\infty, c)$ *and* (c, ∞).

(f) *The function* $f(t) = \frac{t}{c-t}$ *with* $c \geq 0$ *is SOC-monotone on* $(-\infty, c)$ *and* (c, ∞).

Example 2.14. *The following functions are SOC-convex.*

(a) *The function* $f(t) = t^r$ *with* $r \geq 0$ *is SOC-convex on* $[0, \infty)$ *if and only if* $r \in [1, 2]$. *Particularly,* $f(t) = t^2$ *is SOC-convex on* $\mathbb{R}$.

(b) *The function* $f(t) = t^{-r}$ *with* $r > 0$ *is SOC-convex on* $(0, \infty)$ *if and only if* $r \in [0, 1]$.

(c) *The function* $f(t) = t^r$ *with* $r \geq 0$ *is SOC-concave if and only if* $r \in [0, 1]$.

(d) *The entropy function* $f(t) = t \ln t$ *is SOC-convex on* $[0, \infty)$.

(e) *The logarithmic function* $f(t) = -\ln t$ *is SOC-convex on* $(0, \infty)$.

(f) *The function* $f(t) = \frac{t}{t-\sigma}$ *with* $\sigma \geq 0$ *is SOC-convex on* (σ, ∞).

(g) *The function* $f(t) = -\frac{t}{t+\sigma}$ *with* $\sigma \geq 0$ *is SOC-convex on* $(-\sigma, \infty)$.

(h) *The function* $f(t) = \frac{t^2}{1-t}$ *is SOC-convex on* $(-1, 1)$.

Next we illustrate the applications of the SOC-monotonicity and SOC-convexity of certain functions in establishing some important inequalities. For example, by the SOC-monotonicity of $-t^{-r}$ and t^r with $r \in [0, 1]$, one can get the order-reversing inequality and the Löwner-Heinz inequality, and by the SOC-monotonicity and SOC-concavity of $-t^{-1}$, one may obtain the general harmonic-arithmetic mean inequality.

Proposition 2.29. *For any* $x, y \in \mathbb{H}$ *and* $0 \leq r \leq 1$, *the following inequalities hold:*

(a) $y^{-r} \succeq_{\mathcal{K}^n} x^{-r}$ *if* $x \succeq_{\mathcal{K}^n} y \succ_{\mathcal{K}^n} 0$;

(b) $x^r \succeq_{\mathcal{K}^n} y^r$ *if* $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$;

(c) $\left[\beta x^{-1} + (1-\beta) y^{-1}\right]^{-1} \preceq_K \beta x + (1-\beta) y$ *for any* $x, y \succ_{\mathcal{K}^n} 0$ *and* $\beta \in (0, 1)$.

From the second inequality of Proposition 2.29, we particularly have the following result which generalizes [64, Eq.(3.9)], and is often used when analyzing the properties of the generalized Fischer-Burmeister (FB) SOC complementarity function $\phi_p(x, y) := (|x|^p + |y|^p)^{1/p} - (x + y)$. To know more about this function ϕ_p, please refer to [122].

Proposition 2.30. *For any $x, y \in \mathbb{H}$, let $z(x,y) := (|x|^p + |y|^p)^{1/p}$ for any $p > 1$. Then,*

$$z(x,y) \succeq_{\mathcal{K}^n} |x| \succeq_{\mathcal{K}^n} x \quad \text{and} \quad z(x,y) \succeq_{\mathcal{K}^n} |y| \succeq_{\mathcal{K}^n} y.$$

The SOC-convexity can also be used to establish some matrix inequalities. From (2.51) we see that, when $\mathbb{H}$ reduces to the n-dimensional Euclidean space $\mathbb{R}^n$, the differential operator $(f^{\mathrm{soc}})'(x)$ becomes the following $n \times n$ symmetric matrix:

$$\begin{bmatrix} b_1(x) & c_1(x)\overline{x}_e^T \\ c_1(x)\overline{x}_e & a_0(x)I + (b_1(x) - a_0(x))\overline{x}_e\overline{x}_e^T \end{bmatrix}$$

where $a_0(x)$, $b_1(x)$ and $c_1(x)$ are same as before, and I is an identity matrix. Thus, from Proposition 2.28, we have the following result which is hard to get by direct calculation.

Proposition 2.31. *If $f \in C^2(J)$ is SOC-convex on the open interval J, then for any $x, y \in S$ with $x \succeq_{\mathcal{K}^n} y$,*

$$\begin{bmatrix} b_1(x) & c_1(x)\overline{x}_e^T \\ c_1(x)\overline{x}_e & a_0(x)I + (b_1(x) - a_0(x))\overline{x}_e\overline{x}_e^T \end{bmatrix} \succeq \begin{bmatrix} b_1(y) & c_1(y)\overline{x}_e^T \\ c_1(y)\overline{x}_e & a_0(y)I + (b_1(y) - a_0(y))\overline{x}_e\overline{x}_e^T \end{bmatrix}.$$

Particularly, when $f(t) = t^2$ $(t \in \mathbb{R})$, this conclusion reduces to the following implication

$$x \succeq_{\mathcal{K}^n} y \Longrightarrow \begin{bmatrix} x_0 & \overline{x}_e^T \\ \overline{x}_e & x_0 I \end{bmatrix} \succeq \begin{bmatrix} y_0 & \overline{y}_e^T \\ \overline{y}_e & y_0 I \end{bmatrix}.$$

As mentioned earlier, with certain SOC-monotone and SOC-convex functions, one can easily establish some determinant inequalities. Below is a stronger version of Proposition 1.8(b).

Proposition 2.32. *For any $x, y \in K$ and any real number $p \geq 1$, it holds that*

$$\sqrt[p]{\det(x+y)} \geq 2^{\frac{2}{p}-2}\left(\sqrt[p]{\det(x)} + \sqrt[p]{\det(y)}\right).$$

Proof. In light of Example 2.12(b), we see that $f(t) = t^{1/p}$ is SOC-concave on $[0, \infty)$, which says

$$\left(\frac{x+y}{2}\right)^{1/p} \succeq_{\mathcal{K}^n} \frac{x^{1/p} + y^{1/p}}{2}.$$

This together with the fact that $\det(x) \geq \det(y)$ whenever $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ implies

$$2^{-\frac{2}{p}} \det\left(\sqrt[p]{x+y}\right) = \det\left(\sqrt[p]{\frac{x+y}{2}}\right) \geq \det\left(\frac{\sqrt[p]{x} + \sqrt[p]{y}}{2}\right) \geq \frac{\det(\sqrt[p]{x}) + \det(\sqrt[p]{y})}{4},$$

where $\det(x+y) \geq \det(x) + \det(y)$ for $x, y \in K$ is used for the last inequality. In addition, by the definition of $\det(x)$, it is clear that $\det\left(\sqrt[p]{x}\right) = \sqrt[p]{\det(x)}$. Thus, from the last equation, we obtain the desired inequality. The proof is complete. $\square$

Comparing Example 2.13 with Example 2.14, we observe that there are some relations between SOC-monotone and SOC-convex functions. For example, $f(t) = t\ln t$ and $f(t) = -\ln t$ are SOC-convex on $(0,\infty)$, and its derivative functions are SOC-monotone on $(0,\infty)$. This is similar to the case for matrix-convex and matrix-monotone functions. However, it is worthwhile to point out that they can not inherit all relations between matrix-convex and matrix-monotone functions, since the class of continuous SOC-monotone (SOC-convex) functions coincides with the class of continuous matrix-monotone (matrix-convex) functions of order 2 only, and there exist gaps between matrix-monotone (matrix-convex) functions of different orders (see [70, 114]). Then, a question occurs to us: which relations for matrix-convex and matrix-monotone functions still hold for SOC-convex and SOC-monotone functions.

Lemma 2.10. *Assume that $f : J \to \mathbb{R}$ is three times differentiable on the open interval J. Then, f is a non-constant SOC-monotone function if and only if f' is strictly positive and $(f')^{-1/2}$ is concave.*

Proof. "$\Leftarrow$". Clearly, f is a non-constant function. Also, by [69, Proposition 2.2], we have

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1} \le \sqrt{f'(t_2)f'(t_1)}, \qquad \forall t_2, t_1 \in J.$$

This, by the strict positivity of f' and Proposition 2.21, shows that f is SOC-monotone.

"$\Rightarrow$". The result is direct by [59, Theorem III] and Proposition 2.23. $\Box$

Using Lemma 2.10, we may verify that SOC-monotone and SOC-convex functions inherit the following relation of matrix-monotone and matrix-convex functions.

Proposition 2.33. *If $f : J \to \mathbb{R}$ is a continuous SOC-monotone function, then the function $g(t) = \int_a^t f(s)ds$ with some $a \in J$ is SOC-convex.*

Proof. It suffices to consider the case where f is a non-constant SOC-monotone function. Due to Proposition 2.22, we may assume $f \in C^3(J)$. By Lemma 2.10, $f'(t) > 0$ for all $t \in J$ and $(f')^{-1/2}$ is concave. Since $g \in C^4(J)$ and $g''(t) = f'(t) > 0$ for all $t \in J$, in order to prove that g is SOC-convex, we only need to argue

$$\frac{g''(t)g^{(4)}(t)}{48} \ge \frac{\left[g^{(3)}(t)\right]^2}{36} \iff \frac{f'(t)f^{(3)}(t)}{48} \ge \frac{\left[f''(t)\right]^2}{36}, \qquad \forall t \in J. \tag{2.71}$$

Since $(f')^{-1/2}$ is concave, its second-order derivative is nonpositive. From this, we have

$$\frac{1}{32}\left(f''(t)\right)^2 \le \frac{1}{48} f'(t)f^{(3)}(t), \qquad \forall t \in J,$$

which implies the inequality (2.71). The proof is complete. $\Box$

Similar to matrix-monotone and matrix-convex functions, the converse of Proposition 2.33 does not hold. For example, $f(t) = \frac{t^2}{1-t}$ on $(-1,1)$ is SOC-convex by Example

2.14(g), but its derivative $g'(t) = \frac{1}{(1-t)^2} - 1$ is not SOC-monotone by Proposition 2.21. As a consequence of Proposition 2.33 and Proposition 2.28, we have the following result.

Proposition 2.34. *Let $f \in C^2(J)$. If f' is SOC-monotone, then f is SOC-convex. This is equivalent to saying that for any $x, y \in S$ with $x \succeq_{\mathcal{K}^n} y$,*

$$(f')^{\rm soc}(x) \succeq_{\mathcal{K}^n} (f')^{\rm soc}(y) \implies (f^{\rm soc})'(x) - (f^{\rm soc})'(y) \succeq 0.$$

From [22, Theorem V. 2.5], we know that a continuous function f mapping $[0, \infty)$ into itself is matrix-monotone if and only if it is matrix-concave. However, for such f we cannot prove that f is SOC-concave when it is SOC-monotone, but f is SOC-concave under a little stronger condition than SOC-monotonicity, i.e., the matrix-monotonicity of order 4.

Proposition 2.35. *Let $f : [0, \infty) \to [0, \infty)$ be continuous. If f is matrix-monotone of order 4, then f is SOC-concave.*

Proof. By [108, Theorem 2.1], if f is continuous and matrix-monotone of order $2n$, then f is matrix-concave of order n. This together with Proposition 2.27 gives the result. □

Note that Proposition 2.35 verifies Conjecture 2.2 partially and also can be viewed as the converse of Proposition 2.8. From [22], we know that the functions in Example 2.13(a)-(c) are all matrix-monotone, and so they are SOC-concave by Proposition 2.35(b). In addition, using Proposition 2.35(b) and noting that $-t^{-1}$ $(t > 0)$ is SOC-monotone and SOC-concave on $(0, \infty)$, we readily have the following consequence.

Proposition 2.36. *Let $f : (0, \infty) \to (0, \infty)$ be continuous. If f is matrix-monotone of order 4, then the function $g(t) = \frac{1}{f(t)}$ is SOC-convex.*

Proposition 2.37. *Let f be a continuous real function on the interval $[0, \alpha)$. If f is SOC-convex, then the indefinite integral of $\frac{f(t)}{t}$ is also SOC-convex.*

Proof. The result follows directly by [115, Proposition 2.7] and Proposition 2.27. □

For a continuous real function f on the interval $[0, \alpha)$, [22, Theorem V. 2.9] states that the following two conditions are equivalent:

(i) f is matrix-convex and $f(0) \le 0$;

(ii) The function $g(t) = \frac{f(t)}{t}$ is matrix-monotone on $(0, \alpha)$.

At the end of this section, let us look back to Conjecture 2.1. By looking into Example 2.13(a)-(c) and (f)-(g), we find that these functions are continuous, nondecreasing and concave. Then, one naturally asks whether such functions are SOC-monotone or not, which recalls Conjecture 2.1(b). The following counterexample shows that Conjecture 2.1(b) does not hold generally. To the contrast, Conjecture 2.1(a) remains open.

Example 2.15. *Let* $f : (0,\infty) \to \mathbb{R}$ *be* $f(t) = \begin{cases} -t\ln t + t & \text{if} \quad t \in (0,1), \\ 1 & \text{if} \quad t \in [1,+\infty). \end{cases}$ *Then, the function* $f(t)$ *is not SOC-monotone.*

Solution. This function is continuously differentiable, nondecreasing and concave on $(0,+\infty)$. However, letting $t_1 = 0.1$ and $t_2 = 3$,

$$f'(t_1)f'(t_2) - \left(\frac{f(t_1) - f(t_2)}{t_1 - t_2}\right)^2 = -\left(\frac{-t_1\ln t_1 + t_1 - 1}{t_1 - t_2}\right)^2 = -0.0533.$$

By Proposition 2.21, we know that the function f is not SOC-monotone. ■

Chapter 3

Algorithmic Applications

In this Chapter, we will see details about how the characterizations established in Chapter 2 be applied in real algorithms. In particular, the SOC-convexity are often involved in the solution methods of convex SOCPs; for example, the proximal-like methods. We present three types of proximal-like algorithms, and refer the readers to [116, 117, 119] for their numerical performance.

3.1 Proximal-like algorithm for SOCCP

In this section, we focus on the convex second-order cone program (CSOCP) whose mathematical format is

$$\begin{array}{ll} \min & f(\zeta) \\ \text{s.t.} & A\zeta + b \succeq_{\mathcal{K}^n} 0, \end{array} \tag{3.1}$$

where A is an $n \times m$ matrix with $n \geq m$, $b \in \mathbb{R}^n$, $f : \mathbb{R}^m \to (-\infty, \infty]$ is a closed proper convex function. Here $\mathcal{K}^n$ is the second-order cone given as in (1.1), i.e.,

$$\mathcal{K}^n := \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1 \right\},$$

and $x \succeq_{\mathcal{K}^n} 0$ means $x \in \mathcal{K}^n$. Note that a function is closed if and only if it is lower semi-continuous (l.s.c. for short) and a function is proper if $f(\zeta) < \infty$ for at least one $\zeta \in \mathbb{R}^m$ and $f(\zeta) > -\infty$ for all $\zeta \in \mathbb{R}^m$. The CSOCP, as an extension of the standard second-order cone programming (SOCP), has applications in a broad range of fields including engineering, control, data science, finance, robust optimization, and combinatorial optimization; see [1, 28, 46, 49, 80, 99, 103, 105, 136] and references therein.

Recently, the SOCP has received much attention in optimization, particularly in the context of solutions methods. Note that the CSOCP is a special class of convex programs, and therefore it can be solved via general convex programming methods. One of these methods is the proximal point algorithm for minimizing a convex function $f(\zeta)$ defined on $\mathbb{R}^m$ which replaces the problem $\min_{\zeta\in\mathbb{R}^m} f(\zeta)$ by a sequence of minimization problems with strictly convex objectives, generating a sequence $\{\zeta^k\}$ defined by

$$\zeta^k = \operatorname{argmin}_{\zeta\in\mathbb{R}^m}\left\{f(\zeta) + \frac{1}{2\mu_k}\|\zeta - \zeta^{k-1}\|^2\right\}, \tag{3.2}$$

where $\{\mu_k\}$ is a sequence of positive numbers and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^m$. The method was due to Martinet [106] who introduced the above proximal minimization problem based on the Moreau proximal approximation [111] of f. The proximal point algorithm was then further developed and studied by Rockafellar [132, 133]. Later, several researchers [35, 40, 60, 61, 144] proposed and investigated nonquadratic proximal point algorithm for the convex programming with nonnegative constraints, by replacing the quadratic distance in (3.2) with other distance-like functions. Among others, Censor and Zenios [35] replaced the method (3.2) by a method of the form

$$\zeta^k = \operatorname{argmin}_{\zeta\in\mathbb{R}^m}\left\{f(\zeta) + \frac{1}{\mu_k}D(\zeta, \zeta^k)\right\}, \tag{3.3}$$

where $D(\cdot,\cdot)$, called D-function, is a measure of distance based on a Bregman function. Recall that, given a differentiable function φ, it is called a *Bregman function* [34, 55] if it satisfies the properties listed in Definition 3.1 below, and the induced *D-function* is given as follows:

$$D(\zeta,\xi) := \varphi(\zeta) - \varphi(\xi) - \langle\nabla\varphi(\xi),\ \zeta - \xi\rangle, \tag{3.4}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbb{R}^m$ and $\nabla\varphi$ denotes the gradient of φ.

Definition 3.1. *Let $S \subseteq \mathbb{R}^m$ be an open set and $\bar{S}$ be its closure. The function $\varphi : \bar{S} \to \mathbb{R}$ is called a Bregman function with zone S if the following properties hold:*

(i) *φ is continuously differentiable on S;*

(ii) *φ is strictly convex and continuous on $\bar{S}$;*

(iii) *For each $\gamma \in \mathbb{R}$, the level sets $L_D(\xi,\gamma) = \{\zeta \in \bar{S} : D(\zeta,\xi) \le \gamma\}$ and $L_D(\zeta,\gamma) = \{\xi \in S : D(\zeta,\xi) \le \gamma\}$ are bounded for any $\xi \in S$ and $\zeta \in \bar{S}$, respectively;*

(iv) *If $\{\xi^k\} \subset S$ converges to ξ^*, then $D(\xi^*,\xi^k) \to 0$;*

(v) *If $\{\zeta^k\}$ and $\{\xi^k\}$ are sequences such that $\xi^k \to \xi^* \in \bar{S}$, $\{\zeta^k\}$ is bounded and if $D(\zeta^k,\xi^k) \to 0$, then $\zeta^k \to \xi^*$.*

The Bregman proximal minimization (BPM) method described in (3.3) was further extended by Kiwiel [90] with generalized Bregman functions, called B-functions. Compared with Bregman functions, these functions are possibly nondifferentiable and infinite on the boundary of their domain. For the detailed definition of B-functions and the convergence of BPM method using B-functions, please refer to [90].

Next, we present a class of distance measures on SOC and discuss its relations with the D-function and the double-regularized Bregman distance [137]. To the end, we need a class of functions $\phi : [0,\infty) \to \mathbb{R}$ satisfying

(T1) ϕ is continuously differentiable on $\mathbb{R}_{++}$;

(T2) ϕ is strictly convex and continuous on $\mathbb{R}_{+}$;

(T3) For each $\gamma \in \mathbb{R}$, the level sets $\{s \in \mathbb{R}_+ \,|\, d(s,t) \le \gamma\}$ and $\{t \in \mathbb{R}_{++} \,|\, d(s,t) \le \gamma\}$ are bounded for any $t \in \mathbb{R}_{++}$ and $s \in \mathbb{R}_+$, respectively;

(T4) If $\{t^k\} \subset \mathbb{R}_{++}$ is a sequence such that $\lim_{k\to+\infty} t^k = 0$, then for all $s \in \mathbb{R}_{++}$, $\lim_{k\to+\infty} \phi'(t^k)(s-t^k) = -\infty$;

where the function $d : [0,\infty) \times (0,\infty) \to \mathbb{R}$ is defined by

$$d(s,t) = \phi(s) - \phi(t) - \phi'(t)(s-t), \quad \forall s \in \mathbb{R}_+,\ t \in \mathbb{R}_{++}. \tag{3.5}$$

The function ϕ satisfying (T4) is said in [81–83] to be *boundary coercive*. If setting $\phi(x) = +\infty$ when $x \notin \mathbb{R}_+$, then ϕ becomes a closed proper strictly convex function on $\mathbb{R}$. Furthermore, by [90, Lemma 2.4(d)] and (T3), it is not difficult to see that $\phi(x)$ and $\sum_{i=1}^n \phi(x_i)$ are an B-function on $\mathbb{R}$ and $\mathbb{R}^n$, respectively. Unless otherwise stated, in the rest of this section, we always assume that ϕ satisfies (T1)-(T4).

Using (1.8), the corresponding SOC functions of ϕ and ϕ' are given by

$$\phi^{\rm soc}(x) = \phi\left(\lambda_1(x)\right) u_x^{(1)} + \phi\left(\lambda_2(x)\right) u_x^{(2)}, \tag{3.6}$$

and

$$(\phi')^{\rm soc}(x) = \phi'\left(\lambda_1(x)\right) u_x^{(1)} + \phi'\left(\lambda_2(x)\right) u_x^{(2)}, \tag{3.7}$$

which are well-defined over $\mathcal{K}^n$ and ${\rm int}(\mathcal{K}^n)$, respectively. In view of this, we define

$$H(x,y) := \begin{cases} {\rm tr}\left[\phi^{\rm soc}(x) - \phi^{\rm soc}(y) - (\phi')^{\rm soc}(y)\circ(x-y)\right] & \forall x \in \mathcal{K}^n, y \in {\rm int}(\mathcal{K}^n), \\ \infty & \text{otherwise.} \end{cases} \tag{3.8}$$

In what follows, we will show that the function $H : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, +\infty]$ enjoys some favorable properties similar to those of the D-function. Particularly, we prove that $H(x,y) \ge 0$ for any $x \in \mathcal{K}^n, y \in {\rm int}(\mathcal{K}^n)$, and moreover, $H(x,y) = 0$ if and only if $x = y$. Consequently, it can be regarded as a distance measure on the SOC.

We first start with a technical lemma that will be used in the subsequent analysis.

Lemma 3.1. *Suppose that* $\phi : [0, \infty) \to \mathbb{R}$ *satisfies (T1)-(T4). Let* $\phi^{\rm soc}(x)$ *and* $(\phi')^{\rm soc}(x)$ *be given as in (3.6) and (3.7), respectively. Then, the following hold.*

(a) $\phi^{\rm soc}(x)$ *is continuously differentiable on* $\mathrm{int}(\mathcal{K}^n)$ *with the gradient* $\nabla\phi^{\rm soc}(x)$ *satisfying* $\nabla\phi^{\rm soc}(x)e = (\phi')^{\rm soc}(x)$.

(b) $\mathrm{tr}[\phi^{\rm soc}(x)] = \sum_{i=1}^{2} \phi[\lambda_i(x)]$ *and* $\mathrm{tr}[(\phi')^{\rm soc}(x)] = \sum_{i=1}^{2} \phi'[\lambda_i(x)]$.

(c) $\mathrm{tr}[\phi^{\rm soc}(x)]$ *is continuously differentiable on* $\mathrm{int}(\mathcal{K}^n)$ *with* $\nabla \mathrm{tr}[\phi^{\rm soc}(x)] = 2\nabla\phi^{\rm soc}(x)e$.

(d) $\mathrm{tr}[\phi^{\rm soc}(x)]$ *is strictly convex and continuous on* $\mathrm{int}(\mathcal{K}^n)$.

(e) *If* $\{y^k\} \subset \mathrm{int}(\mathcal{K}^n)$ *is a sequence such that* $\lim_{k\to+\infty} y^k = \bar{y} \in \mathrm{bd}(\mathcal{K}^n)$, *then*

$$\lim_{k\to+\infty} \langle \nabla \mathrm{tr}[\phi^{\rm soc}(y^k)],\ x - y^k\rangle = -\infty \quad \text{for all } x \in \mathrm{int}(\mathcal{K}^n).$$

In other words, the function $\mathrm{tr}[\phi^{\rm soc}(x)]$ *is boundary coercive.*

Proof. (a) The first part follows directly from Proposition 1.14. Now we prove the second part. If $x_2 \neq 0$, then by formulas (1.28)-(1.29) it is easy to compute that

$$\nabla\phi^{\rm soc}(x)e = \begin{bmatrix} \dfrac{\phi'(\lambda_2(x)) + \phi'(\lambda_1(x))}{2} \\ \dfrac{\phi'(\lambda_2(x)) - \phi'(\lambda_1(x))}{2}\dfrac{x_2}{\|x_2\|} \end{bmatrix}.$$

In addition, using equations (1.4) and (3.7), we can prove that the vector in the right hand side is exactly $(\phi')^{\rm soc}(x)$. Therefore, $\nabla\phi^{\rm soc}(x)e = (\phi')^{\rm soc}(x)$. If $x_2 = 0$, then using (1.27) and (1.4), we can also prove that $\nabla\phi^{\rm soc}(x)e = (\phi')^{\rm soc}(x)$.

(b) The result follows directly from Property 1.1(d) and equations (3.6)-(3.7).

(c) From part(a) and the fact that $\mathrm{tr}[\phi^{\rm soc}(x)] = \mathrm{tr}[\phi^{\rm soc}(x) \circ e] = 2\langle\phi^{\rm soc}(x), e\rangle$, clearly, $\mathrm{tr}[\phi^{\rm soc}(x)]$ is continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$. Applying the chain rule for inner product of two functions immediately yields that $\nabla\mathrm{tr}[\phi^{\rm soc}(x)] = 2\nabla\phi^{\rm soc}(x)e$.

(d) It is clear that $\phi^{\rm soc}(x)$ is continuous on $\mathcal{K}^n$. We next prove that it is strictly convex on $\mathrm{int}(\mathcal{K}^n)$. For any $x, y \in \mathcal{K}^n$ with $x \neq y$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$, we have

$$\begin{aligned} \lambda_1(\alpha x + \beta y) &= \alpha x_1 + \beta y_1 - \|\alpha x_2 + \beta y_2\| \geq \alpha\lambda_1(x) + \beta\lambda_1(y), \\ \lambda_2(\alpha x + \beta y) &= \alpha x_1 + \beta y_1 + \|\alpha x_2 + \beta y_2\| \leq \alpha\lambda_2(x) + \beta\lambda_2(y), \end{aligned}$$

which implies that

$$\alpha\lambda_1(x) + \beta\lambda_1(y) \leq \lambda_1(\alpha x + \beta y) \leq \lambda_2(\alpha x + \beta y) \leq \alpha\lambda_2(x) + \beta\lambda_2(y).$$

On the other hand,

$$\lambda_1(\alpha x + \beta y) + \lambda_2(\alpha x + \beta y) = 2\alpha x_1 + 2\beta y_1 = [\alpha\lambda_1(x) + \beta\lambda_1(y)] + [\alpha\lambda_2(x) + \beta\lambda_2(y)].$$

The last two equations imply that there exists $\rho \in [0,1]$ such that

$$\begin{aligned}
\lambda_1(\alpha x + \beta y) &= \rho[\alpha\lambda_1(x) + \beta\lambda_1(y)] + (1-\rho)[\alpha\lambda_2(x) + \beta\lambda_2(y)], \\
\lambda_2(\alpha x + \beta y) &= (1-\rho)[\alpha\lambda_1(x) + \beta\lambda_1(y)] + \rho[\alpha\lambda_2(x) + \beta\lambda_2(y)].
\end{aligned}$$

Thus, from Property 1.1, it follows that

$$\begin{aligned}
\mathrm{tr}[\phi^{\mathrm{soc}}(\alpha x + \beta y)] &= \phi[\lambda_1(\alpha x + \beta y)] + \phi[\lambda_2(\alpha x + \beta y)] \\
&= \phi\Big[\rho(\alpha\lambda_1(x) + \beta\lambda_1(y)) + (1-\rho)(\alpha\lambda_2(x) + \beta\lambda_2(y))\Big] \\
&\quad + \phi\Big[(1-\rho)(\alpha\lambda_1(x) + \beta\lambda_1(y)) + \rho(\alpha\lambda_2(x) + \beta\lambda_2(y))\Big] \\
&\leq \rho\phi\big(\alpha\lambda_1(x) + \beta\lambda_1(y)\big) + (1-\rho)\phi\big(\alpha\lambda_2(x) + \beta\lambda_2(y)\big) \\
&\quad + (1-\rho)\phi\big(\alpha\lambda_1(x) + \beta\lambda_1(y)\big) + \rho\phi\big(\alpha\lambda_2(x) + \beta\lambda_2(y)\big) \\
&= \phi\big(\alpha\lambda_1(x) + \beta\lambda_1(y)\big) + \phi\big(\alpha\lambda_2(x) + \beta\lambda_2(y)\big) \\
&< \alpha\phi\big(\lambda_1(x)\big) + \beta\phi\big(\lambda_1(y)\big) + \alpha\phi\big(\lambda_2(x)\big) + \beta\phi\big(\lambda_2(y)\big) \\
&= \alpha\mathrm{tr}[\phi^{\mathrm{soc}}(x)] + \beta\mathrm{tr}[\phi^{\mathrm{soc}}(y)],
\end{aligned}$$

where the first equality and the last one follow from part(b), and the two inequalities are due to the strict convexity of ϕ on $\mathbb{R}_{++}$. From the definition of strict convexity, we thus prove that the conclusion holds.

(e) From part(a) and part(c), we can readily obtain the following equality

$$\nabla\mathrm{tr}[\phi^{\mathrm{soc}}(x)] = 2(\phi')^{\mathrm{soc}}(x), \quad \forall x \in \mathrm{int}(\mathcal{K}^n). \tag{3.9}$$

Using the relation and Proposition 1.3, we then have

$$\begin{aligned}
\langle\nabla\mathrm{tr}[\phi^{\mathrm{soc}}(y^k)],\ x - y^k\rangle &= 2\langle(\phi')^{\mathrm{soc}}(y^k),\ x - y^k\rangle \\
&= \mathrm{tr}[(\phi')^{\mathrm{soc}}(y^k) \circ (x - y^k)] \\
&= \mathrm{tr}[(\phi')^{\mathrm{soc}}(y^k) \circ x] - \mathrm{tr}[(\phi')^{\mathrm{soc}}(y^k) \circ y^k] \\
&\leq \sum_{i=1}^{2} \phi'[\lambda_i(y^k)]\lambda_i(x) - \mathrm{tr}[(\phi')^{\mathrm{soc}}(y^k) \circ y^k].
\end{aligned} \tag{3.10}$$

In addition, by Property 1.1, for any $y \in \mathrm{int}(\mathcal{K}^n)$, we can compute that

$$\begin{aligned}
(\phi')^{\mathrm{soc}}(y) \circ y &= \Big[\phi'(\lambda_1(y))u_y^{(1)} + \phi'(\lambda_2(y))u_y^{(2)}\Big] \circ \Big[\lambda_1(y)u_y^{(1)} + \lambda_2(y)u_y^{(2)}\Big] \\
&= \phi'(\lambda_1(y))\lambda_1(y)u_y^{(1)} + \phi'(\lambda_2(y))\lambda_2(y)u_y^{(2)},
\end{aligned} \tag{3.11}$$

which implies that

$$\mathrm{tr}[(\phi')^{\mathrm{soc}}(y^k) \circ y^k] = \sum_{i=1}^{2} \phi'[\lambda_i(y^k)]\lambda_i(y^k). \tag{3.12}$$

Combining with (3.10) and (3.12) immediately yields that

$$\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y^k)],\ x - y^k \rangle \le \sum_{i=1}^{2} \phi'[\lambda_i(y^k)][\lambda_i(x) - \lambda_i(y^k)]. \tag{3.13}$$

Note that $\lambda_2(\bar{y}) \ge \lambda_1(\bar{y}) = 0$ and $\lambda_2(x) \ge \lambda_1(x) > 0$ since $\bar{y} \in \mathrm{bd}(\mathcal{K}^n)$ and $x \in \mathrm{int}(K^n)$. Hence, if $\lambda_2(\bar{y}) = 0$, then by (T4) and the continuity of $\lambda_i(\cdot)$ for $i = 1, 2$,

$$\lim_{k \to +\infty} \phi'[\lambda_i(y^k)][\lambda_i(x) - \lambda_i(y^k)] = -\infty, \quad i = 1, 2,$$

which means that

$$\lim_{k \to +\infty} \sum_{i=1}^{2} \phi'[\lambda_i(y^k)][\lambda_i(x) - \lambda_i(y^k)] = -\infty. \tag{3.14}$$

If $\lambda_2(\bar{y}) > 0$, then $\lim_{k \to +\infty} \phi'[\lambda_2(y^k)][\lambda_2(x) - \lambda_2(y^k)]$ is finite and

$$\lim_{k \to +\infty} \phi'[\lambda_1(y^k)][\lambda_1(x) - \lambda_1(y^k)] = -\infty,$$

and therefore the result in (3.14) also holds under such case. Combining (3.14) with (3.13), we prove that the conclusion holds. □

Using the relation in (3.9), we have that for any $x \in \mathcal{K}^n$ and $y \in \mathrm{int}(K^n)$,

$$\mathrm{tr}\Big[(\phi')^{\mathrm{soc}}(y) \circ (x - y)\Big] = 2\Big\langle (\phi')^{\mathrm{soc}}(y),\ x - y \Big\rangle = \Big\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)],\ x - y \Big\rangle.$$

As a consequence, the function $H(x, y)$ in (3.8) can be rewritten as

$$H(x,y) = \begin{cases} \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - \langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)], x - y \rangle & \forall x \in \mathcal{K}^n, y \in \mathrm{int}(\mathcal{K}^n), \\ \infty & \text{otherwise.} \end{cases} \tag{3.15}$$

By the representation, we next investigate several important properties of $H(x, y)$.

Proposition 3.1. *Let $H(x, y)$ be the function defined as in (3.8) or (3.15). Then, the following hold.*

(a) *$H(x, y)$ is continuous on $\mathcal{K}^n \times \mathrm{int}(\mathcal{K}^n)$, and for any $y \in \mathrm{int}(\mathcal{K}^n)$, the function $H(\cdot, y)$ is strictly convex on $\mathcal{K}^n$.*

(b) *For any given $y \in \mathrm{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$ with*

$$\nabla_x H(x, y) = \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)] = 2\big[(\phi')^{\mathrm{soc}}(x) - (\phi')^{\mathrm{soc}}(y)\big]. \tag{3.16}$$

(c) *$H(x, y) \ge \sum_{i=1}^{2} d(\lambda_i(x), \lambda_i(y)) \ge 0$ for any $x \in \mathcal{K}^n$ and $y \in \mathrm{int}(\mathcal{K}^n)$, where $d(\cdot, \cdot)$ is defined by (3.5). Moreover, $H(x, y) = 0$ if and only if $x = y$.*

(d) *For every $\gamma \in \mathbb{R}$, the partial level sets of $L_H(y,\gamma) = \{x \in \mathcal{K}^n : H(x,y) \leq \gamma\}$ and $L_H(x,\gamma) = \{y \in \mathrm{int}(\mathcal{K}^n) : H(x,y) \leq \gamma\}$ are bounded for any $y \in \mathrm{int}(\mathcal{K}^n)$ and $x \in \mathcal{K}^n$, respectively.*

(e) *If $\{y^k\} \subset \mathrm{int}(\mathcal{K}^n)$ is a sequence converging to $y^* \in \mathrm{int}(\mathcal{K}^n)$, then $H(y^*, y^k) \to 0$.*

(f) *If $\{x^k\} \subset \mathrm{int}(\mathcal{K}^n)$ and $\{y^k\} \subset \mathrm{int}(\mathcal{K}^n)$ are sequences such that $\{y^k\} \to y^* \in \mathrm{int}(\mathcal{K}^n)$, $\{x^k\}$ is bounded, and $H(x^k, y^k) \to 0$, then $x^k \to y^*$.*

Proof. (a) Note that $\phi^{\mathrm{soc}}(x)$, $(\phi')^{\mathrm{soc}}(y)$, $(\phi')^{\mathrm{soc}}(y) \circ (x-y)$ are continuous for any $x \in \mathcal{K}^n$ and $y \in \mathrm{int}(\mathcal{K}^n)$ and the trace function $\mathrm{tr}(\cdot)$ is also continuous, and hence $H(x,y)$ is continuous on $\mathcal{K}^n \times \mathrm{int}(\mathcal{K}^n)$. From Lemma 3.1(d), $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ is strictly convex over $\mathcal{K}^n$, whereas $-\mathrm{tr}[\phi^{\mathrm{soc}}(y)] - \langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)], x-y\rangle$ is clearly convex in $\mathcal{K}^n$ for fixed $y \in \mathrm{int}(\mathcal{K}^n)$. This means that $H(\cdot, y)$ is strictly convex for any $y \in \mathrm{int}(\mathcal{K}^n)$.

(b) By Lemma 3.1(c), the function $H(\cdot, y)$ for any given $y \in \mathrm{int}(\mathcal{K}^n)$ is continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$. The first equality in (3.16) is obvious and the second is due to (3.9).

(c) The result follows directly from the following equalities and inequalities:

$$\begin{aligned}
H(x,y) &= \mathrm{tr}\big[\phi^{\mathrm{soc}}(x)\big] - \mathrm{tr}\big[\phi^{\mathrm{soc}}(y)\big] - \mathrm{tr}\big[(\phi')^{\mathrm{soc}}(y) \circ (x-y)\big] \\
&= \mathrm{tr}\big[\phi^{\mathrm{soc}}(x)\big] - \mathrm{tr}\big[\phi^{\mathrm{soc}}(y)\big] - \mathrm{tr}\big[(\phi')^{\mathrm{soc}}(y) \circ x\big] + \mathrm{tr}\big[(\phi')^{\mathrm{soc}}(y) \circ y\big] \\
&\geq \mathrm{tr}\big[\phi^{\mathrm{soc}}(x)\big] - \mathrm{tr}\big[\phi^{\mathrm{soc}}(y)\big] - \sum_{i=1}^{2} \phi'(\lambda_i(y))\lambda_i(x) + \mathrm{tr}\big[(\phi')^{\mathrm{soc}}(y) \circ y\big] \\
&= \sum_{i=1}^{2} \Big[\phi(\lambda_i(x)) - \phi(\lambda_i(y)) - \phi'(\lambda_i(y))\lambda_i(x) + \phi'(\lambda_i(y))\lambda_i(y)\Big] \\
&= \sum_{i=1}^{2} \Big[\phi(\lambda_i(x)) - \phi(\lambda_i(y)) - \phi'(\lambda_i(y))(\lambda_i(x) - \lambda_i(y))\Big] \\
&= \sum_{i=1}^{2} d(\lambda_i(x), \lambda_i(y)) \geq 0,
\end{aligned}$$

where the first equality is due to (3.8), the second and fourth are obvious, the third follows from Lemma 3.1(b) and (3.11), the last one is from (3.5), and the first inequality follows from Proposition 1.3 and the last one is due to the strict convexity of ϕ on $\mathbb{R}_+$. Note that $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ is strictly convex for any $x \in \mathcal{K}^n$ by Lemma 3.1(d), and therefore $H(x,y) = 0$ if and only if $x = y$ by (3.15).

(d) From part(c), we have that $L_H(y,\gamma) \subseteq \{x \in \mathcal{K}^n |\ \sum_{i=1}^{2} d(\lambda_i(x), \lambda_i(y)) \leq \gamma\}$. By (T3), the set in the right hand side is bounded. Thus, $L_H(y,\gamma)$ is bounded for $y \in \mathrm{int}(\mathcal{K}^n)$. Similarly, $L_H(x,\gamma)$ is bounded for $x \in \mathcal{K}^n$.

From part(a)-(d), we immediately obtain the results in (e) and (f). $\square$

Remark 3.1. **(i)** *From (3.8), it is not difficult to see that* $H(x,y)$ *is exactly a distance measure induced by* $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ *via formula (3.4). Therefore, if* $n=1$ *and* ϕ *is a Bregman function with zone* $\mathbb{R}_{++}$*, i.e.,* ϕ *also satisfies the property:*

(e) *if* $\{s^k\} \subseteq \mathbb{R}_+$ *and* $\{t^k\} \subset \mathbb{R}_{++}$ *are sequences such that* $t^k \to t^*$*,* $\{s^k\}$ *is bounded, and* $d(s^k,t^k) \to 0$*, then* $s^k \to t^*$*;*

then $H(x,y)$ *reduces to the Bregman distance function* $d(x,y)$ *in (3.5).*

(ii) *When* $n>1$*,* $H(x,y)$ *is generally not a Bregman distance even if* ϕ *is a Bregman function with zone* $\mathbb{R}_{++}$*, by noting that Proposition 3.1(e) and (f) do not hold for* $\{x^k\} \subseteq \mathrm{bd}(\mathcal{K}^n)$ *and* $y^* \in \mathrm{bd}(\mathcal{K}^n)$*. By the proof of Proposition 3.1(c), the main reason is that in order to guarantee that*

$$\mathrm{tr}[(\phi')^{\mathrm{soc}}(y) \circ x] = \sum_{i=1}^{2} \phi'(\lambda_i(y))\lambda_i(x)$$

for any $x \in \mathcal{K}^n$ *and* $y \in \mathrm{int}(\mathcal{K}^n)$*, the relation* $[(\phi')^{\mathrm{soc}}(y)]_2 = \alpha x_2$ *with some* $\alpha > 0$ *is required, where* $[(\phi')^{\mathrm{soc}}(y)]_2$ *is a vector composed of the last* $n-1$ *elements of* $(\phi')^{\mathrm{soc}}(y)$*. It is very stringent for* ϕ *to satisfy such relation. By this,* $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ *is not a B-function [90] on* $\mathbb{R}^n$*, either, even if* ϕ *itself is a B-function.*

(iii) *We observe that* $H(x,y)$ *is inseparable, whereas the double-regularized distance function proposed by [137] belongs to the separable class of functions. In view of this,* $H(x,y)$ *can not become a double-regularized distance function in* $\mathcal{K}^n \times \mathrm{int}(\mathcal{K}^n)$*, even when* ϕ *is such that* $\tilde{d}(s,t) = d(s,t)/\phi''(t) + \frac{\mu}{2}(s-t)^2$ *is a double regularized component (see [137]).*

In view of Proposition 3.1 and Remark 3.1, we call $H(x,y)$ a *quasi D-function.* In the following, we present several specific examples of quasi D-functions.

Example 3.1. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be* $\phi(t) = t\ln t - t$ *with the convention* $0 \ln 0 = 0$*.*

Solution. It is easy to verify that ϕ satisfies (T1)-(T4). By [64, Proposition 3.2 (b)] and (3.6)-(3.7), we can compute that for any $x \in \mathcal{K}^n$ and $y \in \mathrm{int}(\mathcal{K}^n)$,

$$\phi^{\mathrm{soc}}(x) = x \circ \ln x - x \quad \text{and} \quad (\phi')^{\mathrm{soc}}(y) = \ln y.$$

Therefore, we obtain

$$H(x,y) = \begin{cases} \mathrm{tr}(x \circ \ln x - x \circ \ln y + y - x), & \forall x \in \mathcal{K}^n, y \in \mathrm{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise}, \end{cases}$$

which is a quasi D-function. ■

Example 3.2. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be* $\phi(t) = t^2 - \sqrt{t}$.

Solution. It is not hard to verify that ϕ satisfies (T1)-(T4). From Property 1.2, we have that for any $x \in \mathcal{K}^n$,

$$x^2 = x \circ x = \lambda_1^2(x)u_x^{(1)} + \lambda_2^2(x)u_x^{(2)} \quad \text{and} \quad x^{1/2} = \sqrt{\lambda_1(x)}u_x^{(1)} + \sqrt{\lambda_2(x)}u_x^{(2)}.$$

By a direct computation, we then obtain for any $x \in \mathcal{K}^n$ and $y \in \text{int}(\mathcal{K}^n)$,

$$\phi^{\text{soc}}(x) = x \circ x - x^{1/2} \quad \text{and} \quad (\phi')^{\text{soc}}(y) = 2y - \frac{\text{tr}(y^{1/2})e - y^{1/2}}{2\sqrt{\det(y)}}.$$

This yields

$$H(x,y) = \begin{cases} \text{tr}\left[(x-y)^2 - (x^{1/2} - y^{1/2}) + \dfrac{(\text{tr}(y^{1/2})e - y^{1/2}) \circ (x-y)}{2\sqrt{\det(y)}}\right], & \forall x \in \mathcal{K}^n, y \in \text{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise}, \end{cases}$$

which is a quasi D-function. ■

Example 3.3. *Let* $\phi : [0,\infty) \to \mathbb{R}$ *be* $\phi(t) = t\ln t - (1+t)\ln(1+t) + (1+t)\ln 2$ *with the convention* $0 \ln 0 = 0$.

Solution. It is easily shown that ϕ satisfies (T1)-(T4) Using Property 1.1, we know that for any $x \in \mathcal{K}^n$ and $y \in \text{int}(\mathcal{K}^n)$,

$$\phi^{\text{soc}}(x) = x \circ \ln x - (e+x) \circ \ln(e+x) + (e+x)\ln 2$$

and

$$(\phi')^{\text{soc}}(y) = \ln y - \ln(e+y) + e\ln 2.$$

Consequently, we obtain

$$H(x,y) = \begin{cases} \text{tr}\big[x \circ (\ln x - \ln y) - (e+x) \circ (\ln(e+x) - \ln(e+y))\big], & \forall x \in \mathcal{K}^n, y \in \text{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise}, \end{cases}$$

which is a quasi D-function. ■

In addition, from [81, 83, 144], it follows that $\sum_{i=1}^m \phi(\zeta_i)$ generated by ϕ in the above examples is a Bregman function with zone $S = \mathbb{R}_+^m$, and consequently $\sum_{i=1}^m d(\zeta_i, \xi_i)$ defined as in (3.5) is a D-function induced by $\sum_{i=1}^m \phi(\zeta_i)$.

Proposition 3.2. *Let* $H(x,y)$ *be defined as in (3.8) or (3.15). Then, for all* $x, y \in \text{int}(\mathcal{K}^n)$ *and* $z \in \mathcal{K}^n$*, the following three-points identity holds:*

$$\begin{aligned} H(z,x) + H(x,y) - H(z,y) &= \Big\langle \nabla\text{tr}[\phi^{\text{soc}}(y)] - \nabla\text{tr}[\phi^{\text{soc}}(x)],\ z - x \Big\rangle \\ &= \text{tr}\left[\Big((\phi')^{\text{soc}}(y) - (\phi')^{\text{soc}}(x)\Big) \circ (z-x)\right]. \end{aligned}$$

Proof. Using the definition of H given as in (3.15), we have

$$\begin{aligned}
\Big\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(x)],\ z-x \Big\rangle &= \mathrm{tr}[\phi^{\mathrm{soc}}(z)] - \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - H(z,x), \\
\Big\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)],\ x-y \Big\rangle &= \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - H(x,y), \\
\Big\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)],\ z-y \Big\rangle &= \mathrm{tr}[\phi^{\mathrm{soc}}(z)] - \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - H(z,y).
\end{aligned}$$

Subtracting the first two equations from the last one gives the first equality. By (3.9),

$$\Big\langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(x)],\ z-x \Big\rangle = 2\Big\langle (\phi')^{\mathrm{soc}}(y) - (\phi')^{\mathrm{soc}}(x),\ z-y \Big\rangle.$$

This together with the fact that $\mathrm{tr}(x \circ y) = \langle x, y\rangle$ leads to the second equality. □

In this section, we propose a proximal-like algorithm for solving the CSOCP based on the quasi D-function $H(x,y)$. For the sake of notation, we denote $\mathcal{F}$ by the set

$$\mathcal{F} = \Big\{ \zeta \in \mathbb{R}^m \mid A\zeta + b \succeq_{\mathcal{K}^n} 0 \Big\}. \tag{3.17}$$

It is easy to verify that $\mathcal{F}$ is convex and its interior $\mathrm{int}(\mathcal{F})$ is given by

$$\mathrm{int}(\mathcal{F}) = \Big\{ \zeta \in \mathbb{R}^m \mid A\zeta + b \succ_{\mathcal{K}^n} 0 \Big\}. \tag{3.18}$$

Let $\psi : \mathbb{R}^m \to (-\infty, +\infty]$ be the function defined by

$$\psi(\zeta) = \begin{cases} \mathrm{tr}[\phi^{\mathrm{soc}}(A\zeta + b)] & \text{if } \zeta \in \mathcal{F}, \\ \infty & \text{otherwise.} \end{cases} \tag{3.19}$$

By Lemma 3.1, it is easily shown that the following conclusions hold for $\psi(\zeta)$.

Proposition 3.3. *Let $\psi(\zeta)$ be given as in (3.19). If the matrix A has full rank m, then*

(a) *$\psi(\zeta)$ is continuously differentiable on $\mathrm{int}(\mathcal{F})$ with $\nabla\psi(\zeta) = 2A^T(\phi')^{\mathrm{soc}}(A\zeta + b)$;*

(d) *$\psi(\zeta)$ is strictly convex and continuous on $\mathcal{F}$;*

(c) *$\psi(\zeta)$ is boundary coercive, i.e., if $\{\xi^k\} \subseteq \mathrm{int}(\mathcal{F})$ such that $\lim_{k\to+\infty} \xi^k = \xi \in \mathrm{bd}(\mathcal{F})$, then for all $\zeta \in \mathrm{int}(\mathcal{F})$, there holds that $\lim_{k\to+\infty} \nabla\psi(\xi^k)^T(\zeta - \xi^k) = -\infty$.*

Let $\mathcal{D}(\zeta, \xi)$ be the function induced by the above $\psi(\zeta)$ via formula (3.4), i.e.,

$$\mathcal{D}(\zeta, \xi) = \psi(\zeta) - \psi(\xi) - \langle \nabla\psi(\xi),\ \zeta - \xi \rangle. \tag{3.20}$$

Then, from (3.15) and (3.19), it is not difficult to see that

$$\mathcal{D}(\zeta, \xi) = H(A\zeta + b, A\xi + b). \tag{3.21}$$

Thus, by Proposition 3.1 and Lemma 3.3, we draw the following conclusions.

Proposition 3.4. *Let $\mathcal{D}(\zeta,\xi)$ be given by (3.20) or (3.21). If the matrix A has full rank m, then*

(a) *$\mathcal{D}(\zeta,\xi)$ is continuous on $\mathcal{F}\times\mathrm{int}(\mathcal{F})$, and for any given $\xi\in\mathrm{int}(\mathcal{F})$, the function $\mathcal{D}(\cdot,\xi)$ is strictly convex on $\mathcal{F}$.*

(b) *For any fixed $\xi\in\mathrm{int}(\mathcal{F})$, $\mathcal{D}(\cdot,\xi)$ is continuously differentiable on $\mathrm{int}(\mathcal{F})$ with*

$$\nabla_\zeta\mathcal{D}(\zeta,\xi)=\nabla\psi(\zeta)-\nabla\psi(\xi)=2A^T\Big[(\phi')^{\mathrm{soc}}(A\zeta+b)-(\phi')^{\mathrm{soc}}(A\xi+b)\Big].$$

(c) *$\mathcal{D}(\zeta,\xi)\geq\sum_{i=1}^{2}d(\lambda_i(A\zeta+b),\lambda_i(A\xi+b))\geq 0$ for any $\zeta\in\mathcal{F}$ and $\xi\in\mathrm{int}(\mathcal{F})$, where $d(\cdot,\cdot)$ is defined by (3.5). Moreover, $\mathcal{D}(\zeta,\xi)=0$ if and only if $\zeta=\xi$.*

(d) *For each $\gamma\in\mathbb{R}$, the partial level sets of $L_{\mathcal{D}}(\xi,\gamma)=\{\zeta\in\mathcal{F}:\mathcal{D}(\zeta,\xi)\leq\gamma\}$ and $L_{\mathcal{D}}(\zeta,\gamma)=\{\xi\in\mathrm{int}(\mathcal{F}):\mathcal{D}(\zeta,\xi)\leq\gamma\}$ are bounded for any $\xi\in\mathrm{int}(\mathcal{F})$ and $\zeta\in\mathcal{F}$, respectively.*

The PLA. The first **proximal-like algorithm** that we propose for the CSOCP (3.1) is defined as follows:

$$\left\{\begin{array}{lcl}\zeta^0 & \in & \mathrm{int}(\mathcal{F}),\\ \zeta^k & = & \mathrm{argmin}_{\zeta\in\mathcal{F}}\left\{f(\zeta)+(1/\mu_k)\mathcal{D}(\zeta,\zeta^{k-1})\right\}\quad(k\geq 1),\end{array}\right. \tag{3.22}$$

where $\{\mu_k\}_{k\geq1}$ is a sequence of positive numbers. To establish the convergence of the algorithm, we make the following Assumptions for the CSOCP:

(A1) $\inf\Big\{f(\zeta)\mid\zeta\in\mathcal{F}\Big\}:=f_*>-\infty$ and $\mathrm{dom}(f)\cap\mathrm{int}(\mathcal{F})\neq\emptyset$.

(A2) The matrix A is of maximal rank m.

Remark 3.2. *Assumption (A1) is elementary for the solution of the CSOCP. Assumption (A2) is common in the solution of SOCPs and it is obviously satisfied when $\mathcal{F}=\mathcal{K}^n$. Moreover, if we consider the standard SOCP*

$$\begin{array}{ll}\min & c^Tx\\ \text{s.t.} & Ax=b,\ x\in\mathcal{K}^n,\end{array}$$

where $A\in\mathbb{R}^{m\times n}$ with $m\leq n$, $b\in\mathbb{R}^m$, and $c\in\mathbb{R}^n$, the assumption that A has full row rank m is standard. Consequently, its dual problem, given by

$$\begin{array}{ll}\max & b^Ty\\ \text{s.t.} & c-A^Ty\succeq_{\mathcal{K}^n}0,\end{array} \tag{3.23}$$

satisfies assumption (A2). This shows that we can solve the SOCP by applying the proximal-like algorithm (PLA) defined as in (3.22) to the dual problem (3.23).

Now, we show the algorithm PLA given by (3.22) is well-defined under assumptions (A1) and (A2).

Proposition 3.5. *Suppose that assumptions (A1)-(A2) hold. Then, the algorithm PLA given by (3.22) generates a sequence* $\{\zeta^k\} \subset \mathrm{int}(\mathcal{F})$ *such that*

$$-2\mu_k^{-1}A^T\left[(\phi')^{\mathrm{soc}}(A\zeta^k+b)-(\phi')^{\mathrm{soc}}(A\zeta^{k-1}+b)\right]\in\partial f(\zeta^k).$$

Proof. The proof proceeds by induction. For $k=0$, it clearly holds. Assume that $\zeta^{k-1}\in\mathrm{int}(\mathcal{F})$. Let $f_k(\zeta):=f(\zeta)+\mu_k^{-1}\mathcal{D}(\zeta,\zeta^{k-1})$. Then assumption (A1) and Proposition 3.4(d) imply that f_k has bounded level sets in $\mathcal{F}$. By the lower semi-continuity of f and Proposition 3.4(a), the minimization problem $\min_{\zeta\in\mathcal{F}} f_k(\zeta)$, i.e., the subproblem in (3.22), has solutions. Moreover, the solution ζ^k is unique due to the convexity of f and the strict convexity of $\mathcal{D}(\cdot,\xi)$. In the following, we prove that $\zeta^k\in\mathrm{int}(\mathcal{F})$.

By [131, Theorem 23.8] and the definition of $\mathcal{D}(\zeta,\xi)$ given by (3.20), we can verify that ζ^k is the only $\zeta\in\mathrm{dom}(f)\cap\mathcal{F}$ such that

$$2\mu_k^{-1}A^T(\phi')^{\mathrm{soc}}(A\zeta^{k-1}+b)\in\partial\left(f(\zeta)+\mu_k^{-1}\psi(\zeta)+\delta(\zeta|\mathcal{F})\right), \tag{3.24}$$

where $\delta(\zeta|\mathcal{F})=0$ if $\zeta\in\mathcal{F}$ and $+\infty$ otherwise. We will show that

$$\partial\left(f(\zeta)+\mu_k^{-1}\psi(\zeta)+\delta(\zeta|\mathcal{F})\right)=\emptyset \quad \text{for all } \zeta\in\mathrm{bd}(\mathcal{F}), \tag{3.25}$$

which by (3.24) implies that $\zeta^k\in\mathrm{int}(\mathcal{F})$. Take $\zeta\in\mathrm{bd}(\mathcal{F})$ and assume that there exists $w\in\partial\Big(f(\zeta)+\mu_k^{-1}\psi(\zeta)\Big)$. Take $\widehat{\zeta}\in\mathrm{dom}(f)\cap\mathrm{int}(\mathcal{F})$ and let

$$\zeta^l=(1-\epsilon_l)\zeta+\epsilon_l\widehat{\zeta} \tag{3.26}$$

with $\lim_{l\to+\infty}\epsilon_l=0$. From the convexity of $\mathrm{int}(\mathcal{F})$ and $\mathrm{dom}(f)$, it then follows that $\zeta^l\in\mathrm{dom}(f)\cap\mathrm{int}(\mathcal{F})$, and moreover, $\lim_{l\to+\infty}\zeta^l=\zeta$. Consequently,

$$\begin{aligned}
\epsilon_l w^T(\widehat{\zeta}-\zeta) &= w^T(\zeta^l-\zeta)\\
&\le f(\zeta^l)-f(\zeta)+\mu_k^{-1}\Big[\psi(\zeta^l)-\psi(\zeta)\Big]\\
&\le f(\zeta^l)-f(\zeta)+\mu_k^{-1}\Big\langle 2A^T(\phi')^{\mathrm{soc}}(A\zeta^l+b),\ \zeta^l-\zeta\Big\rangle\\
&\le \epsilon_l(f(\widehat{\zeta})-f(\zeta))+\mu_k^{-1}\frac{\epsilon_l}{1-\epsilon_l}\mathrm{tr}\Big[(\phi')^{\mathrm{soc}}(A\zeta^l+b)\circ(A\widehat{\zeta}-A\zeta^l)\Big],
\end{aligned}$$

where the first equality is due to (3.26), the first inequality follows from the definition of subdifferential and the convexity of $f(\zeta)+\mu_k^{-1}\psi(\zeta)$ in $\mathcal{F}$, the second one is due to the convexity and differentiability of $\psi(\zeta)$ in $\mathrm{int}(\mathcal{F})$, and the last one is from (3.26) and the convexity of f. Using Proposition 1.3 and (3.11), we then have

$$\begin{aligned}
& \mu_k(1-\epsilon_l)[f(\zeta)-f(\widehat{\zeta})+w^T(\widehat{\zeta}-\zeta)] \\
\le\ & \mathrm{tr}\Big[(\phi')^{\mathrm{soc}}(A\zeta^l+b)\circ(A\widehat{\zeta}+b)\Big]-\mathrm{tr}\Big[(\phi')^{\mathrm{soc}}(A\zeta^l+b)\circ(A\zeta^l+b)\Big] \\
\le\ & \sum_{i=1}^{2}\Big[\phi'(\lambda_i(A\zeta^l+b))\lambda_i(A\widehat{\zeta}+b)-\phi'(\lambda_i(A\zeta^l+b))\lambda_i(A\zeta^l+b)\Big] \\
=\ & \sum_{i=1}^{2}\phi'(\lambda_i(A\zeta^l+b))\Big[\lambda_i(A\widehat{\zeta}+b)-\lambda_i(A\zeta^l+b)\Big].
\end{aligned}$$

Since $\zeta\in\mathrm{bd}(\mathcal{F})$, i.e., $A\zeta+b\in\mathrm{bd}(\mathcal{K}^n)$, it follows that $\lim_{l\to+\infty}\lambda_1(A\zeta^l+b)=0$. Thus, using (T4) and following the same line as the proof of Lemma 3.1(d), we can prove that the right hand side of the last inequality goes to $-\infty$ when l tends to ∞, whereas the left-hand side has a finite limit. This gives a contradiction. Hence, the equation (3.25) follows, which means that $\zeta^k\in\mathrm{int}(\mathcal{F})$.

Finally, let us prove $\partial\delta(\zeta^k|\ \mathcal{F})=\{0\}$. From [131, page 226], it follows that

$$\partial\delta(z|\mathcal{K}^n)=\{\upsilon\in\mathbb{R}^n\,|\,\upsilon\preceq_{\mathcal{K}^n}0,\ \mathrm{tr}(\upsilon\circ z)=0\}.$$

Using [131, Theorem 23.9] and the assumption $\mathrm{dom}(f)\cap\mathrm{int}(\mathcal{F})\neq\emptyset$, we have

$$\partial\delta(\zeta|\ \mathcal{F})=\left\{A^T\upsilon\in\mathbb{R}^n\,|\,\upsilon\preceq_{\mathcal{K}^n}0,\ \mathrm{tr}(\upsilon\circ(A\zeta+b))=0\right\}.$$

In addition, from the self-dual property of symmetric cone $\mathcal{K}^n$, we know that $\mathrm{tr}(x\circ y)=0$ for any $x\succeq_{\mathcal{K}^n}0$ and $y\succ_{\mathcal{K}^n}0$ implies $x=0$. Thus, we obtain $\partial\delta(\zeta^k|\mathcal{F})=\{0\}$. This together with (3.24) and [131, Theorem 23.8] yields the desired result. □

Proposition 3.5 implies that the second-order cone constrained subproblem in (3.22) is actually equivalent to an unconstrained one

$$\zeta^k=\mathrm{argmin}_{\zeta\in\mathbb{R}^m}\left\{f(\zeta)+\frac{1}{\mu_k}\mathcal{D}(\zeta,\zeta^{k-1})\right\},$$

which is obviously simpler than the original CSOCP. This shows that the proximal-like algorithm proposed transforms the CSOCP into the solution of a sequence of simpler problems. We next present some properties satisfied by $\{\zeta^k\}$. For convenience, we denote the optimal set of the CSOCP by $\mathcal{X}:=\{\zeta\in\mathcal{F}\,|\,f(\zeta)=f_*\}$.

Proposition 3.6. *Let $\{\zeta^k\}$ be the sequence generated by the algorithm PLA given by (3.22), and let $\sigma_N=\sum_{k=1}^N\mu_k$. Then, the following hold.*

(a) *$\{f(\zeta^k)\}$ is a nonincreasing sequence.*

(b) *$\mu_k(f(\zeta^k)-f(\zeta))\le\mathcal{D}(\zeta,\zeta^{k-1})-\mathcal{D}(\zeta,\zeta^k)$ for all $\zeta\in\mathcal{F}$.*

(c) $\sigma_N(f(\zeta^N) - f(\zeta)) \le \mathcal{D}(\zeta, \zeta^0) - \mathcal{D}(\zeta, \zeta^N)$ *for all* $\zeta \in \mathcal{F}$.

(d) $\mathcal{D}(\zeta, \zeta^k)$ *is nonincreasing for any* $\zeta \in \mathcal{X}$ *if the optimal set* $\mathcal{X} \neq \emptyset$.

(e) $\mathcal{D}(\zeta^k, \zeta^{k-1}) \to 0$ *if the optimal set* $\mathcal{X} \neq \emptyset$.

Proof. (a) By the definition of ζ^k given as in (3.22), we have

$$f(\zeta^k) + \mu_k^{-1}\mathcal{D}(\zeta^k, \zeta^{k-1}) \le f(\zeta^{k-1}) + \mu_k^{-1}\mathcal{D}(\zeta^{k-1}, \zeta^{k-1}).$$

Since $\mathcal{D}(\zeta^k, \zeta^{k-1}) \ge 0$ and $\mathcal{D}(\zeta^{k-1}, \zeta^{k-1}) = 0$ by Proposition 3.4(c), it follows that

$$f(\zeta^k) \le f(\zeta^{k-1}) \quad (k \ge 1).$$

(b) By Proposition 3.5, $2\mu_k^{-1}A^T[(\phi')^{\rm soc}(A\zeta^{k-1} + b) - (\phi')^{\rm soc}(A\zeta^k + b)] \in \partial f(\zeta^k)$. Hence, from the definition of subdifferential, it follows that for any $\zeta \in \mathcal{F}$,

$$\begin{aligned} f(\zeta) &\ge f(\zeta^k) + 2\mu_k^{-1}\Big\langle (\phi')^{\rm soc}(A\zeta^{k-1} + b) - (\phi')^{\rm soc}(A\zeta^k + b),\ A\zeta - A\zeta^k \Big\rangle \\ &= f(\zeta^k) + \mu_k^{-1}{\rm tr}\Big[[(\phi')^{\rm soc}(A\zeta^{k-1} + b) - (\phi')^{\rm soc}(A\zeta^k + b)] \circ [(A\zeta + b) - (A\zeta^k + b)]\Big] \\ &= f(\zeta^k) + \mu_k^{-1}\left[H(A\zeta + b, A\zeta^k + b) + H(A\zeta^k + b, A\zeta^{k-1} + b) - H(A\zeta + b, A\zeta^{k-1} + b)\right] \\ &= f(\zeta^k) + \mu_k^{-1}\Big[\mathcal{D}(\zeta, \zeta^k) + \mathcal{D}(\zeta^k, \zeta^{k-1}) - D(\zeta, \zeta^{k-1})\Big], \end{aligned} \tag{3.27}$$

where the first equality is due to the definition of determinant and the second follows from Proposition 3.2. From this inequality and the nonnegativity of $\mathcal{D}(\zeta^k, \zeta^{k-1})$, we readily obtain the conclusion.

(c) From the result in part(b), we have

$$\mu_k[f(\zeta^{k-1}) - f(\zeta^k)] \ge \mathcal{D}(\zeta^{k-1}, \zeta^k) - \mathcal{D}(\zeta^{k-1}, \zeta^{k-1}) = \mathcal{D}(\zeta^{k-1}, \zeta^k).$$

Multiplying this inequality by σ_{k-1} and noting that $\sigma_k = \sigma_{k-1} + \mu_k$, one has

$$\sigma_{k-1}f(\zeta^{k-1}) - (\sigma_k - \mu_k)f(\zeta^k) \ge \sigma_{k-1}\mu_k^{-1}\mathcal{D}(\zeta^{k-1}, \zeta^k). \tag{3.28}$$

Summing up the inequalities in (3.28) for $k = 1, 2, \cdots, N$ and using $\sigma_0 = 0$ yields

$$-\sigma_N f(\zeta^N) + \sum_{k=1}^{N} \mu_k f(x^k) \ge \sum_{k=1}^{N} \sigma_{k-1}\mu_k^{-1}\mathcal{D}(\zeta^{k-1}, \zeta^k). \tag{3.29}$$

On the other hand, summing the inequality in part (b) over $k = 1, 2, \cdots, N$, we get

$$-\sigma_N f(\zeta) + \sum_{k=1}^{N} \mu_k f(\zeta^k) \le \mathcal{D}(\zeta, \zeta^0) - \mathcal{D}(\zeta, \zeta^N). \tag{3.30}$$

Now subtracting (3.29) from (3.30) yields that

$$\sigma_N[f(\zeta^N) - f(\zeta)] \le \mathcal{D}(\zeta, \zeta^0) - \mathcal{D}(\zeta, \zeta^N) - \sum_{k=1}^{N} \sigma_{k-1}\mu_k^{-1}\mathcal{D}(\zeta^{k-1}, \zeta^k).$$

This together with the nonnegativity of $\mathcal{D}(\zeta^{k-1}, \zeta^k)$ implies the conclusion.

(d) Note that $f(\zeta^k) - f(\zeta) \ge 0$ for all $\zeta \in \mathcal{X}$. Thus, the result follows from part(b) directly.

(e) From part(d), we know that $\mathcal{D}(\zeta, \zeta^k)$ is nonincreasing for any $\zeta \in \mathcal{X}$. This together with $\mathcal{D}(\zeta, \zeta^k) \ge 0$ for any k implies that $\mathcal{D}(\zeta, \zeta^k)$ is convergent. Thus, we have

$$\mathcal{D}(\zeta, \zeta^{k-1}) - \mathcal{D}(\zeta, \zeta^k) \to 0. \tag{3.31}$$

On the other hand, from (3.27) it follows that

$$0 \le \mu_k[f(\zeta^k) - f(\zeta)] \le \mathcal{D}(\zeta, \zeta^{k-1}) - \mathcal{D}(\zeta, \zeta^k) - \mathcal{D}(\zeta^k, \zeta^{k-1}), \quad \forall \zeta \in \mathcal{X},$$

which implies

$$\mathcal{D}(\zeta^k, \zeta^{k-1}) \le \mathcal{D}(\zeta, \zeta^{k-1}) - \mathcal{D}(\zeta, \zeta^k), \quad \forall \zeta \in \mathcal{X}.$$

This together with (3.31) and the nonnegativity of $\mathcal{D}(\zeta^k, \zeta^{k-1})$ yields the result. □

We have proved that the proximal-like algorithm (PLA) defined as in (3.22) is well-defined and satisfies some favorable properties. By this, we next establish its convergence.

Proposition 3.7. *Let $\{\zeta^k\}$ be the sequence generated by the algorithm PLA given by in (3.22), and let $\sigma_N = \sum_{k=1}^N \mu_k$. Then, under Assumptions (A1)-(A2),*

(a) *if $\sigma_N \to \infty$, then $\lim_{N\to+\infty} f(\zeta^N) \to f_*$;*

(b) *if $\sigma_N \to \infty$ and the optimal set $\mathcal{X} \neq \emptyset$, then the sequence $\{x^k\}$ is bounded and every accumulation point is a solution of the CSOCP.*

Proof. (a) From the definition of f_*, there exists a $\widehat{\zeta} \in \mathcal{F}$ such that

$$f(\widehat{\zeta}) < f_* + \epsilon, \quad \forall \epsilon > 0.$$

However, from Proposition 3.6(c) and the nonnegativity of $\mathcal{D}(\zeta, \zeta^N)$, we have that

$$f(\zeta^N) - f(\zeta) \le \sigma_N^{-1}\mathcal{D}(\zeta, \zeta^0), \quad \forall \zeta \in \mathcal{F}.$$

Let $\zeta = \widehat{\zeta}$ in the above inequality and take the limit with $\sigma_N \to +\infty$, we then obtain

$$\lim_{N\to+\infty} f(\zeta^N) < f_* + \epsilon.$$

Considering that ϵ is arbitrary and $f(\zeta^N) \ge f_*$, we thus have the desired result.

(b) Suppose that $\zeta^* \in \mathcal{X}$. Then, from Proposition 3.6(d), $\mathcal{D}(\zeta^*, \zeta^k) \le \mathcal{D}(\zeta^*, \zeta^0)$ for any k. This implies that $\{\zeta^k\} \subseteq L_{\mathcal{D}}(\zeta^*, \mathcal{D}(\zeta^*, \zeta^0))$. By Proposition 3.6(d), the sequence $\{\zeta^k\}$ is then bounded. Let $\bar{\zeta} \in \mathcal{F}$ be an accumulation point of $\{\zeta^k\}$ with subsequence $\{\zeta^{k_j}\} \to \bar{\zeta}$. Then, from part(a), it follows that $f(\zeta^{k_j}) \to f_*$. On the other hand, since f is lower-semicontinuous, we have $f(\bar{\zeta}) = \liminf_{k_j\to+\infty} f(\zeta^{k_j})$. The two sides show that $f(\bar{\zeta}) \le f(\zeta^*)$. Consequently, $\bar{\zeta}$ is a solution of the CSOCP. □

3.2 Interior proximal-like algorithms for SOCCP

In Section 3.1, we present a proximal-like algorithm based on Bregman-type functions for the CSOCP (3.1). In this section, we focus on another proximal-like algorithm, which is similar to entropy-like proximal algorithm. We will illustrate how to construct the distance measure needed for tacking the CSOCP (3.1).

The entropy-like proximal algorithm was designed for minimizing a convex function $f(\zeta)$ subject to nonnegative constraints $\zeta \geq 0$. In [61], Eggermont first introduced the Kullback-Leibler relative entropy, defined by

$$d(\zeta, \xi) = \sum_{i=1}^{m} \zeta_i \ln(\zeta_i/\xi_i) + \zeta_i - \xi_i, \quad \forall \zeta \geq 0,\ \xi > 0,$$

where we adopt the convention of $0 \ln 0 = 0$. The original entropy-like proximal point algorithm is as below:

$$\begin{cases} \zeta^0 > 0 \\ \zeta^k = \underset{\zeta > 0}{\operatorname{argmin}} \left\{ f(\zeta) + {\mu_k}^{-1} d(\zeta^{k-1}, \zeta) \right\}. \end{cases} \tag{3.32}$$

Later, Teboulle [144] proposed to replace the usual Kullback-Leibler relative entropy with a new type of distance-like function, called φ-divergence, to define the entropy-like proximal map. Let $\varphi : \mathbb{R} \to (-\infty, \infty]$ be a closed proper convex function satisfying certain conditions (see [81, 144]). The φ-divergence induced by φ is defined as

$$d_\varphi(\zeta, \xi) := \sum_{i=1}^{m} \xi_i \varphi(\zeta_i/\xi_i).$$

Based on the φ-divergence, Isume et al [81–83] generalized Eggermont's algorithm as

$$\begin{cases} \zeta^0 > 0 \\ \zeta^k = \underset{\zeta > 0}{\operatorname{argmin}} \left\{ f(\zeta) + {\mu_k}^{-1} d_\varphi(\zeta, \zeta^{k-1}) \right\} \end{cases} \tag{3.33}$$

and obtained the convergence theorems under weaker assumptions. Clearly, when

$$\varphi(t) = -\ln t + t - 1 \quad (t > 0),$$

we have that $d_\varphi(\zeta, \xi) = d(\xi, \zeta)$, and consequently the algorithm reduces to Eggermont's (3.32).

Observing that the proximal-like algorithm (3.33) associated with $\varphi(t) = -\ln t + t - 1$ inherits the features of the interior point method as well as the proximal point method, Auslender [8] extended the algorithm to general linearly constrained convex minimization problems and variational inequalities on polyhedra. Then, is it possible to extend the algorithm to nonpolyhedra symmetric conic optimization problems and establish the corresponding convergence results? In this section, we will explore its extension to the setting of second-order cones and establish a class of interior proximal-like algorithms for the CSOCP. We should mention that the algorithm (3.33) with the entropy function $t\ln t - t + 1$ ($t \geq 0$) was recently extended to convex semidefinite programming [58].

Again as defined in (3.17) and (3.18), we denote $\mathcal{F}$ the constraint set of the CSOCP, i.e.,

$$\mathcal{F} := \{\zeta \in \mathbb{R}^m \mid A\zeta + b \succeq_{\mathcal{K}^n} 0\},$$

and denote its interior by $\mathrm{int}(\mathcal{F})$, i.e.,

$$\mathrm{int}(\mathcal{F}) := \{\zeta \in \mathbb{R}^m \mid A\zeta + b \succ_{\mathcal{K}^n} 0\}.$$

Accordingly, the 2nd proximal-like algorithm that we propose for the CSOCP is defined as follows:

$$\begin{cases} \zeta^0 \in \mathrm{int}(\mathcal{F}) \\ \zeta^k = \underset{\zeta \in \mathrm{int}(\mathcal{F})}{\mathrm{argmin}} \left\{ f(\zeta) + \mu_k^{-1} D(A\zeta + b, A\zeta^{k-1} + b) \right\}, \end{cases} \tag{3.34}$$

where $D : \mathbb{R}^n \times \mathbb{R}^n \to (-\infty, +\infty]$ is a closed proper convex function generated by a class of twice continuously differentiable and strictly convex functions on $(0, +\infty)$, and the specific expression is given later. The class of distance measures includes as a special case the natural extension of $d_\varphi(x, y)$ with $\varphi(t) = -\ln t + t - 1$ to the second-order cones. For the proximal-like algorithm (3.34), we particularly consider an approximate version which allows inexact minimization of the subproblem (3.34) and establish its global convergence results under some mild assumptions.

Throughout this section, for a differentiable function h on $\mathbb{R}$, we denote h', h'' and h''' by its first, second and third derivative, respectively. Recall that a function is closed if and only if it is lower semi-continuous and a function is proper if $f(\zeta) < \infty$ for at least one $\zeta \in \mathbb{R}^m$ and $f(\zeta) > -\infty$ for all $\zeta \in \mathbb{R}^m$. For a closed proper convex function $f : \mathbb{R}^m \to (-\infty, \infty]$, we denote its domain by $\mathrm{dom} f := \{\, \zeta \in \mathbb{R}^m \mid f(\zeta) < \infty\}$ and the subdifferential of f at $\widehat{\zeta}$ by

$$\partial f(\widehat{\zeta}) := \left\{ w \in \mathbb{R}^m \mid f(\zeta) \geq f(\widehat{\zeta}) + \langle w, \zeta - \widehat{\zeta}\rangle,\ \forall \zeta \in \mathbb{R}^m \right\}.$$

As usual, if f is differentiable at ζ, the notation $\nabla f(\zeta)$ represents the gradient at ζ of f.

Next, we present the definition of the distance-like function $D(x, y)$ involved in the proximal-like algorithm (3.34) and some specific examples. Let $\phi : \mathbb{R} \to (-\infty, \infty]$ be a closed proper convex function with $\mathrm{dom}\phi = [0, \infty)$ and assume that

(C1) ϕ is strictly convex on its domain.

(C2) ϕ is twice continuously differentiable on int(domϕ) with $\lim_{t\to 0^+}\phi''(t) = +\infty$.

(C3) $\phi'(t)t - \phi(t)$ is convex on int(domϕ).

(C4) ϕ' is SOC-concave on int(domϕ).

In the sequel, we denote by Φ the class of functions satisfying conditions (C1)-(C4).

Given a $\phi \in \Phi$, let $\phi^{\rm soc}$ and $(\phi')^{\rm soc}$ be the vector-valued function given as in (1.8). We define $D(x,y)$ involved in the proximal-like algorithm (3.34) by

$$D(x,y) := \begin{cases} {\rm tr}\Big[\phi^{\rm soc}(y) - \phi^{\rm soc}(x) - (\phi')^{\rm soc}(x)\circ(y-x)\Big], & \forall x\in {\rm int}(\mathcal{K}^n), y \in \mathcal{K}^n, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.35}$$

The function, as will be shown later, possesses some favorable properties. Particularly, $D(x,y) \ge 0$ for any $x, y \in {\rm int}(\mathcal{K}^n)$, and $D(x,y) = 0$ if and only if $x = y$. Hence, $D(x,y)$ can be used to measure the distance between any two points in ${\rm int}(\mathcal{K}^n)$.

In the following, we concentrate on the examples of the distance-like function $D(x,y)$. For this purpose, we first give another characterization for condition (C3).

Lemma 3.2. *Let $\phi : \mathbb{R} \to (-\infty,\infty]$ be a closed proper convex function with* dom$(\phi) = [0,+\infty)$. *If ϕ is thrice continuously differentiable on* int(domϕ)*, then ϕ satisfies condition (C3) if and only if its derivative function ϕ' is exponentially convex (which means the function $\phi'(\exp(\cdot)) : \mathbb{R}\to\mathbb{R}$ is convex on $\mathbb{R}$), or*

$$\phi'(t_1t_2) \le \frac{1}{2}\Big(\phi'(t_1^2) + \phi'(t_2^2)\Big), \quad \forall t_1, t_2 > 0. \tag{3.36}$$

Proof. Since the function ϕ is thrice continuously differentiable on int(domϕ), ϕ satisfies condition (C3) if and only if

$$\phi''(t) + t\phi'''(t) \ge 0, \quad \forall t > 0.$$

Observe that the inequality is also equivalent to

$$t\phi''(t) + t^2\phi'''(t) \ge 0, \quad \forall t > 0,$$

and hence substituting by $t = \exp(\theta)$ for $\theta \in \mathbb{R}$ into the inequality yields that

$$\exp(\theta)\phi''(\exp(\theta)) + \exp(2\theta)\phi'''(\exp(\theta)) \ge 0, \quad \forall \theta\in\mathbb{R}.$$

Since the left hand side of this inequality is exactly $[\phi'(\exp(\theta))]''$, it means that $\phi'(\exp(\cdot))$ is convex on $\mathbb{R}$. Consequently, the first part of the conclusions follows.

Note that the convexity of $\phi'(\exp(\cdot))$ on $\mathbb{R}$ is equivalent to saying for any $\theta_1, \theta_2 \in \mathbb{R}$,

$$\phi'(\exp(r\theta_1 + (1-r)\theta_2)) \le r\phi'(\exp(\theta_1)) + (1-r)\phi'(\exp(\theta_2)), \quad r \in [0,1],$$

which, by letting $t_1 = \exp(\theta_1)$ and $t_2 = \exp(\theta_2)$, can be rewritten as

$$\phi'(t_1^r t_2^{1-r}) \le r\phi'(t_1) + (1-r)\phi'(t_2), \quad \forall t_1, t_2 > 0 \text{ and } r \in [0,1].$$

This is clearly equivalent to the statement in (3.36) due to the continuity of ϕ'. □

Remark 3.3. *The exponential convexity was also used in the definition of the self-regular function [124], in which the authors denote Ω by the set of functions whose elements are twice continuously differentiable and exponentially convex on $(0, +\infty)$. By Lemma 3.2, clearly, if $h \in \Omega$, then the function $\int_0^t h(\theta)d\theta$ necessarily satisfies condition (C3). For example, $\ln t$ belongs to Ω, and hence $\int_0^t \ln\theta d\theta = t\ln t$ satisfies condition (C3).*

Now we present several examples showing how to construct $D(x,y)$. From these examples, we see that the conditions required by $\phi \in \Phi$ are not so strict and the construction of the distance-like functions in SOCs can be completed by selecting a class of single variate convex functions.

Example 3.4. *Let $\phi_1 : \mathbb{R} \to (-\infty, \infty]$ be given by*

$$\phi_1(t) = \begin{cases} t\ln t - t + 1 & \text{if } t \ge 0, \\ \infty & \text{if } t < 0. \end{cases}$$

Solution. It is easy to verify that ϕ_1 satisfies conditions (C1)-(C3). In addition, by Example 2.10 and 2.12, the function $\ln t$ is SOC-concave and SOC-monotone on $(0, \infty)$, hence the condition (C4) also holds. From formula (1.8), it follows that for any $y \in \mathcal{K}^n$ and $x \in \mathrm{int}(\mathcal{K}^n)$,

$$\phi^{\mathrm{soc}}(y) = y \circ \ln y - y + e \quad\text{and}\quad (\phi')^{\mathrm{soc}}(x) = \ln x.$$

Consequently, the distance-like function induced by ϕ_1 is given by

$$D_1(x,y) = \mathrm{tr}\left(y \circ \ln y - y \circ \ln x + x - y\right), \quad \forall x \in \mathrm{int}(\mathcal{K}^n), y \in \mathcal{K}^n.$$

This function is precisely the natural extension of the entropy-like distance $d_\varphi(\cdot,\cdot)$ with $\varphi(t) = -\ln t + t - 1$ to the second-order cones. In addition, comparing $D_1(x,y)$ with the distance-like function $H(x,y)$ in Example 3.1 of [116] (see Section 3.1), we note that $D_1(x,y) = H(y,x)$, but the proximal-like algorithms corresponding to them are completely different. ■

Example 3.5. *Let* $\phi_2 : \mathbb{R} \to (-\infty, \infty]$ *be given by*

$$\phi_2(t) = \begin{cases} t \ln t + (1+t)\ln(1+t) - (1+t)\ln 2 & \text{if } t \geq 0, \\ \infty & \text{if } t < 0. \end{cases}$$

Solution. By computing, we can verify that ϕ_2 satisfies conditions (C1)-(C3). Furthermore, from earlier examples, we learn that ϕ_2 also satisfies condition (C4). This means that $\phi_2 \in \Phi$. For any $y \in \mathcal{K}^n$ and $x \in \text{int}(\mathcal{K}^n)$, we can compute that

$$\begin{aligned} \phi^{\text{soc}}(y) &= y \circ \ln y + (e+y) \circ \ln(e+y) - \ln 2(e+y), \\ (\phi')^{\text{soc}}(x) &= (2 - \ln 2)e + \ln x + \ln(e+x). \end{aligned}$$

Therefore, the distance-like function generated by such a ϕ is given by

$$D_2(x,y) = \text{tr}\big[-\ln(e+x) \circ (e+y) + y \circ (\ln y - \ln x) + (e+y) \circ \ln(e+y) - 2(y-x) \big]$$

for any $x \in \text{int}(\mathcal{K}^n)$ and $y \in \mathcal{K}^n$. It should be pointed out that $D_2(x,y)$ is not the extension of $d_\varphi(\cdot,\cdot)$ with $\varphi(t) = \phi_2(t)$ given by [81] to the second-order cones. ■

Example 3.6. *For any* $0 \leq r < \frac{1}{2}$, *let* $\phi_3 : \mathbb{R} \to (-\infty, \infty]$ *be given by*

$$\phi_3(t) = \begin{cases} t^{\frac{2r+3}{2}} + t^2 & \text{if } t \geq 0, \\ \infty & \text{if } t < 0. \end{cases}$$

Solution. It is easy to verify that ϕ_3 satisfies conditions (C1)-(C3). Furthermore, from Examples 2.10-2.12, it follows that ϕ_3 satisfies condition (C4). Thus, $\phi_3 \in \Phi$. By a simple computation,

$$\phi^{\text{soc}}(y) = y^{\frac{2r+3}{2}} + y^2 \quad \forall y \in \mathcal{K}^n \quad \text{and} \quad (\phi')^{\text{soc}}(x) = \frac{2r+3}{2} x^{\frac{2r+1}{2}} + 2x \quad \forall x \in \text{int}(\mathcal{K}^n).$$

Hence, the distance-like function induced by ϕ_3 has the following expression

$$D_3(x,y) = \text{tr}\left[\frac{2r+1}{2} x^{\frac{2r+3}{2}} + x^2 - y \circ \left(\frac{2r+3}{2} x^{\frac{2r+1}{2}} + 2x \right) + y^{\frac{2r+3}{2}} + y^2 \right].$$

■

Example 3.7. *For any* $0 < a \leq 1$, *let* $\phi_4 : \mathbb{R} \to (-\infty, \infty]$ *be given by*

$$\phi_4(t) = \begin{cases} t^{a+1} + at \ln t - at & \text{if } t \geq 0, \\ \infty & \text{if } t < 0. \end{cases}$$

Solution. It is easily shown that ϕ_4 satisfies conditions (C1)-(C3). By Examples 2.11-2.14, $(\phi')_4$ is SOC-concave on $(0,\infty)$. Hence, $\phi_4 \in \Phi$. For any $y \in \mathcal{K}^n$ and $x \in \mathrm{int}(\mathcal{K}^n)$,

$$\phi^{\mathrm{soc}}(y) = y^{a+1} + ay \circ \ln y - ay \quad \text{and} \quad (\phi')^{\mathrm{soc}}(x) = (a+1)x^a + a\ln x.$$

Consequently, the distance-like function induced by ϕ_4 has the following expression

$$D_4(x,y) = \mathrm{tr}\Big[ax^{a+1} + ax - y \circ \Big((a+1)x^a + a\ln x\Big) + y^{a+1} + ay \circ \ln y - ay\Big].$$

■

In what follows, we study some favorable properties of the function $D(x,y)$. We begin with some technical lemmas that will be used in the subsequent analysis.

Lemma 3.3. *Suppose that $\phi : \mathbb{R} \to (-\infty,\infty]$ belongs to the class of Φ, i.e., satisfying (C1)-(C4). Let ϕ^{soc} and $(\phi')^{\mathrm{soc}}$ be the corresponding SOC-functions of ϕ and ϕ' given as in (1.8). Then, the following hold.*

(a) *$\phi^{\mathrm{soc}}(x)$ and $(\phi')^{\mathrm{soc}}(x)$ are well-defined on $\mathcal{K}^n$ and $\mathrm{int}(\mathcal{K}^n)$, respectively, and*

$$\lambda_i[\phi^{\mathrm{soc}}(x)] = \phi[\lambda_i(x)], \quad \lambda_i[(\phi')^{\mathrm{soc}}(x)] = \phi'[\lambda_i(x)], \quad i = 1,2.$$

(b) *$\phi^{\mathrm{soc}}(x)$ and $(\phi')^{\mathrm{soc}}(x)$ are continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$ with the transposed Jacobian at x given as in formulas (1.27)–(1.28).*

(c) *$\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ and $\mathrm{tr}[(\phi')^{\mathrm{soc}}(x)]$ are continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$, and*

$$\begin{aligned} \nabla \mathrm{tr}\left[\phi^{\mathrm{soc}}(x)\right] &= 2\nabla\phi^{\mathrm{soc}}(x)e = 2(\phi')^{\mathrm{soc}}(x), \\ \nabla \mathrm{tr}\left[(\phi')^{\mathrm{soc}}(x)\right] &= 2\nabla(\phi')^{\mathrm{soc}}(x)e = 2(\phi'')^{\mathrm{soc}}(x). \end{aligned}$$

(d) *The function $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ is strictly convex on $\mathrm{int}(\mathcal{K}^n)$.*

Proof. Mimicking the arguments as in Lemma 3.1, in other words, using Propositions 1.13-1.14, Lemma 2.8 and the definition of Φ, the desired results follow. □

Lemma 3.4. *Suppose that $\phi : \mathbb{R} \to (-\infty,\infty]$ belongs to the class of Φ and $z \in \mathbb{R}^n$. Let $\phi_z : \mathrm{int}(\mathcal{K}^n) \to \mathbb{R}$ be defined by*

$$\phi_z(x) := \mathrm{tr}\big[-z \circ (\phi')^{\mathrm{soc}}(x)\big]. \tag{3.37}$$

Then, the function $\phi_z(x)$ possesses the following properties.

(a) *$\phi_z(x)$ is continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$ with $\nabla\phi_z(x) = -2\nabla(\phi')^{\mathrm{soc}}(x) \cdot z$.*

(b) *$\phi_z(x)$ is convex over $\mathrm{int}(\mathcal{K}^n)$ when $z \in \mathcal{K}^n$, and furthermore, it is strictly convex over $\mathrm{int}(\mathcal{K}^n)$ when $z \in \mathrm{int}(\mathcal{K}^n)$.*

Proof. (a) Since $\phi_z(x) = -2\langle (\phi')^{\rm soc}(x), z\rangle$ for any $x \in {\rm int}(\mathcal{K}^n)$, we have that $\phi_z(x)$ is continuously differentiable on ${\rm int}(\mathcal{K}^n)$ by Lemma 3.3(c). Moreover, applying the chain rule for inner product of two functions readily yields $\nabla\phi_z(x) = -2\nabla(\phi')^{\rm soc}(x)\cdot z$.

(b) By the continuous differentiability of $\phi_z(x)$, to prove the convexity of ϕ_z on ${\rm int}(\mathcal{K}^n)$, it suffices to prove the following inequality

$$\phi_z\left(\frac{x+y}{2}\right) \le \frac{1}{2}\Big(\phi_z(x)+\phi_z(y)\Big), \quad \forall x, y \in {\rm int}(\mathcal{K}^n). \tag{3.38}$$

By condition (C4), ϕ' is SOC-concave on $(0,+\infty)$. Therefore, we have

$$-(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) \ \ \preceq_{\mathcal{K}^n} \ \ -\frac{1}{2}\Big[(\phi')^{\rm soc}(x)+(\phi')^{\rm soc}(y)\Big],$$

i.e.,

$$(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y) \ \ \succeq_{\mathcal{K}^n} \ \ 0.$$

Using Property 1.3(d) and the fact that $z \in \mathcal{K}^n$, we then obtain that

$$\left\langle z,\ (\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y)\right\rangle \ge 0, \tag{3.39}$$

which in turn implies that

$$\left\langle -z,\ (\phi')^{\rm soc}\left(\frac{x+y}{2}\right)\right\rangle \le \frac{1}{2}\Big\langle -z,\ (\phi')^{\rm soc}(x)\Big\rangle + \frac{1}{2}\Big\langle -z,\ (\phi')^{\rm soc}(y)\Big\rangle.$$

The last inequality is exactly the one in (3.38). Hence, ϕ_z is convex on ${\rm int}(\mathcal{K}^n)$ for $z \in \mathcal{K}^n$.

To prove the second part of the conclusions, we only need to prove that the inequality in (3.39) holds strictly for any $x, y \in {\rm int}(\mathcal{K}^n)$ and $x \neq y$. By Property 1.3(d), this is also equivalent to proving the vector $(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y)$ is nonzero since

$$(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y) \in \mathcal{K}^n \quad \text{and} \quad z \in {\rm int}(\mathcal{K}^n).$$

From condition (C4), it follows that ϕ' is concave on $(0,+\infty)$ since the SOC-concavity implies the concavity. This together with the strict monotonicity of ϕ' implies that ϕ' is strictly concave on $(0,+\infty)$. Using Lemma 3.3(d), we then have that ${\rm tr}[(\phi')^{\rm soc}(x)]$ is strictly concave on ${\rm int}(\mathcal{K}^n)$. This means that for any $x, y \in {\rm int}(\mathcal{K}^n)$ and $x \neq y$,

$${\rm tr}\left[(\phi')^{\rm soc}\Big(\frac{x+y}{2}\Big)\right] - \frac{1}{2}{\rm tr}\left[(\phi')^{\rm soc}(x)\right] - \frac{1}{2}{\rm tr}\left[(\phi')^{\rm soc}(y)\right] > 0. \tag{3.40}$$

In addition, we note that the first element of $(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y)$ is

$$\frac{\phi'\Big(\lambda_1\Big(\frac{x+y}{2}\Big)\Big) + \phi'\Big(\lambda_2\Big(\frac{x+y}{2}\Big)\Big)}{2} - \frac{\phi'(\lambda_1(x)) + \phi'(\lambda_2(x))}{4} - \frac{\phi'(\lambda_1(y)) + \phi'(\lambda_2(y))}{4},$$

which, by Property 1.1(d), can be rewritten as

$$\frac{1}{2}{\rm tr}\left[(\phi')^{\rm soc}\Big(\frac{x+y}{2}\Big)\right] - \frac{1}{4}{\rm tr}\left[(\phi')^{\rm soc}(x)\right] - \frac{1}{4}{\rm tr}\left[(\phi')^{\rm soc}(y)\right].$$

This together with (3.40) shows that $(\phi')^{\rm soc}\left(\frac{x+y}{2}\right) - \frac{1}{2}(\phi')^{\rm soc}(x) - \frac{1}{2}(\phi')^{\rm soc}(y)$ is nonzero for any $x, y \in {\rm int}(\mathcal{K}^n)$ and $x \neq y$. Consequently, ϕ_z is strictly convex on ${\rm int}(\mathcal{K}^n)$. □

Lemma 3.5. *Let $\mathcal{F}$ be the set defined as in (3.17). Then, its recession cone $0^+\mathcal{F}$ is described by*

$$0^+\mathcal{F} = \Big\{ d \in {\rm I\!R}^m \mid Ad \succeq_{\mathcal{K}^n} 0 \Big\}. \tag{3.41}$$

Proof. Assume that $d \in {\rm I\!R}^m$ such that $Ad \succeq_{\mathcal{K}^n} 0$. Then, for any $\lambda > 0$, $\lambda Ad \succeq_{\mathcal{K}^n} 0$. Considering that $\mathcal{K}^n$ is closed under the "+" operation, we have for any $\zeta \in \mathcal{F}$,

$$A(\zeta + \lambda d) + b = (A\zeta + b) + \lambda(Ad) \quad \succeq_{\mathcal{K}^n} \quad 0. \tag{3.42}$$

By [131, page 61], this shows that every element in the set of the right hand side of (3.41) is a recession direction of $\mathcal{F}$. Consequently, $\{d \in {\rm I\!R}^m \mid Ad \succeq_{\mathcal{K}^n} 0\} \subseteq 0^+\mathcal{F}$.

Now take any $d \in 0^+\mathcal{F}$ and $\zeta \in \mathcal{F}$. Then, for any $\lambda > 0$, equation (3.42) holds. By Property 1.3, we then have $\lambda_1\Big[(A\zeta + b) + \lambda Ad\Big] \geq 0$ for any $\lambda > 0$. This implies that $\lambda_1(Ad) \geq 0$, since otherwise letting $\lambda \to +\infty$ and using the fact that

$$\begin{aligned} \lambda_1\Big[(A\zeta + b) + \lambda Ad\Big] &= (A\zeta + b)_1 + \lambda(Ad)_1 - \|(A\zeta + b)_2 + \lambda(Ad)_2\| \\ &\leq (A\zeta + b)_1 + \lambda(Ad)_1 - \Big(\lambda\|(Ad)_2\| - \|(A\zeta + b)_2\|\Big) \\ &= \lambda\lambda_1(Ad) + \lambda_2(A\zeta + b), \end{aligned}$$

we obtain that $\lambda_1[(A\zeta + b) + \lambda Ad] \to -\infty$. Thus, we prove that $Ad \succeq_{\mathcal{K}^n} 0$, and consequently $0^+\mathcal{F} \subseteq \{d \in {\rm I\!R}^m \mid Ad \succeq_{\mathcal{K}^n} 0\}$. Combining with the above discussions then yields the result. □

Lemma 3.6. *Let $\{a_{nk}\}$ be a sequence of real numbers satisfying*

(i) $a_{nk} \geq 0$, $\forall n = 1, 2, \cdots$ *and* $\forall k = 1, 2, \cdots$.

(ii) $\sum_{k=1}^{\infty} a_{nk} = 1$, $\forall n = 1, 2, \cdots$*; and* $\lim_{n\to\infty} \sum_{k=1}^{n} a_{nk}u_k = u$, $\forall k = 1, 2, \cdots$.

If $\{u_k\}$ *is a sequence such that* $\lim_{k\to+\infty} u_k = u$*, then* $\lim_{k\to+\infty} a_{nk}u_k = u$*.*

Proof. Please see [92, Theorem 2]. □

Now we are in a position to study the properties of the distance-like function $D(x, y)$.

Proposition 3.8. *Given a function* $\phi \in \Phi$*, let* $D(x, y)$ *be defined as in (3.35). Then, the following hold.*

(a) $D(x, y) \geq 0$ *for any* $x \in \mathrm{int}(\mathcal{K}^n)$ *and* $y \in \mathcal{K}^n$*, and* $D(x, y) = 0$ *if and only if* $x = y$*.*

(b) *For any fixed* $y \in \mathcal{K}^n$*,* $D(\cdot, y)$ *is continuously differentiable on* $\mathrm{int}(\mathcal{K}^n)$ *with*

$$\nabla_x D(x, y) = 2\nabla(\phi')^{\mathrm{soc}}(x) \cdot (x - y). \tag{3.43}$$

(c) *For any fixed* $y \in \mathcal{K}^n$*, the function* $D(\cdot, y)$ *is convex over* $\mathrm{int}(\mathcal{K}^n)$*, and for any fixed* $y \in \mathrm{int}(\mathcal{K}^n)$*,* $D(\cdot, y)$ *is strictly convex over* $\mathrm{int}(\mathcal{K}^n)$*.*

(d) *For any fixed* $y \in \mathrm{int}(\mathcal{K}^n)$*, the function* $D(\cdot, y)$ *is essentially smooth.*

(e) *For any fixed* $y \in \mathcal{K}^n$*, the level sets* $L_D(y, \gamma) := \{x \in \mathrm{int}(\mathcal{K}^n) : D(x, y) \leq \gamma\}$ *for all* $\gamma \geq 0$ *are bounded.*

Proof. (a) By Lemma 3.3(c), for any $x \in \mathrm{int}(\mathcal{K}^n)$ and $y \in \mathcal{K}^n$, we can rewrite $D(x, y)$ as

$$D(x, y) = \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - \langle \nabla \mathrm{tr}[\phi^{\mathrm{soc}}(x)],\ y - x\rangle.$$

Notice that $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ is strictly convex on $\mathrm{int}(\mathcal{K}^n)$ by Lemma 3.3 (d), and hence $D(x, y) \geq 0$ for any $x \in \mathrm{int}(\mathcal{K}^n)$ and $y \in \mathcal{K}^n$, and $D(x, y) = 0$ if and only if $x = y$.

(b) By Lemma 3.3(b) and (c), the functions $\mathrm{tr}[\phi^{\mathrm{soc}}(x)]$ and $\langle(\phi')^{\mathrm{soc}}(x), x\rangle$ are continuously differentiable on $\mathrm{int}(\mathcal{K}^n)$. Noting that, for any $x \in \mathrm{int}(\mathcal{K}^n)$ and $y \in \mathcal{K}^n$,

$$D(x, y) = \mathrm{tr}[\phi^{\mathrm{soc}}(y)] - \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - 2\langle(\phi')^{\mathrm{soc}}(x), y - x\rangle,$$

we then have the continuous differentiability of $D(\cdot, y)$ on $\mathrm{int}(\mathcal{K}^n)$. Furthermore,

$$\begin{aligned}
\nabla_x D(x, y) &= -\nabla \mathrm{tr}[\phi^{\mathrm{soc}}(x)] - 2\nabla(\phi')^{\mathrm{soc}}(x) \cdot (y - x) + 2(\phi')^{\mathrm{soc}}(x) \\
&= -2(\phi')^{\mathrm{soc}}(x) + 2\nabla(\phi')^{\mathrm{soc}}(x) \cdot (x - y) + 2(\phi')^{\mathrm{soc}}(x) \\
&= 2\nabla(\phi')^{\mathrm{soc}}(x) \cdot (x - y).
\end{aligned}$$

(c) By the definition of ϕ_z given as in (3.37), $D(x, y)$ can be rewritten as

$$D(x, y) = \mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x)] + \phi_y(x) + \mathrm{tr}[\phi^{\mathrm{soc}}(y)].$$

Thus, to prove the (strict) convexity of $D(\cdot, y)$ on $\mathrm{int}(\mathcal{K}^n)$, it suffices to show that

$$\mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x)] + \phi_y(x)$$

is (strictly) convex on $\mathrm{int}(\mathcal{K}^n)$. Let $\psi : (0, +\infty) \to \mathbb{R}$ be the function defined by

$$\psi(t) := \phi'(t)t - \phi(t). \tag{3.44}$$

Then, the vector-valued function induced by ψ via (1.8) is $(\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x)$, i.e.,

$$\psi^{\mathrm{soc}}(x) = (\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x). \tag{3.45}$$

From condition (C3) and Lemma 3.3(d), it follows that $\mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x)]$ is convex over $\mathrm{int}(\mathcal{K}^n)$. In addition, by Lemma 3.4(b), $\phi_y(x)$ is convex on $\mathrm{int}(\mathcal{K}^n)$ if $y \in \mathcal{K}^n$, and it is strictly convex if $y \in \mathrm{int}(\mathcal{K}^n)$. Thus, we get the desired results.

(d) From [131, page 251] and part(a)-(b), to prove that $D(\cdot, y)$ is essentially smooth for any fixed $y \in \mathrm{int}(\mathcal{K}^n)$, it suffices to show that $\|\nabla_x D(x^k, y)\| \to +\infty$ for any $\{x^k\} \subset \mathrm{int}(\mathcal{K}^n)$ with $x^k \to x \in \mathrm{bd}(\mathcal{K}^n)$. We next prove the conclusion by the two cases: $x_1 > 0$ and $x_1 = 0$. For the sake of notation, let $x^k = (x_1^k, x_2^k) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

<u>Case 1:</u> $x_1 > 0$. In this case, $\|x_2\| = x_1 > 0$ since $x \in \mathrm{bd}(\mathcal{K}^n)$. Noting that $x^k \to x$, we have $x_2^k \neq 0$ for all sufficiently large k. From the gradient formula (3.43),

$$\|\nabla_x D(x^k, y)\| = \|2\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)\| \geq \left|2[\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)]_1\right|, \tag{3.46}$$

where $[\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)]_1$ denotes the first element of the vector $\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)$. By the gradient formula (1.28), we can compute that

$$\begin{aligned}
2[\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)]_1 &= [\phi''(\lambda_2(x^k)) + \phi''(\lambda_1(x^k))](x_1^k - y_1) \\
&\quad + [\phi''(\lambda_2(x^k)) - \phi''(\lambda_1(x^k))]\frac{(x_2^k - y_2)^T x_2^k}{\|x_2^k\|} \\
&= \phi''(\lambda_2(x^k))\left(\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\|\right) \\
&\quad - \phi''(\lambda_1(x^k))\left(y_1 - y_2^T x_2^k/\|x_2^k\| - \lambda_1(x^k)\right). \qquad (3.47)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|2[\nabla(\phi')^{\mathrm{soc}}(x^k) \cdot (x^k - y)]_1\right| &\geq \left|\phi''(\lambda_1(x^k))\left(y_1 - y_2^T x_2^k/\|x_2^k\| - \lambda_1(x^k)\right)\right| \\
&\quad - \left|\phi''(\lambda_2(x^k))\left(\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\|\right)\right| \\
&\geq \left|\phi''(\lambda_1(x^k))\right| \cdot \left(\left|y_1 - y_2^T x_2^k/\|x_2^k\|\right| - \lambda_1(x^k)\right) \\
&\quad - \left|\phi''(\lambda_2(x^k))\right| \cdot \left|\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\|\right| \\
&\geq \left|\phi''(\lambda_1(x^k))\right| \cdot \Big(\lambda_1(y) - \lambda_1(x^k)\Big) \\
&\quad - \left|\phi''(\lambda_2(x^k))\right| \cdot \left|\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\|\right|.
\end{aligned}$$

Noting that $\lambda_1(x^k) \to \lambda_1(x) = 0$, $\lambda_2(x^k) \to \lambda_2(x) > 0$ and $\dfrac{y_2^T x_2^k}{\|x_2^k\|} \to \dfrac{y_2^T x_2}{\|x_2\|}$ as $k \to \infty$, the second term in the right hand side of last inequality converges to a finite value, whereas the first term approaches to ∞ since $|\phi''(\lambda_1(x^k))| \to \infty$ by condition (C2) and $\lambda_1(y) - \lambda_1(x^k) \to \lambda_1(y) > 0$. This implies that as $k \to +\infty$,

$$\left|2[\nabla(\phi')^{\rm soc}(x^k) \cdot (x^k - y)]_1\right| \to \infty.$$

Combining with the inequality (3.46) immediately yields $\|\nabla_x D(x^k, y)\| \to \infty$.

Case 2: $x_1 = 0$. In this case, we necessarily have that $x = 0$ since $x \in \mathcal{K}^n$. Considering that $x^k \to x$, it then follows that $x_2^k = 0$ or $x_2^k > 0$ for all sufficiently large k. If $x_2^k = 0$ for all sufficiently large k, then from (1.27) we have that

$$\|\nabla_x D(x^k, y)\| = \|2\phi''(x_1^k)(x^k - y)\| \geq 2|\phi''(x_1^k)| \cdot |x_1^k - y_1|.$$

Since $y_1 > 0$ by $y \in \mathrm{int}(\mathcal{K}^n)$ and $x_1^k \to x_1 = 0$, applying condition (C2) yields that the right hand side tends to ∞, and consequently $\|\nabla_x D(x^k, y)\| \to +\infty$ when $k \to \infty$.

Next, we consider the case that $x_2^k > 0$ for all sufficiently large k. In this case, the inequalities (3.46)-(3.47) still hold. By Cauchy-Schwartz Inequality,

$$\begin{array}{rcl}
\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\| & \geq & \lambda_2(x^k) - y_1 - \|y_2\| = \lambda_2(x^k) - \lambda_2(y), \\
y_1 - y_2^T x_2^k/\|x_2^k\| - \lambda_1(x^k) & \geq & y_1 - \|y_2\| - \lambda_1(x^k) = \lambda_1(y) - \lambda_1(x^k).
\end{array}$$

Since $\lambda_1(x^k), \lambda_2(x^k) \to 0$ as $k \to +\infty$ and $\lambda_1(y), \lambda_2(y) > 0$ by $y \in \mathrm{int}(\mathcal{K}^n)$, the last two inequalities imply that

$$\begin{array}{rcl}
\lambda_2(x^k) - y_1 - y_2^T x_2^k/\|x_2^k\| & \to & -\lambda_2(y) < 0, \\
y_1 - y_2^T x_2^k/\|x_2^k\| - \lambda_1(x^k) & \to & \lambda_1(y) > 0.
\end{array}$$

On the other hand, by condition (C2), when $k \to \infty$,

$$\phi''(\lambda_2(x^k)) \to \infty, \quad \phi''(\lambda_1(x^k)) \to \infty.$$

The two sides show that the right hand side of (3.47) approaches to $-\infty$ as $k \to +\infty$, and consequently, $2|[\nabla(\phi')^{\rm soc}(x^k) \cdot (x^k - y)]_1| \to +\infty$. Thus, from (3.46), it follows that $\|\nabla_x D(x^k, y)\| \to \infty$ as $k \to \infty$.

(e) From the definition of $D(x, y)$, it follows that for any $x, y \in \mathrm{int}(\mathcal{K}^n)$,

$$\begin{array}{rcl}
D(x,y) & = & \mathrm{tr}[\phi^{\rm soc}(y)] - \mathrm{tr}[\phi^{\rm soc}(x)] - \mathrm{tr}[(\phi')^{\rm soc}(x) \circ y] + \mathrm{tr}[(\phi')^{\rm soc}(x) \circ x] \\
& = & \displaystyle\sum_{i=1}^{2} \phi(\lambda_i(y)) - \sum_{i=1}^{2} \phi(\lambda_i(x)) - \mathrm{tr}[(\phi')^{\rm soc}(x) \circ y] + \mathrm{tr}[(\phi')^{\rm soc}(x) \circ x],
\end{array} \quad (3.48)$$

where the second equality is from Lemma 3.3(a) and Property 1.1. Since

$$\begin{array}{rcl}
(\phi')^{\rm soc}(x) \circ x & = & \left[\phi'(\lambda_1(x))u_x^{(1)} + \phi'(\lambda_2(x))u_x^{(2)}\right] \circ \left[\lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}\right] \\
& = & \phi'(\lambda_1(x))\lambda_1(x)u_x^{(1)} + \phi'(\lambda_2(x))\lambda_2(x)u_x^{(2)},
\end{array}$$

we have from Lemma 3.3(a) that

$$\mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ x] = \sum_{i=1}^{2} \phi'(\lambda_i(x))\lambda_i(x).$$

In addition, by Property 1.1 and Lemma 3.3(a), we have that

$$\mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ y] \le \sum_{i=1}^{2} \phi'(\lambda_i(x))\lambda_i(y).$$

Combining the last two inequalities with (3.48) yields that

$$\begin{aligned} D(x,y) &\ge \sum_{i=1}^{2} \Big[\phi(\lambda_i(y)) - \phi(\lambda_i(x)) - \phi'(\lambda_i(x))\lambda_i(y) + \phi'(\lambda_i(x))\lambda_i(x)\Big] \\ &= \sum_{i=1}^{2} \Big[\phi(\lambda_i(y)) - \phi(\lambda_i(x)) - \phi'(\lambda_i(x))(\lambda_i(y) - \lambda_i(x))\Big] \\ &= \sum_{i=1}^{2} d_B(\lambda_i(y), \lambda_i(x)), \end{aligned}$$

where $d_B : \mathbb{R}_+ \times \mathbb{R}_{++} \to \mathbb{R}$ is the function defined by

$$d_B(s,t) = \phi(s) - \phi(t) - \phi'(t)(s-t).$$

This implies that for any fixed $y \in \mathcal{K}^n$ and $\gamma \ge 0$,

$$L_D(y,\gamma) \subseteq \left\{ x \in \mathrm{int}(\mathcal{K}^n) \,\middle|\, \sum_{i=1}^{2} d_B(\lambda_i(y), \lambda_i(x)) \le \gamma \right\}. \tag{3.49}$$

Note that for any fixed $s \ge 0$, the set $\{t > 0 \,|\, d_B(s,t) \le 0\}$ equals to $\{s\}$ or $\emptyset$, and hence it is bounded. Thus, from [131, Corollary 8.7.1] and condition (C3), it follows that the level sets $\{t > 0 \,|\, d_B(s,t) \le \gamma\}$ for any fixed $s \ge 0$ are bounded. This together with (3.49) implies that the level sets $L_D(y,\gamma)$ are bounded for all $\gamma \ge 0$. □

Proposition 3.9. *Given a function $\phi \in \Phi$, let $D(x,y)$ be defined as in (3.35). Then, for all $x, y \in \mathrm{int}(\mathcal{K}^n)$ and $z \in \mathcal{K}^n$, we have the following inequality*

$$\begin{aligned} D(x,z) - D(y,z) &\ge 2\langle \nabla(\phi')^{\mathrm{soc}}(y) \cdot (z-y),\ y-x \rangle \\ &= 2\langle \nabla(\phi')^{\mathrm{soc}}(y) \cdot (y-x),\ z-y \rangle. \end{aligned} \tag{3.50}$$

Proof. From the definition of $D(x,y)$ and $\phi_z(x)$ and equality (3.45), it follows that

$$\begin{aligned} D(x,z) - D(y,z) &= \mathrm{tr}[(\phi')^{\mathrm{soc}}(x) \circ x - \phi^{\mathrm{soc}}(x)] + \phi_z(x) \\ &\quad - \mathrm{tr}[(\phi')^{\mathrm{soc}}(y) \circ y - \phi^{\mathrm{soc}}(y)] - \phi_z(y) \\ &= \mathrm{tr}[\psi^{\mathrm{soc}}(x)] - \mathrm{tr}[\psi^{\mathrm{soc}}(y)] + \phi_z(x) - \phi_z(y) \\ &\ge \langle \nabla \mathrm{tr}[\psi^{\mathrm{soc}}(y)],\ x-y \rangle + \langle \nabla\phi_z(y),\ x-y \rangle \\ &= \langle 2(\psi')^{\mathrm{soc}}(y),\ x-y \rangle - \langle 2\nabla(\phi')^{\mathrm{soc}}(y) \cdot z,\ x-y \rangle, \end{aligned} \tag{3.51}$$

where the inequality is due to the convexity of $\mathrm{tr}[\psi^{\mathrm{soc}}(x)]$ and $\phi_z(x)$ and the last equality follows from Lemma 3.3(c) and Lemma 3.4(a). From the definition of ψ given as in (3.44), it is easy to compute that

$$\langle (\psi')^{\mathrm{soc}}(y),\ x-y\rangle = \langle (\phi'')^{\mathrm{soc}}(y)\circ y,\ x-y\rangle. \tag{3.52}$$

In addition, by the gradient formulas in (1.27)-(1.28), we can compute that

$$\nabla(\phi')^{\mathrm{soc}}(y)\cdot y = (\phi'')^{\mathrm{soc}}(y)\circ y,$$

which in turn implies that

$$\begin{aligned}
&\langle \nabla(\phi')^{\mathrm{soc}}(y)\cdot z,\ x-y\rangle \\
=\ &\langle \nabla(\phi')^{\mathrm{soc}}(y)\cdot (y+z-y),\ x-y\rangle \\
=\ &\langle \nabla(\phi')^{\mathrm{soc}}(y)\cdot y,\ x-y\rangle + \langle \nabla(\phi')^{\mathrm{soc}}(y)\cdot (z-y),\ x-y\rangle \\
=\ &\langle (\phi'')^{\mathrm{soc}}(y)\circ y,\ x-y\rangle + \langle \nabla(\phi')^{\mathrm{soc}}(y)\cdot (z-y),\ x-y\rangle.
\end{aligned}$$

This, together with (3.52) and (3.51), yields the first inequality in (3.50), whereas the second inequality follows from the symmetry of the matrix $\nabla(\phi')^{\mathrm{soc}}(y)$. □

Propositions 3.8-3.9 indicate that $D(x,y)$ possesses some favorable properties similar to those for d_φ. We will employ these properties to establish the convergence for an approximate version of the proximal-like algorithm (3.34).

The proximal-like algorithm described as (3.34) for the CSOCP consists of a sequence of exact minimization. However, in practical computations, it is impossible to obtain the exact solution of these minimization problems. Therefore, we consider an approximate version of this algorithm which allows the inexact solution of the subproblems (3.34). Throughout this section, we make the following assumptions for the CSOCP:

(A1) $\inf\{f(\zeta)\,|\,\zeta\in\mathcal{F}\} := f_* > -\infty$ and $\mathrm{dom}(f)\cap\mathrm{int}(\mathcal{F})\neq\emptyset$.

(A2) The matrix A is of maximal rank m.

Remark 3.4. *As remarked in Remark 3.2, Assumption (A1) is elementary for the existence of the solution of the CSOCP. Assumption (A2) is common in the solution of the SOCPs, which is clearly satisfied when $\mathcal{F}=\{\zeta\in\mathbb{R}^n\,|\,\zeta\succeq_{\mathcal{K}^n}0\}$. Moreover, if we consider the linear SOCP*

$$\begin{array}{ll} \min & \bar{c}^T x \\ \text{s.t.} & \bar{A}x=\bar{b},\quad x\in\mathcal{K}^n, \end{array} \tag{3.53}$$

where $\bar{A}\in\mathbb{R}^{m\times n}$ with $m\le n$, $\bar{b}\in\mathbb{R}^m$, and $\bar{c}\in\mathbb{R}^n$, the assumption that $\bar{A}$ has full row rank m is standard. Consequently, its dual problem, given by

$$\begin{array}{ll} \max & \bar{b}^T y \\ \text{s.t.} & \bar{c}-\bar{A}^T y\succeq_{\mathcal{K}^n} 0, \end{array} \tag{3.54}$$

satisfies assumption (A2). This shows that we can solve the linear SOCP by applying the approximate proximal-like algorithm described below to the dual problem (3.54). In addition, we know that the recession cone of $\mathcal{F}$ is given by $0^+\mathcal{F} = \{d \in \mathbb{R}^m |\ Ad \succeq_{\mathcal{K}^n} 0\}$. This implies that assumption (A2) is also satisfied when $\mathcal{F}$ is supposed to be bounded, since its recession cone $0^+\mathcal{F}$ now reduces to zero.

For the sake of notation, in the sequel, we denote $\mathcal{D} : \mathrm{int}(\mathcal{F}) \times \mathcal{F} \to \mathbb{R}$ by

$$\mathcal{D}(\zeta, \xi) := D(A\zeta + b, A\xi + b). \tag{3.55}$$

From Proposition 3.8, we readily obtain the following properties of $\mathcal{D}(\zeta, \xi)$.

Proposition 3.10. *Let $\mathcal{D}(\zeta, \xi)$ be defined by (3.55). Then, under Assumption (A2), we have*

(a) *$\mathcal{D}(\zeta, \xi) \geq 0$ for any $\zeta \in \mathrm{int}(\mathcal{F})$ and $\xi \in \mathcal{F}$, and $\mathcal{D}(\zeta, \xi) = 0$ if and only if $\zeta = \xi$;*

(b) *the function $\mathcal{D}(\cdot, \xi)$ for any fixed $\xi \in \mathcal{F}$ is continuously differentiable on $\mathrm{int}(\mathcal{F})$ with*

$$\nabla_\zeta \mathcal{D}(\zeta, \xi) = 2A^T \nabla(\phi')^{\mathrm{soc}}(A\zeta + b)A(\zeta - \xi); \tag{3.56}$$

(c) *for any fixed $\xi \in \mathcal{F}$, the function $\mathcal{D}(\cdot, \xi)$ is convex on $\mathrm{int}(\mathcal{F})$, and for any fixed $\xi \in \mathrm{int}(\mathcal{F})$, then $\mathcal{D}(\cdot, \xi)$ is strictly convex over $\mathrm{int}(\mathcal{F})$;*

(d) *for any fixed $\xi \in \mathrm{int}(\mathcal{F})$, the function $\mathcal{D}(\cdot, \xi)$ is essentially smooth;*

(e) *for any fixed $\xi \in \mathcal{F}$, the level sets $L(\xi, \gamma) = \Big\{\zeta \in \mathrm{int}(\mathcal{F}) : \mathcal{D}(\zeta, \xi) \leq \gamma\Big\}$ for all $\gamma \geq 0$ are bounded.*

Now we describe an approximate version of the proximal-like algorithm (3.34).

The APM. Given a starting point $\zeta^0 \in \mathrm{int}(\mathcal{F})$ and constants $\epsilon_k \geq 0$ and $\mu_k > 0$, generate the sequence $\{\zeta^k\} \subset \mathrm{int}(\mathcal{F})$ satisfying

$$\begin{cases} g^k \in \partial_{\epsilon_k} f(\zeta^k), \\ \mu_k g^k + \nabla_\zeta \mathcal{D}(\zeta^k, \zeta^{k-1}) = 0, \end{cases} \tag{3.57}$$

where $\partial_\epsilon f$ represents the ϵ-subdifferential of f.

Remark 3.5. *The APM can be regarded as an approximate version of the entropy proximal-like algorithm (3.34) in the following sense. From the relation in (3.57) and the convexity of $\mathcal{D}(\cdot, \xi)$ over $\mathrm{int}(\mathcal{F})$ for any fixed $\xi \in \mathrm{int}(\mathcal{F})$, it follows that for any $u \in \mathrm{int}(\mathcal{F})$,*

$$f(u) \geq f(\zeta^k) + \langle u - \zeta^k, g^k \rangle - \epsilon_k$$

and

$$\mu_k^{-1}\mathcal{D}(u,\zeta^{k-1}) \geq \mu_k^{-1}\mathcal{D}(\zeta^k,\zeta^{k-1}) + \mu_k^{-1}\langle \nabla_\zeta \mathcal{D}(\zeta^k,\zeta^{k-1}),\ u-\zeta^k\rangle.$$

Adding the last two inequalities and using (3.57) yields

$$f(u) + \mu_k^{-1}\mathcal{D}(u,\zeta^{k-1}) \geq f(\zeta^k) + \mu_k \mathcal{D}(\zeta^k,\zeta^{k-1}) - \epsilon_k.$$

This implies that

$$\zeta^k \in \epsilon_k - \operatorname{argmin}\left\{ f(\zeta) + \mu_k^{-1}\mathcal{D}(\zeta,\zeta^{k-1})\right\}, \tag{3.58}$$

where for a given function F and $\epsilon \geq 0$, the notation

$$\epsilon - \operatorname{argmin}\ F(\zeta) := \Big\{ \zeta^* : F(\zeta^*) \leq \inf F(\zeta) + \epsilon \Big\}. \tag{3.59}$$

In the rest of this section, we focus on the convergence of the APM defined as in (3.57) under assumptions (A1) and (A2). First, we prove that the APM generates a sequence $\{\zeta^k\} \subset \operatorname{int}(\mathcal{F})$, and consequently the APM is well-defined.

Proposition 3.11. *For any $\xi \in \operatorname{int}(\mathcal{F})$ and $\mu > 0$, we have the following results.*

(a) *The function $F(\cdot) := f(\cdot) + \mu^{-1}\mathcal{D}(\cdot,\xi)$ has bounded level sets under assumption (A1).*

(b) *If, in addition, assumption (A2) holds, then there has a unique $\widehat{\zeta} \in \operatorname{int}(\mathcal{F})$ such that*

$$\widehat{\zeta} = \operatorname*{argmin}_{\zeta \in \operatorname{int}(\mathcal{F})} \left\{ f(\zeta) + \mu^{-1}\mathcal{D}(\zeta,\xi)\right\}, \tag{3.60}$$

and moreover, the minimum in the right hand side is attained at $\widehat{\zeta}$ satisfying

$$-2\mu^{-1}A^T \nabla(\phi')^{\rm soc}(A\widehat{\zeta}+b)A(\widehat{\zeta}-\xi) \in \partial f(\widehat{\zeta}). \tag{3.61}$$

Proof. (a) Fix $\xi \in \operatorname{int}(\mathcal{F})$ and $\mu > 0$. By assumption (A1) and the nonnegativity of $\mathcal{D}(\zeta,\xi)$, to show that $F(\zeta)$ has bounded level sets, it suffices to show that for all $\nu \geq f_*$, the level sets $L(\nu) := \{\zeta \in \operatorname{int}(\mathcal{F}) \,|\, F(\zeta) \leq \nu\}$ are bounded. Notice that $L(\nu) \subseteq L(\xi, \mu(\nu - f_*))$ and $L(\xi,\gamma) := \{\zeta \in \operatorname{int}(\mathcal{F}) \,|\, \mathcal{D}(\zeta,\xi) \leq \gamma\}$ are bounded for all $\gamma \geq 0$ by Proposition 3.10(e). Therefore, the sets $L(\nu)$ all $\nu \geq f_*$ are bounded.

(b) By Proposition 3.10(b), $F(\zeta)$ is a closed proper strictly convex function. Hence, if the minimum exists, it must be unique. From part(a), the minimizer $\widehat{\zeta}$ exists, and so it is unique. Under assumption (A2), using the gradient formula in (3.56) and the optimality conditions for (3.60) then yields that

$$0 \in \partial f(\widehat{\zeta}) + 2\mu^{-1}A^T\nabla(\phi')^{\rm soc}(A\widehat{\zeta}+b)A(\widehat{\zeta}-\xi) + \partial\delta(\widehat{\zeta}\,|\,\mathcal{F}), \tag{3.62}$$

where $\delta(u\,|\,\mathcal{F}) = 0$ if $u \in \mathcal{F}$ and $+\infty$ otherwise. By Proposition3.10(c) and [131, Theorem 26.1], we have $\partial_\zeta \mathcal{D}(\zeta,\xi) = \emptyset$ for all $\zeta \in \operatorname{bd}(\mathcal{F})$. Hence, the relation in (3.62) implies that $\widehat{\zeta} \in \operatorname{int}(\mathcal{F})$. On the other hand, from [131, Page 226], we know that

$$\partial\delta(u\,|\,\mathcal{F}) = \left\{ v \in \mathbb{R}^n \,|\, v \preceq_{\mathcal{K}^n} 0,\ \operatorname{tr}(v \circ u) = 0\right\}.$$

Using Property 1.3, we then obtain $\partial\delta(\widehat{\zeta}\,|\,\mathcal{F}) = \{0\}$. Thus, the proof is completed. □

Next, we investigate the properties of the sequence $\{\zeta^k\}$ generated by the APM defined as in (3.57).

Proposition 3.12. *Let $\{\mu_k\}$ be any sequence of positive numbers and $\sigma_n = \sum_{k=1}^n \mu_k$. Let $\{\zeta^k\}$ be the sequence generated by the APM defined as in (3.57). Then, the following hold.*

(a) $\mu_k[f(\zeta^k) - f(\zeta)] \le \mathcal{D}(\zeta^{k-1},\zeta) - \mathcal{D}(\zeta^k,\zeta) + \mu_k\epsilon_k$ *for all* $\zeta \in \mathcal{F}$.

(b) $\mathcal{D}(\zeta^k,\zeta) \le D(\zeta^{k-1},\zeta) + \mu_k\epsilon_k$ *for all* $\zeta \in \mathcal{F}$ *subject to* $f(\zeta) \le f(\zeta^k)$.

(c) $\sigma_n(f(\zeta^n) - f(\zeta)) \le \mathcal{D}(\zeta^0,\zeta) - D(\zeta^n,\zeta) + \sum_{k=1}^n \sigma_k\epsilon_k$ *for all* $\zeta \in \mathcal{F}$.

Proof. (a) For any $\zeta \in \mathcal{F}$, using the definition of the ϵ-subdifferential, we have

$$f(\zeta) \ge f(\zeta^k) + \langle g^k,\ \zeta - \zeta^k\rangle - \epsilon_k, \tag{3.63}$$

where $g^k \in \partial_{\epsilon_k} f(\zeta^k)$. However, from (3.57) and (3.56), it follows that

$$g^k = -2\mu_k^{-1} A^T \nabla(\phi')^{\rm soc}(A\zeta^k + b)A(\zeta^k - \zeta^{k-1}).$$

Substituting this g^k into (3.63), we then obtain that

$$\mu_k\big[f(\zeta^k) - f(\zeta)\big] \le 2\Big\langle A^T \nabla(\phi')^{\rm soc}(A\zeta^k + b)A(\zeta^k - \zeta^{k-1}),\ \zeta - \zeta^k\Big\rangle + \mu_k\epsilon_k.$$

On the other hand, applying Proposition 3.9 at the points $x = A\zeta^{k-1} + b,\ y = A\zeta^k + b$ and $z = A\zeta + b$ and using the definition of $\mathcal{D}(\zeta,\xi)$ given by (3.55) yields

$$\mathcal{D}(\zeta^{k-1},\zeta) - \mathcal{D}(\zeta^k,\zeta) = 2\Big\langle A^T \nabla(\phi')^{\rm soc}(A\zeta^k + b)A(\zeta^k - \zeta^{k-1}),\ \zeta - \zeta^k\Big\rangle.$$

Combining the last two equations, we immediately obtain the result.

(b) The result follows directly from part (a) for any $\zeta \in \mathcal{F}$ such that $f(\zeta^k) \ge f(\zeta)$.

(c) First, from (3.58), it follows that

$$\zeta^k \in \epsilon_k - \operatorname{argmin}\left\{f(\zeta) + \mu_k^{-1}\mathcal{D}(\zeta,\zeta^{k-1})\right\}.$$

This implies that for any $\zeta \in \operatorname{int}(\mathcal{F})$,

$$f(\zeta) + \mu_k^{-1}\mathcal{D}(\zeta,\zeta^{k-1}) \ge f(\zeta^k) + \mu_k^{-1}\mathcal{D}(\zeta^k,\zeta^{k-1}) - \epsilon_k.$$

Setting $\zeta = \zeta^{k-1}$ in this inequality and using Proposition 3.10(d) then yields that

$$f(\zeta^{k-1}) - f(\zeta^k) \ge \mu_k^{-1}\mathcal{D}(\zeta^k,\zeta^{k-1}) - \epsilon_k \ge -\epsilon_k.$$

Multiplying the above inequality by σ_{k-1} and summing over $k = 1, 2, \cdots, n$, we get

$$\sum_{k=1}^{n} \left[\sigma_{k-1} f(\zeta^{k-1}) - (\sigma_k - \mu_k) f(\zeta^k)\right] \geq -\sum_{k=1}^{n} \sigma_{k-1}\epsilon_k,$$

which, by noting that $\sigma_k = \mu_k + \sigma_{k-1}$ (with $\sigma_0 \equiv 0$), can be reduced to

$$\sigma_n f(\zeta^n) - \sum_{k=1}^{n} \mu_k f(\zeta^k) \leq \sum_{k=1}^{n} \sigma_{k-1}\epsilon_k.$$

On the other hand, using part (a) and summing over $k = 1, 2, \cdots, n$, we have

$$-\sigma_n f(\zeta) + \sum_{k=1}^{n} \mu_k f(\zeta^k) \leq \mathcal{D}(\zeta^0, \zeta) - D(\zeta^n, \zeta) + \sum_{k=1}^{n} \mu_k \epsilon_k, \quad \forall \zeta \in \mathcal{F}.$$

Adding the last two inequalities yields

$$\sigma_n(f(\zeta^n) - f(\zeta)) \leq \mathcal{D}(\zeta^0, \zeta) - D(\zeta^n, \zeta) + \sum_{k=1}^{n} (\mu_k + \sigma_{k-1})\epsilon_k,$$

which proves (c) because $\mu_k + \sigma_{k-1} = \sigma_k$. $\square$

We are now in a position to prove our main convergence result for the APM defined as in (3.57).

Proposition 3.13. *Let $\{\zeta^k\}$ be the sequence generated by the APM defined as in (3.57) and $\sigma_n = \sum_{k=1}^{n} \mu_k$. Then, under assumptions (A1) and (A2), the following hold.*

(a) *If $\sigma_n \to +\infty$ and $\mu_k^{-1}\sigma_k\epsilon_k \to 0$, then $\lim_{n\to+\infty} f(\zeta^n) \to f_*$.*

(b) *If the optimal set $\mathcal{X} \neq \emptyset$, $\sigma_n \to \infty$ and $\sum_{k=1}^{\infty} \mu_k\epsilon_k < \infty$, then the sequence ζ^k is bounded and every accumulation point is a solution of the CSOCP.*

Proof. (a) From Proposition 3.12(c) and the nonnegativity of $\mathcal{D}(\zeta^n, \zeta)$, it follows that

$$f(\zeta^n) - f(\zeta) \leq \sigma_n^{-1}\mathcal{D}(\zeta^0, \zeta) + \sigma_n^{-1} \sum_{k=1}^{n} \sigma_k \epsilon_k, \quad \forall \zeta \in \mathcal{F}.$$

Taking the limit $\sigma_n \to +\infty$ to the two sides of the last inequality, we immediately have that the first term in the right hand side goes to zero. In addition, applying Lemma 3.6 with $a_{nk} := \sigma_n^{-1}\mu_k$ if $k \leq n$ and $a_{nk} := 0$ otherwise and $u_k := \mu_k^{-1}\sigma_k\epsilon_k$, we obtain that the second term in the right hand side

$$\sigma_n^{-1} \sum_{k=1}^{n} \sigma_k \epsilon_k = \sum_{k} a_{nk} u_k \to 0$$

because $\sigma_n \to +\infty$ and $\mu_k^{-1}\sigma_k\epsilon_k \to 0$. Therefore, we have

$$\lim_{n\to+\infty} f(\zeta^n) \le f_*.$$

This, together with the fact that $f(\zeta^n) \ge f_*$, implies the desired result.

(b) Suppose that $\zeta^* \in \mathcal{X}$. For any k, we have $f(\zeta^k) \ge f(\zeta^*)$. From Proposition 3.12(b), it then follows that

$$\mathcal{D}(\zeta^k, \zeta^*) \le \mathcal{D}(\zeta^{k-1}, \zeta^*) + \mu_k\epsilon_k.$$

Since $\sum_{k=1}^{\infty} \mu_k\epsilon_k < +\infty$, using Lemma 3.6 with $v_k := \mathcal{D}(\zeta^k, \zeta^*) \ge 0$ and $\beta_k := \mu_k\epsilon_k \ge 0$ yields that the sequence $\{\mathcal{D}(\zeta^k, \zeta^*)\}$ converges. Thus, by Proposition 3.10(e), the sequence $\{\zeta^k\}$ is bounded and consequently has an accumulation point. Without any loss of generality, let $\widehat{\zeta} \in \mathcal{F}$ be an accumulation point of $\{\zeta^k\}$. Then $\{\zeta^{k_j}\} \to \widehat{\zeta}$ for some $k_j \to +\infty$. Since f is lower semi-continuous, we get $f(\widehat{\zeta}) = \liminf_{k_j\to\infty} f(\zeta^{k_j})$. On the other hand, $f(\zeta^{k_j}) \to f_*$ by part (a). The two sides imply that $f(\widehat{\zeta}) = f_*$. Therefore, $\widehat{\zeta}$ is a solution of the CSOCP. The proof is thus complete. $\square$

3.3 Interior proximal methods for SOCCP

In this section, we consider the below CSOCP which is slightly different from (3.1):

$$\begin{array}{ll} \inf & f(x) \\ \text{s.t.} & Ax = b, \quad x \succeq_{\mathcal{K}} 0, \end{array} \tag{3.64}$$

where $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a closed proper convex function, A is an $m \times n$ matrix with full row rank m, b is a vector in $\mathbb{R}^m$, $x \succeq_{\mathcal{K}} 0$ means $x \in \mathcal{K}$, and $\mathcal{K}$ is the Cartesian product of some second-order cones. In other words,

$$\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_r}$$

where $r, n_1, \ldots, n_r \ge 1$ with $n_1 + \cdots + n_r = n$, and

$$\mathcal{K}^{n_i} := \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_i-1} \,|\, x_1 \ge \|x_2\|\right\}$$

with $\|\cdot\|$ being the Euclidean norm. When f reduces to a linear function, i.e. $f(x) = c^T x$ for some $c \in \mathbb{R}^n$, (3.64) becomes the standard SOCP. Throughout this section, we denote by X_* the optimal set of (3.64), and let $\mathcal{V} := \{x \in \mathbb{R}^n \mid Ax = b\}$. This CSOCP, as an extension of the standard SOCP, has a wide range of applications from engineering, control, finance to robust optimization and combinatorial optimization; see [1, 103] and references therein.

There have proposed various methods for the CSOCP, which include the interior point methods [2, 110, 146], the smoothing Newton methods [52, 64], the smoothing-regularization method [72], the semismooth Newton method [87], and the merit function method [49]. These methods are all developed by reformulating the KKT optimality conditions as a system of equations or an unconstrained minimization problem. This paper will focus on an iterative scheme which is proximal based and handles directly the CSOCP itself. Specifically, the proximal-type algorithm consists of generating a sequence $\{x^k\}$ via

$$x^k := \text{argmin}\left\{\lambda_k f(x) + H(x, x^{k-1}) \mid x \in \mathcal{K} \cap \mathcal{V}\right\}, \quad k = 1, 2, \dots \tag{3.65}$$

where $\{\lambda_k\}$ is a sequence of positive parameters, and $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proximal distance with respect to int $\mathcal{K}$ (see Def. 3.1) which plays the same role as the Euclidean distance $\|x - y\|^2$ in the classical proximal algorithms (see, e.g., [106, 132]), but possesses certain more desirable properties to force the iterates to stay in $\mathcal{K} \cap \mathcal{V}$, thus eliminating the constraints automatically. As will be shown, such proximal distances can be produced with an appropriate closed proper univariate function.

In the rest of this section, we focus on the case where $\mathcal{K} = \mathcal{K}^n$, and all the analysis can be carried over to the case where $\mathcal{K}$ has the direct product structure. Unless otherwise stated, we make the following minimal assumption for the CSOCP (3.64):

(A1) $\text{dom} f \cap (\mathcal{V} \cap \text{int}(\mathcal{K}^n)) \neq \emptyset$ and $f_* := \inf\{f(x) \mid x \in \mathcal{V} \cap \mathcal{K}^n\} > -\infty$.

Definition 3.2. *An extended-valued function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called a proximal distance with respect to* $\text{int}(\mathcal{K}^n)$ *if it satisfies the following properties:*

(P1) $\text{dom} H(\cdot, \cdot) = \mathcal{C}_1 \times \mathcal{C}_2$ *with* $\text{int}(\mathcal{K}^n) \times \text{int}(\mathcal{K}^n) \subset \mathcal{C}_1 \times \mathcal{C}_2 \subseteq \mathcal{K}^n \times \mathcal{K}^n$.

(P2) *For each given $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuous and strictly convex on $\mathcal{C}_1$, and it is continuously differentiable on* $\text{int}(\mathcal{K}^n)$ *with* $\text{dom}\nabla_1 H(\cdot, y) = \text{int}(\mathcal{K}^n)$.

(P3) *$H(x, y) \geq 0$ for all $x, y \in \mathbb{R}^n$, and $H(y, y) = 0$ for all $y \in \text{int}(\mathcal{K}^n)$.*

(P4) *For each fixed $y \in \mathcal{C}_2$, the sets $\{x \in \mathcal{C}_1 : H(x, y) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$.*

Definition 3.2 has a little difference from Definition 2.1 of [10] for a proximal distance w.r.t. $\text{int}(\mathcal{K}^n)$, since here $H(\cdot, y)$ is required to be strictly convex over $\mathcal{C}_1$ for any fixed $y \in \text{int}(\mathcal{K}^n)$. We denote $\mathcal{D}(\text{int}(\mathcal{K}^n))$ by the family of functions H satisfying Definition 3.2. With a given $H \in \mathcal{D}(\text{int}(\mathcal{K}^n))$, we have the following basic iterative algorithm for (3.64).

Interior Proximal Algorithm (IPA). Given $H \in \mathcal{D}(\text{int}(\mathcal{K}^n))$ and $x^0 \in \mathcal{V} \cap \text{int}(\mathcal{K}^n)$. For $k = 1, 2, \dots,$ with $\lambda_k > 0$ and $\varepsilon_k \geq 0$, generate a sequence $\{x^k\} \subset \mathcal{V} \cap \text{int}(\mathcal{K}^n)$ with $g^k \in \partial_{\varepsilon_k} f(x^k)$ via the following iterative scheme:

$$x^k := \text{argmin}\left\{\lambda_k f(x) + H(x, x^{k-1}) \mid x \in \mathcal{V}\right\} \tag{3.66}$$

such that

$$\lambda_k g^k + \nabla_1 H(x^k, x^{k-1}) = A^T u^k \quad \text{for some } u^k \in \mathbb{R}^m. \tag{3.67}$$

The following proposition implies that the IPA is well-defined, and moreover, from its proof we see that the iterative formula (3.66) is equivalent to the iterative scheme (3.65). When $\varepsilon_k > 0$ for any $k \in \mathbb{N}$ (the set of natural numbers), the IPA can be viewed as an approximate interior proximal method, and it becomes exact if $\varepsilon_k = 0$ for all $k \in \mathbb{N}$.

Proposition 3.14. *For any given $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$ and $y \in \mathrm{int}(\mathcal{K}^n)$, consider the problem*

$$f_*(y, \tau) = \inf \{\tau f(x) + H(x, y) \mid x \in \mathcal{V}\} \quad \text{with } \tau > 0. \tag{3.68}$$

Then, for each $\varepsilon \geq 0$, there exist $x(y,\tau) \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)$ and $g \in \partial_\varepsilon f(x(y,\tau))$ such that

$$\tau g + \nabla_1 H(x(y,\tau), y) = A^T u \tag{3.69}$$

for some $u \in \mathbb{R}^m$. Moreover, for such $x(y,\tau)$, we have

$$\tau f(x(y,\tau)) + H(x(y,\tau), y) \leq f_*(y,\tau) + \varepsilon. \tag{3.70}$$

Proof. Set $F(x,\tau) := \tau f(x) + H(x,y) + \delta_{\mathcal{V}\cap\mathcal{K}^n}(x)$, where $\delta_{\mathcal{V}\cap\mathcal{K}^n}(x)$ is the indicator function defined on the set $\mathcal{V} \cap \mathcal{K}^n$. Since $\mathrm{dom}H(\cdot, y) = \mathcal{C}_1 \subset \mathcal{K}^n$, it is clear that

$$f_*(y,\tau) = \inf \{F(x,\tau) \mid x \in \mathbb{R}^n\}. \tag{3.71}$$

Since $f_* > -\infty$, it is easy to verify that for any $\gamma \in \mathbb{R}$ the following relation holds

$$\begin{aligned} \{x \in \mathbb{R}^n \mid F(x,\tau) \leq \gamma\} \quad &\subset \quad \{x \in \mathcal{V} \cap \mathcal{K}^n \mid H(x,y) \leq \gamma - \tau f_*\} \\ &\subset \quad \{x \in \mathcal{C}_1 \mid H(x,y) \leq \gamma - \tau f_*\}, \end{aligned}$$

which together with (P4) implies that $F(\cdot,\tau)$ has bounded level sets. In addition, by (P1)-(P3), $F(\cdot,\tau)$ is a closed proper and strictly convex function. Hence, the problem (3.71) has a unique solution, to say $x(y,\tau)$. From the optimality conditions of (3.71), we get

$$0 \in \partial F(x(y,\tau)) = \tau \partial f(x(y,\tau)) + \nabla_1 H(x(y,\tau), y) + \partial \delta_{\mathcal{V}\cap\mathcal{K}^n}(x(y,\tau))$$

where the equality is due to [131, Theorem 23.8] and $\mathrm{dom} f \cap (\mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)) \neq \emptyset$. Notice that $\mathrm{dom}\, \nabla_1 H(\cdot, y) = \mathrm{int}(\mathcal{K}^n)$ and $\mathrm{dom}\, \partial \delta_{\mathcal{V}\cap\mathcal{K}^n}(\cdot) = \mathcal{V} \cap \mathcal{K}^n$. Therefore, the last equation implies $x(y,\tau) \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)$, and there exists $g \in \partial f(x(y,\tau))$ such that

$$-\tau g - \nabla_1 H(x(y,\tau), y) \in \partial \delta_{\mathcal{V}\cap\mathcal{K}^n}(x(y,\tau)).$$

On the other hand, by the definition of $\delta_{\mathcal{V}\cap\mathcal{K}^n}(\cdot)$, it is not hard to derive that

$$\partial \delta_{\mathcal{V}\cap\mathcal{K}^n}(x) = \mathrm{Im}(A^T), \quad \forall x \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n).$$

The last two equations imply that (3.69) holds for $\varepsilon = 0$. When $\varepsilon > 0$, (3.69) also holds for such $x(y,\tau)$ and g since $\partial f(x(y,\tau)) \subset \partial_\varepsilon f(x(y,\tau))$. Finally, since for each $y \in \mathrm{int}(\mathcal{K}^n)$ the function $H(\cdot,y)$ is strictly convex, and since $g \in \partial_\varepsilon f(x(y,\tau))$, we have

$$\begin{aligned} \tau f(x) + H(x,y) &\geq \tau f(x(y,\tau)) + H(x(y,\tau),y) \\ &\quad + \langle \tau g + \nabla_1 H(x(y,\tau),y), x - x(y,\tau)\rangle - \varepsilon \\ &= \tau f(x(y,\tau)) + H(x(y,\tau),y) + \langle A^T u, x - x(y,\tau)\rangle - \varepsilon \\ &= \tau f(x(y,\tau)) + H(x(y,\tau),y) - \varepsilon \quad \text{for all } x \in \mathcal{V}, \end{aligned}$$

where the first equality is from (3.69) and the last one is by $x, x(y,\tau) \in \mathcal{V}$. Thus, $f_*(y,\tau) = \inf\{\tau f(x) + H(x,y) \mid x \in \mathcal{V}\} \geq \tau f(x(y,\tau)) + H(x(y,\tau),y) - \varepsilon$. □

In the following, we focus on the convergence behaviors of the IPA with H from several subclasses of $\mathcal{D}(\mathrm{int}(\mathcal{K}^n))$, which also satisfy one of the following properties.

(P5) For any $x, y \in \mathrm{int}(\mathcal{K}^n)$ and $z \in \mathcal{C}_1$, $H(z,y) - H(z,x) \geq \langle \nabla_1 H(x,y), z - x\rangle$;

(P5') For any $x, y \in \mathrm{int}(\mathcal{K}^n)$ and $z \in \mathcal{C}_2$, $H(y,z) - H(x,z) \geq \langle \nabla_1 H(x,y), z - x\rangle$.

(P6) For each $x \in \mathcal{C}_1$, the level sets $\{y \in \mathcal{C}_2 \,|\, H(x,y) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$.

Specifically, we denote $\mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$ and $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ by the family of functions $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$ satisfying (P5) and (P5'), respectively. If $\mathcal{C}_1 = \mathcal{K}^n$, we denote $\mathcal{F}_1(\mathcal{K}^n)$ by the family of functions $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$ satisfying (P5) and (P6). If $\mathcal{C}_2 = \mathcal{K}^n$, we write $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ as $\mathcal{F}(\mathcal{K}^n)$. It is easy to see that the class of proximal distance $\mathcal{F}(\mathrm{int}(\mathcal{K}^n))$ (respectively, $\mathcal{F}(\mathcal{K}^n)$) in [10] subsumes the (H,H) with $H \in \mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$ (respectively, $\mathcal{F}_1(\mathcal{K}^n)$), but it does not include any (H,H) with $H \in \mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ (respectively, $\mathcal{F}_2(\mathcal{K}^n)$).

Proposition 3.15. *Let $\{x^k\}$ be the sequence generated by the IPA with $H \in \mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$ or $H \in \mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$. Set $\sigma_\nu = \sum_{k=1}^{\nu} \lambda_k$. Then, the following results hold.*

(a) $f(x^\nu) - f(x) \leq \sigma_\nu^{-1} H(x,x^0) + \sigma_\nu^{-1}\sum_{k=1}^{\nu} \sigma_k \varepsilon_k$ *for any* $x \in \mathcal{V} \cap \mathcal{C}_1$ *if* $H \in \mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$*;* $f(x^\nu) - f(x) \leq \sigma_\nu^{-1} H(x^0,x) + \sigma_\nu^{-1}\sum_{k=1}^{\nu} \sigma_k \varepsilon_k$ *for any* $x \in \mathcal{V} \cap \mathcal{C}_2$ *if* $H \in \mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$.

(b) *If* $\sigma_\nu \to +\infty$ *and* $\varepsilon_k \to 0$*, then* $\liminf_{\nu\to\infty} f(x^\nu) = f_*$.

(c) *The sequence* $\{f(x^k)\}$ *converges to* f_* *whenever* $\sum_{k=1}^{\infty} \varepsilon_k < \infty$.

(d) *If* $X_* \neq \emptyset$*, then* $\{x^k\}$ *is bounded with all limit points in* X_* *under (d1) or (d2) below:*

- **(d1)** X_* *is bounded and* $\sum_{k=1}^{\infty} \varepsilon_k < \infty$*;*
- **(d2)** $\sum_{k=1}^{\infty} \lambda_k \varepsilon_k < \infty$ *and* $H \in \mathcal{F}_1(\mathcal{K}^n)$ *(or* $H \in \mathcal{F}_2(\mathcal{K}^n)$*).*

Proof. The proofs are similar to those of [10, Theorem 4.1]. For completeness, we here take $H \in \mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ for example to prove the results.

(a) Since $g^k \in \partial_{\varepsilon_k} f(x^k)$, from the definition of the subdifferential, it follows that

$$f(x) \geq f(x^k) + \langle g^k, x - x^k \rangle - \varepsilon_k, \quad \forall x \in \mathbb{R}^n.$$

This together with equation (3.67) implies that

$$\lambda_k(f(x^k) - f(x)) \leq \langle \nabla_1 H(x^k, x^{k-1}), x - x^k \rangle + \lambda_k \varepsilon_k, \quad \forall x \in \mathcal{V} \cap \mathcal{C}_2.$$

Using (P5') with $x = x^k, y = x^{k-1}$ and $z = x \in \mathcal{V} \cap \mathcal{C}_2$, it then follows that

$$\lambda_k(f(x^k) - f(x)) \leq H(x^{k-1}, x) - H(x^k, x) + \lambda_k \varepsilon_k, \quad \forall x \in \mathcal{V} \cap \mathcal{C}_2. \tag{3.72}$$

Summing over $k = 1, 2, \ldots, \nu$ in this inequality yields that

$$-\sigma_\nu f(x) + \sum_{k=1}^{\nu} \lambda_k f(x^k) \leq H(x^0, x) - H(x^\nu, x) + \sum_{k=1}^{\nu} \lambda_k \varepsilon_k. \tag{3.73}$$

On the other hand, setting $x = x^{k-1}$ in (3.72), we obtain

$$f(x^k) - f(x^{k-1}) \leq \lambda_k^{-1} \left[H(x^{k-1}, x^{k-1}) - H(x^k, x^{k-1}) \right] + \varepsilon_k \leq \varepsilon_k. \tag{3.74}$$

Multiplying the inequality by σ_{k-1} (with $\sigma_0 \equiv 0$) and summing over $k = 1, \ldots, \nu$, we get

$$\sum_{k=1}^{\nu} \sigma_{k-1} f(x^k) - \sum_{k=1}^{\nu} \sigma_{k-1} f(x^{k-1}) \leq \sum_{k=1}^{\nu} \sigma_{k-1} \varepsilon_k.$$

Noting that $\sigma_k = \lambda_k + \sigma_{k-1}$ with $\sigma_0 \equiv 0$, the above inequality can reduce to

$$\sigma_\nu f(x^\nu) - \sum_{k=1}^{\nu} \lambda_k f(x^k) \leq \sum_{k=1}^{\nu} \sigma_{k-1} \varepsilon_k. \tag{3.75}$$

Adding the inequalities (3.73) and (3.75) and recalling that $\sigma_k = \lambda_k + \sigma_{k-1}$, it follows that

$$f(x^\nu) - f(x) \leq \sigma_\nu^{-1} \left[H(x^0, x) - H(x^\nu, x) \right] + \sigma_\nu^{-1} \sum_{k=1}^{\nu} \sigma_k \varepsilon_k, \quad \forall x \in \mathcal{V} \cap \mathcal{C}_2,$$

which immediately implies the desired result due to the nonnegativity of $H(x^\nu, x)$.

(b) If $\sigma_\nu \to +\infty$ and $\varepsilon_k \to 0$, then applying Lemma 2.2(ii) of [10] with $a_k = \varepsilon_k$ and $b_\nu := \sigma_\nu^{-1} \sum_{k=1}^{\nu} \lambda_k \varepsilon_k$ yields $\sigma_\nu^{-1} \sum_{k=1}^{\nu} \lambda_k \varepsilon_k \to 0$. From part(a), it then follows that

$$\liminf_{\nu \to \infty} f(x^\nu) \leq \inf \left\{ f(x) \mid x \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n) \right\}.$$

This together with $f(x^\nu) \geq \inf\{f(x) \mid x \in \mathcal{V} \cap \mathcal{K}^n\}$ implies that

$$\liminf_{\nu\to\infty} f(x^\nu) = \inf\{f(x) \mid x \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)\} = f_*.$$

(c) From (3.74), $0 \leq f(x^k) - f_* \leq f(x^{k-1}) - f_* + \varepsilon_k$. Using Lemma 2.1 of [10] with $\gamma_k \equiv 0$ and $v_k = f(x^k) - f_*$, we have that $\{f(x^k)\}$ converges to f_* whenever $\sum_{k=1}^{\infty} \varepsilon_k < \infty$.

(d) If the condition (d1) holds, then the sets $\{x \in \mathcal{V} \cap \mathcal{K}^n \mid f(x) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$, since f is closed proper convex and $X_* = \{x \in \mathcal{V} \cap \mathcal{K}^n \mid f(x) \leq f_*\}$. Note that (3.74) implies $\{x^k\} \subset \{x \in \mathcal{V} \cap \mathcal{K}^n \mid f(x) \leq f(x^0) + \sum_{j=1}^{k} \varepsilon_j\}$. Along with $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, clearly, $\{x^k\}$ is bounded. Since $\{f(x^k)\}$ converges to f_* and f is l.s.c., passing to the limit and recalling that $\{x^k\} \subset \mathcal{V} \cap \mathcal{K}^n$ yields that each accumulation point of $\{x^k\}$ is a solution of (3.64).

Suppose that the condition (d2) holds. If $H \in \mathcal{F}_2(\mathcal{K}^n)$, then inequality (3.72) holds for each $x \in \mathcal{V} \cap \mathcal{K}^n$, and particularly for $x_* \in X_*$. Consequently,

$$H(x^k, x_*) \leq H(x^{k-1}, x_*) + \lambda_k \varepsilon_k \quad \forall x_* \in X_*. \tag{3.76}$$

Summing over $k = 1, 2, \ldots, \nu$ for the last inequality, we obtain

$$H(x^\nu, x_*) \leq H(x^0, x_*) + \sum_{k=1}^{\nu} \lambda_k \varepsilon_k.$$

This, by (P4) and $\sum_{k=1}^{\infty} \lambda_k \varepsilon_k < \infty$, implies that $\{x^k\}$ is bounded, and hence has an accumulation point. Without loss of generality, let $\hat{x} \in \mathcal{K}^n$ be an accumulation point of $\{x^k\}$. Then there exists a subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to \hat{x}$ as $j \to +\infty$. From the lower semicontinuity of f and part(c), we get $f(\hat{x}) \leq \lim_{j\to+\infty} f(x^{k_j}) = f_*$, which means that $\hat{x}$ is a solution of (3.64). If $H \in \mathcal{F}_1(\mathcal{K}^n)$, then the last inequality becomes

$$H(x_*, x^\nu) \leq H(x_*, x^0) + \sum_{k=1}^{\nu} \lambda_k \varepsilon_k.$$

By (P6) and $\sum_{k=1}^{\infty} \lambda_k \varepsilon_k < \infty$, we also have that $\{x^k\}$ is bounded, and hence has an accumulation point. Using the same arguments as above, we get the desired result. □

An immediate byproduct of the above analysis yields the following global rate of convergence estimate for the IPA with $H \in \mathcal{F}_1(\mathcal{K}^n)$ or $H \in \mathcal{F}_2(\mathcal{K}^n)$.

Proposition 3.16. *Let $\{x^k\}$ be the sequence given by the IPA with $H \in \mathcal{F}_1(\mathcal{K}^n)$ or $\mathcal{F}_2(\mathcal{K}^n)$. If $X_* \neq \emptyset$ and $\sum_{k=1}^{\infty} \varepsilon_k < \infty$, then $f(x^\nu) - f_* = O(\sigma_\nu^{-1})$.*

Proof. The result is direct by setting $x = x^*$ for some $x^* \in X_*$ in the inequalities of Proposition 3.15(a), and noting that $0 < \frac{\sigma_k}{\sigma_\nu} \leq 1$ for all $k = 1, 2, \cdots, \nu$. □

To establish the global convergence of $\{x^k\}$ to an optimal solution of (3.64), we need to make further assumptions on X_* or the proximal distances in $\mathcal{F}_1(\mathcal{K}^n)$ and $\mathcal{F}_2(\mathcal{K}^n)$. We denote $\widehat{\mathcal{F}}_1(\mathcal{K}^n)$ by the family of functions $H \in \mathcal{F}_1(\mathcal{K}^n)$ satisfying (P7)-(P8) below, $\widehat{\mathcal{F}}_2(\mathcal{K}^n)$ by the family of functions $H \in \mathcal{F}_2(\mathcal{K}^n)$ satisfying (P7')–(P8') below, and $\bar{\mathcal{F}}(\mathcal{K}^n)$ by the family of functions $H \in \mathcal{F}_2(\mathcal{K}^n)$ satisfying (P7')-(P9') below:

(P7) For any $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ converging to $y^* \in \mathcal{K}^n$, we have $H(y^*, y^k) \to 0$;

(P8) For any bounded sequence $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ and any $y^* \in \mathcal{K}^n$ with $H(y^*, y^k) \to 0$, there holds that $\lambda_i(y^k) \to \lambda_i(y^*)$ for $i = 1, 2$;

(P7') For any $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ converging to $y^* \in \mathcal{K}^n$, we have $H(y^k, y^*) \to 0$;

(P8') For any bounded sequence $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ and any $y^* \in \mathcal{K}^n$ with $H(y^k, y^*) \to 0$, there holds that $\lambda_i(y^k) \to \lambda_i(y^*)$ for $i = 1, 2$;

(P9') For any bounded sequence $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ and any $y^* \in \mathcal{K}^n$ with $H(y^k, y^*) \to 0$, there holds that $y^k \to y^*$.

It is easy to see that all previous subclasses of $\mathcal{D}(\mathrm{int}(\mathcal{K}^n))$ have the following relations:

$$\widehat{\mathcal{F}}_1(\mathcal{K}^n) \subseteq \mathcal{F}_1(\mathcal{K}^n) \subseteq \mathcal{F}_1(\mathrm{int}(\mathcal{K}^n)), \quad \bar{\mathcal{F}}_2(\mathcal{K}^n) \subseteq \widehat{\mathcal{F}}_2(\mathcal{K}^n) \subseteq \mathcal{F}_2(\mathcal{K}^n) \subseteq \mathcal{F}_2(\mathrm{int}(\mathcal{K}^n)).$$

Proposition 3.17. *Let $\{x^k\}$ be generated by the IPA with $H \in \mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$ or $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$. Suppose that X_* is nonempty, $\sum_{k=1}^{\infty} \lambda_k \varepsilon_k < \infty$ and $\sum_{k=1}^{\infty} \varepsilon_k < \infty$.*

(a) *If X_* is a single point set, then $\{x^k\}$ converges to an optimal solution of (3.64).*

(b) *If X_* at least includes two elements and for any $x^* = (x_1^*, x_2^*), \bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*) \in X_*$ with $x^* \neq \bar{x}^*$, it holds that $x_1^* \neq \bar{x}_1^*$ or $\|x_2^*\| \neq \|\bar{x}_2^*\|$, then $\{x^k\}$ converges to an optimal solution of (3.64) whenever $H \in \widehat{\mathcal{F}}_1(\mathcal{K}^n)$ (or $H \in \widehat{\mathcal{F}}_2(\mathcal{K}^n)$).*

(c) *If $H \in \bar{\mathcal{F}}_2(\mathcal{K}^n)$, then $\{x^k\}$ converges to an optimal solution of (3.64).*

Proof. Part (a) is direct by Proposition 3.15(d1). We next consider part (b). Assume that $H \in \widehat{\mathcal{F}}_2(\mathcal{K}^n)$. Since $\sum_{k=1}^{\infty} \lambda_k \varepsilon_k < \infty$, from (3.76) and Lemma 2.1 of [10], it follows that the sequence $\{H(x^k, x)\}$ is convergent for any $x \in X_*$. Let $\bar{x}$ be the limit of a subsequence $\{x^{k_l}\}$. By Proposition 3.15(d2), $\bar{x} \in X_*$. Consequently, $\{H(x^k, \bar{x})\}$ is convergent. By (P7'), $H(x^{k_l}, \bar{x}) \to 0$, and so $H(x^k, \bar{x}) \to 0$. Along with (P8'), $\lambda_i(x^k) \to \lambda_i(\bar{x})$ for $i = 1, 2$, i.e.,

$$x_1^k - \|x_2^k\| \to \bar{x}_1 - \|\bar{x}_2\| \;\text{ and }\; x_1^k + \|x_2^k\| \to \bar{x}_1 + \|\bar{x}_2\| \;\text{ as }\; k \to \infty.$$

This implies that $x_1^k \to \bar{x}_1$ and $\|x_2^k\| \to \|\bar{x}_2\|$. Together with the given assumption for X_*, we have that $x^k \to \bar{x}$. Suppose that $H \in \widehat{\mathcal{F}}_1(\mathcal{K}^n)$. The inequality (3.76) becomes

$$H(x_*, x^k) \leq H(x_*, x^{k-1}) + \lambda_k \varepsilon_k, \quad \forall x_* \in X_*,$$

and using (P7)-(P8) and the same arguments as above then yields the result. Part(c) is direct by the arguments above and the property (P9'). □

When all points in the nonempty X_* lie on the boundary of $\mathcal{K}^n$, we must have $x_1^* \neq \bar{x}_1^*$ or $\|x_2^*\| \neq \|\bar{x}_2^*\|$ for any $x^* = (x_1^*, x_2^*), \bar{x}^* = (\bar{x}_1^*, \bar{x}_2^*) \in X_*$ with $x^* \neq \bar{x}^*$, and the assumption for X_* in (b) is automatically satisfied. Since the solutions of (3.64) are generally on the boundary of $\mathcal{K}^n$, the assumption for X_* in Proposition 3.17(b) is much weaker than the one in Proposition 3.17(a).

Up to now, we have studied two types of convergence results for the IPA by the class in which the proximal distance H lies. Proposition 3.15 and Proposition 3.16 show that the largest, and less demanding, classes $\mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$ and $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ provide reasonable convergence properties for the IPA under minimal assumptions on the problem's data. This coincides with interior proximal methods for convex programming over nonnegative orthant cones; see [10]. The smallest subclass $\bar{\mathcal{F}}_2(\mathcal{K}^n)$ of $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$ guarantees that $\{x^k\}$ converges to an optimal solution provided that X_* is nonempty. The smaller class $\widehat{\mathcal{F}}_2(\mathcal{K}^n)$ may guarantee the global convergence of the sequence $\{x^k\}$ to an optimal solution under an additional assumption except the nonempty of X_*. Moreover, we will illustrate that there are indeed examples for the class $\bar{\mathcal{F}}_2(\mathcal{K}^n)$. For the smallest subclass $\widehat{\mathcal{F}}_1(\mathcal{K}^n)$ of $\mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$, the analysis shows that it seems hard to find an example, although it guarantees the convergence of $\{x^k\}$ to an optimal solution by Proposition 3.17(b).

Next, we provide three kinds of ways to construct a proximal distance w.r.t. $\mathrm{int}(\mathcal{K}^n)$ and analyze their own advantages and disadvantages. All of these ways exploit a l.s.c. (lower semi-continuous) proper univariate function to produce such a proximal distance. In addition, with such a proximal distance and the Euclidean distance, we obtain the regularized ones.

The first way produces the proximal distances for the class $\mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$. This way is based on the compound of a univariate function ϕ and the determinant function $\det(\cdot)$, where $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a l.s.c. proper function satisfying the following conditions:

(B1) $\mathrm{dom}\phi \subseteq [0, +\infty)$, $\mathrm{int}(\mathrm{dom}\phi) = (0, +\infty)$, and ϕ is continuous on its domain;

(B2) for any $t_1, t_2 \in \mathrm{dom}\phi$, there holds that

$$\phi(t_1^r t_2^{1-r}) \leq r\phi(t_1) + (1-r)\phi(t_2), \qquad \forall r \in [0,1]; \tag{3.77}$$

(B3) ϕ is continuously differentiable on $\mathrm{int}(\mathrm{dom}\phi)$ with $\mathrm{dom}(\phi') = (0, \infty)$;

(B4) $\phi'(t) < 0$ for all $t \in (0, \infty)$, $\lim_{t \to 0^+} \phi(t) = +\infty$, and $\lim_{t \to +\infty} t^{-1}\phi(t^2) \geq 0$.

With such a univariate ϕ, we define the function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ as in (3.15)

$$H(x,y) := \begin{cases} \phi(\det(x)) - \phi(\det(y)) - \langle \nabla\phi(\det(y)), x - y \rangle, & \forall x, y \in \mathrm{int}(\mathcal{K}^n); \\ \infty. & \text{otherwise.} \end{cases}$$

By the conditions (B1)-(B4), we may prove that H has the following properties.

Proposition 3.18. *Let H be defined as in (3.15) with ϕ satisfying (B1)–(B4). Then, the following hold.*

(a) *For any fixed $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is strictly convex over $\text{int}(\mathcal{K}^n)$.*

(b) *For any fixed $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuously differentiable on $\text{int}(\mathcal{K}^n)$ with*

$$\nabla_1 H(x,y) = 2\phi'(\det(x)) \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix} - 2\phi'(\det(y)) \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix} \tag{3.78}$$

for all $x \in \text{int}(\mathcal{K}^n)$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

(c) *$H(x,y) \geq 0$ for all $x, y \in \mathbb{R}^n$, and $H(y,y) = 0$ for all $y \in \text{int}(\mathcal{K}^n)$.*

(d) *For any $y \in \text{int}(\mathcal{K}^n)$, the sets $\{x \in \text{int}(\mathcal{K}^n) \,|\, H(x,y) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$.*

(e) *For any $x, y \in \text{int}(\mathcal{K}^n)$ and $z \in \text{int}(\mathcal{K}^n)$, the following three point identity holds*

$$H(z,y) = H(z,x) + H(x,y) + \langle \nabla_1 H(x,y), z - x \rangle.$$

Proof. (a) It suffices to prove $\phi(\det(x))$ is strictly convex on $\text{int}(\mathcal{K}^n)$. By Proposition 1.8(a), there has

$$\det(\alpha x + (1-\alpha)z) > (\det(x))^\alpha (\det(z))^{1-\alpha}, \quad \forall \alpha \in (0,1),$$

for all $x, z \in \text{int}(\mathcal{K}^n)$ and $x \neq z$. Since $\phi'(t) < 0$ for all $t \in (0, +\infty)$, we have that ϕ is decreasing on $(0, +\infty)$. This, together with the condition (B2), yields that

$$\begin{aligned} \phi\left[\det(\alpha x + (1-\alpha)z)\right] &< \phi\left[(\det(x))^\alpha (\det(z))^{1-\alpha}\right] \\ &\leq \alpha\phi[\det(x)] + (1-\alpha)\phi[\det(z)], \quad \forall \alpha \in (0,1) \end{aligned}$$

for any $x, z \in \text{int}(\mathcal{K}^n)$ and $x \neq z$. This means that $\phi(\det(x))$ is strictly convex on $\text{int}(\mathcal{K}^n)$.

(b) Since $\det(x)$ is continuously differentiable on $\mathbb{R}^n$ and ϕ is continuously differentiable on $(0, +\infty)$, we have that $\phi(\det(x))$ is continuously differentiable on $\text{int}(\mathcal{K}^n)$. This means that for any fixed $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuously differentiable on $\text{int}(\mathcal{K}^n)$. By a simple computation, we immediately obtain the formula in (3.78).

(c) Since $\phi(\det(x))$ is strictly convex and continuously differentiable on $\text{int}(\mathcal{K}^n)$, we have

$$\phi(\det(x)) > \phi(\det(y)) - \langle \nabla\phi(\det(y)), x - y \rangle,$$

for any $x, y \in \text{int}(\mathcal{K}^n)$ with $x \neq y$. This implies that $H(y,y) = 0$ for all $y \in \text{int}(\mathcal{K}^n)$. In addition, from the inequality and the continuity of ϕ on its domain, it follows that

$$\phi(\det(x)) \geq \phi(\det(y)) - \langle \nabla\phi(\det(y)), x - y \rangle$$

for any $x, y \in \mathrm{int}(\mathcal{K}^n)$. By the definition of H, we have $H(x,y) \geq 0$ for all $x, y \in \mathbb{R}^n$.

(d) Let $\{x^k\} \subset \mathrm{int}(\mathcal{K}^n)$ be a sequence with $\|x^k\| \to \infty$. For any fixed $y = (y_1, y_2) \in \mathrm{int}(\mathcal{K}^n)$, we next prove that the sequence $\{H(x^k, y)\}$ is unbounded by three cases, and then the desired result follows. For convenience, we write $x^k = (x_1^k, x_2^k)$ for each k.

Case 1: the sequence $\{\det(x^k)\}$ has a zero limit point. Without loss of generality, we assume that $\det(x^k) \to 0$ as $k \to \infty$. Together with $\lim_{t \to 0^+} \phi(t) = +\infty$, it readily follows that $\lim_{k\to\infty} \phi(\det(x^k)) \to +\infty$. In addition, for each k we have that

$$\begin{aligned} \langle \nabla\phi(\det(y)), x^k \rangle &= 2\phi'(\det(y))(x_1^k y_1 - (x_2^k)^T y_2) \\ &\leq 2\phi'(\det(y)) y_1 (x_1^k - \|x_2^k\|) \leq 0, \end{aligned} \tag{3.79}$$

where the inequality is true by using $\phi'(t) < 0$ for all $t > 0$, the Cauchy-Schwartz Inequality, and $y \in \mathrm{int}(\mathcal{K}^n)$. Now from (3.15), it then follows that $\lim_{k\to\infty} H(x^k, y) = +\infty$.

Case 2: the sequence $\{\det(x^k)\}$ is unbounded. Noting that $\det(x^k) > 0$ for each k, we must have $\det(x^k) \to +\infty$ as $k \to \infty$. Since ϕ is decreasing on its domain, we have that

$$\frac{\phi(\det(x^k))}{\|x^k\|} = \frac{\sqrt{2}\phi(\lambda_1(x^k)\lambda_2(x^k))}{\sqrt{(\lambda_1(x^k))^2 + (\lambda_2(x^k))^2}} \geq \frac{\phi[(\lambda_2(x^k))^2]}{\lambda_2(x^k)}.$$

Note that $\lambda_2(x^k) \to \infty$ in this case, and from the last equation and (B4) it follows that

$$\lim_{k\to\infty} \frac{\phi(\det(x^k))}{\|x^k\|} \geq \lim_{k\to\infty} \frac{\phi[(\lambda_2(x^k))^2]}{\lambda_2(x^k)} \geq 0.$$

In addition, since $\{\frac{x^k}{\|x^k\|}\}$ is bounded, we without loss of generality assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} = (\hat{x}_1, \hat{x}_2) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Then, $\hat{x} \in \mathcal{K}^n$, $\|\hat{x}\| = 1$, and $\hat{x}_1 > 0$ (if not, $\hat{x} = 0$), and hence

$$\begin{aligned} \lim_{k\to\infty} \left\langle \nabla\phi(\det(y)), \frac{x^k}{\|x^k\|} \right\rangle &= \langle \nabla\phi(\det(y)), \hat{x} \rangle \\ &= 2\phi'(\det(y))(\hat{x}_1 y_1 - \hat{x}_2^T y_2) \\ &\leq 2\phi'(\det(y))\hat{x}_1(y_1 - \|y_2\|) \\ &< 0. \end{aligned}$$

The two sides show that $\lim_{k\to\infty} \frac{H(x^k,y)}{\|x^k\|} > 0$, and consequently $\lim_{k\to\infty} H(x^k, y) = +\infty$.

Case 3: the sequence $\{\det(x^k)\}$ has some limit point ω with $0 < \omega < +\infty$. Without loss of generality, we assume that $\det(x^k) \to \omega$ as $k \to \infty$. Since $\{x^k\}$ is unbounded and $\{x^k\} \subset \mathrm{int}(\mathcal{K}^n)$, we must have $x_1^k \to +\infty$. In addition, by (3.79) and $\phi'(t) < 0$ for $t > 0$,

$$-\langle \nabla\phi(\det(y)), x^k \rangle \geq -2\phi'(\det(y))(x_1^k y_1 - \|x_2^k\|\|y_2\|) \geq -2\phi'(\det(y)) x_1^k (y_1 - \|y_2\|).$$

This along with $y \in \text{int}(\mathcal{K}^n)$ implies that $-\langle \nabla\phi(\det(y)), x^k\rangle \to +\infty$ as $k \to \infty$. Noting that $\phi(\det(x^k))$ is bounded, from (3.15) it follows that $\lim_{k\to\infty} H(x^k, y) \to +\infty$.

(e) For any $x, y \in \text{int}(\mathcal{K}^n)$ and $z \in \text{int}(\mathcal{K}^n)$, from the definition of H it follows that

$$\begin{aligned} H(z,y) - H(z,x) - H(x,y) &= \langle \nabla\phi(\det(x)) - \nabla\phi(\det(y)), z - x\rangle \\ &= \langle \nabla_1 H(x,y), z - x\rangle, \end{aligned}$$

where the last equality is by part (b). The proof is thus complete. □

Proposition 3.18 shows that the function H defined by (3.15) with ϕ satisfying (B1)–(B4) is a proximal distance w.r.t. $\text{int}(\mathcal{K}^n)$ and $\text{dom}\, H = \text{int}(\mathcal{K}^n) \times \text{int}(\mathcal{K}^n)$. Also, $H \in \mathcal{F}_1(\text{int}(\mathcal{K}^n))$. The conditions (B1) and (B3)-(B4) are easy to check, whereas by Lemma 2.2 of [124] we have the following important characterizations for the condition (B2).

Lemma 3.7. *A function $\phi : (0,\infty) \to \mathbb{R}$ satisfies (B2) if and only if one of the following conditions holds:*

(a) *the function $\phi(\exp(\cdot))$ is convex on $\mathbb{R}$;*

(b) $\phi(t_1 t_2) \leq \dfrac{1}{2}\left(\phi(t_1^2) + \phi(t_2^2)\right)$ *for any $t_1, t_2 > 0$;*

(c) $\phi'(t) + t\phi''(t) \geq 0$ *if ϕ is twice differentiable.*

Proof. Please see [124, Lemma 2.2] a proof. □

Example 3.8. *Let $\phi : (0,\infty) \to \mathbb{R}$ be* $\phi(t) = \begin{cases} -\ln t, & \text{if } t > 0, \\ \infty, & \text{otherwise.} \end{cases}$

Solution. It is easy to verify that ϕ satisfies (B1)-(B4). By formula (3.15), the induced proximal distance is

$$H(x,y) := \begin{cases} -\ln \dfrac{\det(x)}{\det(y)} + \dfrac{2x^T J_n y}{\det(y)} - 2, & \forall x, y \in \text{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise}, \end{cases}$$

where J_n is a diagonal matrix with the first entry being 1 and the rest $(n-1)$ entries being -1. This is exactly the proximal distance given by [10]. Since $H \in \mathcal{F}_1(\text{int}(\mathcal{K}^n))$, we have the results of Proposition 3.15(a)-(d1) if the proximal distance is used for the IPA. ■

Example 3.9. *Take $\phi(t) = t^{1-q}/(q-1)$ $(q > 1)$ if $t > 0$, and otherwise $\phi(t) = \infty$.*

Solution. It is not hard to check that ϕ satisfies (B1)-(B4). By (3.15), we compute that

$$H(x,y) := \begin{cases} \dfrac{(\det(x))^{1-q} - (\det(y))^{1-q}}{q-1} + \dfrac{2x^T J_n y}{(\det(y))^q} - (\det(y))^{1-q}, & \forall x, y \in \mathrm{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise,} \end{cases}$$

where J_n is the diagonal matrix same as Example 4.1. Since $H \in \mathcal{F}(\mathrm{int}(\mathcal{K}^n))$, when using the proximal distance for the IPA, the results of Proposition 3.15(a)-(d1) hold. ■

We should emphasize that using the first way can not produce the proximal distances of the class $\mathcal{F}_1(\mathcal{K}^n)$, and so $\widehat{\mathcal{F}}_1(\mathcal{K}^n)$, since the condition $\lim_{t\to 0^+} \phi(t) = +\infty$ is necessary to guarantee that H has the property (P4), but it implies that the domain of $H(\cdot, y)$ for any $y \in \mathrm{int}(\mathcal{K}^n)$ can not be continuously extended to $\mathcal{K}^n$. Thus, when choosing such proximal distances for the IPA, we can not apply Proposition 3.15(d2) and Proposition 3.17.

The other two ways are both based on the compound of the trace function $\mathrm{tr}(\cdot)$ and a vector-valued function induced by a univariate ϕ via (1.8). For convenience, in the sequel, for any l.s.c. proper function $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$, we write $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as

$$d(s,t) := \begin{cases} \phi(s) - \phi(t) - \phi'(t)(s-t), & \text{if } s \in \mathrm{dom}\phi,\ t \in \mathrm{dom}\phi'. \\ \infty, & \text{otherwise.} \end{cases} \tag{3.80}$$

The second way also produces the proximal distances for the class $\mathcal{F}_1(\mathrm{int}(\mathcal{K}^n))$, which requires $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ to be a l.s.c. proper function satisfying the conditions:

(C1) $\mathrm{dom}\phi \subseteq [0, +\infty)$ and $\mathrm{int}(\mathrm{dom}\phi) = (0, \infty)$;

(C2) ϕ is continuous and strictly convex on its domain;

(C3) ϕ is continuously differentiable on $\mathrm{int}(\mathrm{dom}\phi)$ with $\mathrm{dom}(\phi') = (0, \infty)$;

(C4) for any fixed $t > 0$, the sets $\{s \in \mathrm{dom}\phi \,|\, d(s,t) \le \gamma\}$ are bounded with all $\gamma \in \mathbb{R}$; for any fixed $s \in \mathrm{dom}\phi$, the sets $\{t > 0 \,|\, d(s,t) \le \gamma\}$ are bounded with all $\gamma \in \mathbb{R}$.

Let ϕ^{soc} be the vector-valued function induced by ϕ via (1.8) and write $\mathrm{dom}(\phi^{\mathrm{soc}}) = \mathcal{C}_1$. Clearly, $\mathcal{C}_1 \subseteq \mathcal{K}^n$ and $\mathrm{int}\mathcal{C}_1 = \mathrm{int}(\mathcal{K}^n)$. Define the function $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$H(x,y) := \begin{cases} \mathrm{tr}(\phi^{\mathrm{soc}}(x)) - \mathrm{tr}(\phi^{\mathrm{soc}}(y)) - \langle \nabla \mathrm{tr}(\phi^{\mathrm{soc}}(y)), x - y \rangle, & \forall x \in \mathcal{C}_1, y \in \mathrm{int}(\mathcal{K}^n) \\ \infty, & \text{otherwise.} \end{cases} \tag{3.81}$$

Using Property 1.1, Proposition 1.2, Lemma 3.3, the conditions (C1)-(C4), and similar arguments to [117, Proposition 3.1] (also see Section 3.1), it is not difficult to argue that H has the following favorable properties.

Proposition 3.19. *Let H be defined by (3.81) with ϕ satisfying (C1)-(C4). Then, the following hold.*

(a) *For any fixed $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuous and strictly convex on $\mathcal{C}_1$.*

(b) *For any fixed $y \in \text{int}(\mathcal{K}^n)$, $H(\cdot, y)$ is continuously differentiable on $\text{int}(\mathcal{K}^n)$ with*

$$\nabla_1 H(x,y) = \nabla \text{tr}(\phi^{\text{soc}}(x)) - \nabla \text{tr}(\phi^{\text{soc}}(y)) = 2\left[(\phi')^{\text{soc}}(x) - (\phi')^{\text{soc}}(y)\right].$$

(c) *$H(x,y) \geq 0$ for all $x, y \in \mathbb{R}^n$, and $H(y,y) = 0$ for any $y \in \text{int}(\mathcal{K}^n)$.*

(d) *$H(x,y) \geq \sum_{i=1}^{2} d(\lambda_i(x), \lambda_i(y)) \geq 0$ for any $x \in \mathcal{C}_1$ and $y \in \text{int}(\mathcal{K}^n)$.*

(e) *For any fixed $y \in \text{int}(\mathcal{K}^n)$, the sets $\{x \in \mathcal{C}_1 \,|\, H(x,y) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$; for any fixed $x \in \mathcal{C}_1$, the sets $\{y \in \text{int}(\mathcal{K}^n) \,|\, H(x,y) \leq \gamma\}$ are bounded for all $\gamma \in \mathbb{R}$.*

(f) *For any $x, y \in \text{int}(\mathcal{K}^n)$ and $z \in \mathcal{C}_1$, the following three point identity holds:*

$$H(z,y) = H(z,x) + H(x,y) + \langle \nabla_1 H(x,y), z - x \rangle.$$

Proposition 3.19 shows that the function H defined by (3.81) with ϕ satisfying (C1)-(C4) is a proximal distance w.r.t. $\text{int}(\mathcal{K}^n)$ with $\text{dom}\, H = \mathcal{C}_1 \times \text{int}(\mathcal{K}^n)$, and furthermore, such proximal distances belong to the class $\mathcal{F}_1(\text{int}(\mathcal{K}^n))$. In particular, when $\text{dom}\phi = [0, \infty)$, they also belong to the class $\mathcal{F}_1(\mathcal{K}^n)$. We next present some specific examples.

Example 3.10. *Take $\phi(t) = t \ln t - t$ if $t \geq 0$, and otherwise $\phi(t) = \infty$, where we stipulate $0 \ln 0 = 0$.*

Solution. It is easy to verify that ϕ satisfies (C1)-(C4) with $\text{dom}\phi = [0, \infty)$. By formulas (1.8) and (3.81), we compute that H has the following expression:

$$H(x,y) = \begin{cases} \text{tr}(x \circ \ln x - x \circ \ln y + y - x), & \forall x \in \mathcal{K}^n, y \in \text{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise.} \end{cases}$$

■

Example 3.11. *Take $\phi(t) = t^p - t^q$ if $t \geq 0$, and otherwise $\phi(t) = \infty$, where $p \geq 1$ and $0 < q < 1$.*

Solution. We can show that ϕ satisfies the conditions (C1)-(C4) with $\text{dom}(\phi) = [0, \infty)$. When $p = 1$ and $q = 1/2$, from formulas (1.8) and (3.81), we derive that

$$H(x,y) = \begin{cases} \text{tr}\left[y^{\frac{1}{2}} - x^{\frac{1}{2}} + \dfrac{(\text{tr}(y^{\frac{1}{2}})e - y^{\frac{1}{2}}) \circ (x - y)}{2\sqrt{\det(y)}} \right], & \forall x \in \mathcal{K}^n, y \in \text{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise.} \end{cases}$$

■

Example 3.12. *Take* $\phi(t) = -t^q$ *if* $t \geq 0$*, and otherwise* $\phi(t) = \infty$*, where* $0 < q < 1$*.*

Solution. We can show that ϕ satisfies the conditions (C1)-(C4) with dom$\phi = [0, \infty)$. Now

$$H(x,y) = \begin{cases} (1-q)\mathrm{tr}(y^q) - \mathrm{tr}(x^q) + \mathrm{tr}(qy^{q-1} \circ x), & \forall x \in \mathcal{K}^n, y \in \mathrm{int}(\mathcal{K}^n)., \\ \infty, & \text{otherwise.} \end{cases}$$

■

Example 3.13. *Take* $\phi(t) = -\ln t + t - 1$ *if* $t > 0$*, and otherwise* $\phi(t) = \infty$*.*

Solution. It is easy to check that ϕ satisfies (C1)-(C4) with dom$\phi = (0, \infty)$. The induced proximal distance is

$$H(x,y) = \begin{cases} \mathrm{tr}(\ln y) - \mathrm{tr}(\ln x) + 2\langle y^{-1}, x\rangle - 2, & \forall x, y \in \mathrm{int}(\mathcal{K}^n), \\ \infty, & \text{otherwise.} \end{cases}$$

By a simple computation, we have that the proximal distance is same as the one given by Example 3.4, and the one induced by $\phi(t) = -\ln t$ ($t > 0$) via formula (3.81). ■

Clearly, the proximal distances in Examples 3.11-3.13 belong to the class $\mathcal{F}_1(\mathcal{K}^n)$. Also, by Proposition 3.20 below, the proximal distances in Examples 3.14–3.15 also satisfy (P8) since the corresponding ϕ also satisfies the following condition (C5):

(C5) For any bounded sequence $\{a^k\} \subset \mathrm{int}(\mathrm{dom}\phi)$ and $a \in \mathrm{dom}\phi$ such that $\lim\limits_{k\to\infty} d(a, a^k) = 0$, there holds that $a = \lim_{k\to\infty} a^k$, where d is defined as in (3.80).

Proposition 3.20. *Let H be defined as in (3.81) with ϕ satisfying (C1)-(C5) and* $\mathrm{dom}(\phi) = [0, \infty)$*. Then, for any bounded sequence* $\{y^k\} \subseteq \mathrm{int}(\mathcal{K}^n)$ *and* $y^* \in \mathcal{K}^n$ *such that* $H(y^*, y^k) \to 0$*, we have* $\lambda_i(y^k) \to \lambda_i(y^*)$ *for* $i = 1, 2$*.*

Proof. From Proposition 3.19(d) and the nonnegativity of d, for each k we have

$$H(y^*, y^k) \geq d(\lambda_i(y^*), \lambda_i(y^k)) \geq 0, \quad i = 1, 2.$$

This, together with the given assumption $H(y^*, y^k) \to 0$, implies that

$$d(\lambda_i(y^*), \lambda_i(y^k)) \to 0, \quad i = 1, 2.$$

Notice that $\{\lambda_i(y^k)\} \subset \mathrm{int}(\mathrm{dom}\phi)$ and $\lambda_i(y^*) \in \mathcal{K}^n$ for $i = 1, 2$ by Property 1.1(c). From the condition (C5), we immediately obtain $\lambda_i(y^k) \to \lambda_i(y^*)$ for $i = 1, 2$. □

Nevertheless, we should point out that the proximal distance H given by (3.81) with ϕ satisfying (C1)-(C4) and dom$\phi = [0, \infty)$ generally does not have the property (P7), even if ϕ satisfies the condition (C6) below. This fact will be illustrated by Example 3.14.

(C6) For any $\{a^k\} \subset (0, +\infty)$ converging to $a \in [0, \infty)$, $\lim_{k\to\infty} d(a^*, a^k) \to 0$.

Example 3.14. *Let H be the proximal distance induced by the entropy function ϕ in Example 3.10.*

Solution. It is easy to verify that ϕ satisfies the conditions (C1)-(C6). Here we shall present a sequence $\{y^k\} \subset \text{int}(\mathcal{K}^3)$ which converges to $y^* \in \mathcal{K}^3$, but $H(y^*, y^k) \to \infty$. Let

$$y^k = \begin{bmatrix} \sqrt{2(1+e^{-k^3})} \\ \sqrt{1+k^{-1}-e^{-k^3}} \\ \sqrt{1-k^{-1}+e^{-k^3}} \end{bmatrix} \in \text{int}(\mathcal{K}^3) \quad \text{and} \quad y^* = \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix} \in \mathcal{K}^3.$$

By the expression of $H(y^*, y^k)$, i.e., $H(y^*, y^k) = \text{tr}(y^* \circ \ln y^*) - \text{tr}(y^* \circ \ln y^k) + \text{tr}(y^k - y^*)$, it suffices to prove that $\lim_{k\to\infty} -\text{tr}(y^* \circ \ln y^k) = \infty$ since $\lim_{k\to\infty} \text{tr}(y^k - y^*) = 0$ and $\text{tr}(y^* \circ \ln y^*) = \lambda_2(y^*) \ln(\lambda_2(y^*)) < \infty$. By the definition of $\ln y^k$, we have

$$\text{tr}(y^* \circ \ln y^k) = \ln(\lambda_1(y^k)) \left(y_1^* - (y_2^*)^T \bar{y}_2^k\right) + \ln(\lambda_2(y^k)) \left(y_1^* + (y_2^*)^T \bar{y}_2^k\right) \tag{3.82}$$

for $y^* = (y_1^*, y_2^*),\ y^k = (y_1^k, y_2^k) \in \mathbb{R} \times \mathbb{R}^2$ with $\bar{y}_2^k = y_2^k / \|y_2^k\|$. By computing,

$$\begin{aligned} \ln(\lambda_1(y^k)) &= \ln\sqrt{2} - \ln\left(1 + \sqrt{1+e^{-k^3}}\right) - k^3, \\ y_1^* - (y_2^*)^T \bar{y}_2^k &= \frac{1}{\|y_2^k\|}\left(\frac{-k^{-1}+e^{-k^3}}{1+\sqrt{1+k^{-1}-e^{-k^3}}} + \frac{k^{-1}-e^{-k^3}}{1+\sqrt{1-k^{-1}+e^{-k^3}}}\right). \end{aligned}$$

The last two equalities imply that $\lim_{k\to\infty} \ln(\lambda_1(y^k)) \left(y_1^* - (y_2^*)^T \bar{y}_2^k\right) = -\infty$. In addition, by noting that $y_2^k \neq 0$ for each k, we compute that

$$\lim_{k\to\infty} \ln(\lambda_2(y^k)) \left(y_1^* - (y_2^*)^T \bar{y}_2^k\right) = \ln(\lambda_2(y^k)) \left(y_1^* + (y_2^*)^T \frac{y_2^*}{\|y_2^*\|}\right) = \lambda_2(y^*) \ln(\lambda_2(y^*)).$$

From the last two equations, we immediately have $\lim_{k\to\infty} -\text{tr}(y^* \circ \ln y^k) = \infty$. ■

Thus, when the proximal distance in the IPA is chosen as the one given by (3.81) with ϕ satisfying (C1)-(C6) and $\text{dom}(\phi) = [0, \infty)$, Proposition 3.17(b) may not apply, i.e. the global convergence to an optimal solution may not be guaranteed. This is different from interior proximal methods for convex programming over nonnegative orthant cones by noting that ϕ is now a univariate Bregman function. Similarly, it seems hard to find examples for the class $\mathcal{F}_+(\mathcal{K}^n)$ in [10] so that Theorem 2.2 therein can apply for since it also requires (P7).

The third way will produce the proximal distances for the class $\mathcal{F}_2(\text{int}(\mathcal{K}^n))$, which needs a l.s.c. proper function $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ satisfying the following conditions:

(D1) ϕ is strictly convex and continuous on $\text{dom}\phi$, and ϕ is continuously differentiable on a subset of $\text{dom}\phi$, where $\text{dom}(\phi') \subseteq \text{dom}(\phi) \subseteq [0, \infty)$ and $\text{int}(\text{dom}\phi') = (0, \infty)$;

(D2) ϕ is twice continuously differentiable on $\text{int}(\text{dom}\phi)$ and $\lim_{t\to 0^+} \phi''(t) = \infty$;

(D3) $\phi'(t)t - \phi(t)$ is convex on $\text{dom}(\phi')$, and ϕ' is strictly concave on $\text{dom}(\phi')$;

(D4) ϕ' is SOC-concave on $\text{dom}(\phi')$.

With such a univariate ϕ, we define the proximal distance $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$H(x,y) := \begin{cases} \text{tr}(\phi^{\text{soc}}(y)) - \text{tr}(\phi^{\text{soc}}(x)) - \langle \nabla \text{tr}(\phi^{\text{soc}}(x)), y - x \rangle, & \forall x \in \mathcal{C}_1, y \in \mathcal{C}_2, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.83}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are the domain of ϕ^{soc} and $(\phi')^{\text{soc}}$, respectively. By the relation between $\text{dom}(\phi)$ and $\text{dom}(\phi')$, obviously, $\mathcal{C}_2 \subseteq \mathcal{C}_1 \subseteq \mathcal{K}^n$ and $\text{int}\mathcal{C}_1 = \text{int}\mathcal{C}_2 = \text{int}(\mathcal{K}^n)$.

Lemma 3.8. *Let $\phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a l.s.c. proper function satisfying (D1)-(D4). Then, the following hold.*

(a) $\text{tr}\left[(\phi')^{\text{soc}}(x) \circ x - \phi^{\text{soc}}(x)\right]$ *is convex in $\mathcal{C}_1$ and continuously differentiable on* $\text{int}\mathcal{C}_1$.

(b) *For any fixed $y \in \mathbb{R}^n$, $\langle (\phi')^{\text{soc}}(x), y \rangle$ is continuously differentiable on* $\text{int}\mathcal{C}_1$*, and moreover, it is strictly concave over $\mathcal{C}_1$ whenever $y \in \text{int}(\mathcal{K}^n)$.*

Proof. (a) Let $\psi(t) := \phi'(t)t - \phi(t)$. Then, by (D2) and (D3), $\psi(t)$ is convex on $\text{dom}\phi'$ and continuously differentiable on $\text{int}(\text{dom}\phi') = (0, +\infty)$. Since $\text{tr}\left[(\phi')^{\text{soc}}(x) \circ x - \phi^{\text{soc}}(x)\right] = \text{tr}[\psi^{\text{soc}}(x)]$, using Lemma 3.3(b) and (c) immediately yields part(a).

(b) From (D2) and Lemma 3.3(a), $(\phi')^{\text{soc}}(\cdot)$ is continuously differentiable on $\text{int}\,\mathcal{C}_1$. This implies that $\langle y, (\phi')^{\text{soc}}(x) \rangle$ for any fixed y is continuously differentiable on $\text{int}\mathcal{C}_1$. We next show that it is also strictly concave in $\mathcal{C}_1$ whenever $y \in \text{int}(\mathcal{K}^n)$. Note that $\text{tr}[(\phi')^{\text{soc}}(\cdot)]$ is strictly concave on $\mathcal{C}_1$ since ϕ' is strictly concave on $\text{dom}(\phi')$. Consequently,

$$\text{tr}[(\phi')^{\text{soc}}(\beta x + (1-\beta)z)] > \beta \text{tr}[(\phi')^{\text{soc}}(x)] + (1-\beta)\text{tr}[(\phi')^{\text{soc}}(z)], \quad \forall 0 < \beta < 1$$

for any $x, z \in \mathcal{C}_1$ and $x \neq z$. This implies that

$$(\phi')^{\text{soc}}(\beta x + (1-\beta)z) - \beta(\phi')^{\text{soc}}(x) - (1-\beta)(\phi')^{\text{soc}}(z) \neq 0.$$

In addition, since ϕ' is SOC-concave on $\text{dom}\phi'$, it follows that

$$(\phi')^{\text{soc}}[\beta x + (1-\beta)z] - \beta(\phi')^{\text{soc}}(x) - (1-\beta)(\phi')^{\text{soc}}(z) \succeq_{\mathcal{K}^n} 0.$$

Thus, for any fixed $y \in \text{int}(\mathcal{K}^n)$, the last two equations imply that

$$\langle y, (\phi')^{\text{soc}}[\beta x + (1-\beta)z] - \beta(\phi')^{\text{soc}}(x) - (1-\beta)(\phi')^{\text{soc}}(z) \rangle > 0.$$

This shows that $\langle y, (\phi')^{\text{soc}}(x) \rangle$ for any fixed $y \in \text{int}(\mathcal{K}^n)$ is strictly convex on $\mathcal{C}_1$. $\square$

Using the conditions (D1)-(D4) and Lemma 3.8, and following the same arguments as [117, Propositions 4.1 and 4.2], we may prove the following proposition.

Proposition 3.21. *Let H be defined as in (3.83) with ϕ satisfying (D1)-(D4). Then, the following hold.*

(a) $H(x,y) \geq 0$ *for any* $x, y \in \mathbb{R}^n$, *and* $H(y,y) = 0$ *for any* $y \in \mathrm{int}(\mathcal{K}^n)$.

(b) *For any fixed* $y \in \mathcal{C}_2$, $H(\cdot, y)$ *is continuous in* $\mathcal{C}_1$, *and it is strictly convex on* $\mathcal{C}_1$ *whenever* $y \in \mathrm{int}(\mathcal{K}^n)$.

(c) *For any fixed* $y \in \mathcal{C}_2$, $H(\cdot, y)$ *is continuously differentiable on* $\mathrm{int}(\mathcal{K}^n)$ *with*

$$\nabla_1 H(x,y) = 2\nabla(\phi')^{\mathrm{soc}}(x)(x-y).$$

Moreover, $\mathrm{dom}\nabla_1 H(\cdot, y) = \mathrm{int}(\mathcal{K}^n)$ *whenever* $y \in \mathrm{int}(\mathcal{K}^n)$.

(d) $H(x,y) \geq \sum_{i=1}^{2} d(\lambda_i(y), \lambda_i(x)) \geq 0$ *for any* $x \in \mathcal{C}_1$ *and* $y \in \mathcal{C}_2$.

(e) *For any fixed* $y \in \mathcal{C}_2$, *the sets* $\{x \in \mathcal{C}_1 \,|\, H(x,y) \leq \gamma\}$ *are bounded for all* $\gamma \in \mathbb{R}$.

(f) *For all* $x, y \in \mathrm{int}(\mathcal{K}^n)$ *and* $z \in \mathcal{C}_2$, $H(x,z) - H(y,z) \geq 2\langle \nabla_1 H(y,x),\ z-y\rangle$.

Proposition 3.21 demonstrates that the function H defined by (3.83) with ϕ satisfying (D1)-(D4) is a proximal distance w.r.t. the cone $\mathrm{int}(\mathcal{K}^n)$ and possesses the property (P5'), and therefore belongs to the class $\mathcal{F}_2(\mathrm{int}(\mathcal{K}^n))$. If, in addition, $\mathrm{dom}\phi = [0,\infty)$, then H belongs to the class $\mathcal{F}_2(\mathcal{K}^n)$. The conditions (D1)–(D3) are easy to check, and for the condition (D4), we can employ the characterizations in [42, 45] to verify whether ϕ' is SOC-concave or not. Some examples are presented as follows.

Example 3.15. *Let* $\phi(t) = t\ln t - t + 1$ *if* $t \geq 0$, *and otherwise* $\phi(t) = \infty$.

Solution. It is easy to verify that ϕ satisfies (D1)–(D3) with $\mathrm{dom}\phi = [0,\infty)$ and $\mathrm{dom}\phi' = (0,+\infty)$. By Example 2.12(c), ϕ' is SOC-concave on $(0,\infty)$. Using formulas (1.8) and (3.83), we have

$$H(x,y) = \begin{cases} \mathrm{tr}(y \circ \ln y - y \circ \ln x + x - y), & \forall x \in \mathrm{int}(\mathcal{K}^n), y \in \mathcal{K}^n, \\ \infty, & \text{otherwise.} \end{cases}$$

■

Example 3.16. *Take* $\phi(t) = \dfrac{t^{q+1}}{q+1}$ *if* $t \geq 0$, *and otherwise* $\phi(t) = \infty$, *where* $0 < q < 1$.

Solution. It is easy to show that ϕ satisfies (D1)-(D3) with $\mathrm{dom}\phi = [0,\infty)$ and $\mathrm{dom}\phi' = [0,\infty)$. By Example 2.12, ϕ' is also SOC-concave on $[0,\infty)$. By (1.8) and (3.83), we compute that

$$H(x,y) = \begin{cases} \frac{1}{q+1}\mathrm{tr}(y^{q+1}) + \frac{q}{q+1}\mathrm{tr}(x^{q+1}) - \mathrm{tr}(x^q \circ y), & \forall\ x \in \mathrm{int}(\mathcal{K}^n), y \in \mathcal{K}^n, \\ \infty, & \text{otherwise.} \end{cases}$$

■

Example 3.17. *Take* $\phi(t) = (1+t)\ln(1+t) + \dfrac{t^{q+1}}{q+1}$ *if* $t \geq 0$, *and otherwise* $\phi(t) = \infty$, *where* $0 < q < 1$.

Solution. We can verify that ϕ satisfies (D1)-(D3) with $\mathrm{dom}(\phi) = \mathrm{dom}(\phi') = [0, \infty)$. From Example 2.12, ϕ' is also SOC-concave on $[0, \infty)$. Using (1.8) and (3.83), it is not hard to compute that for any $x, y \in \mathcal{K}^n$,

$$\begin{aligned} H(x,y) &= \mathrm{tr}\left[(e+y)\circ(\ln(e+y) - \ln(e+x))\right] - \mathrm{tr}(y-x) \\ &\quad + \frac{1}{q+1}\mathrm{tr}(y^{q+1}) + \frac{q}{q+1}\mathrm{tr}(x^{q+1}) - \mathrm{tr}(x^q \circ y). \end{aligned}$$

■

Note that the proximal distances in Example 3.16 and Example 3.17 belong to the class $\mathcal{F}_2(\mathcal{K}^n)$. By Proposition 3.22 below, the ones in Example 3.16 and Example 3.17 also belong to the class $\widehat{\mathcal{F}}_2(\mathcal{K}^n)$.

Proposition 3.22. *Let* H *be defined as in (3.83) with* ϕ *satisfying (D1)-(D4). Suppose that* $\mathrm{dom}(\phi) = \mathrm{dom}(\phi') = [0, \infty)$. *Then,* H *possesses the properties (P7') and (P8').*

Proof. By the given assumption, $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{K}^n$. From Proposition 3.21(b), the function $H(\cdot, y^*)$ is continuous on $\mathcal{K}^n$. Consequently, $\lim_{k\to\infty} H(y^k, y^*) = H(y^*, y^*) = 0$.

From Proposition 3.21(d), $H(y^k, y^*) \geq d(\lambda_i(y^*), \lambda_i(y^k)) \geq 0$ for $i = 1, 2$. This together with the assumption $H(y^k, y^*) \to 0$ implies $d(\lambda_i(y^*), \lambda_i(y^k)) \to 0$ for $i = 1, 2$. From this, we necessarily have $\lambda_i(y^k) \to \lambda_i(y^*)$ for $i = 1, 2$. Suppose not, then the bounded sequence $\{\lambda_i(y^k)\}$ must have another limit point $\nu_i^* \geq 0$ such that $\nu_i^* \neq \lambda_i(y^*)$. Without loss of generality, we assume that $\lim_{k\in K, k\to\infty} \lambda_i(y^k) = \nu_i^*$. Then, we have

$$d(\nu_i^*, \lambda_i(y^*)) = \lim_{k\to\infty} d(\nu_i^*, \lambda_i(y^k)) = \lim_{k\in K, k\to\infty} d(\nu_i^*, \lambda_i(y^k)) = d(\nu_i^*, \nu_i^*) = 0,$$

where the first equality is due to the continuity of $d(s, \cdot)$ for any fixed $s \in [0, +\infty)$, and the second one is by the convergence of $\{d(\nu_i^*, \lambda_i(y^k))\}$ implied by the first equality. This contradicts the fact that $d(\nu_i^*, \lambda_i(y^*)) > 0$ since $\nu_i^* \neq \lambda_i(y^*)$. □

As illustrated by the following example, the proximal distance generated by (3.83) with ϕ satisfying (D1)-(D4) generally does not belong to the class $\bar{\mathcal{F}}_2(\mathcal{K}^n)$.

Example 3.18. *Let* H *be the proximal distance as in Example 3.15.*

Solution. Let

$$y^k = \begin{bmatrix} \sqrt{2} \\ (-1)^k \frac{k}{k+1} \\ (-1)^k \frac{k}{k+1} \end{bmatrix} \text{ for each } k \text{ and } y^* = \begin{bmatrix} \sqrt{2} \\ 1 \\ 1 \end{bmatrix}.$$

It is not hard to check that the sequence $\{y^k\} \subseteq \operatorname{int}(\mathcal{K}^3)$ satisfies $H(y^k, y^*) \to 0$. Clearly, the sequence $y^k \nrightarrow y^*$ as $k \to \infty$, but $\lambda_1(y^k) \to \lambda_1(y^*) = 0$ and $\lambda_2(y^k) \to \lambda_2(y^*) = 2\sqrt{2}$.

Finally, let H_1 be a proximal distance produced via one of the ways above, and define

$$H_\alpha(x, y) := H_1(x, y) + \frac{\alpha}{2}\|x - y\|^2, \tag{3.84}$$

where $\alpha > 0$ is a fixed parameter. Then, by Propositions 3.18, 3.19 and 3.21 and the identity

$$\|z - x\|^2 = \|z - y\|^2 + \|y - x\|^2 + 2\langle z - y, y - x\rangle, \quad \forall x, y, z \in \mathbb{R}^n,$$

it is easily shown that H_α is also a proximal distance w.r.t. $\operatorname{int}(\mathcal{K}^n)$. Particularly, when H_1 is given by (3.83) with ϕ satisfying (D1)-(D4) and $\operatorname{dom}(\phi) = \operatorname{dom}(\phi') = [0, \infty)$ (for example the distances in Examples 3.16 and and Example 3.17), the regularized proximal distance H_α satisfies (P7') and (P9'), and hence $H_\alpha \in \bar{F}_2(\mathcal{K}^n)$. With such a regularized proximal distance, the sequence generated by the IPA converges to an optimal solution of (3.64) if $X_* \neq \emptyset$. ■

To sum up, we may construct a proximal distance w.r.t. the cone $\operatorname{int}(\mathcal{K}^n)$ via three ways with an appropriate univariate function. The first way in (3.15) can only produce a proximal distance belonging to $\mathcal{F}_1(\operatorname{int}(\mathcal{K}^n))$, the second way in (3.81) produces a proximal distance of $\mathcal{F}_1(\mathcal{K}^n)$ if $\operatorname{dom}(\phi) = [0, \infty)$, whereas the third way in (3.83) produces a proximal distance of the class $\widehat{\mathcal{F}}_2(\mathcal{K}^n)$ if $\operatorname{dom}(\phi) = \operatorname{dom}(\phi') = [0, \infty)$. Particularly, the regularized proximal distances H_α in (3.84) with H_1 given by (3.83) with $\operatorname{dom}(\phi) = \operatorname{dom}(\phi') = [0, \infty)$ belong to the smallest class $\bar{F}_2(\mathcal{K}^n)$. With such regularized proximal distances, we have the convergence result of Proposition 3.17(c) for the general convex SOCP with $X_* \neq \emptyset$.

For the linear SOCP, we will obtain some improved convergence results for the IPA by exploring the relations between the sequence generated by the IPA and the central path associated to the corresponding proximal distances.

Given a l.s.c. proper strictly convex function Φ with $\operatorname{dom}(\Phi) \subseteq \mathcal{K}^n$ and $\operatorname{int}(\operatorname{dom}\Phi) = \operatorname{int}(\mathcal{K}^n)$, the *central path* of (3.64) associated to Φ is the set $\{x(\tau) \,|\, \tau > 0\}$ defined by

$$x(\tau) := \operatorname{argmin}\Big\{\tau f(x) + \Phi(x) \,|\, x \in \mathcal{V} \cap \mathcal{K}^n\Big\} \quad \text{for } \tau > 0. \tag{3.85}$$

In what follows, we will focus on the central path of (3.64) w.r.t. a distance-like function $H \in \mathcal{D}(\operatorname{int}(\mathcal{K}^n))$. From Proposition 3.2, we immediately have the following result.

Proposition 3.23. *For any given $H \in \mathcal{D}(\operatorname{int}(\mathcal{K}^n))$ and $\bar{x} \in \operatorname{int}(\mathcal{K}^n)$, the central path $\{x(\tau) \,|\, \tau > 0\}$ associated to $H(\cdot, \bar{x})$ is well defined and is in $\mathcal{V} \cap \operatorname{int}(\mathcal{K}^n)$. For each $\tau > 0$, there exists $g_\tau \in \partial f(x(\tau))$ such that $\tau g_\tau + \nabla_1 H(x(\tau), \bar{x}) = A^T y(\tau)$ for some $y(\tau) \in \mathbb{R}^m$.*

We next study the favorable properties of the central path associated to $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$.

Proposition 3.24. *For any given $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$ and $\bar{x} \in \mathrm{int}(\mathcal{K}^n)$, let $\{x(\tau) \,|\, \tau > 0\}$ be the central path associated to $H(\cdot, \bar{x})$. Then, the following results hold.*

(a) *The function $H(x(\tau), \bar{x})$ is nondecreasing in τ.*

(b) *The set $\{x(\tau) \,|\, \hat{\tau} \le \tau \le \tilde{\tau}\}$ is bounded for any given $0 < \hat{\tau} < \tilde{\tau}$.*

(c) *$x(\tau)$ is continuous at any $\tau > 0$.*

(d) *The set $\{x(\tau) \,|\, \tau \ge \bar{\tau}\}$ is bounded for any $\bar{\tau} > 0$ if $X_* \neq \emptyset$ and $\mathrm{dom} H(\cdot, \bar{x}) = \mathcal{K}^n$.*

(e) *All cluster points of $\{x(\tau) \,|\, \tau > 0\}$ are solutions of (3.64) if $X_* \neq \emptyset$.*

Proof. The proofs are similar to those of Propositions 3–5 of [83].

(a) Take $\tau_1, \tau_2 > 0$ and let $x^i = x(\tau_i)$ for $i = 1, 2$. Then, from Proposition 3.23, we know $x^1, x^2 \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)$ and there exist $g^1 \in \partial f(x^1)$ and $g^2 \in \partial f(x^2)$ such that

$$\nabla_1 H(x^1, \bar{x}) = -\tau_1 g^1 + A^T y^1 \quad \text{and} \quad \nabla_1 H(x^2, \bar{x}) = -\tau_2 g^2 + A^T y^2 \tag{3.86}$$

for some $y^1, y^2 \in \mathbb{R}^m$. This together with the convexity of $H(\cdot, \bar{x})$ yields that

$$\begin{aligned} \tau_1^{-1} \left(H(x^1, \bar{x}) - H(x^2, \bar{x}) \right) &\le \tau_1^{-1} \langle \nabla_1 H(x^1, \bar{x}), x^1 - x^2 \rangle = \langle g^1, x^2 - x^1 \rangle, \\ \tau_2^{-1} \left(H(x^2, \bar{x}) - H(x^1, \bar{x}) \right) &\le \tau_2^{-1} \langle \nabla_1 H(x^2, \bar{x}), x^2 - x^1 \rangle = \langle g^2, x^1 - x^2 \rangle. \end{aligned} \tag{3.87}$$

Adding the two inequalities and using the convexity of f, we obtain

$$\left(\tau_1^{-1} - \tau_2^{-1} \right) \left(H(x^1, \bar{x}) - H(x^2, \bar{x}) \right) \le \langle g^1 - g^2, x^2 - x^1 \rangle \le 0.$$

Thus, $H(x^1, \bar{x}) \le H(x^2, \bar{x})$ whenever $\tau_1 \le \tau_2$. Particularly, from the last two equations,

$$\begin{aligned} 0 &\le \tau_1^{-1} \left[H(x^1, \bar{x}) - H(x^2, \bar{x}) \right] \\ &\le \tau_1^{-1} \langle \nabla_1 H(x^1, \bar{x}), x^1 - x^2 \rangle \\ &\le \langle g^2, x^2 - x^1 \rangle \\ &\le \tau_2^{-1} \left[H(x^1, \bar{x}) - H(x^2, \bar{x}) \right], \quad \forall \tau_1 \ge \tau_2 > 0. \end{aligned} \tag{3.88}$$

(b) By part(a), $H(x(\tau), \bar{x}) \le H(x(\tilde{\tau}), \bar{x})$ for any $\tau \le \tilde{\tau}$, which implies that

$$\{x(\tau) \colon \tau \le \tilde{\tau}\} \subseteq L_1 = \{x \in \mathrm{int}(\mathcal{K}^n) \mid H(x, \bar{x}) \le H(x(\tilde{\tau}), \bar{x})\}.$$

Noting that $\{x(\tau) \colon \hat{\tau} \le \tau \le \tilde{\tau}\} \subseteq \{x(\tau) \colon \tau \le \tilde{\tau}\} \subseteq L_1$, the desired result follows by (P4).

(c) Fix $\bar{\tau} > 0$. To prove that $x(\tau)$ is continuous at $\bar{\tau}$, it suffices to prove that $\lim_{k\to\infty} x(\tau_k) = x(\bar{\tau})$ for any sequence $\{\tau_k\}$ such that $\lim_{k\to\infty} \tau_k = \bar{\tau}$. Given such a sequence $\{\tau_k\}$, and take $\hat{\tau}, \tilde{\tau}$ such that $\hat{\tau} > \bar{\tau} > \tilde{\tau}$. Then, $\{x(\tau) : \hat{\tau} \le \tau \le \tilde{\tau}\}$ is bounded by part (b), and

$\tau_k \in (\hat{\tau}, \tilde{\tau})$ for sufficiently large k. Consequently, the sequence $\{x(\tau_k)\}$ is bounded. Let $\bar{y}$ be a cluster point of $\{x(\tau_k)\}$, and without loss of generality assume that $\lim_{k\to\infty} x(\tau_k) = \bar{y}$. Let $K_1 := \{k : \tau_k \leq \bar{\tau}\}$ and take $k \in K_1$. Then, from (3.88) with $\tau_1 = \bar{\tau}$ and $\tau_2 = \tau_k$,

$$\begin{aligned} 0 &\leq \bar{\tau}^{-1}\left[H(x(\bar{\tau}), \bar{x}) - H(x(\tau_k), \bar{x})\right] \\ &\leq \bar{\tau}^{-1}\langle \nabla_1 H(x(\bar{\tau}), \bar{x}), x(\bar{\tau}) - x(\tau_k)\rangle \\ &\leq \tau_k^{-1}\left[H(x(\bar{\tau}), \bar{x}) - H(x(\tau_k), \bar{x})\right]. \end{aligned}$$

If K_1 is infinite, taking the limit $k \to \infty$ with $k \in K_1$ in the last inequality and using the continuity of $H(\cdot, \bar{x})$ on $\text{int}(\mathcal{K}^n)$ yields that

$$H(x(\bar{\tau}), \bar{x}) - H(\bar{y}, \bar{x}) = \langle \nabla_1 H(x(\bar{\tau}), \bar{x}), x(\bar{\tau}) - \bar{y}\rangle.$$

This together with the strict convexity of $H(\cdot, \bar{x})$ implies $x(\bar{\tau}) = \bar{y}$. If K_1 is finite, then $K_2 := \{k : \tau_k \geq \bar{\tau}\}$ must be infinite. Using the same arguments, we also have $x(\bar{\tau}) = \bar{y}$.

(d) By (P3) and Proposition 3.23, there exists $g_\tau \in \partial f(x(\tau))$ such that for any $z \in \mathcal{V} \cap \mathcal{K}^n$,

$$H(x(\tau), \bar{x}) - H(z, \bar{x}) \leq \tau^{-1}\langle \nabla_1 H(x(\tau), \bar{x}), x(\tau) - z\rangle = \langle g_\tau, z - x(\tau)\rangle. \tag{3.89}$$

In particular, taking $z = x^* \in X_*$ in the last equality and using the fact

$$0 \geq f(x^*) - f(x(\tau)) \geq \langle g_\tau, x^* - x(\tau)\rangle,$$

we have $H(x(\tau), \bar{x}) - H(x^*, \bar{x}) \leq 0$. Hence, $\{x(\tau) \,|\, \tau > \bar{\tau}\} \subset \{x \in \text{int}(\mathcal{K}^n) \,|\, H(x, \bar{x}) \leq H(x^*, \bar{x})\}$. By (P4), the latter is bounded, and the desired result then follows.

(e) Let $\hat{x}$ be a cluster point of $\{x(\tau)\}$ and $\{\tau_k\}$ be a sequence such that $\lim_{k\to\infty} \tau_k = +\infty$ and $\lim_{k\to\infty} x(\tau_k) = \hat{x}$. Write $x^k := x(\tau_k)$ and take $x^* \in X_*$ and $z \in \mathcal{V} \cap \text{int}(\mathcal{K}^n)$. Then, for any $\varepsilon > 0$, we have $x(\varepsilon) := (1-\varepsilon)x^* + \varepsilon z \in \mathcal{V} \cap \text{int}(\mathcal{K}^n)$. From the property (P3),

$$\langle \nabla_1 H(x(\varepsilon), \bar{x}) - \nabla_1 H(x^k, \bar{x}), x^k - x(\varepsilon)\rangle \leq 0.$$

On the other hand, taking $z = x(\varepsilon)$ in (3.89), we readily have

$$\tau_k^{-1}\langle \nabla_1 H(x^k, \bar{x}), x^k - x(\varepsilon)\rangle = \langle g^k, x(\varepsilon) - x^k\rangle$$

with $g^k \in \partial f(x^k)$. Combining the last two equations, we obtain

$$\tau_k^{-1}\langle \nabla_1 H(x(\varepsilon), \bar{x}), x^k - x(\varepsilon)\rangle \leq \langle g^k, x(\varepsilon) - x^k\rangle.$$

Since the subdifferential set $\partial f(x^k)$ for each k is compact and $g^k \in \partial f(x^k)$, the sequence $\{g^k\}$ is bounded. Taking the limit in the last inequality yields $0 \leq \langle \hat{g}, x(\varepsilon) - \hat{x}\rangle$, where $\hat{g}$ is a limit point of $\{g^k\}$, and by [131, Theorem 24.4], $\hat{g} \in \partial f(\hat{x})$. Taking the limit $\varepsilon \to 0$ in the inequality, we get $0 \leq \langle \hat{g}, x^* - \hat{x}\rangle$. This implies that $f(\hat{x}) \leq f(x^*)$ since $x^* \in X_*$ and $\hat{g} \in \partial f(\hat{x})$. Consequently, $\hat{x}$ is a solution of the CSOCP (3.64). □

Particularly, from the following proposition, we also have that the central path is convergent if $H \in \mathcal{D}(\text{int}(\mathcal{K}^n))$ satisfies $\text{dom}H(\cdot, \bar{x}) = \mathcal{K}^n$, where $\bar{x} \in \text{int}(\mathcal{K}^n)$ is a given point. Notice that $H(\cdot, \bar{x})$ is continuous on $\text{dom}H(\cdot, \bar{x})$ by (P2), and hence the assumption for H is equivalent to saying that $H(\cdot, \bar{x})$ is continuous at the boundary of the cone $\mathcal{K}^n$.

Proposition 3.25. *For any given $\bar{x} \in \text{int}(\mathcal{K}^n)$ and $H \in \mathcal{D}(\text{int}(\mathcal{K}^n))$ with $\text{dom}H(\cdot, \bar{x}) = \mathcal{K}^n$, let $\{x(\tau) : \tau > 0\}$ be the central path associated to $H(\cdot, \bar{x})$. If X_* is nonempty, then $\lim_{\tau\to\infty} x(\tau)$ exists and is the unique solution of $\min\{H(x, \bar{x}) \mid x \in X_*\}$.*

Proof. Let $\hat{x}$ be a cluster point of $\{x(\tau)\}$ and $\{\tau_k\}$ be such that $\lim_{k\to\infty} \tau_k = \infty$ and $\lim_{k\to\infty} x(\tau_k) = \hat{x}$. Then, for any $x \in X_*$, using (3.88) with $x^1 = x(\tau_k)$ and $x^2 = x$, we obtain

$$[H(x(\tau_k), \bar{x}) - H(x, \bar{x})] \leq \tau_k \langle g^k, x - x(\tau_k)\rangle \leq \tau_k [f(x) - f(x(\tau_k))] \leq 0,$$

where the second inequality is since $g^k \in \partial f(x(\tau_k))$, and the last one is due to $x \in X_*$. Taking the limit $k \to \infty$ in the last inequality and using the continuity of $H(\cdot, \bar{x})$, we have $H(\hat{x}, \bar{x}) \leq H(x, \bar{x})$ for all $x \in X_*$. Since $\hat{x} \in X_*$ by Proposition 3.27(e), this shows that any cluster point of $\{x(\tau) \mid \tau > 0\}$ is a solution of $\min\{H(x, \bar{x}) \mid x \in X_*\}$. By the uniqueness of the solution of $\min\{H(x, \bar{x}) \mid x \in X_*\}$, we have $\lim_{\tau\to\infty} x(\tau) = x^*$. □

For the linear SOCP, we may establish the relations between the sequence generated by the IPA and the central path associated to the corresponding distance-like functions.

Proposition 3.26. *For the linear SOCP, let $\{x^k\}$ be the sequence generated by the IPA with $H \in \mathcal{D}(\text{int}(\mathcal{K}^n))$, $x^0 \in \mathcal{V} \cap \text{int}(\mathcal{K}^n)$ and $\varepsilon_k \equiv 0$, and $\{x(\tau) \mid \tau > 0\}$ be the central path associated to $H(\cdot, x^0)$. Then, $x^k = x(\tau_k)$ for $k = 1, 2, \ldots$ under either of the conditions:*

(a) *H is constructed via (3.15) or (3.81), and $\{\tau_k\}$ is given by $\tau_k = \sum_{j=0}^k \lambda_j$ for $k = 1, 2, \ldots$;*

(b) *H is constructed via (3.83), the mapping $\nabla(\phi')^{\text{soc}}(\cdot)$ defined on $\text{int}(\mathcal{K}^n)$ maps any vector $\mathbb{R}^n$ into $\text{Im}A^T$, and the sequence $\{\tau_k\}$ is given by $\tau_k = \lambda_k$ for $k = 1, 2, \cdots$.*

Moreover, for any positive increasing sequence $\{\tau_k\}$, there exists a positive sequence $\{\lambda_k\}$ with $\sum_{k=1}^\infty \lambda_k = \infty$ such that the proximal sequence $\{x^k\}$ satisfies $x_k = x(\tau_k)$.

Proof. (a) Suppose that H is constructed via (3.15). From (3.67) and Proposition 3.18(b), we have

$$\lambda_j c + \nabla\phi(\det(x^j)) - \nabla\phi(\det(x^{j-1})) = A^T u^j \quad \text{for } j = 0, 1, 2, \ldots. \tag{3.90}$$

Summing the equality from $j = 0$ to k and taking $\tau_k = \sum_{j=0}^k \lambda_j$, $y^k = \sum_{j=0}^k u^j$, we get

$$\tau_k c + \nabla\phi(\det(x^k)) - \nabla\phi(\det(x^0)) = A^T y^k.$$

This means that x^k satisfies the optimal conditions of the problem

$$\min\left\{\tau_k f(x) + H(x, x^0) \mid x \in \mathcal{V} \cap \text{int}(\mathcal{K}^n)\right\}, \tag{3.91}$$

and so $x^k = x(\tau_k)$. Now let $\{x(\tau)\colon \tau > 0\}$ be the central path. Take a positive increasing sequence $\{\tau_k\}$ and let $x^k \equiv x(\tau_k)$. Then from Proposition 3.23 and Proposition 3.18(b), it follows that

$$\tau_k c + \nabla\phi(\det(x^k)) - \nabla\phi(\det(x^0)) = A^T y^k \quad \text{for some } y^k \in \mathbb{R}^m.$$

Setting $\lambda_k = \tau_k - \tau_{k-1}$ and $u^k = y^k - y^{k-1}$, from the last equality it follows that

$$\lambda_k c + \nabla\phi(\det(x^k)) - \nabla\phi(\det(x^{k-1})) = A^T u^k.$$

This shows that $\{x^k\}$ is the sequence generated by the IPA with $\varepsilon_k \equiv 0$. If H is given by (3.81), using Proposition 3.19(b) and the same arguments, we also have the result holds.

(b) Under this case, by Proposition 3.21(c), the above (3.90) becomes

$$\lambda_j c + \nabla(\phi')^{\rm soc}(x^j)\cdot(x^j - x^{j-1}) = A^T u^j \quad \text{for } j = 0, 1, 2, \ldots.$$

Since $\phi''(t) > 0$ for all $t \in (0, \infty)$ by (D1) and (D2), from [64, Proposition 5.2] it follows that $\nabla(\phi')^{\rm soc}(x)$ is positive definite on $\mathrm{int}(\mathcal{K}^n)$. Thus, the last equality is equivalent to

$$\left[\nabla(\phi')^{\rm soc}(x^j)\right]^{-1}\lambda_j c + (x^j - x^{j-1}) = \left[\nabla(\phi')^{\rm soc}(x^j)\right]^{-1} A^T u^j \quad \text{for } j = 0, 1, 2, \ldots. \tag{3.92}$$

Summing the equality (3.92) from $j = 0$ to k and making suitable arrangement, we get

$$\lambda_k c + \nabla(\phi')^{\rm soc}(x^k)(x^k - x^0) = A^T u^k + \nabla(\phi')^{\rm soc}(x^k)\sum_{j=0}^{k-1}\left[\nabla(\phi')^{\rm soc}(x^j)\right]^{-1}(A^T u^j - \lambda_j c),$$

which, using the given assumptions and setting $\tau_k = \lambda_k$, reduces to

$$\tau_k c + \nabla(\phi')^{\rm soc}(x^k)(x^k - x^0) = A^T \bar{y}^k \quad \text{for some } \bar{y}^k \in \mathbb{R}^m.$$

This means that x^k is the unique solution of (3.91), and hence $x^k = x(\tau_k)$ for any k. Let $\{x(\tau)\colon \tau > 0\}$ be the central path. Take a positive increasing sequence $\{\tau_k\}$ and define the sequence $x^k = x(\tau_k)$. Then, from Proposition 3.23 and Proposition 3.21(c),

$$\tau_k c + \nabla(\phi')^{\rm soc}(x^k)(x^k - x^0) = A^T y^k \quad \text{for some } y^k \in \mathbb{R}^m,$$

which, by the positive definiteness of $\nabla(\phi')^{\rm soc}(\cdot)$ on $\mathrm{int}(\mathcal{K}^n)$, implies that

$$[\nabla(\phi')^{\rm soc}(x^k)]^{-1}(\tau_k c - A^T y^k) + [\nabla(\phi')^{\rm soc}(x^{k-1})]^{-1}(\tau_{k-1}c - A^T y^{k-1}) + (x^k - x^{k-1}) = 0.$$

Consequently,

$$\tau_k c + \nabla(\phi')^{\rm soc}(x^k)(x^k - x^{k-1}) = \nabla(\phi')^{\rm soc}(x^k)[\nabla(\phi')^{\rm soc}(x^{k-1})]^{-1}(A^T y^{k-1} - \tau_{k-1}c).$$

Using the given assumptions and setting $\lambda_k = \tau_k$, we have

$$\lambda_k c + \nabla(\phi')^{\rm soc}(x^k)(x^k - x^{k-1}) = A^T u^k \quad \text{for some } u^k \in \mathbb{R}^m.$$

for some $u^k \in \mathbb{R}^m$. This implies that $\{x^k\}$ is the sequence generated by the IPA and the sequence $\{\lambda_k\}$ satisfies $\sum_{k=1}^{\infty} \lambda_k = \infty$ since $\{\tau_k\}$ is a positive increasing sequence. $\square$

From Proposition 3.25 and Proposition 3.26, we readily have the following improved convergence results of the sequence generated by the IPA for the linear SOCP.

Proposition 3.27. *For the linear SOCP, let $\{x^k\}$ be the sequence generated by the IPA with $H \in \mathcal{D}(\mathrm{int}(\mathcal{K}^n))$, $x^0 \in \mathcal{V} \cap \mathrm{int}(\mathcal{K}^n)$ and $\varepsilon_k \equiv 0$. If one of the conditions is satisfied:*

(a) *H is constructed via (3.81) with $\mathrm{dom}H(\cdot, x^0) = \mathcal{K}^n$ and $\sum_{k=0}^{\infty} \lambda_k = \infty$;*

(b) *H is constructed via (3.83) with $\mathrm{dom}H(\cdot, x^0) = \mathcal{K}^n$, the mapping $\nabla(\phi')^{\mathrm{soc}}(\cdot)$ defined on $\mathrm{int}(\mathcal{K}^n)$ maps any vector in $\mathbb{R}^n$ into $\mathrm{Im}A^T$, and $\lim_{k\to\infty} \lambda_k = \infty$;*

and $X_* \neq \emptyset$, then $\{x^k\}$ converges to the unique solution of $\min\{H(x, x^0) \mid x \in X_*\}$.

Chapter 4

SOC means and SOC inequalities

In this chapter, we present some other types of applications of the aforementioned SOC-functions, SOC-convexity, and SOC-monotonicity. These include so-called SOC means, SOC weighted means, and a few SOC trace versions of Young, Hölder, Minkowski inequalities, and Powers-Størmer's inequality. We believe that these results will be helpful in convergence analysis of optimizations involved with SOC. Many materials of this chapter are extracted from [37, 78, 79], the readers can look into them for more details.

4.1 SOC means

From Chapter 3, we have seen that the SOC-monotonicity and SOC-convexity are often involved in the solution methods of convex SOCPs. What other applications does SOC-momotone functions hold besides the algorithmic aspect? Surprisingly, some other applications of SOC-monotone functions lie in different areas from those for SOC-convex functions. In particular, the SOC-monotone functions can be employed to establish the concepts of various SOC-means, which are natural extensions of traditional means. It also helps on achieving some important inequalities. To see these, we start with recalling the definitions of means.

A *mean* is a binary map $m : (0, \infty) \times (0, \infty) \to (0, \infty)$ satisfying the following:

(a) $m(a, b) > 0$;

(b) $\min\{a, b\} \leq m(a, b) \leq \max\{a, b\}$;

(c) $m(a, b) = m(b, a)$;

(d) $m(a, b)$ is increasing in a, b;

(e) $m(\alpha a, \alpha b) = \alpha m(a, b)$, for all $\alpha > 0$;

(f) $m(a,b)$ is continuous in a,b.

Many types of means have been investigated in the literature, to name a few, the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, identric mean, contra-harmonic mean, quadratic (or root-square) mean, first Seiffert mean, second Seiffert mean, and Neuman-Sandor mean, etc.. In addition, many inequalities describing the relationship among different means have been established. For instance, for any two positive real number a,b, it is well-known that

$$\min\{a,b\} \leq H(a,b) \leq G(a,b) \leq L(a,b) \leq A(a,b) \leq \max\{a,b\}, \tag{4.1}$$

where

$$\begin{aligned}
H(a,b) &= \frac{2ab}{a+b}, \\
G(a,b) &= \sqrt{ab}, \\
L(a,b) &= \begin{cases} \dfrac{a-b}{\ln a - \ln b} & \text{if} \quad a \neq b, \\ a & \text{if} \quad a = b, \end{cases} \\
A(a,b) &= \frac{a+b}{2},
\end{aligned}$$

represents the harmonic mean, geometric mean, logarithmic mean, and arithmetic mean, respectively. For more details regarding various means and their inequalities, please refer to [32, 66].

Recently, the matrix version of means have been generalized from the classical means, see [23, 25–27]. In particular, the matrix version of Arithmetic Geometric Mean Inequality (AGM) is proved in [23, 24], and has attracted much attention. Indeed, let A and B be two $n \times n$ positive definite matrices, the following inequalities hold under the partial order induced by positive semidefinite matrices cone $\mathcal{S}^n_+$:

$$(A : B) \preceq A \# B \preceq \frac{1}{2}(A + B), \tag{4.2}$$

where

$$\begin{aligned}
A : B &= 2\left(A^{-1} + B^{-1}\right)^{-1}, \\
A \# B &= A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^{1/2} A^{1/2},
\end{aligned}$$

denote the matrix harmonic mean and the matrix geometric mean, respectively. For more details about matrix means and their related inequalities, please see [23, 25–27, 89] and references therein.

Note that the nonnegative orthant, the cone of positive semidefinite matrices, and the second-order cone all belong to the class of symmetric cones [62]. This motivates us to consider further extension of means, that is, the means associated with SOC. More specifically, in this section, we generalize some well-known means to the SOC setting and build up some inequalities under the partial order induced by $\mathcal{K}^n$. One trace inequality is established as well. For achieving these results, the SOC-monotonicity contributes a lot in the analysis. That is the application aspect of SOC-monotone function that we want to illustrate.

The relation $\succeq_{\mathcal{K}^n}$ is not a linear ordering. Hence, it is not possible to compare any two vectors (elements) via $\succeq_{\mathcal{K}^n}$. Nonetheless, we note that for any $a, b \in \mathbb{R}$

$$\begin{aligned} \max\{a,b\} &= b + [a-b]_+ = \frac{1}{2}(a+b+|a-b|), \\ \min\{a,b\} &= a - [a-b]_+ = \frac{1}{2}(a+b-|a-b|). \end{aligned}$$

This motivates us to define the supremum and infimum of $\{x, y\}$, denoted by $x \vee y$ and $x \wedge y$ respectively, in the SOC setting as follows. For any $x, y \in \mathbb{R}^n$, we let

$$\begin{aligned} x \vee y &:= y + [x-y]_+ = \frac{1}{2}(x+y+|x-y|), \\ x \wedge y &:= \begin{cases} x - [x-y]_+ = \frac{1}{2}(x+y-|x-y|), & \text{if } x+y \succeq_{\mathcal{K}^n} |x-y|; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

In view of the above expressions, we define the SOC means in a similar way.

Definition 4.1. *A binary operation $(x, y) \mapsto M(x, y)$ defined on $\mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n)$ is called an SOC mean if the following conditions are satisfied:*

(i) $M(x, y) \succ_{\mathcal{K}^n} 0$;

(ii) $x \wedge y \preceq_{\mathcal{K}^n} M(x, y) \preceq_{\mathcal{K}^n} x \vee y$;

(iiii) *$M(x, y)$ is monotone in x, y;*

(iv) $M(\alpha x, \alpha y) = \alpha M(x, y)$, $\alpha > 0$;

(v) *$M(x, y)$ is continuous in x, y.*

We start with the simple SOC arithmetic mean $A(x, y) : \mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n) \to \mathrm{int}(\mathcal{K}^n)$, which is defined by

$$A(x, y) = \frac{x+y}{2}. \tag{4.3}$$

It is clear that $A(x,y)$ satisfies all the above conditions. Besides, it is not hard to verify that the SOC harmonic mean of x and y, $H(x,y) : \mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n) \to \mathrm{int}(\mathcal{K}^n)$, can be defined as

$$H(x,y) = \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}. \tag{4.4}$$

The relation between $A(x,y)$ and $H(x,y)$ is described as below.

Proposition 4.1. *Let $A(x,y)$, $H(x,y)$ be defined as in (4.3) and (4.4), respectively. For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, there holds*

$$x \wedge y \preceq_{\mathcal{K}^n} H(x,y) \preceq_{\mathcal{K}^n} A(x,y) \preceq_{\mathcal{K}^n} x \vee y.$$

Proof. (i) To verify the first inequality, if $\frac{1}{2}(x+y-|x-y|) \notin \mathcal{K}^n$, the inequality holds clearly. Suppose $\frac{1}{2}(x+y-|x-y|) \succeq_{\mathcal{K}^n} 0$, we note that $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} x$ and $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} y$. Then, using the SOC-monotonicity of $f(t) = -t^{-1}$ shown in Proposition 2.3, we obtain

$$x^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1} \quad \text{and} \quad y^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1},$$

which imply

$$\frac{x^{-1}+y^{-1}}{2} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1}.$$

Next, applying the SOC-monotonicity again, we conclude that

$$\frac{x+y-|x-y|}{2} \preceq_{\mathcal{K}^n} \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}.$$

(ii) To see the second inequality, we first observe that

$$\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} \preceq_{\mathcal{K}^n} \frac{1}{2}(x^{-1})^{-1} + \frac{1}{2}(y^{-1})^{-1} = \frac{x+y}{2},$$

where the inequality comes from the SOC-convexity of $f(t) = t^{-1}$.

(iii) To check the last inequality, we observe that

$$\frac{x+y}{2} \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2} \quad \Longleftrightarrow \quad 0 \preceq_{\mathcal{K}^n} \frac{|x-y|}{2},$$

where it is clear $|x-y| \succeq_{\mathcal{K}^n} 0$ always holds for any element x, y. Then, the desired result follows. $\Box$

Now, we consider the SOC geometric mean, denoted by $G(x,y)$, which can be borrowed from the geometric mean of symmetric cone, see [102]. More specifically, let V

be a Euclidean Jordan algebra, $\mathcal{K}$ be the set of all square elements of V (the associated symmetric cone), and $\Omega := \text{int}\mathcal{K}$ (the interior symmetric cone). For $x \in V$, let $\mathcal{L}(x)$ denote the linear operator given by $\mathcal{L}(x)y := x \circ y$, and let

$$P(x) := 2\mathcal{L}(x)^2 - \mathcal{L}(x^2). \tag{4.5}$$

The mapping P is called the *quadratic representation* of V. If x is invertible, then we have

$$P(x)\mathcal{K} = \mathcal{K} \quad \text{and} \quad P(x)\Omega = \Omega.$$

Suppose that $x, y \in \Omega$, the geometric mean of x and y, denoted by $x\#y$, is

$$x\#y := P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}}.$$

On the other hand, it turns out that the cone Ω admits a $G(\Omega)$-invariant Riemannian metric [62]. The unique geodesic curve joining x and y is

$$t \mapsto x\#_t y := P(x^{\frac{1}{2}}) \left(P(x^{-\frac{1}{2}})y\right)^t,$$

and the geometric mean $x\#y$ is the midpoint of the geodesic curve. In addition, Lim establishes the arithmetic-geometric-harmonic means inequalities [102, Theorem 2.8],

$$\left(\frac{x^{-1} + y^{-1}}{2}\right)^{-1} \preceq_{\mathcal{K}} x\#y \preceq_{\mathcal{K}} \frac{x+y}{2}, \tag{4.6}$$

where $\preceq_{\mathcal{K}}$ is the partial order induced by the closed convex cone $\mathcal{K}$. The inequality (4.6) includes the inequality (4.2) as a special case. For more details, please refer to [102]. As an example of Euclidean Jordan algebra, for any x and y in $\text{int}(\mathcal{K}^n)$, we therefore adopt the geometric mean $G(x, y)$ as

$$G(x, y) := P(x^{\frac{1}{2}}) \left(P(x^{-\frac{1}{2}})y\right)^{\frac{1}{2}}. \tag{4.7}$$

Then, we immediately have the following parallel properties of SOC geometric mean.

Proposition 4.2. *Let $A(x, y)$, $H(x, y)$, $G(x, y)$ be defined as in (4.3), (4.4) and (4.7), respectively. Then, for any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, we have*

(a) $G(x, y) = G(y, x)$.

(b) $G(x, y)^{-1} = G(x^{-1}, y^{-1})$.

(c) $H(x, y) \preceq_{\mathcal{K}^n} G(x, y) \preceq_{\mathcal{K}^n} A(x, y)$.

Next, we look into another type of SOC mean, the SOC logarithmic mean $L(x, y)$. First, for any two positive real numbers a, b, Carlson [33] has set up the integral representation:

$$L(a, b) = \left[\int_0^1 \frac{dt}{ta + (1-t)b} \right]^{-1},$$

whereas Neuman [113] has also provided an alternative integral representation:

$$L(a, b) = \int_0^1 a^{1-t} b^t dt.$$

Moreover, Bhatia [23, page 229] proposes the matrix logarithmic mean of two positive definite matrices A and B as

$$L(A, B) = A^{1/2} \int_0^1 \left(A^{-1/2} B A^{-1/2}\right)^t dt \; A^{1/2}.$$

In other words,

$$L(A, B) = \int_0^1 A \#_t B \; dt,$$

where $A \#_t B =: A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} = P(A^{1/2})(P(A^{-1/2})B)^t$ is called the t-weighted geometric mean. We remark that $A \#_t B = A^{1-t} B^t$ for $AB = BA$, and the definition of logarithmic mean coincides with the one of real numbers. This integral representation motivates us to define the SOC logarithmic mean on $\text{int}(\mathcal{K}^n) \times \text{int}(\mathcal{K}^n)$ as

$$L(x, y) = \int_0^1 x \#_t y \; dt. \tag{4.8}$$

To verify it is an SOC mean, we need the following technical lemmas. The first lemma is the symmetric cone version of Bernoulli inequality.

Lemma 4.1. *Let V be a Euclidean Jordan algebra, $\mathcal{K}$ be the associated symmetric cone, and e be the Jordan identity. Then,*

$$(e + s)^t \preceq_{\mathcal{K}} e + ts,$$

where $0 \le t \le 1$, $s \succeq_{\mathcal{K}} -e$, and the partial order is induced by the closed convex cone $\mathcal{K}$.

Proof. For any $s \in V$, we denote the spectral decomposition of s as $\sum_{i=1}^{r} \lambda_i c_i$. Since $s \succeq_{\mathcal{K}} -e$, we obtain that each eigenvalue $\lambda_i \ge -1$. Then, we have

$$\begin{aligned} (e + s)^t &= (1 + \lambda_1)^t c_1 + (1 + \lambda_2)^t c_2 + \cdots + (1 + \lambda_r)^t c_r \\ &\preceq_{\mathcal{K}} (1 + t\lambda_1) c_1 + (1 + t\lambda_2) c_2 + \cdots + (1 + t\lambda_r) c_r \\ &= e + ts, \end{aligned}$$

where the inequality holds by the real number version of Bernoulli inequality. $\square$

Lemma 4.1 is the Bernoulli Inequality associated with symmetric cone although we will use it only in the SOC setting.

Lemma 4.2. *Suppose that $u(t) : \mathbb{R} \to \mathbb{R}^n$ is integrable on $[a, b]$.*

(a) *If $u(t) \succeq_{\mathcal{K}^n} 0$ for any $t \in [a, b]$, then $\int_a^b u(t)dt \succeq_{\mathcal{K}^n} 0$.*

(b) *If $u(t) \succ_{\mathcal{K}^n} 0$ for any $t \in [a, b]$, then $\int_a^b u(t)dt \succ_{\mathcal{K}^n} 0$.*

Proof. (a) Consider the partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ with $t_k = a + k(b - a)/n$ and some $\bar{t}_k \in [t_{k-1}, t_k]$, we have

$$\int_a^b u(t)dt = \lim_{n\to\infty} \sum_{k=1}^n u(\bar{t}_k)\frac{b-a}{n} \succeq_{\mathcal{K}^n} 0$$

because $u(t) \succeq_{\mathcal{K}^n} 0$ and $\mathcal{K}^n$ is closed.

(b) For convenience, we write $u(t) = (u_1(t), u_2(t)) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and let

$$\begin{array}{rcl} \bar{u}(t) & = & (\|u_2(t)\|, u_2(t)) \,, \\ \tilde{u}(t) & = & (u_1(t) - \|u_2(t)\|, \mathbf{0}) \,. \end{array}$$

Then, we have

$$u(t) = \bar{u}(t) + \tilde{u}(t) \quad \text{ and } \quad \left\{ \begin{array}{l} \bar{u}(t) \succeq_{\mathcal{K}^n} 0, \\ u_1(t) - \|u_2(t)\| > 0. \end{array} \right.$$

Note that $\int_a^b \tilde{u}(t)dt = (\int_a^b (u_1(t) - \|u_2(t)\|)dt, \mathbf{0}) \succ_{\mathcal{K}^n} 0$ since $u_1(t) - \|u_2(t)\| > 0$. This together with $\int_a^b \bar{u}(t)dt \succeq_{\mathcal{K}^n} 0$ yields that

$$\int_a^b u(t)dt = \int_a^b \bar{u}(t)dt + \int_a^b \tilde{u}(t)dt \succ_{\mathcal{K}^n} 0.$$

Thus, the proof is complete. $\square$

Proposition 4.3. *Suppose that $u(t) : \mathbb{R} \to \mathbb{R}^n$ and $v(t) : \mathbb{R} \to \mathbb{R}^n$ are integrable on $[a, b]$.*

(a) *If $u(t) \succeq_{\mathcal{K}^n} v(t)$ for any $t \in [a, b]$, then $\int_a^b u(t)dt \succeq_{\mathcal{K}^n} \int_a^b v(t)dt$.*

(b) *If $u(t) \succ_{\mathcal{K}^n} v(t)$ for any $t \in [a, b]$, then $\int_a^b u(t)dt \succ_{\mathcal{K}^n} \int_a^b v(t)dt$.*

Proof. It is an immediate consequence of Lemma 4.2. $\square$

Proposition 4.4. *Let $A(x,y)$, $G(x,y)$, and $L(x,y)$ be defined as in (4.3), (4.7), and (4.8), respectively. For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, there holds*

$$G(x,y) \preceq_{\mathcal{K}^n} L(x,y) \preceq_{\mathcal{K}^n} A(x,y),$$

and hence $L(x,y)$ is an SOC mean.

Proof. (i) To verify the first inequality, we first note that

$$G(x,y) = P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}} = \int_0^1 P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}} dt.$$

Let $s = P(x^{-\frac{1}{2}})y = \lambda_1 u_s^{(1)} + \lambda_2 u_s^{(2)}$. Then, we have

$$\begin{aligned}
& L(x,y) - G(x,y) \\
= \;& \int_0^1 P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^t \, dt - P(x^{\frac{1}{2}})(P(x^{-\frac{1}{2}})y)^{\frac{1}{2}} \\
= \;& \int_0^1 P(x^{\frac{1}{2}}) \left(\lambda_1^t u_s^{(1)} + \lambda_2^t u_s^{(2)}\right) \, dt - P(x^{\frac{1}{2}}) \left(\sqrt{\lambda_1} u_s^{(1)} + \sqrt{\lambda_2} u_s^{(2)}\right) \\
= \;& \left[\int_0^1 \lambda_1^t dt\right] P(x^{\frac{1}{2}}) u_s^{(1)} + \left[\int_0^1 \lambda_2^t dt\right] P(x^{\frac{1}{2}}) u_s^{(2)} - P(x^{\frac{1}{2}}) \left(\sqrt{\lambda_1} u_s^{(1)} + \sqrt{\lambda_2} u_s^{(2)}\right) \\
= \;& \left[\frac{\lambda_1 - 1}{\ln \lambda_1 - \ln 1} - \sqrt{\lambda_1}\right] P(x^{\frac{1}{2}}) u_s^{(1)} + \left[\frac{\lambda_2 - 1}{\ln \lambda_2 - \ln 1} - \sqrt{\lambda_2}\right] P(x^{\frac{1}{2}}) u_s^{(2)} \\
= \;& \left[L(\lambda_1, 1) - G(\lambda_1, 1)\right] P(x^{\frac{1}{2}}) u_s^{(1)} + \left[L(\lambda_2, 1) - G(\lambda_2, 1)\right] P(x^{\frac{1}{2}}) u_s^{(2)} \\
\succeq_{\mathcal{K}^n} \;& 0,
\end{aligned}$$

where last inequality holds by (4.1) and $P(x^{\frac{1}{2}}) u_s^{(i)} \in \mathcal{K}^n$. Thus, we obtain the first inequality.

(ii) To see the second inequality, we let $s = P(x^{-\frac{1}{2}})y - e$. Then, we have $s \succeq_{\mathcal{K}^n} -e$, and applying Lemma 4.1 gives

$$\left(e + P(x^{-\frac{1}{2}})y - e\right)^t \preceq_{\mathcal{K}^n} e + t\left[P(x^{-\frac{1}{2}})y - e\right],$$

which is equivalent to

$$0 \preceq_{\mathcal{K}^n} (1-t)e + t\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^t.$$

Since $P(x^{\frac{1}{2}})$ is invariant on $\mathcal{K}^n$, we have

$$\begin{aligned}
0 \;\preceq_{\mathcal{K}^n}\; & P(x^{\frac{1}{2}}) \left((1-t)e + t\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^t\right) \\
= \;& (1-t)x + ty - x \#_t y.
\end{aligned}$$

Hence, by Proposition 4.3, we obtain

$$L(x,y) = \int_0^1 x\#_t y\ dt \preceq_{\mathcal{K}^n} \int_0^1 [(1-t)x + ty]\ dt = A(x,y).$$

The proof is complete. $\square$

Finally, for SOC quadratic mean, it is natural to consider the following

$$Q(x,y) := \left(\frac{x^2+y^2}{2}\right)^{1/2}.$$

It is easy to verify $A(x,y) \preceq_{\mathcal{K}^n} Q(x,y)$. However, $Q(x,y)$ does not satisfy the property(ii) mentioned in the definition of SOC mean. Indeed, taking $x = \begin{bmatrix} 31 \\ 10 \\ -20 \end{bmatrix} \in \mathcal{K}^n$ and $y = \begin{bmatrix} 10 \\ 9 \\ 0 \end{bmatrix} \in \mathcal{K}^n$, it is obvious that $x \succ_{\mathcal{K}^n} y$. In addition, by simple calculation, we have

$$\left(\frac{x^2+y^2}{2}\right)^{1/2} = \begin{bmatrix} s \\ \frac{400}{2s} \\ \frac{-620}{2s} \end{bmatrix} \approx \begin{bmatrix} 24.30 \\ 8.23 \\ -12.76 \end{bmatrix},$$

where $s = \sqrt{\frac{1}{2}\left(821 + \sqrt{821^2 - (400^2 + 620^2)}\right)} \approx 24.30$. However,

$$x \vee y - \left(\frac{x^2+y^2}{2}\right)^{1/2} \approx \begin{bmatrix} 6.7 \\ 1.77 \\ -7.24 \end{bmatrix}$$

is not in $\mathcal{K}^n$. Hence, this definition of $Q(x,y)$ cannot officially serve as an SOC mean.

To sum up, we already have the following inequalities

$$x \wedge y \preceq_{\mathcal{K}^n} H(x,y) \preceq_{\mathcal{K}^n} G(x,y) \preceq_{\mathcal{K}^n} L(x,y) \preceq_{\mathcal{K}^n} A(x,y) \preceq_{\mathcal{K}^n} x \vee y,$$

but we do not have SOC quadratic mean. Nevertheless, we still can generalize all the means inequalities as in (4.1) to SOC setting when the dimension is 2. To see this, the Jordan product on second-order cone of order 2 satisfies the associative law and closedness such that the geometric mean

$$G(x,y) = x^{1/2} \circ y^{1/2}$$

and the logarithmic mean

$$L(x,y) = \int_0^1 x^{1-t} \circ y^t\ dt$$

are well-defined (note this is true only when $n = 2$) and coincide with the definition (4.7), (4.8). Then, the following inequalities

$$x \wedge y \preceq_{\mathcal{K}^2} H(x,y) \preceq_{\mathcal{K}^2} G(x,y) \preceq_{\mathcal{K}^2} L(x,y) \preceq_{\mathcal{K}^2} A(x,y) \preceq_{\mathcal{K}^2} Q(x,y) \preceq_{\mathcal{K}^2} x \vee y$$

hold as well.

By applying Proposition 1.1(a), we immediately obtain one trace inequality for SOC mean.

Proposition 4.5. *Let $A(x,y)$, $H(x,y)$, $G(x,y)$ and $L(x,y)$ be defined as in* (4.3)-(4.4), (4.7)-(4.8)*, respectively. For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, there holds*

$$\mathrm{tr}(x \wedge y) \le \mathrm{tr}(H(x,y)) \le \mathrm{tr}(G(x,y)) \le \mathrm{tr}(L(x,y)) \le \mathrm{tr}(A(x,y)) \le \mathrm{tr}(x \vee y).$$

4.2 SOC Inequalities

It is well-known that the Young inequality, the Hölder inequality, and the Minkowski inequality are powerful tools in analysis and are widely applied in many fields. There exist many kinds of variants, generalizations, and refinements, which provide a variety of applications. Here, we explore the trace versions of Young inequality, Hölder inequality, Minkowski inequality in the setting of second-order cone. We start with recalling these three classical inequalities [18, 67] briefly.

Suppose that $a, b \ge 0$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, the Young inequality is expressed by

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

The Young inequality is a special case of the weighted AM-GM (Arithmetic Mean-Geometric Mean) inequality and very useful in real analysis. In particular, it can be employed as a tool to prove the Hölder inequality:

$$\sum_{k=1}^{n} |a_k b_k| \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}},$$

where $a_1, a_2, \cdots, a_n$, $b_1, b_2, \cdots, b_n$ are real (or complex) numbers. In light of the Hölder inequality, one can deduce the Minkowski inequality as below:

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{\frac{1}{p}}.$$

In 1995, Ando [3] showed the singular value version of Young inequality that

$$s_j(AB) \leq s_j\left(\frac{A^p}{p} + \frac{B^q}{q}\right) \quad \text{for all } 1 \leq j \leq n, \tag{4.9}$$

where A and B are positive definite matrices. Note that both positive semidefinite cone and second-order cone belong to symmetric cones [62]. It is natural to ask whether there is a similar version in the setting of second-order cone. First, in view of the classical Young inequality, one may conjecture that the Young inequality in the SOC setting is in form of

$$x \circ y \preceq_{\mathcal{K}^n} \frac{x^p}{p} + \frac{y^q}{q}.$$

However, this inequality does not hold in general (a counterexample is presented later). Here "$\circ$" is the Jordan product associated with second-order cone. Next, according to Ando's inequality (4.9), we naively make another conjecture that the eigenvalue version of Young inequality in the SOC setting may look like

$$\lambda_j(x \circ y) \leq \lambda_j\left(\frac{x^p}{p} + \frac{y^q}{q}\right), \quad j = 1, 2. \tag{4.10}$$

Although we believe it is true, it is very complicated to prove the inequality directly due to the algebraic structure of $\frac{x^p}{p} + \frac{x^q}{q}$. Eventually, we seek another variant and establish the SOC trace version of Young inequality. Accordingly, we further deduce the SOC trace versions of Hölder and Minkowski inequalities.

As mentioned earlier, one may conjecture that the Young inequality in the SOC setting is in form of

$$x \circ y \preceq_{\mathcal{K}^n} \frac{x^p}{p} + \frac{y^q}{q}.$$

However, this inequality does not hold in general. For example, taking $p = 3$, $q = \frac{3}{2}$, $x = (\frac{1}{8}, \frac{1}{8}, 0)$, and $y = (\frac{1}{8}, 0, \frac{1}{8})$, we obtain $x^3 = (\frac{1}{128}, \frac{1}{128}, 0)$, $y^{\frac{3}{2}} = (\frac{1}{16}, 0, \frac{1}{16})$. Hence,

$$x \circ y = \left(\frac{1}{64}, \frac{1}{64}, \frac{1}{64}\right) \quad \text{and} \quad \frac{x^3}{3} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}} = \left(\frac{17}{384}, \frac{1}{384}, \frac{16}{384}\right),$$

which says

$$\frac{x^3}{3} + \frac{y^{\frac{3}{2}}}{\frac{3}{2}} - x \circ y = \left(\frac{11}{384}, \frac{-5}{384}, \frac{10}{384}\right) \notin \mathcal{K}^n.$$

In view of this and motivated by the Ando's singular value version of Young inequality as in (4.9), we turn to derive the eigenvalue version of Young inequality in the setting of second-order cone. But, we do not succeed in achieving such type inequality. Instead, we consider the SOC trace version of the Young inequality.

Proposition 4.6. (Young inequality-Type I) *For any $x, y \in \mathcal{K}^n$, there holds*

$$\mathrm{tr}(x \circ y) \leq \mathrm{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right),$$

where $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. First, we note $x \circ y = (x_1 y_1 + \langle x_2, y_2 \rangle, x_1 y_2 + y_1 x_2)$ and denote $\frac{x^p}{p} + \frac{y^q}{q} := (w_1, w_2)$ where

$$\begin{aligned} w_1 &= \frac{\lambda_1(x)^p + \lambda_2(x)^p}{2p} + \frac{\lambda_1(y)^q + \lambda_2(y)^q}{2q}, \\ w_2 &= \frac{\lambda_2(x)^p - \lambda_1(x)^p}{2p} \frac{x_2}{\|x_2\|} + \frac{\lambda_2(y)^q - \lambda_1(y)^q}{2q} \frac{y_2}{\|y_2\|}. \end{aligned}$$

Then, the desired result follows by

$$\begin{aligned} \mathrm{tr}(x \circ y) &\leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y) \\ &\leq \left(\frac{\lambda_1(x)^p}{p} + \frac{\lambda_1(y)^q}{q}\right) + \left(\frac{\lambda_2(x)^p}{p} + \frac{\lambda_2(y)^q}{q}\right) \\ &= \mathrm{tr}\left(\frac{x^p}{p} + \frac{y^q}{q}\right), \end{aligned}$$

where the last inequality is due to the Young inequality on real number setting. □

Remark 4.1. *When $p = q = 2$, the Young inequality in Proposition 4.6 reduces to*

$$2\langle x, y \rangle = \mathrm{tr}(x \circ y) \leq \mathrm{tr}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = \|x\|^2 + \|y\|^2,$$

which is equivalent to $0 \leq \|x - y\|^2$. As a matter of fact, for any $x, y \in \mathbb{R}^n$, the inequality $(x - y)^2 \succeq_{\mathcal{K}^n} 0$ always holds, which implies $2x \circ y \preceq_{\mathcal{K}^n} x^2 + y^2$. Therefore, by Proposition 1.1(a), we obtain $\mathrm{tr}(x \circ y) \leq \mathrm{tr}\left(\frac{x^2}{2} + \frac{y^2}{2}\right)$ as well.

We note that the classical Young inequality can be extended to nonnegative real numbers, that is,

$$|ab| = |a| \cdot |b| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \quad \forall a, b \in \mathbb{R}.$$

This motivates us to consider further generalization of the SOC trace version of Young inequality as in Proposition 4.6. However, $|x| \circ |y|$ and $|x \circ y|$ are unequal in general; and no relation between them. To see this, taking $x = (\sqrt{2}, 1, 1) \in \mathcal{K}^3$ and $y = (\sqrt{2}, 1, -1) \in \mathcal{K}^3$, yields $x \circ y = (2, 2\sqrt{2}, 0) \notin \mathcal{K}^3$. In addition, it implies

$$|x| \circ |y| = (2, 2\sqrt{2}, 0) \preceq_{\mathcal{K}^n} (2\sqrt{2}, 2, 0) = |x \circ y|.$$

On the other hand, let $x = (0,1,0)$, $y = (0,1,1)$, which give $|x| = (1,0,0)$, $|y| = (\sqrt{2},0,0)$. However, we see that

$$|x \circ y| = (1,0,0) \preceq_{\mathcal{K}^n} (\sqrt{2},0,0) = |x| \circ |y|.$$

From these two examples, it also indicates that there is no relationship between $\mathrm{tr}(|x| \circ |y|)$ and $\mathrm{tr}(|x \circ y|)$. In other words, there are two possible extensions of Proposition 4.6:

$$\mathrm{tr}(|x| \circ |y|) \leq \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right) \quad \text{or} \quad \mathrm{tr}(|x \circ y|) \leq \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right).$$

Fortunately, these two types of generalizations are both true.

Proposition 4.7. (Young inequality-Type II) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathrm{tr}(|x| \circ |y|) \leq \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right),$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Following the proof of Proposition 4.6, we have

$$\begin{aligned}
& \mathrm{tr}(|x| \circ |y|) \\
\leq\; & \lambda_1(|x|)\lambda_1(|y|) + \lambda_2(|x|)\lambda_2(|y|) \\
=\; & \min_i\{|\lambda_i(x)|\} \min_i\{|\lambda_i(y)|\} + \max_i\{|\lambda_i(x)|\} \max_i\{|\lambda_i(y)|\} \\
\leq\; & \frac{(\min_i\{|\lambda_i(x)|\})^p}{p} + \frac{(\min_i\{|\lambda_i(y)|\})^q}{q} + \frac{(\max_i\{|\lambda_i(x)|\})^p}{p} + \frac{(\max_i\{|\lambda_i(y)|\})^q}{q} \\
=\; & \left(\frac{|\lambda_1(x)|^p}{p} + \frac{|\lambda_2(x)|^p}{p}\right) + \left(\frac{|\lambda_1(y)|^q}{q} + \frac{|\lambda_2(y)|^q}{q}\right) \\
=\; & \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right),
\end{aligned}$$

where the last inequality holds by the Young inequality on real number setting. $\square$

We point out that Proposition 4.7 is more general than Proposition 4.6 because it is true for all $x, y \in \mathbb{R}^n$, not necessary restricted to $x, y \in \mathcal{K}^n$. For real numbers, it is clear that $ab \leq |a| \cdot |b|$. It is natural to ask whether $\mathrm{tr}(x \circ y)$ is less than $\mathrm{tr}(|x| \circ |y|)$ or not. Before establishing the relationship, we need the following technical lemma.

Lemma 4.3. *For $0 \preceq_{\mathcal{K}^n} u \preceq_{\mathcal{K}^n} x$ and $0 \preceq_{\mathcal{K}^n} v \preceq_{\mathcal{K}^n} y$, there holds*

$$0 \leq \mathrm{tr}(u \circ v) \leq \mathrm{tr}(x \circ y).$$

Proof. Suppose $0 \preceq_{\mathcal{K}^n} u \preceq_{\mathcal{K}^n} x$ and $0 \preceq_{\mathcal{K}^n} v \preceq_{\mathcal{K}^n} y$, we have

$$\begin{aligned} & \operatorname{tr}(x \circ y) - \operatorname{tr}(u \circ v) \\ = \; & \operatorname{tr}(x \circ y - u \circ v) \\ = \; & \operatorname{tr}(x \circ y - x \circ v + x \circ v - u \circ v) \\ = \; & \operatorname{tr}(x \circ (y - v) + (x - u) \circ v) \\ = \; & \operatorname{tr}(x \circ (y - v)) + \operatorname{tr}((x - u) \circ v) \\ \geq \; & 0, \end{aligned}$$

where the inequality holds by Property 1.3(d). $\square$

Proposition 4.8. *For any $x, y \in \mathbb{R}^n$, there holds* $\operatorname{tr}(x \circ y) \leq \operatorname{tr}(|x| \circ |y|)$.

Proof. For any $x \in \mathbb{R}^n$, it can be expressed by $x = [x]_+ + [x]_-$, and then

$$\begin{aligned} \operatorname{tr}(x \circ y) &= \operatorname{tr}(([x]_+ + [x]_-) \circ y) \\ &= \operatorname{tr}([x]_+ \circ y) + \operatorname{tr}((-[x]_-) \circ (-y)) \\ &\leq \operatorname{tr}([x]_+ \circ |y|) + \operatorname{tr}((-[x]_-) \circ |y|) \\ &= \operatorname{tr}(([x]_+ - [x]_-) \circ |y|) \\ &= \operatorname{tr}(|x| \circ |y|), \end{aligned}$$

where the inequality holds by Lemma 4.3. $\square$

There is some interpretation from geometric view for Proposition 4.8. More specifically, by the definition of trace in second-order cone, we notice

$$\operatorname{tr}(x \circ y) = 2\langle x, y \rangle = 2\|x\| \cdot \|y\| \cos\theta$$

where θ is the angle between the vectors x and y. According to the definition of absolute value associated with second-order cone, we know the equality in Proposition 4.8 holds whenever $x, y \in \mathcal{K}^n$ or $x, y \in -\mathcal{K}^n$. Otherwise, it can be observed that the angle between $|x|$ and $|y|$ is smaller than the angle between x and y since the vector $x, |x|$ and the axis of second-order cone are in a hyperplane.

Proposition 4.9. *For any $x, y \in \mathbb{R}^n$, the following inequalities hold.*

(a) $\operatorname{tr}((x+y)^2) \leq \operatorname{tr}((|x| + |y|)^2)$, *i.e.*, $\|x + y\| \leq \||x| + |y|\|$.

(b) $\operatorname{tr}((x-y)^2) \geq \operatorname{tr}((|x| - |y|)^2)$, *i.e.*, $\|x - y\| \geq \||x| - |y|\|$.

Proof. (a) From Proposition 4.8, we have

$$\operatorname{tr}\left((x+y)^2\right) = \operatorname{tr}\left(x^2 + 2x \circ y + y^2\right) \leq \operatorname{tr}\left(|x|^2 + 2|x| \circ |y| + |y|^2\right) = \operatorname{tr}\left((|x| + |y|)^2\right).$$

This is equivalent to $\|x+y\|^2 \leq \||x|+|y|\|^2$, which implies $\|x+y\| \leq \||x|+|y|\|$.

(b) The proof is similar to part(a). $\square$

In contrast to Proposition 4.8, applying Proposition 1.1(a), it is clear that $\mathrm{tr}(x \circ y) \leq \mathrm{tr}\,(|x \circ y|)$ because $x \circ y \preceq_{\mathcal{K}^n} |x \circ y|$. In view of this, we try to achieve another extension as below.

Proposition 4.10. (Young inequality-Type III) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathrm{tr}(|x \circ y|) \leq \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right),$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. For analysis needs, we write $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Note that if $x \circ y \in \mathcal{K}^n \cup (-\mathcal{K}^n)$, the desired inequality holds immediately by Proposition 4.7 and Proposition 4.8. Thus, it suffices to show the inequality holds for $x \circ y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. In fact, we only need to show the inequality for the case of $x_1 \geq 0$ and $y_1 \geq 0$. The other cases can be derived by suitable changing variable like

$$|x \circ y| = |-(x \circ y)| = |(-x) \circ y| = |x \circ (-y)| = |(-x) \circ (-y)|.$$

To proceed, we first claim the following inequality

$$2\|x_1 y_2 + y_1 x_2\| \leq |\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|, \tag{4.11}$$

which is also equivalent to $4\|x_1 y_2 + y_1 x_2\|^2 \leq (|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|)^2$. Indeed, we observe that

$$4\|x_1 y_2 + y_1 x_2\|^2 = 4\left(x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 + 2x_1 y_1 \langle x_2, y_2\rangle\right).$$

On the other hand,

$$\begin{aligned}
& \left(|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|\right)^2 \\
= \; & [\lambda_1(x)\lambda_1(y)]^2 + [\lambda_2(x)\lambda_2(y)]^2 + 2\,|\lambda_1(x)\lambda_1(y)\lambda_2(x)\lambda_2(y)| \\
= \; & 2(x_1 y_1 + \|x_2\|\|y_2\|)^2 + 2(x_1\|y_2\| + y_1\|x_2\|)^2 + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
= \; & 2\left(x_1^2 y_1^2 + \|x_2\|^2\|y_2\|^2 + x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2\right) + 8x_1 y_1\|x_2\|\|y_2\| \\
& + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right|.
\end{aligned}$$

Therefore, we conclude that (4.11) is satisfied by checking

$$\begin{aligned}
& \left(|\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|\right)^2 - 4\|x_1y_2 + y_1x_2\|^2 \\
= \ & 2\left(x_1^2y_1^2 + \|x_2\|^2\|y_2\|^2 + x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2\right) + 8x_1y_1\|x_2\|\|y_2\| \\
& + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| - 4\left(x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 + 2x_1y_1\langle x_2, y_2\rangle\right) \\
= \ & 2\left(x_1^2y_1^2 + \|x_2\|^2\|y_2\|^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2\right) + 8x_1y_1\left(\|x_2\|\|y_2\| - \langle x_2, y_2\rangle\right) \\
& +2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
= \ & 2\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right) + 2\left|\left(x_1^2 - \|x_2\|^2\right)\left(y_1^2 - \|y_2\|^2\right)\right| \\
& +8x_1y_1(\|x_2\|\|y_2\| - \langle x_2, y_2\rangle) \\
\geq \ & 0,
\end{aligned}$$

where the last inequality is due to the Cauchy-Schwarz Inequality.

Suppose that $x \circ y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. From the simple calculation, we have

$$|x \circ y| = \left(\|x_1y_2 + y_1x_2\|, \ \frac{x_1y_1 + \langle x_2, y_2\rangle}{\|x_1y_2 + y_1x_2\|}(x_1y_2 + y_1x_2)\right),$$

which says $\mathrm{tr}(|x \circ y|) = 2\|x_1y_2 + y_1x_2\|$. Using inequality (4.11), we obtain

$$\begin{aligned}
\mathrm{tr}(|x \circ y|) &\leq |\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)| \\
&\leq \left(\frac{|\lambda_1(x)|^p}{p} + \frac{|\lambda_1(y)|^q}{q}\right) + \left(\frac{|\lambda_2(x)|^p}{p} + \frac{|\lambda_2(y)|^q}{q}\right) \\
&= \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right),
\end{aligned}$$

where the last inequality holds by the classical Young inequality on real number setting. □

There also exist some trace versions of Young inequalities in the setting of Euclidean Jordan algebra, please see [14, Theorem 23] and [79, Theorem 3.5-3.6]. Using the SOC trace versions of Young inequalities, we can derive the SOC trace versions of Hölder inequalities as below.

Proposition 4.11. (Hölder inequality-Type I) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathrm{tr}(|x| \circ |y|) \leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathrm{tr}(|x|^q)]^{\frac{1}{q}},$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Let $\alpha = [\mathrm{tr}(|x|^p)]^{\frac{1}{p}}$ and $\beta = [\mathrm{tr}(|x|^q)]^{\frac{1}{q}}$. By Proposition 4.7, we have

$$\mathrm{tr}\left(\frac{|x|}{\alpha} \circ \frac{|y|}{\beta}\right) \leq \mathrm{tr}\left(\frac{|\frac{|x|}{\alpha}|^p}{p} + \frac{|\frac{|y|}{\beta}|^q}{q}\right) = \frac{1}{p}\mathrm{tr}\left(\frac{|x|^p}{\alpha^p}\right) + \frac{1}{q}\mathrm{tr}\left(\frac{|y|^q}{\beta^q}\right) = \frac{1}{p} + \frac{1}{q} = 1.$$

Therefore, we conclude that

$$\mathrm{tr}(|x| \circ |y|) \leq \alpha \cdot \beta = [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathrm{tr}(|x|^q)]^{\frac{1}{q}}$$

because $\alpha, \beta > 0$. $\square$

Proposition 4.12. (Hölder inequality-Type II) *For any $x, y \in \mathbb{R}^n$, there holds*

$$\mathrm{tr}(|x \circ y|) \leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} \cdot [\mathrm{tr}(|x|^q)]^{\frac{1}{q}},$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. The proof is similar to Proposition 4.11 by using Proposition 4.10. $\square$

Remark 4.2. *When $p = q = 2$, both inequalities in Proposition 4.11 and Proposition 4.12 deduce*

$$\left|2\langle x, y\rangle\right| = \mathrm{tr}(|x \circ y|) \leq \left[\mathrm{tr}(|x|^2)\right]^{\frac{1}{2}} \cdot \left[\mathrm{tr}(|x|^2)\right]^{\frac{1}{2}} = 2\|x\| \cdot \|y\|,$$

which is equivalent to the Cauchy-Schwarz inequality in $\mathbb{R}^n$.

Next, we present the SOC trace version of Minkowski inequality.

Proposition 4.13. (Minkowski inequality) *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and $p > 1$, there holds*

$$[\mathrm{tr}(|x + y|^p)]^{\frac{1}{p}} \leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}}.$$

Proof. We partition the proof into three parts. Let $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.
(i) For $x + y \in \mathcal{K}^n$, we have $|x + y| = x + y$, then we have

$$\begin{aligned}
\mathrm{tr}(|x + y|^p) &= \mathrm{tr}(|x + y| \circ |x + y|^{p-1}) = \mathrm{tr}((x + y) \circ |x + y|^{p-1}) \\
&= \mathrm{tr}(x \circ |x + y|^{p-1}) + \mathrm{tr}(y \circ |x + y|^{p-1}) \\
&\leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} \cdot \left[\mathrm{tr}(|x + y|^{(p-1)q})\right]^{\frac{1}{q}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}} \cdot \left[\mathrm{tr}(|x + y|^{(p-1)q})\right]^{\frac{1}{q}} \\
&= \left([\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}}\right) \cdot [\mathrm{tr}(|x + y|^p)]^{\frac{1}{q}},
\end{aligned}$$

which implies $[\mathrm{tr}(|x + y|^p)]^{\frac{1}{p}} \leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}}$.
(ii) For $x + y \in -\mathcal{K}^n$, we have $|x + y| = -x - y$, then we have

$$\begin{aligned}
\mathrm{tr}(|x + y|^p) &= \mathrm{tr}((-x) \circ |x + y|^{p-1}) + \mathrm{tr}((-y) \circ |x + y|^{p-1}) \\
&\leq [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} \cdot \left[\mathrm{tr}(|x + y|^{(p-1)q})\right]^{\frac{1}{q}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}} \cdot \left[\mathrm{tr}(|x + y|^{(p-1)q})\right]^{\frac{1}{q}} \\
&= \left([\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}}\right) \cdot [\mathrm{tr}(|x + y|^p)]^{\frac{1}{q}},
\end{aligned}$$

which also implies $[\mathrm{tr}(|x+y|^p)]^{\frac{1}{p}} \le [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}}$.

(iii) For $x+y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we note that $\lambda_1(x+y) < 0$ and $\lambda_2(x+y) > 0$, which says,

$$\begin{aligned}
|\lambda_1(x+y)| &= |x_1+y_1-\|x_2+y_2\|| = \|x_2+y_2\| - x_1 - y_1 \le \|x_2\| + \|y_2\| - x_1 - y_1,\\
|\lambda_2(x+y)| &= |x_1+y_1+\|x_2+y_2\|| = x_1+y_1+\|x_2+y_2\| \le x_1+y_1+\|x_2\|+\|y_2\|.
\end{aligned}$$

This yields

$$\begin{aligned}
[\mathrm{tr}(|x+y|^p)]^{\frac{1}{p}} &= [|\lambda_1(x+y)|^p + |\lambda_2(x+y)|^p]^{\frac{1}{p}}\\
&\le [(\|x_2\|+\|y_2\|-x_1-y_1)^p + (\|x_2\|+\|y_2\|+x_1+y_1)^p]^{\frac{1}{p}}\\
&= [(-\lambda_1(x)-\lambda_1(y))^p + (\lambda_2(x)+\lambda_2(y))^p]^{\frac{1}{p}}\\
&= [|\lambda_1(x)+\lambda_1(y)|^p + |\lambda_2(x)+\lambda_2(y)|^p]^{\frac{1}{p}}\\
&\le [|\lambda_1(x)|^p + |\lambda_2(x)|^p]^{\frac{1}{p}} + [|\lambda_1(y)|^p + |\lambda_2(y)|^p]^{\frac{1}{p}}\\
&= [\mathrm{tr}(|x|^p)]^{\frac{1}{p}} + [\mathrm{tr}(|y|^p)]^{\frac{1}{p}},
\end{aligned}$$

where the last inequality holds by the classical Minkowski inequality on real number setting. □

Remark 4.3. *We elaborate more about Proposition 4.13. We can define a norm* $|||\cdot|||_p$ *on* $\mathbb{R}^n$ *by*

$$|||x|||_p := [\mathrm{tr}(|x|^p)]^{\frac{1}{p}},$$

and hence it induces a distance $d(x,y) = |||x-y|||_p$ *on* $\mathbb{R}^n$*. In particular, this norm will deduce the Euclidean-norm when* $p=2$*, and the inequality reduces to the triangular inequality. In addition, this norm is similar to Schatten p-norm, which arise when applying the p-norm to the vector of singular values of a matrix. For more details, please refer to [22].*

According to the arguments in Proposition 4.13, if we wish to establish the SOC trace version of Minkowski inequality in general case without any restriction, the crucial key is verifying the SOC triangular inequality

$$|x+y| \preceq_{\mathcal{K}^n} |x| + |y|.$$

Unfortunately, this inequality does not hold. To see this, checking $x = (\sqrt{2}, 1, -1)$ and $y = (-\sqrt{2}, -1, 0)$ will lead to a counterexample. More specifically, $x \in \mathcal{K}^n, y \in -\mathcal{K}^n$, and $x+y = (0,0,-1) \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, which says $|x+y| = (1,0,0)$ and $|x|+|y| = x+(-y) = (2\sqrt{2}, 2, -1)$. Hence,

$$|x|+|y|-|x+y| = (2\sqrt{2}-1, 2, -1) \notin \mathcal{K}^n \cup (-\mathcal{K}^n).$$

Moreover, we have

$$\begin{aligned}
\lambda_1(|x+y|) = 1 > 2\sqrt{2} - \sqrt{5} = \lambda_1(|x|+|y|),\\
\lambda_2(|x+y|) = 1 < 2\sqrt{2} + \sqrt{5} = \lambda_2(|x|+|y|).
\end{aligned}$$

Nonetheless, we build another SOC trace version of triangular inequality as below.

Proposition 4.14. (Triangular inequality) *For any* $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ *and* $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, *there holds*

$$\mathrm{tr}(|x+y|) \leq \mathrm{tr}(|x|) + \mathrm{tr}(|y|).$$

Proof. In order to complete the proof, we discuss three cases.
(i) If $x+y \in \mathcal{K}^n$, then $|x+y| = x+y \preceq_{\mathcal{K}^n} |x|+|y|$, and hence

$$\mathrm{tr}(|x+y|) \leq \mathrm{tr}(|x|) + \mathrm{tr}(|y|)$$

by Proposition 1.1(a).
(ii) If $x+y \in -\mathcal{K}^n$, then $|x+y| = -x-y \preceq_{\mathcal{K}^n} |x|+|y|$, and hence

$$\mathrm{tr}(|x+y|) \leq \mathrm{tr}(|x|) + \mathrm{tr}(|y|).$$

(iii) Suppose $x+y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have $|x+y| = \left(\|x_2+y_2\|, \frac{x_1+y_1}{\|x_2+y_2\|}(x_2+y_2)\right)$ from simple calculation, then

$$\mathrm{tr}(|x+y|) = 2\|x_2+y_2\|.$$

If one of x, y is in $\mathcal{K}^n$ (for convenience, we let $x \in \mathcal{K}^n$), we have two subcases: $y \in -\mathcal{K}^n$ and $y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. For $y \in -\mathcal{K}^n$, we have $|y| = -y$ and $-y_1 \geq \|y_2\|$, and hence

$$\mathrm{tr}(|x|+|y|) = \mathrm{tr}(x-y) = 2(x_1-y_1) \geq 2(\|x_2\|+\|y_2\|) \geq 2\|x_2+y_2\| = \mathrm{tr}(|x+y|).$$

For $y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, we have $|y| = \left(\|y_2\|, \frac{y_1}{\|y_2\|}y_2\right)$, and hence

$$\mathrm{tr}(|x|+|y|) = 2(x_1+\|y_2\|) \geq 2(\|x_2\|+\|y_2\|) \geq 2\|x_2+y_2\| = \mathrm{tr}(|x+y|).$$

If one of x, y is in $-\mathcal{K}^n$, then the argument is similar. To complete the proof, it remains to show the inequality holds for $x, y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$. Indeed, in this case, we have

$$\mathrm{tr}(|x|+|y|) = 2(\|x_2\|+\|y_2\|) \geq 2\|x_2+y_2\| = \mathrm{tr}(|x+y|).$$

Hence, we complete the proof. □

To close this section, we comment a few words about the aforementioned inequalities. In real analysis, Young inequality is the main tool to derive the Hölder inequality, and then the Minkowski inequality can be derived by applying Hölder inequality. Tao et al. [143] establish a trace p-norm in the setting of Euclidean Jordan algebra. In particular, they directly show the trace version of Minkowski inequality, see [143, Theorem 4.1]. As an application of trace versions of Young inequalities, we use the approach which follows the same idea as in real analysis to derive the trace versions of Hölder inequalities. Furthermore, the SOC trace version of Minkowski inequality is also deduced. On the other hand, the trace version of Triangular inequality holds for any Euclidean Jordan algebra, see [97, Proposition 4.3] and [143, Corollary 3.1]. In the setting of second-order cone, we prove the inequality by discussing three cases directly.

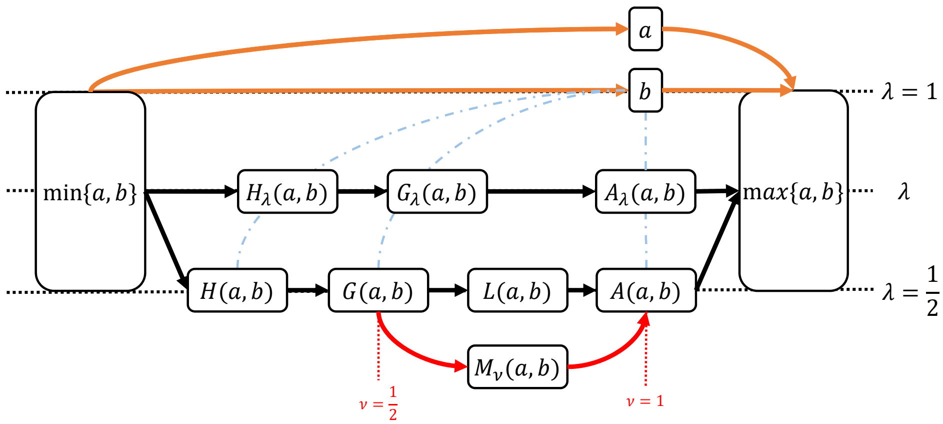


Figure 4.1: Relationship between means defined on real number.

4.3 SOC weighted means and trace inequalities

In this section, we further investigate the *weighted means* and their induced inequalities associated with SOC. More specifically, we set up the concepts of some weighted means in the SOC setting. Then, we achieve a few inequalities on the new-extended weighted means and their corresponding traces associated with second-order cone. As a byproduct, a version of Powers-Størmers inequality is established. Indeed, for real numbers, there exits a diagraph regarding the weighted means and the weighted Arithmetic-Geometric-Mean inequality, see Figure 4.1. The direction of arrow in Figure 4.1 represents the ordered relationship. We shall define these weighted means in the setting of second-order cone and build up the relationship among these SOC weighted means.

Lemma 4.4. *Suppose that V is a Jordan algebra with an identity element e. Let $P(x)$ be defined as in (4.5). Then, $P(x)$ possesses the following properties.*

(a) *An element x is invertible if and only if $P(x)$ is invertible. In this case, $P(x)x^{-1} = x$ and $P(x)^{-1} = P(x^{-1})$.*

(b) *If x and y are invertible, then $P(x)y$ is invertible and $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.*

(c) *For any elements x and y, $P(P(x)y) = P(x)P(y)P(x)$. In particular, $P(x^2) = P(x)^2$.*

(d) *For any elements $x, y \in \overline{\Omega}$, $x \preceq y$ if and only if $P(x) \preceq P(y)$, which means $P(y) - P(x)$ is a positive semidefinite matrix.*

Proof. Please see Proposition II.3.1, Proposition II.3.2, and Proposition II.3.4 of [62] and Lemma 2.3 of [100]. □

Recall that a binary operation $(x, y) \mapsto M(x, y)$ defined on $\text{int}(\mathcal{K}^n) \times \text{int}(\mathcal{K}^n)$ is called an SOC mean if it satisfies all conditions in Definition 4.1. In light of $A(x, y)$, $H(x, y)$,

$G(x,y)$, we consider their corresponding SOC weighted means as below. For $0 \le \lambda \le 1$, we let

$$\begin{aligned} A_\lambda(x,y) &:= (1-\lambda)x + \lambda y, && (4.12)\\ H_\lambda(x,y) &:= \left((1-\lambda)x^{-1} + \lambda y^{-1}\right)^{-1}, && (4.13)\\ G_\lambda(x,y) &:= P\left(x^{\frac{1}{2}}\right)\left(P(x^{-\frac{1}{2}})y\right)^{\lambda}, && (4.14)\end{aligned}$$

denote the SOC weighted arithmetic mean, the SOC weighted harmonic mean, and the SOC weighted geometric mean, respectively. According to the definition, it is clear that

$$\begin{aligned} A_{1-\lambda}(x,y) &= A_\lambda(y,x),\\ H_{1-\lambda}(x,y) &= H_\lambda(y,x),\\ G_{1-\lambda}(x,y) &= G_\lambda(y,x). \end{aligned}$$

We note that when $\lambda = 1/2$, these SOC weighted means coincide with the SOC arithmetic mean $A(x,y)$, the SOC harmonic mean $H(x,y)$, and the SOC geometric mean $G(x,y)$, respectively.

Proposition 4.15. *Suppose $0 \le \lambda \le 1$. Let $A_\lambda(x,y)$, $H_\lambda(x,y)$, and $G_\lambda(x,y)$ be defined as in (4.12), (4.13), and (4.14), respectively. Then, for any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, there holds*

$$x \wedge y \preceq_{\mathcal{K}^n} H_\lambda(x,y) \preceq_{\mathcal{K}^n} G_\lambda(x,y) \preceq_{\mathcal{K}^n} A_\lambda(x,y) \preceq_{\mathcal{K}^n} x \vee y.$$

Proof. (i) To verify the first inequality, we discuss two cases. For $\frac{1}{2}(x+y-|x-y|) \notin \mathcal{K}^n$, the inequality holds automatically. For $\frac{1}{2}(x+y-|x-y|) \in \mathcal{K}^n$, we note that $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} x$ and $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} y$. Then, using the SOC-monotonicity of $f(t) = -t^{-1}$ shown in Proposition 2.3, we obtain

$$x^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1} \quad \text{and} \quad y^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1},$$

which imply

$$(1-\lambda)x^{-1} + \lambda y^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1}.$$

Next, applying the SOC-monotonicity again to this inequality, we conclude that

$$\frac{x+y-|x-y|}{2} \preceq_{\mathcal{K}^n} \left((1-\lambda)x^{-1} + \lambda y^{-1}\right)^{-1}.$$

(ii) For the second and third inequalities, it suffices to verify the third inequality (the second one can be deduced thereafter). Let $s = P(x^{-\frac{1}{2}})y - e$, which gives $s \succeq_{\mathcal{K}^n} -e$. Then, applying Lemma 4.1 yields

$$\left(e + P(x^{-\frac{1}{2}})y - e\right)^{\lambda} \preceq_{\mathcal{K}^n} e + \lambda\left[P(x^{-\frac{1}{2}})y - e\right],$$

which is equivalent to

$$0 \preceq_{\mathcal{K}^n} (1-\lambda)e + \lambda \left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^{\lambda}.$$

Since $P(x^{\frac{1}{2}})$ is invariant on $\mathcal{K}^n$, we have

$$\begin{aligned} 0 \;&\preceq_{\mathcal{K}^n}\; P(x^{\frac{1}{2}})\left((1-\lambda)e + \lambda \left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^{\lambda}\right) \\ &= \;(1-\lambda)x + \lambda y - P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\lambda}, \end{aligned}$$

and hence

$$P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\lambda} \preceq_{\mathcal{K}^n} (1-\lambda)x + \lambda y. \tag{4.15}$$

For the second inequality, replacing x and y in (4.15) by x^{-1} and y^{-1}, respectively, gives

$$P(x^{-\frac{1}{2}})\left(P(x^{\frac{1}{2}})y^{-1}\right)^{\lambda} \preceq_{\mathcal{K}^n} (1-\lambda)x^{-1} + \lambda y^{-1}.$$

Using the SOC-monotonicity again, we conclude

$$\left((1-\lambda)x^{-1} + \lambda y^{-1}\right)^{-1} \preceq_{\mathcal{K}^n} \left(P(x^{-\frac{1}{2}})\left(P(x^{\frac{1}{2}})y^{-1}\right)^{\lambda}\right)^{-1} = P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\lambda},$$

where the equality holds by Lemma 4.4(b).

(iii) To see the last inequality, we observe that $x \preceq_{\mathcal{K}^n} \frac{1}{2}(x+y+|x-y|)$ and $y \preceq_{\mathcal{K}^n} \frac{1}{2}(x+y+|x-y|)$, which imply

$$(1-\lambda)x + \lambda y \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}.$$

Then, the desired result follows. □

In Section 4.2, we have established three SOC trace versions of Young inequalities. Based on Proposition 4.15, we provide the SOC determinant version of Young inequality.

Proposition 4.16. (Determinant Young inequality) *For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, there holds*

$$\det(x \circ y) \le \det\left(\frac{x^p}{p} + \frac{y^q}{q}\right),$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. Since $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} G_{\frac{1}{q}}(x^p, y^q) = P(x^{\frac{p}{2}})\left(P(x^{-\frac{p}{2}})y^q\right)^{\frac{1}{q}}$, and hence

$$\det\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \ge \det\left(P(x^{\frac{p}{2}})\left(P(x^{-\frac{p}{2}})y^q\right)^{\frac{1}{q}}\right) = \det(x)\det(y) \ge \det(x \circ y)$$

by [62, Proposition III.4.2] and Proposition 1.2(b). □

Now, we consider the family of Heinz means

$$M_\nu(a,b) := \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}$$

for $a, b > 0$ and $0 \le \nu \le 1$. Following the idea of Kubo-Ando extension in [96], the SOC Heinz mean can be defined as

$$M_\nu(x,y) := \frac{G_\nu(x,y) + G_\nu(y,x)}{2}, \tag{4.16}$$

where $x, y \succ_{\mathcal{K}^n} 0$ and $0 \le \nu \le 1$. We point out that an obvious "naive" extension could be

$$B_\nu(x,y) := \frac{x^\nu \circ y^{1-\nu} + x^{1-\nu} \circ y^\nu}{2}. \tag{4.17}$$

Unfortunately, B_ν may not always satisfy the definition of SOC mean. Although it is not an SOC mean, we still are interested in seeking the trace or norm inequality about B_ν and other SOC means, and it will be discussed later.

For any positive numbers a, b, it is well-known that

$$\sqrt{ab} \le M_\nu(a,b) \le \frac{a+b}{2}. \tag{4.18}$$

Together with the proof of Proposition 4.15, we can obtain the following inequality accordingly.

Proposition 4.17. *Suppose* $0 \le \nu \le 1$ *and* $\lambda = \frac{1}{2}$. *Let* $A_{\frac{1}{2}}(x,y)$, $G_{\frac{1}{2}}(x,y)$, *and* $M_\nu(x,y)$ *be defined as in (4.12), (4.14), and (4.16), respectively. Then, for any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *there holds*

$$G_{\frac{1}{2}}(x,y) \preceq_{\mathcal{K}^n} M_\nu(x,y) \preceq_{\mathcal{K}^n} A_{\frac{1}{2}}(x,y).$$

Proof. Consider $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$ and $0 \le \nu \le 1$, from Proposition 4.15, we have

$$\begin{aligned} M_\nu(x,y) &= \frac{G_\nu(x,y) + G_\nu(y,x)}{2} \\ &\preceq_{\mathcal{K}^n} \frac{A_\nu(x,y) + A_\nu(y,x)}{2} \\ &= A_{\frac{1}{2}}(x,y). \end{aligned}$$

On the other hand, we note that

$$\begin{aligned}
M_\nu(x,y) &= \frac{G_\nu(x,y)+G_{1-\nu}(x,y)}{2} \\
&= \frac{P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\nu}+P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{1-\nu}}{2} \\
&= P(x^{\frac{1}{2}})\left(\frac{\left(P(x^{-\frac{1}{2}})y\right)^{\nu}+\left(P(x^{-\frac{1}{2}})y\right)^{1-\nu}}{2}\right) \\
&\succeq_{\mathcal{K}^n} P(x^{\frac{1}{2}})\left(\left(P(x^{-\frac{1}{2}})y\right)^{\frac{\nu}{2}}\circ\left(P(x^{-\frac{1}{2}})y\right)^{\frac{1-\nu}{2}}\right) \\
&= G_{\frac{1}{2}}(x,y),
\end{aligned}$$

where the inequality holds due to the fact $\frac{u+v}{2}\succeq_{\mathcal{K}^n} u^{\frac{1}{2}}\circ v^{\frac{1}{2}}$ for any $u,v\in\mathcal{K}^n$ and the invariant property of $P(x^{\frac{1}{2}})$ on $\mathcal{K}^n$. □

Over all, we could have a picture regarding the ordered relationship of these SOC weighted means as depicted in Figure 4.2.

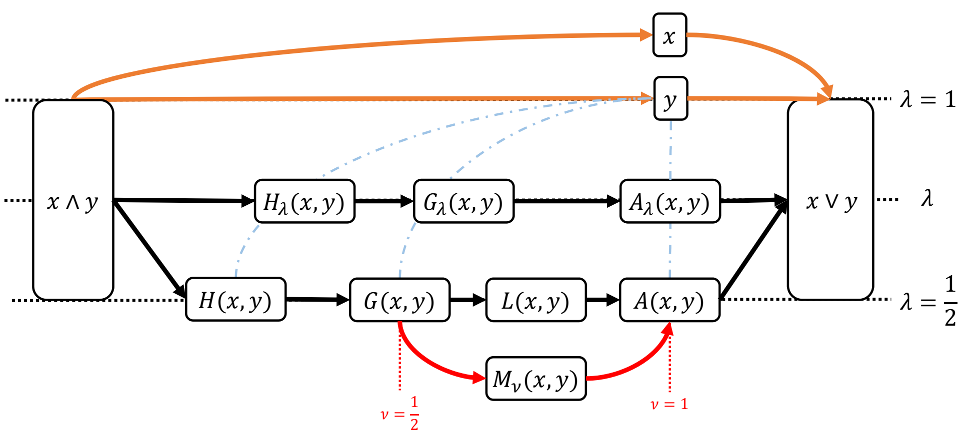


Figure 4.2: Relationship between means defined on second-order cone.

Up to now, we have extended the weighted harmonic mean, weighted geometric mean, weighted Heinz mean, and weighted arithmetic mean to second-order cone setting. As below, we explore some other inequalities associated with traces of these SOC weighted means. First, by applying Proposition 1.1(b), we immediately obtain the following trace inequalities for SOC weighted means.

Proposition 4.18. *Suppose $0\le\lambda\le 1$. Let $A_\lambda(x,y)$, $H_\lambda(x,y)$, and $G_\lambda(x,y)$ be defined as in (4.12), (4.13), and (4.14), respectively. For, any $x\succ_{\mathcal{K}^n}0$ and $y\succ_{\mathcal{K}^n}0$, there holds*

$$\operatorname{tr}(x\wedge y)\le\operatorname{tr}(H_\lambda(x,y))\le\operatorname{tr}(G_\lambda(x,y))\le\operatorname{tr}(A_\lambda(x,y))\le\operatorname{tr}(x\vee y).$$

Proposition 4.19. *Suppose* $0 \leq \nu \leq 1$ *and* $\lambda = \frac{1}{2}$. *Let* $A_{\frac{1}{2}}(x, y)$, $H_{\frac{1}{2}}(x, y)$, $G_{\frac{1}{2}}(x, y)$, *and* $M_\nu(x, y)$ *be defined as in (4.12), (4.13), (4.14), and (4.16), respectively. Then, for any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *there holds*

$$\operatorname{tr}(x \wedge y) \leq \operatorname{tr}(H_{\frac{1}{2}}(x, y)) \leq \operatorname{tr}(G_{\frac{1}{2}}(x, y)) \leq \operatorname{tr}(M_\nu(x, y)) \leq \operatorname{tr}(A_{\frac{1}{2}}(x, y)) \leq \operatorname{tr}(x \vee y).$$

As mentioned earlier, there are some well-known means, like Heinz mean

$$M_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2}, \qquad \text{for } 0 < \nu < 1,$$

which cannot serve as SOC means albeit it is a natural extension. Even though they are not SOC means, it is still possible to derive some trace or norm inequality about these means.

Next, we pay attention to another special inequality. The Powers-Størmers inequality asserts that for $s \in [0, 1]$ the following inequality

$$2\mathbf{Tr}\left(A^s B^{1-s}\right) \geq \mathbf{Tr}\left(A + B - |A - B|\right)$$

holds for any pair of positive definite matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [4]. In [73, 74], Hoa, Osaka and Tomiyama investigate the generalized Powers-Størmer inequality. More specifically, for any positive matrices A, B and matrix-concave function f, they prove that

$$\mathbf{Tr}(A) + \mathbf{Tr}(B) - \mathbf{Tr}(|A - B|) \leq 2\mathbf{Tr}\left(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}\right),$$

where $g(t) = \begin{cases} \frac{t}{f(t)}, & t \in (0, \infty) \\ 0, & t = 0 \end{cases}$. Moreover, Hoa et al. also shows that the Powers-Størmers Inequality characterizes the trace property for a normal linear positive functional on a von Neumann algebras and for a linear positive functional on a C^*-algebra. Motivated by the above facts, we establish a version of the Powers-Størmers inequality for SOC-monotone function on $[0, \infty)$ in the SOC setting.

Proposition 4.20. *For any* $x, y, z \in \mathbb{R}^n$, *there holds* $\operatorname{tr}((x \circ y) \circ z) = \operatorname{tr}(x \circ (y \circ z))$.

Proof. From direct computation, we have $x \circ y = (x_1 y_1 + \langle x_2, y_2 \rangle, x_1 y_2 + y_1 x_2)$ and

$$\operatorname{tr}((x \circ y) \circ z) = 2\left(x_1 y_1 z_1 + z_1 \langle x_2, y_2 \rangle + x_1 \langle y_2, z_2 \rangle + y_1 \langle x_2, z_2 \rangle\right).$$

Similarly, we also have $y \circ z = (y_1 z_1 + \langle y_2, z_2 \rangle, y_1 z_2 + z_1 y_2)$ and

$$\operatorname{tr}(x \circ (y \circ z)) = 2\left(x_1 y_1 z_1 + x_1 \langle y_2, z_2 \rangle + y_1 \langle x_2, z_2 \rangle + z_1 \langle x_2, y_2 \rangle\right).$$

Therefore, we conclude the desired result. □

According to the proof in [73, 74], the crucial point is under what conditions of $f(t)$, there holds the SOC-monotonicity of $\frac{t}{f(t)}$. For establishing the SOC version of Powers-Størmers Inequality, it is also a key, which is answered in next proposition.

Proposition 4.21. *Let f be a strictly positive, continuous function on $[0,\infty)$. The function $g(t) := \dfrac{t}{f(t)}$ is SOC-monotone if one of the following conditions holds.*

(a) *f is matrix-monotone of order 4;*

(b) *f is matrix-concave of order 3;*

(c) *For any contraction $T : \mathcal{K}^n \mapsto \mathcal{K}^n$ and $z \in \mathcal{K}^n$, there holds*

$$f^{\rm soc}(Tz) \succeq_{\mathcal{K}^n} Tf^{\rm soc}(z). \tag{4.19}$$

Proof. (a) According to [73, Proposition 2.1], the 4-matrix-monotonicity of f would imply the 2-matrix-monotonicity of g, which coincides with the SOC-monotonicity by Proposition 2.23.

(b) From [74, Theorem 2.1], the 3-matrix-concavity of f implies the 2-matrix-monotonicity of g, which coincides with the SOC-monotonicity as well.

(c) Suppose $0 \prec_{\mathcal{K}^n} x \preceq_{\mathcal{K}^n} y$, we have $P(x^{\frac{1}{2}}) \preceq P(y^{\frac{1}{2}})$ by SOC-monotonicity of $t^{1/2}$ and Lemma 4.4, which implies $\|P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})\| \leq 1$. Hence, $P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})$ is an contraction. Then

$$\begin{aligned}
& x = P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})y) \\
\Longrightarrow\quad & f^{\rm soc}(x) = f^{\rm soc}(P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})y)) \\
\Longrightarrow\quad & f^{\rm soc}(x) \succeq_{\mathcal{K}^n} P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})f^{\rm soc}(y)) \\
\Longleftrightarrow\quad & P(x^{-\frac{1}{2}})f^{\rm soc}(x) \succeq_{\mathcal{K}^n} P(y^{-\frac{1}{2}})f^{\rm soc}(y) \\
\Longleftrightarrow\quad & x^{-1} \circ f^{\rm soc}(x) \succeq_{\mathcal{K}^n} y^{-1} \circ f^{\rm soc}(y) \\
\Longleftrightarrow\quad & x \circ (f^{\rm soc}(x))^{-1} \preceq_{\mathcal{K}^n} y \circ (f^{\rm soc}(y))^{-1} \\
\Longleftrightarrow\quad & g^{\rm soc}(x) \preceq_{\mathcal{K}^n} g^{\rm soc}(y),
\end{aligned}$$

where the second implication holds by setting $T = P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})$ and the first equivalence holds by the invariant property of $P(x^{-\frac{1}{2}})$ on $\mathcal{K}^n$. $\square$

Remark 4.4. *We elaborate more about Proposition 4.21. We notice that the SOC-monotonicity and SOC-concavity of f are not strong enough to guarantee the SOC-monotonicity of g. Indeed, the SOC-monotonicity and SOC-concavity only coincides with the 2-matrix-monotonicity and 2-matrix-concavity, respectively. Hence, we need stronger condition on f to assure the SOC-monotonicity of g. Another point to mention is that the condition (4.19) in Proposition 4.21(c) is a similar idea for SOC setting parallel to the following condition:*

$$C^* f(A) C \preceq f(C^*AC) \tag{4.20}$$

for any positive semidefinite A and a contraction C in the space of matrices. This inequality (4.20) plays a key role in proving matrix-monotonicity and matrix-convexity.

For more details about this condition, please refer to [73, 74]. To the contrast, it is not clear about how to define $(\cdot)^*$ associated with SOC. Nonetheless, we figure out that the condition (4.19) may act as a role like (4.20).

Proposition 4.22. *Let $f : [0, \infty) \longrightarrow (0, \infty)$ be SOC-monotone and satisfy one of the conditions in Proposition 4.21. Then, for any $x, y \in \mathcal{K}^n$, there holds*

$$\mathrm{tr}(x + y) - \mathrm{tr}(|x - y|) \leq 2\mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right), \tag{4.21}$$

where $g(t) = \frac{t}{f(t)}$ if $t > 0$, and $g(0) = 0$.

Proof. For any $x, y \in \mathcal{K}^n$, it is known that $x - y$ can be expressed as $[x - y]_+ - [x - y]_-$. Let us denote by $\mathbf{p} := [x - y]_+$ and $\mathbf{q} := [x - y]_-$. Then we have

$$x - y = \mathbf{p} - \mathbf{q} \quad \text{and} \quad |x - y| = \mathbf{p} + \mathbf{q}$$

and the inequality (4.21) is equivalent to the following

$$\mathrm{tr}(x) - \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) \leq \mathrm{tr}(\mathbf{p}).$$

Since $y + \mathbf{p} \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ and $y + \mathbf{p} = x + \mathbf{q} \succeq_{\mathcal{K}^n} x \succeq_{\mathcal{K}^n} 0$, we have $g^{\mathrm{soc}}(x) \preceq_{\mathcal{K}^n} g^{\mathrm{soc}}(y + \mathbf{p})$ and by Proposition 4.20

$$\begin{aligned}
& \mathrm{tr}(x) - \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) \\
= \; & \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(x) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) - \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) \\
\leq \; & \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y + \mathbf{p}) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) - \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) \\
= \; & \mathrm{tr}\left(f^{\mathrm{soc}}(x)^{\frac{1}{2}} \circ (g^{\mathrm{soc}}(y + \mathbf{p}) - g^{\mathrm{soc}}(y)) \circ f^{\mathrm{soc}}(x)^{\frac{1}{2}}\right) \\
\leq \; & \mathrm{tr}\left(f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}} \circ (g^{\mathrm{soc}}(y + \mathbf{p}) - g^{\mathrm{soc}}(y)) \circ f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}}\right) \\
= \; & \mathrm{tr}\left(f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y + \mathbf{p}) \circ f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}}\right) \\
& - \mathrm{tr}\left(f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(y + \mathbf{p})^{\frac{1}{2}}\right) \\
\leq \; & \mathrm{tr}(y + \mathbf{p}) - \mathrm{tr}\left(f^{\mathrm{soc}}(y)^{\frac{1}{2}} \circ g^{\mathrm{soc}}(y) \circ f^{\mathrm{soc}}(y)^{\frac{1}{2}}\right) \\
= \; & \mathrm{tr}(y + \mathbf{p}) - \mathrm{tr}(y) \\
= \; & \mathrm{tr}(\mathbf{p}).
\end{aligned}$$

Hence, we prove the assertion. $\square$

As an application we achieve the SOC version of Powers-Størmer's inequality.

Proposition 4.23. *For any $x, y \in \mathcal{K}^n$ and $0 \le \lambda \le 1$, there holds*

$$\mathrm{tr}\left(x + y - |x-y|\right) \le 2\mathrm{tr}\left(x^\lambda \circ y^{1-\lambda}\right) \le \mathrm{tr}\left(x + y + |x-y|\right).$$

Proof. (i) For the first inequality, taking $f(t) = t^\lambda$ for $0 \le \lambda \le 1$ and applying Proposition 4.22. It is known that f is matrix-monotone with $f((0,\infty)) \subseteq (0,\infty)$ and $g(t) = \frac{t}{f(t)} = t^{1-\lambda}$. Then, the inequality follows from (4.21) in Proposition 4.22.
(ii) For the second inequality, we note that

$$\begin{aligned} 0 \ &\preceq_{\mathcal{K}^n} \ x \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}, \\ 0 \ &\preceq_{\mathcal{K}^n} \ y \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}. \end{aligned}$$

Moreover, for $0 \le \lambda \le 1$, $f(t) = t^\lambda$ is SOC-monotone on $[0,\infty)$. This implies that

$$\begin{aligned} 0 \ &\preceq_{\mathcal{K}^n} \ x^\lambda \preceq_{\mathcal{K}^n} \left(\frac{x+y+|x-y|}{2}\right)^\lambda, \\ 0 \ &\preceq_{\mathcal{K}^n} \ y^{1-\lambda} \preceq_{\mathcal{K}^n} \left(\frac{x+y+|x-y|}{2}\right)^{1-\lambda}. \end{aligned}$$

Then, applying Lemma 4.3 gives

$$\mathrm{tr}\left(x^\lambda \circ y^{1-\lambda}\right) \le \mathrm{tr}\left(\frac{x+y+|x-y|}{2}\right),$$

which is the desired result. $\Box$

According to the definition of B_λ, we observe that

$$B_0(x,y) = B_1(x,y) = \frac{x+y}{2} = A_{\frac{1}{2}}(x,y).$$

This together with Proposition 4.22 leads to

$$\mathrm{tr}(x \wedge y) \le \mathrm{tr}\left(B_\lambda(x,y)\right) \le \mathrm{tr}(x \vee y).$$

In fact, we can sharpen the upper bound of $\mathrm{tr}(B_\lambda(x,y))$ as shown in the following proposition, which also shows when the maximum occurs. Moreover, the inequality (4.18) remains true for second-order cone, in the following trace version.

Proposition 4.24. *For any $x, y \in \mathcal{K}^n$ and $0 \le \lambda \le 1$, there holds*

$$2\mathrm{tr}\left(x^{\frac{1}{2}} \circ y^{\frac{1}{2}}\right) \le \mathrm{tr}\left(x^\lambda \circ y^{1-\lambda} + x^{1-\lambda} \circ y^\lambda\right) \le \mathrm{tr}(x+y),$$

which is equivalent to $\mathrm{tr}\left(x^{\frac{1}{2}} \circ y^{\frac{1}{2}}\right) \le \mathrm{tr}\left(B_\lambda(x,y)\right) \le \mathrm{tr}(A_{\frac{1}{2}}(x,y))$. *In particular,*

$$\mathrm{tr}\left(x^{1-\lambda} \circ y^\lambda\right) \le \mathrm{tr}\left(A_\lambda(x,y)\right).$$

Proof. It is clear that the inequalities hold when $\lambda = 0, 1$. Suppose that $\lambda \neq 0, 1$, we set $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$.

For the first inequality, we write $x = \xi_1 u_x^{(1)} + \xi_2 u_x^{(2)}$, $y = \mu_1 u_y^{(1)} + \mu_2 u_y^{(2)}$ by spectral decomposition (1.2)-(1.4). We note that $\xi_i, \mu_j \geq 0$ and $u_x^{(i)}, u_y^{(j)} \in \mathcal{K}^n$ for all $i, j = 1, 2$. Then

$$x^\lambda \circ y^{1-\lambda} + x^{1-\lambda} \circ y^\lambda - 2x^{\frac{1}{2}} \circ y^{\frac{1}{2}} = \sum_{i,j=1}^{2} \left(\xi_i^\lambda \mu_j^{1-\lambda} + \xi_i^{1-\lambda} \mu_j^\lambda - 2\sqrt{\xi_i \mu_j} \right) u_x^{(i)} \circ u_y^{(j)},$$

which implies

$$\begin{aligned}
& \operatorname{tr}\left(x^\lambda \circ y^{1-\lambda} + x^{1-\lambda} \circ y^\lambda - 2x^{\frac{1}{2}} \circ y^{\frac{1}{2}} \right) \\
= & \operatorname{tr}\left(\sum_{i,j=1}^{2} \left(\xi_i^\lambda \mu_j^{1-\lambda} + \xi_i^{1-\lambda} \mu_j^\lambda - 2\sqrt{\xi_i \mu_j} \right) u_x^{(i)} \circ u_y^{(j)} \right) \\
= & \sum_{i,j=1}^{2} \operatorname{tr}\left(\left(\xi_i^\lambda \mu_j^{1-\lambda} + \xi_i^{1-\lambda} \mu_j^\lambda - 2\sqrt{\xi_i \mu_j} \right) u_x^{(i)} \circ u_y^{(j)} \right) \\
= & \sum_{i,j=1}^{2} \left(\xi_i^\lambda \mu_j^{1-\lambda} + \xi_i^{1-\lambda} \mu_j^\lambda - 2\sqrt{\xi_i \mu_j} \right) \operatorname{tr}\left(u_x^{(i)} \circ u_y^{(j)} \right) \\
\geq & \ 0,
\end{aligned}$$

where the inequality holds by (4.18) and Property 1.3(d).

For the second inequality, by the trace version of Young inequality in Proposition 4.10, we have

$$\begin{aligned}
\operatorname{tr}\left(x^\lambda \circ y^{1-\lambda} \right) & \leq \operatorname{tr}\left(\frac{(x^\lambda)^p}{p} + \frac{(y^{1-\lambda})^q}{q} \right) = \operatorname{tr}\left(\frac{x}{p} + \frac{y}{q} \right), \\
\operatorname{tr}\left(x^{1-\lambda} \circ y^\lambda \right) & \leq \operatorname{tr}\left(\frac{(x^{1-\lambda})^q}{q} + \frac{(y^\lambda)^p}{p} \right) = \operatorname{tr}\left(\frac{x}{q} + \frac{y}{p} \right).
\end{aligned}$$

Adding up these two inequalities together yields the desired result. □

Chapter 5

Possible Extensions

It is known that the concept of convexity plays a central role in many applications including mathematical economics, engineering, management science, and optimization theory. Moreover, much attention has been paid to its generalization, to the associated generalization of the results previously developed for the classical convexity, and to the discovery of necessary and/or sufficient conditions for a function to have generalized convexities. Some of the known extensions are quasiconvex functions, r-convex functions [11, 151], and SOC-convex functions as introduced in Chapter 2. Other further extensions can be found in [127, 149]. For a single variable continuous, the midpoint-convex function on $\mathbb{R}$ is also a convex function. This result was generalized in [148] by relaxing continuity to lower-semicontinuity and replacing the number $\frac{1}{2}$ with an arbitrary parameter $\alpha \in (0,1)$. An analogous consequence was obtained in [112, 149] for quasiconvex functions.

To understand the main idea behind r-convex function, we recall some concepts that were independently defined by Martos [107] and Avriel [12], and has been studied by the latter author. Indeed, this concept relies on the classical definition of convex functions and some well-known results from analysis dealing with weighted means of positive numbers. Let $w = (w_1, ..., w_m) \in \mathbb{R}^m$, $q = (q_1, ..., q_m) \in \mathbb{R}^m$ be vectors whose components are positive and nonnegative numbers, respectively, such that $\sum_{i=1}^m q_i = 1$. Given the vector of weights q, the *weighted* r*-mean* of the numbers $w_1, ..., w_m$ is defined as below (see [71]):

$$M_r(w;q) = M_r(w_1, ..., w_m; q) := \begin{cases} \left(\sum_{i=1}^m q_i (w_i)^r \right)^{1/r} & \text{if} \quad r \neq 0, \\ \prod_{i=1}^m (w_i)^{q_i} & \text{if} \quad r = 0. \end{cases} \tag{5.1}$$

It is well-known from [71] that for $s > r$, there holds

$$M_s(w_1, ..., w_m; q) \geq M_r(w_1, ..., w_m; q) \tag{5.2}$$

for all $q_1, ..., q_m \geq 0$ with $\sum_{i=1}^m q_i = 1$. The r-convexity is built based on the aforementioned weighted r-mean. For a convex set $S \subseteq \mathbb{R}^n$, a real-valued function $f : S \subseteq \mathbb{R}^n \to$

$\mathbb{R}$ is said to be *r-convex* if, for any $x, y \in S$, $\lambda \in [0,1]$, $q_1 = \lambda$, $q_2 = 1 - \lambda$, $q = (q_1, q_2)$, there has

$$f(q_1 x + q_2 y) \le \ln\left\{M_r\left(e^{f(x)}, e^{f(y)}; q\right)\right\}.$$

From (5.1), it can be verified that the above inequality is equivalent to

$$f(\lambda x + (1-\lambda)y) \le \begin{cases} \ln\left[\lambda e^{rf(x)} + (1-\lambda)e^{rf(y)}\right]^{1/r} & \text{if} \quad r \ne 0, \\ \lambda f(x) + (1-\lambda) f(y) & \text{if} \quad r = 0. \end{cases} \tag{5.3}$$

Similarly, f is said to be *r-concave* on S if the inequality (5.3) is reversed. It is clear from the above definition that a real-valued function is convex (concave) if and only if it is 0-convex (0-concave). Besides, for $r < 0$ ($r > 0$), an r-convex (r-concave) function is called *superconvex* (*superconcave*); while for $r > 0$ ($r < 0$), it is called *subconvex* (*subconcave*). In addition, it can be verified that the r-convexity of f on C with $r > 0$ ($r < 0$) is equivalent to the convexity (concavity) of e^{rf} on S.

A function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *quasiconvex* on S if, for all $x, y \in S$,

$$f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}, \quad 0 \le \lambda \le 1.$$

Analogously, f is said to be *quasiconcave* on S if, for all $x, y \in S$,

$$f(\lambda x + (1-\lambda)y) \ge \min\{f(x), f(y)\}, \quad 0 \le \lambda \le 1.$$

From [71], we know that

$$\begin{aligned} \lim_{r\to\infty} M_r(w_1, ..., w_m; q) &\equiv M_\infty(w_1, ..., w_m) = \max\{w_1, ..., w_m\}, \\ \lim_{r\to-\infty} M_r(w_1, \cdots, w_m; q) &\equiv M_{-\infty}(w_1, ..., w_m) = \min\{w_1, \cdots, w_m\}. \end{aligned}$$

Then, it follows from (5.2) that $M_\infty(w_1, ..., w_m) \ge M_r(w_1, ..., w_m; q) \ge M_{-\infty}(w_1, ..., w_m)$ for every real number r. Thus, if f is r-convex on S, it is also $(+\infty)$-convex, that is, $f(\lambda x + (1-\lambda)y) \le \max\{f(x), f(y)\}$ for every $x, y \in S$ and $\lambda \in [0,1]$. Similarly, if f is r-concave on S, it is also $(-\infty)$-concave, i.e., $f(\lambda x + (1-\lambda)y) \ge \min\{f(x), f(y)\}$.

The following review some basic properties regarding r-convex function from [11] that will be used in the subsequent analysis.

Property 5.1. *Let $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$. Then, the followings hold.*

(a) *If f is r-convex (r-concave) on S, then f is also s-convex (s-concave) on S for $s > r$ ($s < r$).*

(b) *Suppose that f is twice continuously differentiable on S. For any $(x, r) \in S \times \mathbb{R}$, we define*

$$\phi(x, r) = \nabla^2 f(x) + r\nabla f(x)\nabla f(x)^T.$$

Then, f is r-convex on S if and only if ϕ is positive semidefinite for all $x \in S$.

(c) *Every r-convex (r-concave) function on a convex set S is also quasiconvex (quasiconcave) on S.*

(d) *f is r-convex if and only if $(-f)$ is $(-r)$-concave.*

(e) *Let f be r-convex (r-concave), $\alpha \in \mathbb{R}$ and $k > 0$. Then $f + \alpha$ is r-convex (r-concave) and $k \cdot f$ is $(\frac{r}{k})$-convex ($(\frac{r}{k})$-concave).*

(f) *Let $\phi, \psi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be r-convex (r-concave) and $\alpha_1, \alpha_2 > 0$. Then, the function θ defined by*

$$\theta(x) = \begin{cases} \ln\left[\alpha_1 e^{r\phi(x)} + \alpha_2 e^{r\psi(x)}\right]^{1/r} & \text{if} \quad r \neq 0, \\ \alpha_1\phi(x) + \alpha_2\psi(x) & \text{if} \quad r = 0, \end{cases}$$

is also r-convex (r-concave).

(g) *Let $\phi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ be r-convex (r-concave) such that $r \leq 0$ ($r \geq 0$) and let the real valued function ψ be nondecreasing s-convex (s-concave) on $\mathbb{R}$ with $s \in \mathbb{R}$. Then, the composite function $\theta = \psi \circ \phi$ is also s-convex (s-concave).*

(h) *$\phi : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is r-convex (r-concave) if and only if, for every $x, y \in S$, the function ψ given by*

$$\psi(\lambda) = \phi\left((1-\lambda)x + \lambda y\right)$$

is an r-convex (r-concave) function of λ for $0 \leq \lambda \leq 1$.

(i) *Let ϕ be a twice continuously differentiable real quasiconvex function on an open convex set $S \subseteq \mathbb{R}^n$. If there exists a real number r^* satisfying*

$$r^* = \sup_{x \in S,\ \|z\|=1} \frac{-z^T \nabla^2 \phi(x) z}{[z^T \nabla \phi(x)]^2} \tag{5.4}$$

whenever $z^T \nabla \phi(x) \neq 0$, then ϕ is r-convex for every $r \geq r^*$. We obtain the r-concave analog of the above theorem by replacing supremum in (5.4) by infimum.

5.1 Examples of r-functions

In this section, we try to discover some new r-convex functions which can be verified by applying Property 5.1. With these examples, we have a more complete picture about characterizations of r-convex functions. Moreover, for any given r, we also provide examples which are r-convex functions.

Example 5.1. *For any real number p, let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(t) = t^p$.*

(a) *If $p > 0$, then f is convex for $p \geq 1$, and $(+\infty)$-convex for $0 < p < 1$.*

(b) *If $p < 0$, then f is convex.*

Solution. To see this, we first note that $f'(t) = pt^{p-1}$, $f''(t) = p(p-1)t^{p-2}$ and

$$\sup_{s \cdot f'(t) \neq 0, |s|=1} \frac{-s \cdot f''(t) \cdot s}{[s \cdot f'(t)]^2} = \sup_{p \neq 0} \frac{(1-p)t^{-p}}{p} = \begin{cases} \infty & \text{if } \ 0 < p < 1, \\ 0 & \text{if } \ p > 1 \text{ or } p < 0. \end{cases}$$

Then, applying Property 5.1 yields the desired result. ■

Example 5.2. *Suppose that f is defined on $(-\frac{\pi}{2}, \frac{\pi}{2})$.*

(a) *The function $f(t) = \sin t$ is ∞-convex.*

(b) *The function $f(t) = \tan t$ is 1-convex.*

(c) *The function $f(t) = \ln(\sec t)$ is (-1)-convex.*

(d) *The function $f(t) = \ln|\sec t + \tan t|$ is 1-convex.*

Solution. (a) We note that $f'(t) = \cos t$, $f''(t) = -\sin t$, and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}, |s|=1} \frac{-s \cdot f''(t) \cdot s}{[s \cdot f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{\sin t}{\cos^2 t} = \infty.$$

Hence $f(t) = \sin t$ is ∞-convex.

(b) Using $f'(t) = \sec^2 t$, $f''(t) = 2\sec^2 t \cdot \tan t$, and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-2\sec^2 t \cdot \tan t}{\sec^4 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\sin 2t) = 1,$$

which says that $f(t) = \tan t$ is 1-convex.

(c) Note that $f'(t) = \tan t$, $f''(t) = \sec^2 t$, and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-k\sec^2 t}{\tan^2 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\csc^2 t) = -1.$$

Then, it is clear to see that $f(t) = \ln(\sec t)$ is (-1)-convex.

(d) Note that $f'(t) = \sec t$, $f''(t) = \sec t \cdot \tan t$, and

$$\sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} \frac{-\sec t \cdot \tan t}{\sec^2 t} = \sup_{-\frac{\pi}{2} < t < \frac{\pi}{2}} (-\sin t) = 1.$$

Thus, $f(t) = \ln|\sec t + \tan t|$ is 1-convex. ■

In light of Example 5.2(b)-(c) and Property 5.1(e), the next example indicates that for any given $r \in \mathbb{R}$ (no matter positive or negative), we can always construct an r-convex function accordingly. The graphs of various r-convex functions are depicted in Figure 5.1.

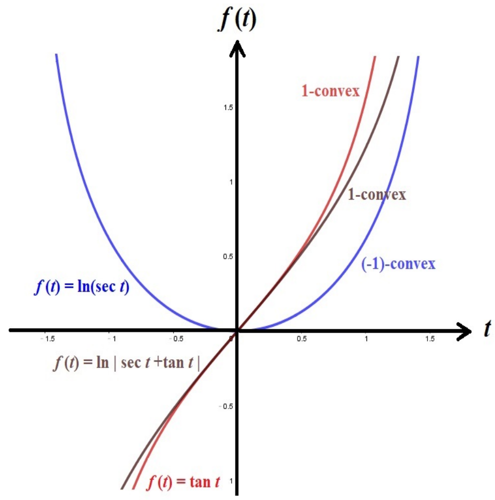


Figure 5.1: Graphs of r-convex functions with various values of r.

Example 5.3. *For any* $r \neq 0$*, let* f *be defined on* $(-\frac{\pi}{2}, \frac{\pi}{2})$.

(a) *The function* $f(t) = \dfrac{\tan t}{r}$ *is* $|r|$*-convex.*

(b) *The function* $f(t) = \dfrac{\ln(\sec t)}{r}$ *is* $(-r)$*-convex.*

Solution. (a) First, we compute that $f'(t) = \dfrac{\sec^2 t}{r}$, $f''(t) = \dfrac{2\sec^2 t \cdot \tan t}{r}$, and

$$\sup_{-\frac{\pi}{2}<t<\frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2}<t<\frac{\pi}{2}} (-r \sin 2t) = |r|.$$

This says that $f(t) = \dfrac{\tan t}{r}$ is $|r|$-convex.

(b) Similarly, from $f'(t) = \dfrac{\tan t}{r}$, $f''(t) = \dfrac{\sec^2 t}{r}$, and

$$\sup_{-\frac{\pi}{2}<t<\frac{\pi}{2}} \frac{-f''(t)}{[f'(t)]^2} = \sup_{-\frac{\pi}{2}<t<\frac{\pi}{2}} (-r \csc^2 t) = -r.$$

Then, it is easy to see that $f(t) = \dfrac{\ln(\sec t)}{r}$ is $(-r)$-convex. ■

Example 5.4. *The function* $f(x) = \frac{1}{2}\ln(\|x\|^2 + 1)$ *defined on* $\mathbb{R}^2$ *is* 1*-convex.*

Solution. For $x = (s,t) \in \mathbb{R}^2$, and any real number $r \neq 0$, we consider the function

$$\begin{aligned}
\phi(x,r) &= \nabla^2 f(x) + r\nabla f(x)\nabla f(x)^T \\
&= \frac{1}{(\|x\|^2+1)^2}\begin{bmatrix} t^2-s^2+1 & -2st \\ -2st & s^2-t^2+1\end{bmatrix} + \frac{r}{(\|x\|^2+1)^2}\begin{bmatrix} s^2 & st \\ st & t^2\end{bmatrix} \\
&= \frac{1}{(\|x\|^2+1)^2}\begin{bmatrix} (r-1)s^2+t^2+1 & (r-2)st \\ (r-2)st & s^2+(r-1)t^2+1\end{bmatrix}.
\end{aligned}$$

Applying Property 5.1(b), we know that f is r-convex if and only if ϕ is positive semidefinite, which is equivalent to

$$(r-1)s^2+t^2+1 \geq 0 \tag{5.5}$$

$$\begin{vmatrix} (r-1)s^2+t^2+1 & (r-2)st \\ (r-2)st & s^2+(r-1)t^2+1\end{vmatrix} \geq 0. \tag{5.6}$$

It is easy to verify the inequality (5.5) holds for all $x \in \mathbb{R}^2$ if and only if $r \geq 1$. Moreover, we note that

$$\begin{aligned}
& \begin{vmatrix} (r-1)s^2+t^2+1 & (r-2)st \\ (r-2)st & s^2+(r-1)t^2+1\end{vmatrix} \geq 0 \\
\iff\ & s^2t^2+s^2+t^2+1+(r-1)^2s^2t^2+(r-1)(s^4+s^2+t^4+t^2)-(r-2)^2s^2t^2 \geq 0 \\
\iff\ & s^2+t^2+1+(2r-2)s^2t^2+(r-1)(s^4+s^2+t^4+t^2) \geq 0,
\end{aligned}$$

and hence the inequality (5.6) holds for all $x \in \mathbb{R}^2$ whenever $r \geq 1$. Thus, we conclude by Property 5.1(b) that f is 1-convex on $\mathbb{R}^2$. ■

5.2 SOC-r-convex functions

In this section, we define the so-called SOC-r-convex functions [77], which can be viewed as the natural extension of r-convex functions to the setting associated with SOC.

Lemma 5.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = e^t$ and $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. If $x_1 - y_1 \geq \|x_2\| + \|y_2\|$, then $e^x \succeq_{\mathcal{K}^n} e^y$. In particular, if $x \in \mathcal{K}^n$, then $e^x \succeq_{\mathcal{K}^n} e^{(0,0)}$.*

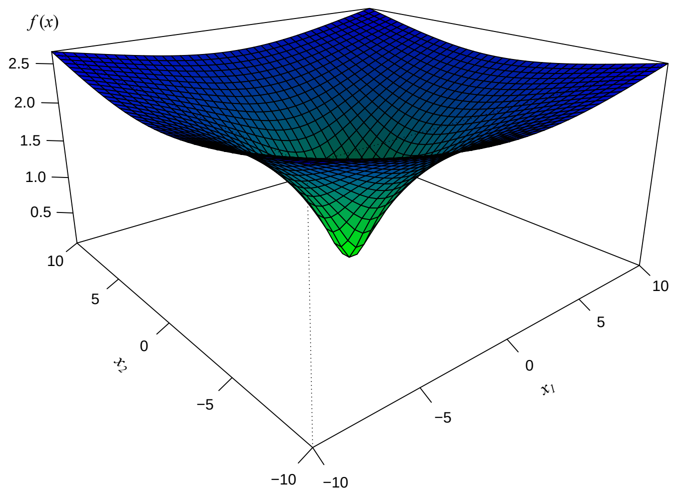


Figure 5.2: Graphs of 1-convex function $f(x) = \frac{1}{2}\ln(\|x\|^2 + 1)$.

Proof. First, we analyze that

$$\begin{aligned}
& e^x \succeq_{\mathcal{K}^n} e^y \\
\iff\ & e^{x_1}\cosh(\|x_2\|) - e^{y_1}\cosh(\|y_2\|) \geq \left\| e^{x_1}\sinh(\|x_2\|)\frac{x_2}{\|x_2\|} - e^{y_1}\sinh(\|y_2\|)\frac{y_2}{\|y_2\|}\right\| \\
\iff\ & \left[e^{x_1}\cosh(\|x_2\|) - e^{y_1}\cosh(\|y_2\|)\right]^2 - \left\| e^{x_1}\sinh(\|x_2\|)\frac{x_2}{\|x_2\|} - e^{y_1}\sinh(\|y_2\|)\frac{y_2}{\|y_2\|}\right\|^2 \\
& = e^{2x_1} + e^{2y_1} - 2e^{x_1+y_1}\left[\cosh(\|x_2\|)\cosh(\|y_2\|) - \sinh(\|x_2\|)\sinh(\|y_2\|)\frac{\langle x_2, y_2\rangle}{\|x_2\|\|y_2\|}\right] \\
& \geq 0.
\end{aligned}$$

Looking into the above terms and the goal, it suffices to show that

$$e^{2x_1} + e^{2y_1} - 2e^{x_1+y_1}\cosh(\|x_2\| + \|y_2\|) \geq 0.$$

This is true under the assumption because

$$\begin{aligned}
& e^{2x_1} + e^{2y_1} - 2e^{x_1+y_1}\cosh(\|x_2\| + \|y_2\|) \geq 0 \\
\iff\ & \cosh(\|x_2\| + \|y_2\|) \leq \frac{e^{2x_1} + e^{2y_1}}{2e^{x_1+y_1}} = \frac{e^{x_1-y_1} + e^{y_1-x_1}}{2} = \cosh(x_1 - y_1) \\
\iff\ & x_1 - y_1 \geq \|x_2\| + \|y_2\|.
\end{aligned}$$

Thus, the proof is complete. $\square$

In general, to verify the SOC-convexity of e^t, we observe that the following fact

$$0 \prec_{\mathcal{K}^n} e^{rf^{\rm soc}(\lambda x+(1-\lambda)y)} \preceq_{\mathcal{K}^n} w \quad \Longrightarrow \quad rf^{\rm soc}(\lambda x+(1-\lambda)y) \preceq_{\mathcal{K}^n} \ln(w)$$

is important and often needed. Note for $x_2 \neq 0$, we also have some observations as below.

(a) $e^x \succ_{\mathcal{K}^n} 0 \iff \cosh(\|x_2\|) \geq |\sinh(\|x_2\|)| \iff e^{-\|x_2\|} > 0$.

(b) $0 \prec_{\mathcal{K}^n} \ln(x) \iff \ln(x_1^2 - \|x_2\|^2) > \left|\ln\left(\frac{x_1+\|x_2\|}{x_1-\|x_2\|}\right)\right| \iff \ln(x_1 - \|x_2\|) > 0 \iff x_1 - \|x_2\| > 1$. Hence $(1,0) \prec_{\mathcal{K}^n} x$ implies $0 \prec_{\mathcal{K}^n} \ln(x)$.

(c) $\ln(1,0) = (0,0)$ and $e^{(0,0)} = (1,0)$.

Definition 5.1. *Suppose that $r \in \mathbb{R}$ and $f : C \subseteq \mathbb{R} \to \mathbb{R}$ where C is a convex subset of $\mathbb{R}$. Let $f^{\rm soc} : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be its corresponding SOC-function defined as in (1.8). The function f is said to be SOC-r-convex of order n on C if, for $x, y \in S$ and $\lambda \in [0,1]$, there holds*

$$f^{\rm soc}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \begin{cases} \frac{1}{r}\ln\left(\lambda e^{rf^{\rm soc}(x)} + (1-\lambda)e^{rf^{\rm soc}(y)}\right) & r \neq 0, \\ \lambda f^{\rm soc}(x) + (1-\lambda)f^{\rm soc}(y) & r = 0. \end{cases} \tag{5.7}$$

Similarly, f is said to be SOC-r-concave of order n on C if the inequality (5.7) is reversed. We say f is SOC-r-convex (respectively, SOC-r-concave) on C if f is SOC-r-convex of all order n (respectively, SOC-r-concave of all order n) on C.

It is clear from the above definition that a real function is SOC-convex (SOC-concave) if and only if it is SOC-0-convex (SOC-0-concave). In addition, a function f is SOC-r-convex if and only if $-f$ is SOC-$(-r)$-concave. From [11, Theorem 4.1], it is shown that $\phi : \mathbb{R} \to \mathbb{R}$ is r-convex with $r \neq 0$ if and only if $e^{r\phi}$ is convex whenever $r > 0$ and concave whenever $r < 0$. However, we observe that the exponential function e^t is not SOC-convex for $n \geq 3$ by Example 2.11. This is a hurdle to build parallel result for general n in the setting of SOC case. As seen in Proposition 5.3, the parallel result is true only for $n = 2$. Indeed, for $n \geq 3$, only one direction holds which can be viewed as a weaker version of [11, Theorem 4.1].

Proposition 5.1. *Let $f : [0,\infty) \to [0,\infty)$ be continuous. If f is SOC-r-concave with $r \geq 0$, then f is SOC-monotone.*

Proof. For any $0 < \lambda < 1$, we can write $\lambda x = \lambda y + \frac{(1-\lambda)\lambda}{(1-\lambda)}(x-y)$.

(i) If $r = 0$, then f is SOC-concave. Hence, it is SOC-monotone by Proposition 2.8.

(ii) If $r > 0$, then

$$\begin{aligned} f^{\mathrm{soc}}(\lambda x) &\succeq_{\mathcal{K}^n} \frac{1}{r}\ln\left(\lambda e^{rf^{\mathrm{soc}}(y)} + (1-\lambda)e^{rf^{\mathrm{soc}}(\frac{\lambda}{1-\lambda}(x-y))}\right) \\ &\succeq_{\mathcal{K}^n} \frac{1}{r}\ln\left(\lambda e^{r(0,0)} + (1-\lambda)e^{r(0,0)}\right) \\ &= \frac{1}{r}\ln\left(\lambda(1,0) + (1-\lambda)(1,0)\right) \\ &= 0, \end{aligned}$$

where the second inequality is due to $x - y \succeq_{\mathcal{K}^n} 0$, Lemma 5.1 and Examples 2.9-2.10. Letting $\lambda \to 1$, we obtain that $f^{\mathrm{soc}}(x) \succeq_{\mathcal{K}^n} f^{\mathrm{soc}}(y)$, which says that f is SOC-monotone. □

In fact, in light of Lemma 5.1 and Examples 2.9-2.10, we have the following Lemma which is useful for subsequent analysis.

Lemma 5.2. *Let $z \in \mathbb{R}^n$ and $w \in \mathrm{int}(\mathcal{K}^n)$. Then, the following hold.*

(a) *For $n = 2$ and $r > 0$, $z \preceq_{\mathcal{K}^n} \ln(w)/r \Longleftrightarrow rz \preceq_{\mathcal{K}^n} \ln(w) \Longleftrightarrow e^{rz} \preceq_{\mathcal{K}^n} w$.*

(b) *For $n = 2$ and $r < 0$, $z \preceq_{\mathcal{K}^n} \ln(w)/r \Longleftrightarrow rz \succeq_{\mathcal{K}^n} \ln(w) \Longleftrightarrow e^{rz} \succeq_{\mathcal{K}^n} w$.*

(c) *For $n \geq 2$, if $e^{rz} \preceq_{\mathcal{K}^n} w$, then $rz \preceq_{\mathcal{K}^n} \ln(w)$.*

Proposition 5.2. *For $n = 2$ and let $f : \mathbb{R} \to \mathbb{R}$. Then, the following hold.*

(a) *The function $f(t) = t$ is SOC-r-convex (SOC-r-concave) on $\mathbb{R}$ for $r > 0$ ($r < 0$).*

(b) *If f is SOC-convex, then f is SOC-r-convex (SOC-r-concave) for $r > 0$ ($r < 0$).*

Proof. (a) For $r > 0$, $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$, we note that the corresponding vector-valued SOC-function of $f(t) = t$ is $f^{\mathrm{soc}}(x) = x$. Therefore, to prove the desired result, we need to verify that

$$f^{\mathrm{soc}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \frac{1}{r}\ln\left(\lambda e^{rf^{\mathrm{soc}}(x)} + (1-\lambda)e^{rf^{\mathrm{soc}}(y)}\right).$$

To this end, we see that

$$\begin{aligned} & \lambda x + (1-\lambda)y \preceq_{\mathcal{K}^n} \frac{1}{r}\ln\left(\lambda e^{rx} + (1-\lambda)e^{ry}\right) \\ \Longleftrightarrow\ & \lambda rx + (1-\lambda)ry \preceq_{\mathcal{K}^n} \ln\left(\lambda e^{rx} + (1-\lambda)e^{ry}\right) \\ \Longleftrightarrow\ & e^{\lambda rx + (1-\lambda)ry} \preceq_{\mathcal{K}^n} \lambda e^{rx} + (1-\lambda)e^{ry}, \end{aligned}$$

where the first "$\Longleftrightarrow$" is true due to Lemma 5.2, whereas the second "$\Longleftrightarrow$" holds because e^t and $\ln t$ are SOC-monotone of order 2 by Lemma 5.1 and Example 2.9. Then, using the fact that e^t is SOC-convex of order 2 gives the desired result.

(b) For any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, it can be verified that

$$\begin{aligned} f^{\text{soc}}(\lambda x + (1-\lambda)y) &\preceq_{\mathcal{K}^n} \lambda f^{\text{soc}}(x) + (1-\lambda) f^{\text{soc}}(y) \\ &\preceq_{\mathcal{K}^n} \frac{1}{r} \ln \left(\lambda e^{r f^{\text{soc}}(x)} + (1-\lambda) e^{r f^{\text{soc}}(y)} \right), \end{aligned}$$

where the second inequality holds according to the proof of (a). Thus, the desired result follows. □

Proposition 5.3. *Let $f : \mathbb{R} \to \mathbb{R}$. Then f is SOC-r-convex if e^{rf} is SOC-convex (SOC-concave) for $n \geq 2$ and $r > 0$ ($r < 0$). For $n = 2$, we can replace "if" by "if and only if".*

Proof. Suppose that e^{rf} is SOC-convex. For any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, using that fact that $\ln t$ is SOC-monotone (see Example 2.13) yields

$$\begin{aligned} & e^{r f^{\text{soc}}(\lambda x + (1-\lambda)y)} \preceq_{\mathcal{K}^n} \lambda e^{r f^{\text{soc}}(x)} + (1-\lambda) e^{r f^{\text{soc}}(y)} \\ \Longrightarrow \quad & r f^{\text{soc}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \ln \left(\lambda e^{r f^{\text{soc}}(x)} + (1-\lambda) e^{r f^{\text{soc}}(y)} \right) \\ \Longleftrightarrow \quad & f^{\text{soc}}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \frac{1}{r} \ln \left(\lambda e^{r f^{\text{soc}}(x)} + (1-\lambda) e^{r f^{\text{soc}}(y)} \right). \end{aligned}$$

When $n = 2$, e^t is SOC-monotone as well, which implies that the "$\Longrightarrow$" can be replaced by "$\Longleftrightarrow$". Thus, the proof is complete. □

Combining with Proposition 2.16, we can characterize the SOC-r-convexity as follows.

Proposition 5.4. *Let $f \in C^{(2)}(J)$ with J being an open interval in $\mathbb{R}$ and dom$(f^{soc}) \subseteq \mathbb{R}^n$. Then, for $r > 0$, the followings hold.*

(a) *f is SOC-r-convex of order 2 if and only if e^{rf} is convex;*

(b) *f is SOC-r-convex of order $n \geq 3$ if e^{rf} is convex and satisfies the inequality (2.36). for any $t_0, t \in J$ and $t_0 \neq t$.*

Next, we present several examples of SOC-r-convex and SOC-r-concave functions of order 2. For examples of SOC-r-convex and SOC-r-concave functions (of order n), we are still unable to discover them.

Example 5.5. *For $n = 2$, the following hold.*

(a) *The function $f(t) = t^2$ is SOC-r-convex on $\mathbb{R}$ for $r \geq 0$.*

(b) *The function $f(t) = t^3$ is SOC-r-convex on $[0, \infty)$ for $r > 0$, while it is SOC-r-concave on $(-\infty, 0]$ for $r < 0$.*

(c) *The function* $f(t) = \dfrac{1}{t}$ *is SOC-r-convex on* $[-r/2, 0)$ *or* $(0, \infty)$ *for* $r > 0$*, while it is SOC-r-concave on* $(-\infty, 0)$ *or* $(0, -r/2]$ *for* $r < 0$.

(d) *The function* $f(t) = \sqrt{t}$ *is SOC-r-convex on* $[1/r^2, \infty)$ *for* $r > 0$*, while it is SOC-r-concave on* $[0, \infty)$ *for* $r < 0$.

(e) *The function* $f(t) = \ln t$ *is SOC-r-convex (SOC-r-concave) on* $(0, \infty)$ *for* $r > 0$ $(r < 0)$.

Solution. (a) First, we denote $h(t) := e^{rt^2}$. Then, we have $h'(t) = 2rte^{rt^2}$ and $h''(t) = (1 + 2rt^2)2re^{rt^2}$. We know h is convex if and only if $h''(t) \geq 0$. Thus, the desired result holds by applying Proposition 2.16 and Proposition 5.4. The arguments for other cases are similar and we omit them. ■

5.3 SOC-quasiconvex Functions

In this section, we define the so-called SOC-quasiconvex functions which is a natural extension of quasiconvex functions to the setting associated with second-order cone.

Recall that a function $f : S \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *quasiconvex* on S if, for any $x, y \in S$ and $0 \leq \lambda \leq 1$, there has

$$f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.$$

We point out that the relation $\succeq_{\mathcal{K}^n}$ is not a linear ordering. Hence, it is not possible to compare any two vectors (elements) via $\succeq_{\mathcal{K}^n}$. Nonetheless, we note that

$$\max\{a, b\} = b + [a - b]_+ = \frac{1}{2}(a + b + |a - b|), \quad \text{for any } a, b \in \mathbb{R}.$$

This motivates us to define SOC-quasiconvex functions in the setting of second-order cone.

Definition 5.2. *Let* $f : C \subseteq \mathbb{R} \to \mathbb{R}$ *and* $0 \leq \lambda \leq 1$. *The function* f *is said to be SOC-quasiconvex of order* n *on* C *if, for any* $x, y \in \mathbb{R}^n$*, there has*

$$f^{\rm soc}(\lambda x + (1 - \lambda)y) \preceq_{\mathcal{K}^n} f^{\rm soc}(y) + [f^{\rm soc}(x) - f^{\rm soc}(y)]_+ ,$$

where

$$\begin{aligned} & f^{\rm soc}(y) + [f^{\rm soc}(x) - f^{\rm soc}(y)]_+ \\ = & \left\{ \begin{array}{cl} f^{\rm soc}(x) & \text{if} \quad f^{\rm soc}(x) \succeq_{\mathcal{K}^n} f^{\rm soc}(y), \\ f^{\rm soc}(y) & \text{if} \quad f^{\rm soc}(x) \prec_{\mathcal{K}^n} f^{\rm soc}(y), \\ \frac{1}{2}\left(f^{\rm soc}(x) + f^{\rm soc}(y) + |f^{\rm soc}(x) - f^{\rm soc}(y)|\right) & \text{if} \quad f^{\rm soc}(x) - f^{\rm soc}(y) \notin \mathcal{K}^n \cup (-\mathcal{K}^n). \end{array} \right. \end{aligned}$$

Similarly, f is said to be SOC-quasiconcave of order n if

$$f^{\rm soc}(\lambda x + (1-\lambda)y) \succeq_{\mathcal{K}^n} f^{\rm soc}(x) - \left[f^{\rm soc}(x) - f^{\rm soc}(y)\right]_+ .$$

The function f is called SOC-quasiconvex (SOC-quasiconcave) if it is SOC-quasiconvex of all order n (SOC-quasiconcave of all order n).

Proposition 5.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be $f(t) = t$. Then, f is SOC-quasiconvex on $\mathbb{R}$.*

Proof. First, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, and $0 \le \lambda \le 1$, we have

$$\begin{aligned} f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(x) &\iff (1-\lambda) f^{\rm soc}(y) \preceq_{\mathcal{K}^n} (1-\lambda) f^{\rm soc}(x) \\ &\iff \lambda f^{\rm soc}(x) + (1-\lambda) f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(x). \end{aligned}$$

Recall that the corresponding SOC-function of $f(t) = t$ is $f^{\rm soc}(x) = x$. Thus, for all $x \in \mathbb{R}^n$, this implies $f^{\rm soc}(\lambda x + (1-\lambda)y) = \lambda f^{\rm soc}(x) + (1-\lambda) f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(x)$ under this case: $f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(x)$. The argument is similar to the case of $f^{\rm soc}(x) \preceq_{\mathcal{K}^n} f^{\rm soc}(y)$. Hence, it remains to consider the case of $f^{\rm soc}(x) - f^{\rm soc}(y) \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, i.e., it suffices to show that

$$\lambda f^{\rm soc}(x) + (1-\lambda) f^{\rm soc}(y) \preceq_{\mathcal{K}^n} \frac{1}{2}\left(f^{\rm soc}(x) + f^{\rm soc}(y) + |f^{\rm soc}(x) - f^{\rm soc}(y)|\right).$$

To this end, we note that

$$|f^{\rm soc}(x) - f^{\rm soc}(y)| \succeq_{\mathcal{K}^n} f^{\rm soc}(x) - f^{\rm soc}(y) \quad \text{and} \quad |f^{\rm soc}(x) - f^{\rm soc}(y)| \succeq_{\mathcal{K}^n} f^{\rm soc}(y) - f^{\rm soc}(x),$$

which respectively implies

$$\frac{1}{2}\left(f^{\rm soc}(x) + f^{\rm soc}(y) + |f^{\rm soc}(x) - f^{\rm soc}(y)|\right) \succeq_{\mathcal{K}^n} x, \tag{5.8}$$

$$\frac{1}{2}\left(f^{\rm soc}(x) + f^{\rm soc}(y) + |f^{\rm soc}(x) - f^{\rm soc}(y)|\right) \succeq_{\mathcal{K}^n} y. \tag{5.9}$$

Then, adding up (5.8) $\times\lambda$ and (5.9) $\times(1-\lambda)$ yields the desired result. □

Proposition 5.6. *If $f : C \subseteq \mathbb{R} \to \mathbb{R}$ is SOC-convex on C, then f is also SOC-quasiconvex on C.*

Proof. For any $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$, it can be verified that

$$f^{\rm soc}(\lambda x + (1-\lambda)y) \preceq_{\mathcal{K}^n} \lambda f^{\rm soc}(x) + (1-\lambda) f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(y) + \left[f^{\rm soc}(x) - f^{\rm soc}(y)\right]_+ ,$$

where the second inequality holds according to the proof of Proposition 5.5. Thus, the desired result follows. □

From Proposition 5.6, we can easily construct examples of SOC-quasiconvex functions. More specifically, all the SOC-convex functions which were verified in [42] are SOC-quasiconvex functions, for instances, t^2 on $\mathbb{R}$, and t^3, $\frac{1}{t}$, $t^{1/2}$ on $(0, \infty)$. Nonetheless, the characterizations of SOC-quasiconvex functions are very limited, more investigations are desired.

Bibliography

[1] F. Alizadeh and D. Goldfarb, *Seond-order cone programming*, Mathematical Programming, vol. 95, pp. 3-51, 2003.

[2] E. D. Andersen, C. Roos, and T. Terlaky, *On implementing a primal-dual interior-point method for conic quadratic optimization*, Mathematical Programming, vol. 95, pp. 249-277, 2003.

[3] T. Ando, *Matrix Young inequalities*, Operator Theory: Advances and Applications vol. 75, pp. 33-38, 1995.

[4] K.M. Audenaert, J. Calsamiglia, R. Muoz-Tapia, E. Bagan, L. Masanes, A. Acin, and F. Verstraete, *Discriminating states: The quantum Chernoff bound*, Physical Review Letters, vol. 98, 160-501, 2007.

[5] J.S. Aujla and H.L. Vasudeva, *Convex and monotone operator functions*, Annales Polonici Mathematici, vol. 62, pp. 1-11, 1995.

[6] A. Auslender, *Penalty and barrier methods: a unified framework*, SIAM Journal on Optimization, vol. 10, pp. 211-230, 1999.

[7] A. Auslender, *Variational inequalities over the cone of semidefinite positive symmetric matrices and over the Lorentz cone*, Optimization Methods and Software, vol. 18, pp. 359-376, 2003.

[8] A. Auslender and M. Haddou, *An interior-proximal method for convex linearly constrained problems and its extension to variational inequalities*, Mathematical Programming, vol. 71, pp. 77-100, 1995.

[9] A. Auslender and H. Ramirez, *Penalty and barrier methods for convex semidefinite programming*, Mathematical Methods of Operations Research, vol. 63, pp. 195-219, 2006.

[10] A. Auslender and M. Teboulle, *Interior gradient and proximal methods for convex and conic optimization*, SIAM Journal on Optimization, vol. 16, pp. 697-725, 2006.

[11] M. Avriel, *r-convex functions*, Mathematical Programming, vol. 2, pp. 309-323, 1972.

[12] M. Avriel, *Solution of certain nonlinear programs involving r-convex functions* Journal of Optimization Theory and Applications, vol. 11, pp. 159-174, 1973.

[13] J.S. Aujla and H.L. Vasudeva, *Convex and monotone operator functions*, Annales Polonici Mathematici, vol. 62, pp. 1-11, 1995.

[14] M. Baes, *Convexity and differentiability properties of spectral functions and spectral mappings on Euclidean Jordan algebras*, Linear Algebra and Its Applications, vol. 422, pp. 664-700, 2007.

[15] Y.-Q. Bai and G. Q. Wang, *Primal-dual interior-point algorithms for second-order cone optimization based on a new parametric kernel function*, Acta Mathematica Sinica, vol. 23, pp. 2027-2042, 2007.

[16] H. Bauschke, O. Güler, A.S. Lewis, and S. Sendow, *Hyperbolic polynomial and convex analysis*, Canadian Journal of Mathematics, vol. 53, pp. 470-488, 2001.

[17] M.S. Bazaraa, H.D. Sherali, and C.M. Shetty, *Nonlinear Programming*, John Wiley and Sons, 3rd edition, 2006.

[18] E.F. Beckenbach and R. Bellman, *Inequalities*, Springer, Berlin-Gottingen, 1961.

[19] H. Y. Benson and R. J. Vanderbei, *Solving problems with semidefinite and related constraints using interior-point methods for nonlinear programming*, Mathematical Programming, vol. 95, pp. 279-302, 2003.

[20] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms and Engineering Applications*, MPS-SIAM Series on Optimization, SIAM, Philadelphia, USA, 2001.

[21] D.P. Bertsekas, *Nonlinear Programming*, 2nd edition, Athena Scientific, Massachusetts, 1999.

[22] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[23] R. Bhatia, *Positive definite matrices*, Princeton University Press, 2005.

[24] R. Bhatia, *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra and Its Applications, vol. 413, pp. 355-363, 2006.

[25] R. Bhatia and C. Davis, *More matrix forms of the arithmeticgeometric mean inequality*, SIAM Journal on Matrix Analysis and Applications, vol. 14, pp. 132-136, 1993.

[26] R. Bhatia and F. Kittaneh, *Notes on matrix arithmeticgeometric mean inequalities*, Linear Algebra and Its Applications, vol. 308, pp. 203-211, 2000.

[27] P. Bhatia and K.P. Parthasarathy, *Positive definite functions and operator inequalities*, Bulletin of the London Mathematical Society, vol. 32, pp. 214-228, 2000.

[28] C. Bhattacharyya, *Second-order cone programming formulation for feature selection*, Journal of Machine Learning Research, vol. 5, pp. 1417-1433, 2004.

[29] J.-F. Bonnans, H.C. Ramírez, *Perturbation analysis of second-order cone programming problems*, Mathematical Programming, vol. 104, pp. 205-227, 2005.

[30] J.M. Borwein and A.S. Lewis, *Convex analysis and nonlinear optimization: theory and examples*, Springer-Verlag, New York, 2000.

[31] J. Brinkhuis, Z.-Q. Luo, and S. Zhang, *Matrix convex functions with applications to weighted centers for semi-definite programming*, submitted manuscript, 2006.

[32] P.S. Bullen, *Handbook of Means and Their Inequalities*, Mathematics and Its Applications, volume 560, Kluwer Academic Publishers, 2003.

[33] B.C. Carlson, *Some inequalities for hypergeometric functions*, Proceedings of the American Mathematical Society, vol. 17, pp. 32-39, 1966.

[34] Y. Censor and A. Lent, *An interval row action method for interval convex programming*, Journal of Optimization Theory and Applications, vol. 34, pp. 321-353, 1981.

[35] Y. Censor and S. A. Zenios, *The proximal minimization algorithm with D-functions*, Journal of Optimization Theory and Applications, vol. 73, pp. 451-464, 1992.

[36] Y.-L. Chang and J.-S. Chen, *Convexity of symmetric cone trace functions in Euclidean Jordan algebras*, Journal of Nonlinear and Convex Analysis, vol. 14, pp. 53-61, 2013.

[37] Y.-L. Chang, C.-H. Huang, J.-S. Chen, and C.-C. Hu, *Some inequalities for means defined on the Lorentz cone*, *Mathematical Inequalities and Applications*, vol. 21, no. 4, pp. 1015-1028, 2018.

[38] Y.-L. Chang and C.-Y. Yang, *Some useful inequalities via trace function method in Euclidean Jordan algebras*, Numerical Algebra, Control and Optimization, vol. 4, pp. 39-48, 2014.

[39] B. Chen and P. Harker, *A non-interior point continuation method for linear complementarity problems*, SIAM Journal on Matrix Analysis and Application, vol. 14, pp. 1168-1190, 1993.

[40] G. Chen and M. Teboulle, *Convergence analysis of a proximal-like minimization algorithm using Bregman functions*, SIAM Journal on Optimization, vol. 3, pp. 538-543, 1993.

[41] J.-S. Chen, *Alternative proofs for some results of vector-valued functions associated with second-order cone*, Journal of Nonlinear and Convex Analysis, vol. 6, no. 2, pp. 297-325, 2005.

[42] J.-S. Chen, *The convex and monotone functions associated with second-order cone*, Optimization, vol. 55, pp. 363-385, 2006.

[43] J.-S. Chen, *A new merit function and its related properties for the second-order cone complementarity problem*, Pacific Journal of Optimization, vol. 2, pp. 167-179, 2006.

[44] J.-S. Chen, *Two classes of merit functions for the second-order cone complementarity problem*, Mathematical Methods of Operations Research, vol. 64, no. 3, pp. 495-519, 2006.

[45] J.-S. Chen, X. Chen, S.-H. Pan, and J. Zhang, *Some characterizations for SOC-monotone and SOC-convex functions*, Journal of Global Optimization, vol. 45, pp. 259-279, 2009.

[46] J.-S. Chen, X. Chen, and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Mathematical Programming, vol. 101, pp. 95-117, 2004.

[47] J.-S. Chen, T.-K. Liao, and S.-H. Pan, *Using Schur Complement Theorem to prove convexity of some SOC-functions*, Journal of Nonlinear and Convex Analysis, vol. 13, no. 3, pp. 421-431, 2012.

[48] J.-S. Chen and S.-H. Pan, *A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs*, Pacific Journal of Optimization, vol. 8, no. 1, pp. 33-74, 2012.

[49] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of the second-order cone complementarity problem*, Mathematical Programming, vol. 104, pp. 293-327, 2005.

[50] X. Chen and P. Tseng, *Non-interior continuation methods for solving semidefinite complementarity problems*, Mathematical Programming, vol. 95, pp. 431-474, 2003.

[51] X. Chen, H. Qi, and P. Tseng, *Analysis of nonsmooth symmetric-matrix-valued functions with applications to semidefinite complementarity problems*, SIAM Journal on Optimization, vol. 13, pp. 960-985, 2003.

[52] X.D. Chen, D. Sun, and J. Sun, *Complementarity functions and numerical experiments for second-order cone complementarity problems*, Computational Optimization and Applications, vol. 25, pp. 39-56, 2003.

[53] Y.-Y. Chiang, S.-H. Pan, and J.-S. Pan, *A merit function method for infinite-dimensional SOCCPs*, Journal of Mathematical Analysis and Applications, vol. 383, pp. 159-178, 2011.

[54] F.H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[55] A. DePierro and A. Iusem, *A relaxed version of Bregman's method for convex programming*, Journal of Optimization Theory and Applications, vol. 5, pp. 421-440, 1986.

[56] C. Ding, D. Sun, and K.-C. Toh, *An introduction to a class of matrix cone programming*, Mathematical Programming, vol. 144, pp. 141-179, 2014.

[57] C. Ding, D. Sun, J. Sun, and K.-C. Toh, *Spectral operators of matrices*, Mathematical Programming, vol. 168, pp. 509-531, 2018.

[58] M. Doljansky and M. Teboulle, *An interior proximal algorithm and the exponential multiplier method for semidefinite programming*, SIAM Journal on Optimization, vol. 9, pp. 1-13, 1998.

[59] W. Donoghue, *Monotone matrix functions and analytic continuation*, Springer, Berlin, Heidelberg, New York, 1974.

[60] J. Eckstein, *Nonlinear proximal point algorithms using Bregman functions with applications to convex programming*, Mathematics of Operations Research, vol. 18, pp. 202-226, 1993.

[61] P. P. B. Eggermont, *Multiplicative iterative algorithms for convex programming*, Linear Algebra and Its Applications, vol. 130, pp. 25-42, 1990.

[62] J. Faraut and A. Koranyi, *Analysis on Symmetric cones*, Oxford Mathematical Monographs, Oxford University Press, New York, 1994.

[63] A. Fischer, *Solution of monotone complementarity problems with locally Lipschitzian functions*, Mathematical Programming, vol. 76, pp. 513-532, 1997.

[64] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second-order cone complementarity problems*, SIAM Journal on Optimization, vol. 12, pp. 436-460, 2002.

[65] E. Galewska and M. Galewski, *r-convex transformability in nonlinear programming problems*, Commentationes Mathematicae Universitatis Carolinae, vol. 46, pp. 555-565, 2005.

[66] M. Hajja, P.S. Bullen, J. Matkowski, E. Neuman, and S. Simic, *Means and Their Inequalities*, International Journal of Mathematics and Mathematical Sciences, vol. 2013, 2013.

[67] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 2nd edition, London, 1952.

[68] F. Hansen and G.K. Pedersen, *Jensen's inequality for operators and Löwner's theorem*, Mathematische Annalen, vol. 258, pp. 229-241, 1982.

[69] F. Hansen and J. Tomiyama, *Differential analysis of matrix convex functions*, Linear Algebra and Its Applications, vol. 420, pp. 102-116, 2007.

[70] F. Hansen, G.-X. Ji, and J. Tomiyama, *Gaps between classes of matrix monotone functions*, Bulletin of the London Mathematical Society, vol. 36, pp. 53-58, 2004.

[71] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, 1967.

[72] S. Hayashi, N. Yamashita, and M. Fukushima, *A combined smoothing and regularization method for monotone second-order cone complementarity problems*, SIAM Journal on Optimization, vol. 15, pp. 593-615, 2005.

[73] D.T. Hoa, H. Osaka, and H.M. Toan, *On generalized Powers-Størmers inequality*, Linear Algebra and Its Applications, vol. 438, pp. 242-249, 2013.

[74] D.T. Hoa, H. Osaka, and H.M. Toan, *Characterization of operator monotone functions by Powers-Størmer type inequalities*, Linear and Multilinear Algebra, vol. 63, pp. 1577-1589, 2015.

[75] R.A. Horn and C.R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1985.

[76] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Press, Cambridge, United Kingdom, 1991.

[77] C.-H. Huang, H.-L. Huang, and J.-S. Chen, *Examples of r-convex functions and characterizations of r-convex functions associated with second-order cone*, Linear and Nonlinear Analysis, vol. 3, no. 3, pp. 367-384, 2017.

[78] C.-H. Huang, Y.-L. Chang, and J.-S. Chen, *Some inequalities on weighted means and traces defined on second-order cone*, submitted manuscript, 2018.

[79] C.-H. Huang, J.-S. Chen, and C.-C. Hu, *Trace versions of Young inequality associated with second-order cone and its applications*, submitted manuscript, 2018.

[80] G. Huang, S. Song, J.N.D. Gupta, and C. Wu, *A second-order cone programming approach for semi-supervised learning*, Pattern Recognition, vol. 46, pp. 3548-3558, 2013.

[81] A.N. Iusem, *Some properties of generalized proximal point methods for quadratic and linear programming*, Journal of Optimization Theory and Applications, vol. 85, pp. 593-612, 1995.

[82] A.N. Iusem, B.F. Svaiter and M. Teboulle, *Entropy-like proximal methods in convex programming*, Mathematics of Operations Research, vol. 19, pp. 790-814, 1994.

[83] A.N. Iusem and M. Teboulle, *Convergence rate analysis of nonquadratic proximal and augmented Lagrangian methods for convex and linear programming*, Mathematics of Operations Research, vol. 20, pp. 657-677, 1995.

[84] A.N. Iusem, B.F. Svaiter, and J.X. da Cruz Neto, *Central paths, generalzied proximal point methods and Cauchy trajectories in Riemannian manifolds*, SIAM Journal on Control and Optimization, vol. 2, pp. 566-588, 1999.

[85] C. Kanzow, *Some non-interior continuation method for linear complementarity problems*, SIAM Journal on Matrix Analysis and Application, vol. 17, pp. 851-868, 1996.

[86] C. Kanzow and C. Nagel, *Semidefinite programs: new search directions, smoothing-type methods, and numerical results*, SIAM Journal on Optimization, vol. 13, pp. 1-23, 2002.

[87] C. Kanzow, I. Ferenczi, and M. Fukushima, *On the local convergence of semismooth Newton methods for linear and nonlinear second-order cone programs without strict complementarity*, SIAM Journal on Optimization, vol. 20, pp. 297-320, 2009.

[88] C. Kanzow, Y. Yamashita, and M. Fukushima, *New NCP functions and their properties*, Journal of Optimization Theory and Applications, vol. 97, pp. 115-135, 1997.

[89] F. Kittaneh and M. Krnic, *Refined Heinz operator inequalities*, Linear and Multilinear Algebra, vol. 61, pp. 1148-1157, 2013.

[90] K.C. Kiwiel, *Proximal minimization methods with generalized Bregman functions*, SIAM Journal on Control and Optimization, vol. 35, pp. 1142-1168, 1997.

[91] A. Klinger and O.L. Mangasarian, *Logarithmic convexity and geometric programming*, Journal of Mathematical Analysis and Applications, vol. 24, pp. 388-408, 1968.

[92] K. Knopp, *Infinite Sequences and Series*, Dover Publications, New York., 1956.

[93] F. Kraus, *Über konvekse Matrixfunktionen*, Mathematische Zeitschrift, vol. 41, pp. 18-42, 1936.

[94] A. Korányi, *Monotone functions on formally real Jordan algebras*, Mathematische Annalen, vol. 269, pp. 73-76, 1984.

[95] M. Kŏvara and M. Stingl, *On the solution of large-scale SDP problems by the modified barrier method using iterative solvers*, Mathematical Programming, Series B, vol. 109, pp. 413–444, 2007.

[96] F. Kubo and T. Ando, *Means of positive linear operators*, Mathematische Annalen, vol. 246, pp. 205-224, 1980.

[97] S. Kum and Y. Lim *Penalized complementarity functions on symmetric cones*, Journal of Global Optimization, vol. 46, pp. 475-485, 2010.

[98] M. K. Kwong, *Some results on matrix monotone functions*, Linear Algebra and Its Applications, vol. 118, pp. 129-153, 1989.

[99] Y.-J. Kuo and H. D. Mittelmann, *Interior point methods for second-order cone programming and OR applications*, Computational Optimization and Applications, vol. 28, pp. 255-285, 2004.

[100] H. Lee and Y. Lim, *Metric and spectral geometric means on symmetric cones*, Kyungpook Mathematical Journal, vol. 47, pp. 133-150, 2007.

[101] A. S. Lewis and H. S. Sendov, *Twice differentiable spectral functions*, SIAM Journal on Matrix Analysis and Application, vol. 23, pp. 368-386, 2001.

[102] Y. Lim, *Geometric means on symmetric cones*, Archiv der Mathematik, vol. 75, pp. 39-45, 2000.

[103] M. S. Lobo, L. Vandenberghe, S. Boyd, and H. Lebret, *Application of second-order cone programming*, Linear Algebra and Its Applications, vol. 284, pp. 193-228, 1998.

[104] K. Löwner, *Über monotone matrixfunktionen*, Mathematische Zeitschrift, vol. 38, pp. 177-216, 1934.

[105] S. Maldonad and J. López, *Imbalanced data classification using second-order cone programming support vector machines*, Pattern Recognition, vol. 47, no. 5, pp. 2070-2079, 2014.

[106] B. Martinet, *Perturbation des methodes d'Optimisation*, Application, R A.I.R.O. Numerical Analysis, vol. 12, pp. 153-171, 1978.

[107] B. Martos, *The Power of Nonlinear Programming Methods* (In Hungarian). MTA Közgazdaságtudományi Intézetének Közleményei, No. 20, Budapest, Hungary, 1966.

[108] R. Mathias, *Concavity of monotone matrix functions of finite orders*, Linear and Multilinear Algebra, vol. 27, pp. 129-138, 1990.

[109] R. Mifflin, *Semismooth and semiconvex functions in constrained optimization*, SIAM Journal on Control and Optimization, vol. 15, pp. 959-972, 1977.

[110] R.D.C. Monteiro and T. Tsuchiya, *Polynomial convergence of primal-dual algorithms for the second-order cone programs based on the MZ-family of directions*, Mathematical Programming, vol. 88, pp. 61-83, 2000.

[111] J.J. Moreau, *Promimité et dualité dans un espace Hilbertien*, Bulletin de la Société Mathématique de France, vol. 93, pp. 273-299, 1965.

[112] R.N. Mukherjee and L.V. Keddy, *Semicontinuity and quasiconvex functions*, Journal of Optimization Theory and Applications, vol. 94, pp. 715-720, 1997.

[113] E. Neuman, *The weighted logarithmic mean*, Journal of Mathematical Analysis and Applications, vol. 188, pp. 885-900, 1994.

[114] H. Osaka, S. Silvestrov, and J. Tomiyama, *Monotone operator functions, gaps and power moment problem*, Mathematical Scandinavica, vol. 100, pp. 161-183, 2007.

[115] H. Osaka and J. Tomiyama, *Double piling structure of matrix monotone functions and of matrix convex functions*, Linear Algebra and Its Applications, vol. 431, pp. 1825-1832, 2009.

[116] S.-H. Pan and J.-S. Chen, *Proximal-like algorithnm using quasi D-function for convex second-order cone programming*, Journal of Optimization Theory and Applications, vol. 138, pp. 95-113, 2008.

[117] S.-H. Pan and J.-S. Chen, *A class of interior proximal-like algorithms for convex second-order cone programming*, SIAM Journal on Optimization, vol. 19, no. 2, pp. 883-910, 2008.

[118] S.-H. Pan and J.-S. Chen, *A damped Gauss-Newton method for the second-order cone complementarity problem*, Applied Mathematics and Optimization, vol. 59, pp. 293-318, 2009.

[119] S.-H. Pan and J.-S. Chen, *Interior proximal methods and central paths for convex second-order cone programming*, Nonlinear Analysis: Theory, Methods and Applications, vol. 73, no. 9, pp. 3083-3100, 2010.

[120] S.-H. Pan and J.-S. Chen, *A proximal gradient descent method for the extended second-order cone linear complementarity problem*, Journal of Mathematical Analysis and Applications, vol. 366, no. 1, pp. 164-180, 2010.

[121] S.-H. Pan and J.-S. Chen, *A semismooth Newton method for SOCCPs based on a one-parametric class of complementarity functions*, Computational Optimization and Applications, vol. 45, no. 1, pp. 59-88, 2010.

[122] S.-H. Pan, S. Kum, Y. Lim, and J.-S. Chen, *On the generalized Fischer-Burmeister merit function for the second-order cone complementarity problem*, Mathematics of Computation, vol. 83, no. 287, pp. 1143-1171, 2014.

[123] J.-S. Pang, D. Sun, and J. Sun, *Semismooth homeomorphisms and strong stability of semidefinite and Lorentz cone complementarity problems*, Mathematics of Operations Research, vol. 28, pp. 39-63, 2003.

[124] J. Peng, C. Roos, and T. Terlaky, *Self-Rugularity: A New Paradigm for Primal-Dual Interior-Point Algorithms*, Princeton University Press, 2002.

[125] I. Polik and T. Terlaky, *A comprehensive study of the S-Lemma*, Advanced Optimization On-Line, Report No. 2004/14, 2004.

[126] R. Polyak, *Modified barrier functions: Theory and methods*, Mathematical Programming, vol. 54, pp. 177-222, 1992.

[127] J. Ponstein, *Seven kinds of convexity*, SIAM Review, vol. 9, pp. 115-119, 1967.

[128] H.-D. Qi and X. Chen, *On stationary points of merit functions for semifefinite complementarity problems*, Report, Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing, 1997.

[129] L. Qi, *Convergence analysis of some algorithms for solving nonsmooth equations*, Mathematics of Operations Research, vol. 18, pp. 227-244, 1993.

[130] L. Qi and J. Sun, *A nonsmooth version of Newton's method*, Mathematical Programming, vol. 58, pp. 353-367, 1993.

[131] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey, 1970.

[132] R.T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, vol. 14, pp. 877-898, 1976.

[133] R.T. Rockafellar, *Augmented Lagrangians and applications of proximial point algorithm in convex programming*, Mathematics of Operations Research, vol. 1, pp. 97-116, 1976.

[134] R.T. Rockafellar and R.J.-B. Wets, *Variational Analysis*, Springer-Verlag, Berlin, 1998.

[135] J. Schott, *Matrix Analysis for Statistics*, John Wiley and Sons, 2nd edition, 2005.

[136] P.K. Shivaswamy, C. Bhattacharyya, and A.J. Smola, *Second-order cone programming approaches for handling missing and uncertain data*, Journal of Machine Learning Research, vol. 7, pp. 1283-1314, 2006.

[137] P.J.S. Silva and J. Eckstein, *Double-regularization proximal methods, with complementarity applications*, Computational Optimization and Applications, vol. 33, pp. 115-156, 2006.

[138] S. Smale, *Algorithms for solving equations*, in Proceedings of the International Congress of Mathematics, American Mathematical Society, Providence, pp. 172-195, 1987.

[139] D. Sun and L. Qi, *On NCP functions*, Computational Optimization and Applications, vol. 13, pp. 201-220, 1999.

[140] D. Sun and J. Sun, *Semismooth matrix valued functions*, Mathematics of Operationa Research, vol. 27, pp. 150-169, 2002.

[141] D. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Mathematics of Operations Research, vol. 33, pp. 421-445, 2008.

[142] D. Sun and J. Sun, *Strong semismoothness of Fischer-Burmeister SDC and SOC function*, Mathematical Programming, vol. 103, pp. 575-581, 2005.

[143] J. Tao, L. Kong, Z. Luo, and N. Xiu, *Some majorization inequalities in Euclidean Jordan algebras*, Linear Algebra and Its Applications, vol. 461, pp. 92-122, 2014.

[144] M. Teboulle, *Entropic proximal mappings with applications to nonlinear programming*, Mathematics of Operations Research, vol. 17, pp. 670–690, 1992.

[145] P. Tseng, *Merit function for semidefinite complementarity problems*, Mathematical Programming, vol. 83, pp. 159-185, 1998.

[146] T. Tsuchiya, *A convergence analysis of the scaling-invariant primal-dual path-following algorithms for second-order cone programming*, Optimization Methods and Software, vol. 11, pp. 141-182, 1999.

[147] M. Uchiyama and M. Hasumi, *On some operator monotone functions* , Integral Equations and Operator Theory, vol. 42, pp. 243-251, 2011.

[148] X.M. Yang, *Convexity of semicontinuous functions*, OPSEARCH, Operational Research Society of India, vol. 31, pp. 309-317, 1994.

[149] Y.M. Yang and S.Y. Liu, *Three kinds of generalized convexity*, Journal of Optimization Theory and Applications, vol. 86, pp. 501-513, 1995.

[150] A. Yoshise, *Interior point trajectories and a homogenous model for nonlinear complementarity problems over symmetric cones*, SIAM Journal on Optimization, vol. 17, pp. 1129-1153, 2006.

[151] Y.X. Zhao, S.Y. Wang, and U.L. Coladas, *Characterizations of r-convex functions*, Journal of Optimization Theory and Applications, vol. 145, pp. 186-195, 2010.