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On the existence of saddle points for nonlinear second-order cone programming problems

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Abstract In this paper, we study the existence of local and global saddle points for nonlinear second-order cone programming problems. The existence of local saddle points is developed by using the second-order sufficient conditions, in which a sigma-term is added to reflect the curvature of second-order cone. Furthermore, by dealing with the perturbation of the primal problem, we establish the existence of global saddle points, which can be applicable for the case of multiple optimal solutions. The close relationship between global saddle points and exact penalty representations are discussed as well.

Keywords Local and global saddle points · Second-order sufficient conditions · Augmented Lagrangian · Exact penalty representations

Mathematics Subject Classification 90C26 · 90C46

1 Introduction

Recall that the *second-order cone* (SOC), also called the Lorentz cone or ice-cream cone, in $\mathbb{R}^{m+1}$ is defined as

$$\mathcal{K}_{m+1} = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^m \mid \|x_2\| \leq x_1\},$$

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where $\|\cdot\|$ denotes the Euclidean norm. The order relation induced by this pointed closed convex cone $\mathcal{K}_{m+1}$ is given by

$$x \succeq_{\mathcal{K}_{m+1}} 0 \iff x \in \mathbb{R}^{m+1}, \quad x_1 \geq \|x_2\|.$$

In this paper, we consider the following nonlinear second-order cone programming (NSOCP)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & g_j(x) \succeq_{\mathcal{K}_{m_j+1}} 0, \quad j = 1, 2, \ldots, J, \\ & h(x) = 0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, $g_j : \mathbb{R}^n \to \mathbb{R}^{m_j+1}$ are twice continuously differentiable functions, and $\mathcal{K}_{m_j+1}$ is the second-order cone in $\mathbb{R}^{m_j+1}$ for $j = 1, 2, \ldots, J$.

For a given nonlinear programming problem, we can define another programming problem associated with it by using traditional Lagrangian functions. The original problem is called the primal problem, and the latter one is called the dual problem. Since the weak duality property always holds, our concern is on how to obtain the strong duality property (or zero duality gap property). In other words, we want to know when the primal and dual problems have the same optimal values, which provides the theoretical foundation for many primal-dual type methods. However, if we employ the traditional Lagrangian functions, then some convexity is necessary for achieving strong duality property. To overcome this drawback, we need to resort to the *augmented Lagrangian* functions, whose main advantage is ensuring the strong duality property without requiring convexity. In addition, the zero duality gap property coincides with the existence of global saddle points, provided that the optimal solution sets of the primal and dual problems are nonempty, respectively. Many researchers have studied the properties of augmented Lagrangian and the existence of saddle points. For example, Rockafellar and Wets [13] proposed a class of augmented Lagrangian where augmented function is required to be convex functions. This was extended by Huang and Yang [6] where convexity condition is replaced by level-boundedness, and it was further generalized by Zhou and Yang [21] where level-boundedness condition is replaced by so-called valley-at-zero property; see also [14] for more details. These important works give an unified frame for the augmented Lagrangian function and its duality theory. Meanwhile, Floudas and Jongen [5] pointed out the crucial role of saddle points for the minimization of smooth functions with a finite number of stationary points. The necessary and/or sufficient conditions to ensure the existence of local and/or global saddle points were investigated by many researchers. For example, the existence of local and global saddle points of Rockafellar's augmented Lagrangian function was studied in [12]. Local saddle points of the generalized Mangasarian's augmented Lagrangian were analyzed in [19]. The existences of local and global saddle points of pth power nonlinear Lagrangian were discussed in [7,8,18]. For more references, please see [9,10,14,16,17,20,22].

All the results mentioned above are focused on either the standard nonlinear programming or the generalized minimizing problems [13]. The main purpose of this paper is to establish the existences of local and global saddle points of NSOCP (1) by sufficiently exploiting the special structure of SOC. As shown in nonlinear programming, the positive definiteness of $\nabla^2_{xx} L$ over the critical cone is a sufficient condition for the existence of local saddle points. However, this classical result cannot be extended trivially to NSOCP (1) and the analysis is more complicated because $\mathbb{R}^n_+$ is polyhedral, whereas $\mathcal{K}_{m+1}$ is non-polyhedral. Hence, we particulary study the sigma-term [4], which in some extend stands for the curvature of second-order cone. Our result shows that the local saddle point exists provided that the sum of $\nabla^2_{xx}L$ and $\mathcal{H}$ is positive definite even if $\nabla^2_{xx}L$ is indefinite (see Theorem 2.3). This undoubtedly clarifies the essential role played by the sigma-term. Moreover, by developing the perturbation of the primal problem, we establish the existence of global saddle points without restricting the optimal solution being unique, as required in [12,16]. Furthermore, we study another important concept, exact penalty representation, and develop its new necessary and sufficient conditions. The close relationship between global saddle points and exact penalty representations is established as well.

To end this section, we introduce some basic concepts which will be needed for our subsequent analysis. Let $\mathbb{R}^n$ be n-dimensional real vector space. For $x, y \in \mathbb{R}^n$, the inner product is denoted by $x^T y$ or $\langle x, y\rangle$. Given a convex subset $A \subseteq \mathbb{R}^n$ and a point $x \in A$, the normal cone of A at x, denoted by $N_A(x)$, is defined as

$$N_A(x) := \{v \in \mathbb{R}^n \mid \langle v, z - x\rangle \leq 0,\ \forall z \in A\},$$

and the tangent cone, denoted by $T_A(x)$, is defined as

$$T_A(x) := N_A(x)^\circ,$$

where $N_A(x)^\circ$ means the polar cone of $N_A(x)$. Given $d \in T_A(x)$, the outer second order tangent set is defined as

$$T_A^2(x, d) = \left\{w \in \mathbb{R}^n \;\middle|\; \exists\, t_n \downarrow 0 \text{ such that } \operatorname{dist}\left(x + t_n d + \frac{1}{2}t_n^2 w, A\right) = o(t_n^2)\right\}.$$

The support function of A is

$$\sigma(x \mid A) := \sup\{\langle x, z\rangle \mid z \in A\}.$$

We also write $\operatorname{cl}(A)$, $\operatorname{int}(A)$, and $\partial(A)$ to stand for the closure, interior, and boundary of A, respectively. For the simplicity of notations, let us write $\mathcal{K}_j$ to stand for $\mathcal{K}_{m_j+1}$ and $\mathcal{K}$ be the Cartesian product of these second-order cones, i.e., $\mathcal{K} := \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \mathcal{K}_J$. In addition, we denote $g(x) := (g_1(x), g_2(x), \ldots, g_J(x))$, $p := \sum_{j=1}^J (m_j + 1)$, and S^* means the solution set of NSOCP (1). According to [13, Exercise 11.57], the *augmented Lagrangian function* for NSOCP (1) is written as

$$\begin{aligned}\mathcal{L}_c(x, \lambda, \mu, c) := {}& f(x) + \langle \mu, h(x)\rangle + \frac{c}{2}\|h(x)\|^2 \\ & + \frac{c}{2}\sum_{j=1}^J \left[\operatorname{dist}^2\left(g_j(x) - \frac{\lambda_j}{c}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j}{c}\right\|^2\right]. \end{aligned} \tag{2}$$

Here $c \in \mathbb{R}_{++} := \{\zeta \in \mathbb{R} \mid \zeta > 0\}$ and $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^l$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_J) \in \mathbb{R}^{m_1+1} \times \mathbb{R}^{m_2+1} \times \cdots \times \mathbb{R}^{m_J+1}$.

Definition 1.1 Let $\mathcal{L}_c$ be given as in (2) and $(x^*, \lambda^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^l$.

(a) The triple (x^*, λ^*, μ^*) is said to be a local saddle point of $\mathcal{L}_c$ for some $c > 0$ if there exists $\delta > 0$ such that

$$\mathcal{L}_c(x^*, \lambda, \mu) \leq \mathcal{L}_c(x^*, \lambda^*, \mu^*) \leq \mathcal{L}_c(x, \lambda^*, \mu^*),\ \forall x \in \mathbb{B}(x^*, \delta),\ (\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l, \tag{3}$$

where $\mathbb{B}(x^*, \delta)$ denotes the δ-neighborhood of x^*, i.e., $\mathbb{B}(x^*, \delta) := \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq \delta\}$.

(b) The triple (x^*, λ^*, μ^*) is said to be a global saddle point of $\mathcal{L}_c$ for some $c > 0$ if

$$\mathcal{L}_c(x^*, \lambda, \mu) \le \mathcal{L}_c(x^*, \lambda^*, \mu^*) \le \mathcal{L}_c(x, \lambda^*, \mu^*), \ \forall x \in \mathbb{R}^n, \ (\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l. \quad (4)$$

2 On local saddle points

In this section, we focus on the necessary and sufficient conditions for the existence of local saddle points. For simplicity, we let Q stand for a second-order cone without emphasizing its dimension, while using the notation $Q \subset \mathbb{R}^{m+1}$ to indicate that Q is regarded as a second-order cone in $\mathbb{R}^{m+1}$. In other words, the result holding for Q is also applicable to $\mathcal{K}_i$ for $i = 1, \ldots, J$ in the subsequent analysis. According to [13, Example 6.16] we know for $\mathbf{a} \in Q$,

$$\begin{aligned} -\mathbf{b} \in N_Q(\mathbf{a}) &\Longleftrightarrow \Pi_Q(\mathbf{a} - \mathbf{b}) = \mathbf{a} \\ &\Longleftrightarrow \mathrm{dist}(\mathbf{a} - \mathbf{b}, \mathrm{Q}) = \|\mathbf{b}\| \\ &\Longleftrightarrow \mathbf{a} \in Q, \ \mathbf{b} \in Q, \ \mathbf{a}^T\mathbf{b} = 0, \end{aligned} \quad (5)$$

where the last equivalence comes from the fact that Q is a self-dual cone, i.e., $(Q)^\circ = -Q$.

Lemma 2.1 *Let $\mathcal{L}_c$ be given as in (2). Then, the augmented Lagrangian function $\mathcal{L}_c(x, \lambda, \mu)$ is nondecreasing with respect to $c > 0$.*

Proof See [13, Exercise 11.56]. □

We now discuss the necessary conditions for local saddle points.

Theorem 2.1 *Suppose (x^*, λ^*, μ^*) is a local saddle point of $\mathcal{L}_{c^*}$. Then,*

(a) $-\lambda^* \in N_{\mathcal{K}}(g(x^*))$;
(b) $\mathcal{L}_c(x^*, \lambda^*, \mu^*) = f(x^*)$ for all $c > 0$;
(c) x^* is a local optimal solution to NSOCP (1).

Proof We first show that x^* is a feasible point of NSOCP (1), for which we need to verify two things: (i) $h(x^*) = 0$, (ii) $g_j(x^*) \succeq_{\mathcal{K}_j} 0$ for all $j = 1, 2, \ldots, J$.

(i) Suppose $h(x^*) \ne 0$. Taking $\mu = \gamma h(x^*)$ with $\gamma \to \infty$, and applying the first inequality in (3) yields $\mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*) = \infty$ which is a contradiction. Thus, $h(x^*) = 0$.
(ii) Suppose $g_j(x^*) \notin \mathcal{K}_j$ for some $j = 1, \ldots, J$. Then, there exist $\tilde{\lambda}_j \in \mathcal{K}_j$ such that $\eta := \langle \tilde{\lambda}_j, g_j(x^*) \rangle < 0$. Therefore, for $\beta \in \mathbb{R}$

$$\begin{aligned} &\mathrm{dist}^2\left(g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*}, \mathcal{K}_j\right) - \left\|\frac{\beta\tilde{\lambda}_j}{c^*}\right\|^2 \\ &= \left\|g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*} - \Pi_{\mathcal{K}_j}\left(g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*}\right)\right\|^2 - \left\|\frac{\beta\tilde{\lambda}_j}{c^*}\right\|^2 \\ &= \left\|g_j(x^*) - \Pi_{\mathcal{K}_j}\left(g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*}\right)\right\|^2 - 2\left\langle \frac{\beta\tilde{\lambda}_j}{c^*}, g_j(x^*) - \Pi_{\mathcal{K}_j}\left(g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*}\right)\right\rangle \end{aligned}$$

$$\geq \mathrm{dist}^2\left(g_j(x^*), \mathcal{K}_j\right) - 2\beta\left\langle \frac{\tilde{\lambda}_j}{c^*}, g_j(x^*)\right\rangle$$
$$= \mathrm{dist}^2\left(g_j(x^*), \mathcal{K}_j\right) - 2\beta\frac{\eta}{c^*}. \tag{6}$$

Here the inequality comes from the facts that

$$\left\| g_j(x^*) - \Pi_{\mathcal{K}_j}\left(g_j(x^*) - \frac{\beta\tilde{\lambda}_j}{c^*}\right)\right\| \geq \|g_j(x^*) - \Pi_{\mathcal{K}_j}(g_j(x^*))\| = \mathrm{dist}(g_j(x^*), \mathcal{K}_j)$$

and

$$\left\langle \tilde{\lambda}_j, \Pi_{\mathcal{K}_j}\left(g_j(x^*) - (\beta\tilde{\lambda}_j/c^*)\right)\right\rangle \geq 0$$

because $\tilde{\lambda}_j \in \mathcal{K}_j$ and $\Pi_{\mathcal{K}_j}\left(g_j(x^*) - (\beta\tilde{\lambda}_j/c^*)\right) \in \mathcal{K}_j$. Taking $\beta \to \infty$, it follows from (3) and (6) that $\mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*)$ is unbounded above which is a contradiction.
Plugging $\lambda = 0$ in the first inequality of (3) (i.e., $\mathcal{L}_{c^*}(x^*, 0, \mu^*) \leq \mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*)$), we obtain

$$\sum_{j=1}^{J}\left[\mathrm{dist}^2\left(g_j(x^*) - \frac{\lambda_j^*}{c^*}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j^*}{c^*}\right\|^2\right] \geq 0, \tag{7}$$

where we have used the feasibility of x^* as shown above.
On the other hand, we have

$$\mathrm{dist}\left(g_j(x^*) - \frac{\lambda_j^*}{c^*}, \mathcal{K}_j\right) \leq \left\| g_j(x^*) - \frac{\lambda_j^*}{c^*} - g_j(x^*)\right\| = \left\|\frac{\lambda_j^*}{c^*}\right\|,$$

where the inequality is due to the fact that $g_j(x^*) \in \mathcal{K}_j$ as shown above. This together with (7) ensures that

$$\mathrm{dist}\left(g_j(x^*) - \frac{\lambda_j^*}{c^*}, \mathcal{K}_j\right) = \left\|\frac{\lambda_j^*}{c^*}\right\|. \tag{8}$$

Combining (5) and (8) yields $-\lambda_j^* \in N_{\mathcal{K}_j}(g_j(x^*))$ for all $j = 1, \ldots, J$, i.e., $-\lambda^* \in N_{\mathcal{K}}(g(x^*))$ by [13, Proposition 6.41]. This establishes part (a). Furthermore, it implies

$$\mathrm{dist}\left(g_j(x^*) - \frac{\lambda_j^*}{c}, \mathcal{K}_j\right) = \left\|\frac{\lambda_j^*}{c}\right\|, \quad \forall c > 0, \tag{9}$$

because $-\lambda_j^*/c \in N_{\mathcal{K}_j}(g_j(x^*))$ for all $c > 0$ (since $N_{\mathcal{K}_j}(g_j(x^*))$ is a cone). Hence $\mathcal{L}_c(x^*, \lambda^*, \mu^*) = f(x^*)$ for all $c > 0$. This establishes part (b).
Now, we turn the attention to part (c). Suppose $x \in \mathbb{B}(x^*, \delta)$ is any feasible point of NSOCP (1). Then, from (3), we know

$$f(x) \geq \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) \geq \mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*) = f(x^*),$$

where the first inequality comes from the fact that x is feasible. This means x^* is a local optimal solution to NSOCP (1). The proof is complete. □

For NSOCP (1), we say that *Robinson's constraint qualification* holds at x^* if $\nabla h_i(x^*)$ for $i = 1, \ldots, l$ are linearly independent and there exists $d \in \mathbb{R}^n$ such that

$$\nabla h(x^*)d = 0 \text{ and } g(x^*) + \nabla g(x^*)d \in \text{int}(\mathcal{K}).$$

It is known that if x^* is a local solution to NSOCP (1) and Robinson's constraint qualification holds at x^*, then there exists $(\lambda^*, \mu^*) \in \mathbb{R}^p \times \mathbb{R}^l$ such that the following Karush-Kuhn-Tucker (KKT) conditions

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad h(x^*) = 0, \quad -\lambda^* \in N_{\mathcal{K}}(g(x^*)), \tag{10}$$

or equivalently,

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0, \quad h(x^*) = 0, \quad \lambda^* \in \mathcal{K}, \; g(x^*) \in \mathcal{K}, \quad (\lambda^*)^T g(x^*) = 0,$$

where $L(x, \lambda, \mu)$ is the standard Lagrangian function of NSOCP (1), i.e.,

$$L(x, \lambda, \mu) := f(x) + \langle \mu, h(x) \rangle - \langle \lambda, g(x) \rangle . \tag{11}$$

For convenience of subsequent analysis, we denote by $\Lambda(x^*)$ all Lagrangian multipliers (λ^*, μ^*) satisfying (10).

It is well-known that the second order sufficient conditions are utilized to ensure the existence of local saddle points. In the nonlinear programming, it requires the positive definiteness of $\nabla^2_{xx} L$ over the critical cone. However, due to the non-polyhedric of second-order cone, an additional widely known *sigma*-term (or σ-term), which stands for the curvature of second-order cone, is required. In particular, it was noted in [4, page 177] that the σ-term vanishes when the cone is polyhedral. Due to the important role played by σ-term in the analysis of second-order cone, before developing the sufficient conditions for the existence of local saddle points, we shall study some basic properties of σ-term which will be used in the subsequence analysis. First, based on the arguments given in [1, Theorem 29] we obtain the following result.

Theorem 2.2 *Let $x \in Q$ and $d \in T_Q(x)$. Then, the support function of the outer second order tangent set $T^2_Q(x, d)$ is*

$$\sigma\left(y \mid T^2_Q(x, d)\right) = \begin{cases} -\frac{y_1}{x_1} d^T \begin{bmatrix} 1 & 0 \\ 0 & -I_m \end{bmatrix} d, & \text{for } y \in N_Q(x) \cap \{d\}^\perp, \; x \in \partial Q \backslash \{0\}, \\ 0, & \text{for } y \in N_Q(x) \cap \{d\}^\perp, \; x \notin \partial Q \backslash \{0\}, \\ +\infty, & \text{for } y \notin N_Q(x) \cap \{d\}^\perp. \end{cases}$$

Proof We know from [4, Proposition 3.34] that

$$T^2_Q(x, d) + T_{T_Q(x)}(d) \subset T^2_Q(x, d) \subset T_{T_Q(x)}(d).$$

This implies

$$\begin{aligned} \sigma\left(y \mid T^2_Q(x, d)\right) + \sigma\left(y \mid T_{T_Q(x)}(d)\right) &= \sigma\left(y \mid T^2_Q(x, d) + T_{T_Q(x)}(d)\right) \\ &\le \sigma\left(y \mid T^2_Q(x, d)\right) \le \sigma\left(y \mid T_{T_Q(x)}(d)\right). \end{aligned} \tag{12}$$

Note that

$$\sigma\left(y \mid T_{T_Q(x)}(d)\right) < +\infty \Longleftrightarrow \sigma\left(y \mid T_{T_Q(x)}(d)\right) = 0 \tag{13}$$

$$\Longleftrightarrow y \in N_{T_Q(x)}(d) \tag{14}$$

$$\Longleftrightarrow y \in \left(T_Q(x)\right)^\circ = N_Q(x),\ \ y^T d = 0 \tag{15}$$

where the first and third equivalences come from the fact that $T_{T_Q(x)}(d)$ and $T_Q(x)$ are cones, respectively. Thus, we only need to establish the exact formula of $\sigma\left(y \mid T_Q^2(x,d)\right)$, provided that (15) holds. In addition, it also indicates from (12) that $\sigma\left(y \mid T_Q^2(x,d)\right) = \infty$ whenever $y \notin N_Q(x) \cap \{d\}^\perp$, since $T_Q^2(x,d)$ is nonempty for $x \in Q$ and $d \in T_Q(x)$ by [1, Lemma 27].

In fact, under condition (15), it follows from (12) and (13) that

$$\sigma\left(y \mid T_Q^2(x,d)\right) \le \sigma\left(y \mid T_{T_Q(x)}(d)\right) = 0. \tag{16}$$

Furthermore, in light of condition (15), we discuss the following four cases.

(i) If $x = 0$, then $0 \in T_Q^2(x,d) = T_Q(d)$ where the equality is due to [1, Lemma 27]. Thus,

$$\sigma\left(y \mid T_Q^2(x,d)\right) = \sigma\left(y \mid T_Q(d)\right) \ge 0.$$

This together with (16) implies $\sigma\left(y \mid T_Q^2(x,d)\right) = 0$.

(ii) If $x \in \text{int}(Q)$, then it follows from (15) that $y = 0$. Hence, $\sigma\left(y \mid T_Q^2(x,d)\right) = 0$.

(iii) If $x \in \partial(Q)\backslash\{0\}$ and $d \in \text{int}(T_Q(x))$, then it follows from (14) that $y = 0$ since $d \in \text{int}(T_Q(x))$. Hence $\sigma\left(y \mid T_Q^2(x,d)\right) = 0 = -(y_1/x_1)(d_1^2 - \|d_2\|^2)$.

(iv) If $x \in \partial(Q)\backslash\{0\}$ and $d \in \partial(T_Q(x))$, then the desired result can be obtained by following the arguments given in [1, p. 222]. We provide the proof for the sake of completeness. Note that $\sigma\left(y | T_Q^2(x,d)\right)$ is to maximize $y_1 w_1 + y_2^T w_2$ over all w satisfying $-w_1 x_1 + w_2^T x_2 \le d_1^2 - \|d_2\|^2$ (see [1, Lemma 27]). From $y \in N_Q(x)$, i.e., $-y \in Q$, $x \in Q$, and $x^T y = 0$, we know $-y_1 = \alpha x_1$ and $-y_2 = -\alpha x_2$ with $\alpha = -\frac{y_1}{x_1} \ge 0$, see [1, page 208]. Thus,

$$\langle y, w\rangle = y_1 w_1 + y_2^T w_2 = \alpha\left(w_2^T x_2 - w_1 x_1\right) \le \alpha\left(d_1^2 - \|d_2\|^2\right) = -\frac{y_1}{x_1}\left(d_1^2 - \|d_2\|^2\right).$$

The maximum can be obtained at $(w_1, w_2) = (-\frac{d_1^2}{x_1}, -\frac{\|d_2\|^2}{\|x_2\|^2} x_2)$. This establishes the desired expression. □

Remark 2.1 Let A be a convex subset in $\mathbb{R}^{m+1}$. In the proof of Theorem 2.2, we use the inclusion $T_A^2(x,d) \subset T_{T_A(x)}(d)$. It is known from [4, page 168] that these two sets are the same if A is polyhedral. But, for the non-polyhedral cone Q, the following example shows this inclusion maybe strict.

Example 2.1 For $Q \subset \mathbb{R}^3$, let $\bar{x} = (1,1,0)$ and $\bar{d} = (1,1,1)$. Then,

$$T_Q(\bar{x}) = \{d = (d_1, d_2, d_3) \in \mathbb{R}^3 \mid (d_2, d_3)^T(\bar{x}_2, \bar{x}_3) - d_1\bar{x}_1 \le 0\}$$
$$= \{d = (d_1, d_2, d_3) \mid d_2 - d_1 \le 0\},$$

which implies $\bar{d} \in \partial T_Q(\bar{x})$. Hence,

$$\begin{aligned} T_Q^2(\bar{x}, \bar{d}) &= \{w = (w_1, w_2, w_3) \mid (w_2, w_3)^T (\bar{x}_2, \bar{x}_3) - w_1 \bar{x}_1 \leq \bar{d}_1^2 - \|(\bar{d}_2, \bar{d}_3)\|^2\} \\ &= \{w = (w_1, w_2, w_3) \mid w_2 - w_1 \leq -1\}. \end{aligned}$$

On the other hand, since $T_{T_Q(\bar{x})}(\bar{d}) = \mathrm{cl}(\mathcal{R}_{T_Q(\bar{x})}(\bar{d}))$, where $\mathcal{R}_{T_Q(\bar{x})}(\bar{d})$ denotes the radical (or feasible) cone of $T_Q(\bar{x})$ at $(\bar{d})$, then for each $w \in T_{T_Q(\bar{x})}(\bar{d})$, there exists $w' \in \mathcal{R}_{T_Q(\bar{x})}(\bar{d}) \to w$ such that $\bar{d} + tw' \in T_Q(\bar{x})$ for some $t > 0$, i.e.,

$$\left((\bar{d}_2, \bar{d}_3) + t(w_2', w_3')\right)^T (\bar{x}_2, \bar{x}_3) - (\bar{d}_1 + tw_1')\bar{x}_1 \leq 0,$$

which ensures that $(w_2', w_3')^T (\bar{x}_2, \bar{x}_3) - w_1' \bar{x}_1 \leq 0$. Now, taking limit yields $w_2 - w_1 \leq 0$. Thus, we obtain

$$T_{T_Q(\bar{x})}(\bar{d}) = \{w = (w_1, w_2, w_3) \mid w_2 - w_1 \leq 0\}$$

which says $T_Q^2(\bar{x}, \bar{d}) \subsetneq T_{T_Q(\bar{x})}(\bar{d})$. In fact, $0 \in T_{T_Q(\bar{x})}(\bar{d})$, but $0 \notin T_Q^2(\bar{x}, \bar{d})$.

Corollary 2.1 *For $x \in Q$ and $y \in N_Q(x)$, we define*

$$\Theta(x, y) := T_Q(x) \cap \{y\}^\perp = \{d \mid d \in T_Q(x) \text{ and } y^T d = 0\}.$$

Then, $\sigma\left(y \mid T_Q^2(x, d)\right)$ is nonpositive and continuous with respect to d over $\Theta(x, y)$.

Proof We first show that $\sigma(y \mid T_Q^2(x, d))$ is nonpositive for $d \in \Theta(x, y)$. In fact, we know from Theorem 2.2 that $\sigma\left(y \mid T_Q^2(x, d)\right) = 0$ when $x = 0$, or $x \in \mathrm{int}(Q)$, or $x \in \partial(Q)\backslash\{0\}$ and $d \in \mathrm{int}(T_Q(x))$. If $x \in \partial(Q)\backslash\{0\}$ and $d \in \partial(T_Q(x))$, then we have $x_1 d_1 = x_2^T d_2$ by the formula of $T_Q(x)$, see [1, Lemma 25]. Hence $x_1|d_1| = |x_2^T d_2| \leq \|x_2\|\|d_2\|$ which implies $|d_1| \leq \|d_2\|$ because $x_1 = \|x_2\| > 0$. Note that $-y_1$ is nonnegative since $-y \in Q$. Then, applying Theorem 2.2 yields $\sigma\left(y \mid T_Q^2(x, d)\right) = -(y_1/x_1)(d_1^2 - \|d_2\|^2) \leq 0$. Thus, in any case, we have verified the nonpositivity of $\sigma\left(y \mid T_Q^2(x, d)\right)$ over $\Theta(x, y)$.

Next, we now show the continuity of $\sigma\left(y \mid T_Q^2(x, d)\right)$ with respect to d over $\Theta(x, y)$. Indeed, if $x = 0$ or $x \in \mathrm{int}(Q)$, then $\sigma\left(y \mid T_Q^2(x, d)\right) = 0$ for all $d \in \Theta(x, y)$ which, of course, is continuous. If $x \in \partial Q\backslash\{0\}$, then $\sigma\left(y \mid T_Q^2(x, d)\right) = -(y_1/x_1)(d_1^2 - \|d_2\|^2)$ for $d \in \Theta(x, y)$ which is continuous with respect to d as well. □

Remark 2.2 For a general closed convex cone Ω, $\sigma(y \mid T_\Omega^2(x, d))$ can be a discontinuous function of d; see [4, Page 178] or [15, Page 489]. But, when Ω is the second order cone Q, our result shows that this function is continuous.

For a convex subset A in $\mathbb{R}^{m+1}$, it is well known that the function $\mathrm{dist}^2(x, A)$ is continuously differentiable with $\nabla \mathrm{dist}^2(x, A) = 2(x - \Pi_A(x))$. But, there are very limited results on second order differentiability unless some additional structure is imposed on A, for example, second order regularity, see [2,3,15].

Let $\phi(x) := \mathrm{dist}^2(x, Q)$ for $Q \subset \mathbb{R}^{m+1}$. Since Q is second order regular, then according to [15], ϕ possesses the following nice property: for any $x, d \in \mathbb{R}^{m+1}$, there holds that

$$\lim_{\substack{d' \to d \\ t \downarrow 0}} \frac{\phi(x + td') - \phi(x) - t\phi'(x; d')}{\frac{1}{2}t^2} = \mathcal{V}(x, d) \tag{17}$$

where $\mathcal{V}(x, d)$ is the optimal value of the problem

$$\begin{aligned} \min \quad & \left\{2\|d - z\|^2 - 2\sigma\left(x - \Pi_Q(x) \mid T_Q^2(\Pi_Q(x), z)\right)\right\} \\ \text{s.t.} \quad & z \in \Theta\left(\Pi_Q(x), x - \Pi_Q(x)\right). \end{aligned} \tag{18}$$

With these preparations, the sufficient conditions for the existence of local saddle points are given as below.

Theorem 2.3 *Suppose x^* is a feasible point of the NSOCP (1) satisfying the following:*

(i) x^* is a KKT point and $(\lambda^*, \mu^*) \in \Lambda(x^*)$, i.e.,

$$\nabla_x L(x^*, \lambda^*, \mu^*) = 0 \quad \text{and} \quad -\lambda^* \in N_{\mathcal{K}}(g(x^*)).$$

(ii) the following second order conditions hold

$$\nabla^2_{xx} L(x^*, y^*)(d, d) + d^T \mathcal{H}(x^*, \lambda^*) d > 0, \quad \forall d \in \mathcal{C}(x^*, \lambda^*)\backslash\{0\}, \tag{19}$$

where

$$\mathcal{C}(x^*, \lambda^*) := \left\{d \in \mathbb{R}^n \mid \nabla h(x^*) d = 0, \nabla g(x^*) d \in T_{\mathcal{K}}\left(g(x^*)\right), \left(\nabla g(x^*) d\right)^T (\lambda^*) = 0\right\},$$

and $\mathcal{H}(x^*, \lambda^*) := \sum_{j=1}^J \mathcal{H}^j(x^*, \lambda_j^*)$ with

$$\mathcal{H}^j\left(x^*, \lambda_j^*\right) := \begin{cases} -\frac{(\lambda_j^*)_1}{(g_j(x^*))_1} \nabla g_j(x^*)^T \begin{bmatrix} 1 & 0 \\ 0 & -I_{m_j} \end{bmatrix} \nabla g_j(x^*), & g_j(x^*) \in \partial(\mathcal{K}_j)\backslash\{0\}, \\ 0, & otherwise. \end{cases}$$

Then, (x^*, λ^*, μ^*) is a local saddle point of $\mathcal{L}_c$ for some $c > 0$.

Proof The first inequality in (3) follows from the fact that $\mathcal{L}_c(x^*, \lambda^*, \mu^*) = f(x^*)$ by (5) since $-\lambda^* \in N_{\mathcal{K}}(g(x^*))$, and that $\mathcal{L}_c(x^*, \lambda, \mu) \le f(x^*)$ for all $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$ due to x^* being feasible.

We will prove the second inequality in (3) by contradiction, i.e., we cannot find $c > 0$ and $\delta > 0$ such that $f(x^*) = \mathcal{L}_c(x^*, \lambda^*, \mu^*) \le \mathcal{L}_c(x, \lambda^*, \mu^*)$ for all $x \in \mathbb{B}(x^*, \delta)$. In other words, there exists a sequence $c_n \to \infty$ as $n \to \infty$, and each fixed c_n, we always find a sequence $\{x_k^n\}$ (noting that its sequence is dependent on c_n) such that $x_k^n \to x^*$ as $k \to \infty$ and

$$f(x^*) > \mathcal{L}_{c_n}(x_k^n, \lambda^*, \mu^*). \tag{20}$$

To proceed, we denote $t_k^n := \|x_k^n - x^*\|$ and $d_k^n := (x_k^n - x^*)/\|x_k^n - x^*\|$. Assume, without loss of generality, that $d_k^n \to \tilde{d}^n$ as $k \to \infty$. First, we observe that

$$\begin{aligned} &\phi\left(g_j(x_k^n) - \frac{\lambda_j^*}{c_n}\right) \\ &\quad = \phi\left(g_j(x^*) - \frac{\lambda_j^*}{c_n} + t_k^n \nabla g_j(x^*) d_k^n + \frac{1}{2}(t_k^n)^2 \nabla^2 g_j(x^*)(d_k^n, d_k^n) + o\left((t_k^n)^2\right)\right) \\ &\quad = \phi\left(g_j(x^*) - \frac{\lambda_j^*}{c_n} + t_k^n \left[\nabla g_j(x^*) d_k^n + \frac{1}{2} t_k^n \nabla^2 g_j(x^*)(d_k^n, d_k^n)\right]\right) + o\left((t_k^n)^2\right) \end{aligned}$$

$$= \phi\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}\right) + t_k^n \phi'\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}\right)\left(\nabla g_j(x^*)d_k^n + \frac{1}{2}t_k^n \nabla^2 g_j(x^*)(d_k^n, d_k^n)\right)$$
$$+ \frac{1}{2}(t_k^n)^2 \mathcal{V}\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}, \nabla g_j(x^*)\tilde{d}^n\right) + o\left((t_k^n)^2\right) \tag{21}$$

where the second equality follows from the fact of ϕ being Lipschitz continuous (in fact, ϕ is continuously differentiable) and the last step is due to (17). From (18), $\mathcal{V}\left(g_j(x^*) - \lambda_j^*/c_n, \nabla g_j(x^*)\tilde{d}^n\right)$ is the optimal value of the following problem

$$\begin{aligned} \min \quad & \left\{2\|\nabla g_j(x^*)\tilde{d}^n - z\|^2 - 2\sigma\left(-\frac{\lambda_j^*}{c_n} \,\middle|\, T^2_{\mathcal{K}_j}(g_j(x^*), z)\right)\right\} \\ \text{s.t.} \quad & z \in \Theta(g_j(x^*), -\lambda_j^*) \end{aligned} \tag{22}$$

where we have used the fact that $\Theta\left(g_j(x^*), -\lambda_j^*/c_n\right) = \Theta(g_j(x^*), -\lambda_j^*)$ by definition since $c_n \neq 0$, and $\Pi_{\mathcal{K}_j}\left(g_i(x^*) - (\lambda_j^*/c_n)\right) = g_i(x^*)$ because $-\lambda_j^* \in N_{\mathcal{K}_j}(g_j(x^*))$ by (5).

Note that the optimal value of the above problem (22) is finite since σ is nonpositive by Corollary 2.1, and that the objective function is strongly convex (because $\|\cdot\|^2$ is strongly convex and $-\sigma$ is convex [4, Proposition 3.48]). Hence, the optimal solution of the problem (22) exists and is unique, say z_j^n, i.e.,

$$\mathcal{V}\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}, \nabla g_j(x^*)\tilde{d}^n\right) = 2\left\|\nabla g_j(x^*)\tilde{d}^n - z_j^n\right\|^2 - 2\sigma\left(-\frac{\lambda_j^*}{c_n} \,\middle|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n)\right), \tag{23}$$

where $z_j^n \in \Theta(g_j(x^*), -\lambda_j^*)$. Then, combining (21) and (23) yields

$$\begin{aligned} &\text{dist}^2\left(g_j(x_k^n) - \frac{\lambda_j^*}{c_n}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j^*}{c_n}\right\|^2 \\ &\quad = -2t_k^n\left\langle \frac{\lambda_j^*}{c_n}, \nabla g_j(x^*)d_k^n + \frac{1}{2}t_k^n \nabla^2 g_j(x^*)(d_k^n, d_k^n)\right\rangle \\ &\qquad + (t_k^n)^2\left[\|\nabla g_j(x^*)\tilde{d}^n - z_j^n\|^2 - \sigma\left(-\frac{\lambda_j^*}{c_n} \,\middle|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n)\right)\right] + o((t_k^n)^2), \end{aligned} \tag{24}$$

where we use the fact that $\text{dist}\left(g_j(x^*) - (\lambda_j^*/c_n), \mathcal{K}_j\right) = \|\lambda_j^*/c_n\|$ and

$$\phi'\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}\right) = 2\left[g_j(x^*) - \frac{\lambda_j^*}{c_n} - \Pi_{\mathcal{K}_j}\left(g_j(x^*) - \frac{\lambda_j^*}{c_n}\right)\right] = -2\frac{\lambda_j^*}{c_n}.$$

Since $f(x^*) > \mathcal{L}_{c_n}(x_k^n, \lambda^*, \mu^*)$ by (20), applying the Taylor expansion, we obtain from (24) that

$$\begin{aligned} 0 > \; & f(x_k^n) - f(x^*) + \langle \mu^*, h(x_k^n)\rangle + \frac{c_n}{2}\|h(x_k^n)\|^2 \\ & + \frac{c_n}{2}\sum_{j=1}^{J}\left[\text{dist}^2\left(g_j(x_k^n) - \frac{\lambda_j^*}{c_n}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j^*}{c_k}\right\|^2\right] \end{aligned}$$

$$
\begin{aligned}
&= t_k^n \nabla f(x^*)^T d_k^n + \frac{1}{2}(t_k^n)^2 (d_k^n)^T \nabla^2 f(x^*) d_k^n + o((t_k^n)^2) + \Big\langle \mu^*, t_k^n \nabla h(x^*) d_k^n \\
&\quad + \frac{1}{2}(t_k^n)^2 \nabla h(x^*)(d_k^n, d_k^n) + o((t_k^n)^2) \Big\rangle + \frac{c_n}{2} \| t_k^n \nabla h(x^*) d_k^n + o(t_k^n) \|^2 \\
&\quad + \frac{c_n}{2} \sum_{j=1}^{J} \Bigg[-2 t_k^n \left\langle \frac{\lambda_j^*}{c_n}, \nabla g_j(x^*) d_k^n + \frac{1}{2} t_k^n \nabla^2 g_j(x^*)(d_k^n, d_k^n) \right\rangle \\
&\quad + (t_k^n)^2 \left(\|\nabla g_j(x^*) \tilde{d}^n - z_j^n\|^2 - \sigma\left(-\frac{\lambda_j^*}{c_n} \,\Big|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n) \right) \right) + o((t_k^n)^2) \Bigg].
\end{aligned}
$$

Dividing by $(t_k^n)^2/2$ on both sides and taking limits as $k \to \infty$ give

$$
\begin{aligned}
0 \geq\ & \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}^n, \tilde{d}^n) + c_n \|\nabla h(x^*) \tilde{d}^n\|^2 \\
& + c_n \sum_{j=1}^{J} \left[\|\nabla g_j(x^*) \tilde{d}^n - z_j^n\|^2 - \sigma\left(-\frac{\lambda_j^*}{c_n} \,\Big|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n) \right) \right]
\end{aligned}
\tag{25}
$$

where we use the fact that $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$, the first equality in KKT conditions (10). Since $-\lambda_j^* \in N_{\mathcal{K}_j}(g_j(x^*))$ from (10) and $z_j^n \in \Theta(g_j(x^*), -\lambda_j^*)$, applying Corollary 2.1 yields

$$
\sigma\left(-\frac{\lambda_j^*}{c_n} \,\Big|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n) \right) = \frac{1}{c_n} \sigma\left(-\lambda_j^* \,\big|\, T^2_{\mathcal{K}_j}(g_j(x^*), z_j^n) \right) \leq 0
$$

where the equality is due to the positive homogeneity of the support function, see [11]. Thus, it follows from (25) that

$$
0 \geq \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}^n, \tilde{d}^n) + c_n \|\nabla h(x^*) \tilde{d}^n\|^2 + c_n \sum_{j=1}^{J} \|\nabla g_j(x^*) \tilde{d}^n - z_j^n\|^2.
$$

Due to $\|\tilde{d}^n\| = 1$ for all n, we may assume, taking a subsequence if necessary, that $\tilde{d}^n \to \tilde{d}$. Because c_n can be made sufficiently large as $n \to \infty$, we obtain from the above inequality that $\nabla h(x^*) \tilde{d}^n \to 0$ and $\nabla g_j(x^*) \tilde{d}^n - z_j^n \to 0$. Therefore, $\nabla h(x^*) \tilde{d} = \lim_{n \to \infty} \nabla h(x^*) \tilde{d}^n = 0$ and

$$
\begin{aligned}
\operatorname{dist}\left(\nabla g_j(x^*) \tilde{d}, \Theta(g_j(x^*), -\lambda_j^*) \right) &= \lim_{n \to \infty} \operatorname{dist}\left(\nabla g_j(x^*) \tilde{d}^n, \Theta(g_j(x^*), -\lambda_j^*) \right) \\
&\leq \lim_{n \to \infty} \|\nabla g_j(x^*) \tilde{d}^n - z_j^n\| = 0
\end{aligned}
$$

which implies $\nabla g_j(x^*) \tilde{d} \in \Theta(g_j(x^*), -\lambda_j^*)$ for all $j = 1, 2, \ldots, J$. Thus, we have $\tilde{d} \in \mathcal{C}(x^*, \lambda^*)$. In addition, it follows from (25) again that

$$
\begin{aligned}
0 &\geq \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}^n, \tilde{d}^n) - c_n \sum_{j=1}^{J} \sigma\left(-\frac{\lambda_j^*}{c_n} \,\Big|\, T^2_{\mathcal{K}_j}\left(g_j(x^*), z_j^n\right) \right) \\
&= \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}^n, \tilde{d}^n) - \sum_{j=1}^{J} \sigma\left(-\lambda_j^* \,\Big|\, T^2_{\mathcal{K}_j}\left(g_j(x^*), z_j^n\right) \right).
\end{aligned}
$$

Note that $\sigma\left(-\lambda_j^* \,\middle|\, T^2_{\mathcal{K}_j}(g_j(x^*), \nabla g_j(x^*)\tilde{d}\right) = -\tilde{d}^T \mathcal{H}^j(x^*, \lambda_j^*)\tilde{d}$ by Theorem 2.2. Taking the limits on both sides as $n \to \infty$, using the continuity of σ by Corollary 2.1, and $z_j^n \to \nabla g_j(x^*)\tilde{d}$ (since $\nabla g_j(x^*)\tilde{d}^n - z_j^n \to 0$), we obtain

$$\begin{aligned} 0 &\geq \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}, \tilde{d}) - \sum_{j=1}^{J} \sigma\left(-\lambda_j^* \,\middle|\, T^2_{\mathcal{K}_j}(g_j(x^*), \nabla g_j(x^*)\tilde{d}\right) \\ &= \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}, \tilde{d}) + \sum_{j=1}^{J} \tilde{d}^T \mathcal{H}^j(x^*, \lambda_j^*)\tilde{d} \\ &= \nabla^2_{xx} L(x^*, \lambda^*, \mu^*)(\tilde{d}, \tilde{d}) + \tilde{d}^T \mathcal{H}(x^*, \lambda^*)\tilde{d} \end{aligned}$$

which contradicts (19) since $\tilde{d} \in \mathcal{C}(x^*, \lambda^*)$ and $\tilde{d} \neq 0$. Thus, the proof is complete. □

For convex nonlinear programming, the saddle point has a close relation to the KKT point. Their relationship has been found in [4] by using the traditional Lagrangian functions (11). Here we further discuss their relationship for NSOCP via augmented Lagrangian functions (2).

Definition 2.1 The problem NSOCP (1) is said to be convex if the objective function f is a convex function, h is an affine mapping, and g is a convex mapping with respect to the set $-\mathcal{K}$, i.e., for any $x, y \in \mathbb{R}^n$ and $t \in [0, 1]$, we have

$$g\left(tx + (1-t)y\right) \preceq_{-\mathcal{K}} tg(x) + (1-t)g(y). \tag{26}$$

It is easy to see that g is convex with respect to $-\mathcal{K}$ if and only if g_j is convex with respect to $-\mathcal{K}_j$ for all $j = 1, 2, \ldots, J$. In general, the square of a convex function may not be convex, for example, $(x^2 - 1)^2$ is not convex although $x^2 - 1$ is convex. Nonetheless, the square of the distance function is still convex, i.e., $\mathrm{dist}^2(x, Q)$ is convex. In fact, $\mathrm{dist}^2(x, Q) = \inf\{\|x - y\|^2 + \delta_Q(y) |\, y \in \mathbb{R}^{m+1}\} = \|\cdot\|^2 \Box \delta_Q$, where $\Box$ is the infimal convolution and δ is the indicator function [11]. This conclusion can be also obtained by noting that a differentiable function is convex if and only if its gradient is monotone, see [13]. Hence, it only need to show that $\nabla\,\mathrm{dist}^2(x, Q) = 2(x - \Pi_Q(x))$ is monotone, which is ensured by

$$\begin{aligned} &\left\langle \nabla\,\mathrm{dist}^2(x, Q) - \nabla\,\mathrm{dist}^2(y, Q), x - y \right\rangle \\ &\quad = 2\left\langle x - y - \left(\Pi_Q(x) - \Pi_Q(y)\right), x - y \right\rangle \\ &\quad \geq 2\|x - y\|^2 - 2\left\|\Pi_Q(x) - \Pi_Q(y)\right\| \cdot \|x - y\| \\ &\quad = 2\|x - y\| \cdot \left[\|x - y\| - \left\|\Pi_Q(x) - \Pi_Q(y)\right\|\right] \\ &\quad \geq 0 \end{aligned}$$

where in the last step we use the fact that the metric projection is non-expansive, i.e., $\left\|\Pi_Q(x) - \Pi_Q(y)\right\| \leq \|x - y\|$. □

The following lemma shows that the function $-\mathrm{dist}(\cdot, Q)$ behaves like a monotone function.

Lemma 2.2 *If* $x \succeq_Q y$, *then* $\mathrm{dist}(x, Q) \leq \mathrm{dist}(y, Q)$.

Proof Given x, y with $x \succeq_Q y$, i.e., $x - y \in Q$. Note that $Q + Q = Q$ because Q is a convex cone, see [11]. Hence, we know $Q + (x - y) \subset Q$ since $x - y \in Q$. Then, the desired result follows by

$$\begin{aligned}\mathrm{dist}(x, Q) &= \inf_{z \in Q} \|x - z\| \\ &\le \inf_{z \in Q + (x-y)} \|x - z\| \overset{u := z - x + y}{=\!=\!=\!=} \inf_{u \in Q} \|y - u\| \\ &= \mathrm{dist}(y, Q).\end{aligned}$$

□

The converse of Lemma 2.2 fails, which is illustrated by the following example.

Example 2.2 Consider $\mathcal{K}_2 = \{(x_1, x_2) \mid x_1 \ge |x_2|\}$. Then, for $x = (1, 2)$ and $y = (-1, -1)$, we have

$$\mathrm{dist}(x, \mathcal{K}_2) = \sqrt{2}/2 < \sqrt{2} = \mathrm{dist}(y, \mathcal{K}_2).$$

But, we see $x \not\succeq_{\mathcal{K}_2} y$ since $x - y = (2, 3) \notin \mathcal{K}_2$.

We next show that if the problem NSOCP (1) is convex, then the augmented Lagrangian is also convex.

Theorem 2.4 *If NSOCP (1) is convex, then $\mathcal{L}_c(x, \lambda, \mu)$ is convex with respect to x for all $(c, \lambda, \mu) \in \mathbb{R}_{++} \times \mathbb{R}^p \times \mathbb{R}^l$.*

Proof Since $h : \mathbb{R}^n \to \mathbb{R}^l$ is an affine mapping, then there exists a matrix $M \in \mathbb{R}^{l \times n}$ and $q \in \mathbb{R}^l$ such that $h(x) = Mx + q$. Thus, we know

$$\begin{aligned}&\langle \mu, h(x) \rangle + (c/2)\|h(x)\|^2 \\ &\quad = \langle \mu, Mx + q \rangle + (c/2)\langle Mx + q, Mx + q \rangle \\ &\quad = (c/2)\langle x, M^T M x \rangle + \langle M^T \mu + c M^T q, x \rangle + \langle \mu + (c/2) q, q \rangle\end{aligned}$$

is convex due to $M^T M$ being positive semi-definite. In view of the expression of $\mathcal{L}_c(x, \lambda, \mu)$ given in (2), it remains to show the convexity of $\mathrm{dist}^2\left(g_j(x) - \frac{\lambda_j}{c}, \mathcal{K}_j\right)$. In fact, since g_j is convex with respect to $-\mathcal{K}_j$, it follows from (26) that

$$g_j\,(tx + (1-t)y) - \frac{\lambda_j}{c} \succeq_{\mathcal{K}_j} (t)\left[g_j(x) - \frac{\lambda_j}{c}\right] + (1-t)\left[g_j(y) - \frac{\lambda_j}{c}\right].$$

This together with Lemma 2.2 implies

$$\mathrm{dist}\left(g_j\,(tx + (1-t)y) - \frac{\lambda_j}{c}, \mathcal{K}_j\right) \le \mathrm{dist}\left(t\left[g_j(x) - \frac{\lambda_j}{c}\right] + (1-t)\left[g_j(y) - \frac{\lambda_j}{c}\right], \mathcal{K}_j\right),$$

and hence

$$\begin{aligned}&\mathrm{dist}^2\left(g_j\,(tx + (1-t)y) - \frac{\lambda_j}{c}, \mathcal{K}_j\right) \\ &\le \mathrm{dist}^2\left(t\left[g_j(x) - \frac{\lambda_j}{c}\right] + (1-t)\left[g_j(y) - \frac{\lambda_j}{c}\right], \mathcal{K}_j\right) \\ &\le t\ \mathrm{dist}^2\left(g_j(x) - \frac{\lambda_j}{c}, \mathcal{K}_j\right) + (1-t)\mathrm{dist}^2\left(g_j(y) - \frac{\lambda_j}{c}, \mathcal{K}_j\right)\end{aligned}$$

where the last step is due to the convexity of $\text{dist}^2(x, \mathcal{K}_j)$ as the arguments following (26). □

For convex NSOCP (1), the following result states the relationship between global saddle points and KKT points.

Theorem 2.5 *Suppose that NSOCP (1) is convex. Then, the following hold.*

(a) If (x^*, λ^*, μ^*) satisfies the KKT conditions, then (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_c$ for all $c > 0$.
(b) If (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_c$ for some $c > 0$, then (x^*, λ^*, μ^*) satisfies KKT conditions.

Proof Note first that when x is feasible and $-\lambda \in N_{\mathcal{K}}(g(x))$, we have

$$\begin{aligned}
\nabla_x \mathcal{L}_c(x, \lambda, \mu) &= \nabla f(x) + \nabla h(x)^T \mu + c \nabla h(x)^T h(x) \\
&\quad + c \sum_{j=1}^{J} \nabla g_j(x)^T \left(g_j(x) - \frac{\lambda_j}{c} - \Pi_{\mathcal{K}_j}(g_j(x) - \frac{\lambda_j}{c}) \right) \\
&= \nabla f(x) + \nabla h(x)^T \mu - \sum_{j=1}^{J} \nabla g_j(x)^T \lambda_j \\
&= \nabla f(x) + \nabla h(x)^T \mu - \nabla g(x)^T \lambda \\
&= \nabla_x L(x, \lambda, \mu),
\end{aligned} \tag{27}$$

where in the second equality we use the facts that x is feasible and $\Pi_{\mathcal{K}_j}\big(g_j(x) - (\lambda_j/c)\big) = g_j(x)$ due to $-\lambda_j \in N_{\mathcal{K}_j}(g_j(x))$ by (5).

(a) Suppose (x^*, λ^*, μ^*) satisfies the KKT conditions. For any $c > 0$, since $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$ from (10), then we know $\nabla_x \mathcal{L}_c(x^*, \lambda^*, \mu^*) = 0$ by (27). Besides, $\mathcal{L}_c$ in x is convex by Theorem 2.4 under the hypothesis, therefore we must have x^* being a global optimal solution of $\mathcal{L}_c(x, \lambda^*, \mu^*)$ over $\mathbb{R}^n$, i.e., $\mathcal{L}_c(x^*, \lambda^*, \mu^*) \le \mathcal{L}_c(x, \lambda^*, \mu^*)$ for all $x \in \mathbb{R}^n$. This establishes the second inequality in (4).

On the other hand, we have $\mathcal{L}_c(x^*, \lambda, \mu) \le f(x^*)$ for all $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$ since x^* is a feasible point, and $\mathcal{L}_c(x^*, \lambda^*, \mu^*) = f(x^*)$ by (9). Hence, $\mathcal{L}_c(x^*, \lambda, \mu) \le f(x^*) = \mathcal{L}_c(x^*, \lambda^*, \mu^*)$ for all $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$. This is the first equality in (4).

(b) Suppose that (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_c$ for some $c > 0$. Then, Theorem 2.1 says that x^* is a feasible point and $-\lambda^* \in N_{\mathcal{K}}(g(x^*))$. These means the second and third conditions in (10) hold. In addition, from the second inequality in (4), x^* is an optimal solution of $\mathcal{L}_c(x, \lambda^*, \mu^*)$ over $\mathbb{R}^n$, and hence is a stationary point, i.e., $\nabla_x \mathcal{L}_c(x^*, \lambda^*, \mu^*) = 0$. This together with (27) ensures that $\nabla_x L(x^*, \lambda^*, \mu^*) = 0$, which is just the first condition in (10). □

3 On global saddle points

In this section, we turn our attention to the existence of global saddle point of $\mathcal{L}_c$ for which we need to address the perturbation of NSOCP (1) for subsequent analysis. Given $\alpha \in \mathbb{R}_+ :=$

$\{\zeta \in \mathbb{R} \mid \zeta \geq 0\}$, we define

$$\Gamma(\alpha) := \left\{x \in \mathbb{R}^n \mid \|h(x)\| \leq \alpha,\ \operatorname{dist}(g(x), \mathcal{K}) \leq \alpha\right\} \tag{28}$$

and

$$\Upsilon(\alpha) := \{x \in \mathbb{R}^n \mid f(x) \leq p(0) + \alpha\} \ \text{ where } \ p(\alpha) := \inf_{x \in \Gamma(\alpha)} f(x). \tag{29}$$

It is clear that $\Gamma(0) \cap \Upsilon(0) = S^*$ and $p(0)$ coincides with the optimal value of NSOCP (1). In the sequel, we always assume that f is *bounded below* over $\mathbb{R}^n$.

Lemma 3.1 *For any $\varepsilon > 0$ and $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$, we have*

$$\left\{x \in \mathbb{R}^n \mid \mathcal{L}_c(x, \lambda, \mu) \leq p(0)\right\} \subseteq \Gamma(\varepsilon),$$

whenever $c > 0$ being sufficiently large.

Proof We prove it by contradiction. Suppose on the contrary that there exists $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$, $\varepsilon_0 > 0$, $c_k \to \infty$ as $k \to \infty$, and $\{x^k\}$ such that $\mathcal{L}_{c_k}(x_k, \lambda, \mu) \leq p(0)$, but $x^k \notin \Gamma(\varepsilon_0)$. Since $x^k \notin \Gamma(\varepsilon_0)$, from (28), there have two possible cases: (i) $\|h(x^k)\| > \varepsilon_0$, or (ii) $\operatorname{dist}\big(g(x^k), \mathcal{K}\big) > \varepsilon_0$. Hence, we discuss these two cases, respectively.

(i) If $\|h(x^k)\| > \varepsilon_0$, then as k is large enough, we have

$$\frac{1}{2}\|h(x^k)\| - \frac{\|\mu\|}{c_k} > \frac{1}{4}\varepsilon_0. \tag{30}$$

Therefore, we obtain

$$\begin{aligned} p(0) &\geq f(x_k) + \left\langle \mu, h(x^k)\right\rangle + \frac{c_k}{2}\|h(x^k)\|^2 \\ &\quad + \frac{c_k}{2}\sum_{j=1}^{J}\left[\operatorname{dist}^2\left(g_j(x^k) - \frac{\lambda_j}{c_k},\ \mathcal{K}_j\right) - \left\|\frac{\lambda_j}{c_k}\right\|^2\right] \\ &\geq f(x^k) - \|\mu\| \cdot \|h(x^k)\| + \frac{c_k}{2}\|h(x^k)\|^2 - \frac{c_k}{2}\sum_{j=1}^{J}\left\|\frac{\lambda_j}{c_k}\right\|^2 \\ &= f(x^k) + c_k\|h(x^k)\|\left(\frac{1}{2}\|h(x^k)\| - \frac{\|\mu\|}{c_k}\right) - \frac{\|\lambda\|^2}{2c_k} \\ &\geq f(x^k) + \frac{1}{4}\varepsilon_0^2 c_k - \frac{\|\lambda\|^2}{2c_k} \\ &\to \infty \ \text{ as } \ k \to \infty, \end{aligned}$$

where the last inequality follows from (30), and the last step comes from $\|\lambda\|^2/c_k \to 0$, and the boundedness of f from below. This gives rise to a contradiction.

(ii) If $\operatorname{dist}\big(g(x^k), \mathcal{K}\big) > \varepsilon_0$, then there must exist j_0 such that $\operatorname{dist}\left(g_{j_0}(x^k),\ \mathcal{K}_{j_0}\right) > \frac{\varepsilon_0}{\sqrt{J}}$ for all k, since $\operatorname{dist}^2(g(x^k), \mathcal{K}) = \sum_{j=1}^{J}\operatorname{dist}^2(g_j(x^k), \mathcal{K}_j)$. Thus, as k large enough

$$\operatorname{dist}\left(g_{j_0}(x^k) - \frac{\lambda_{j_0}}{c_k},\ \mathcal{K}_{j_0}\right) \geq \operatorname{dist}\left(g_{j_0}(x^k),\ \mathcal{K}_{j_0}\right) - \frac{\|\lambda_{j_0}\|}{c_k} \geq \frac{\varepsilon_0}{2\sqrt{J}},$$

where the first inequality is due to the fact that the distance function is Lipschitz continuous with modulus one. With this, we further obtain

$$
\begin{aligned}
p(0) &\geq f(x^k) + \frac{c_k}{2}\left(\|h(x^k)\|^2 + \frac{2}{c_k}\langle \mu, h(x^k)\rangle\right) \\
&\quad + \frac{c_k}{2}\sum_{j=1}^{J}\left[\mathrm{dist}^2\left(g_j(x^k) - \frac{\lambda_j}{c_k}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j}{c_k}\right\|^2\right] \\
&\geq f(x^k) + \frac{c_k}{2}\left(\left\|h(x^k) + \frac{\mu}{c_k}\right\|^2 - \left\|\frac{\mu}{c_k}\right\|^2\right) \\
&\quad + \frac{c_k}{2}\,\mathrm{dist}^2\left(g_{j_0}(x^k) - \frac{\lambda_{j_0}}{c_k}, \mathcal{K}_{j_0}\right) - \frac{1}{2}\sum_{j=1}^{J}\frac{\|\lambda_j\|^2}{c_k} \\
&\geq f(x^k) - \frac{\|\mu\|^2 + \|\lambda\|^2}{2c_k} + \frac{1}{8J}c_k\varepsilon_0^2 \\
&\to \infty \ \text{ as } \ k \to \infty,
\end{aligned}
$$

which is a contradiction again. □

Lemma 3.2 *For any $\varepsilon > 0$ and $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$, we have*

$$\left\{x \in \mathbb{R}^n \mid \mathcal{L}_c(x, \lambda, \mu) \leq p(0)\right\} \subseteq \Upsilon(\varepsilon)$$

whenever $c > 0$ being sufficiently large.

Proof For $\varepsilon > 0$, denote $\alpha := \frac{\varepsilon}{2(\|\mu\|+1)}$. According to Lemma 3.1, we can choose $c > 0$ large enough to ensure $\|\lambda\|^2/c \leq \varepsilon$ and

$$\{x \in \mathbb{R}^n \mid \mathcal{L}_c(x, \lambda, \mu) \leq p(0)\} \subseteq \Gamma(\alpha). \tag{31}$$

This implies

$$
\begin{aligned}
f(x) &= \mathcal{L}_c(x, \lambda, \mu) - \langle \mu, h(x)\rangle \\
&\quad - \frac{c}{2}\|h(x)\|^2 - \frac{c}{2}\sum_{j=1}^{J}\left[\mathrm{dist}^2\left(g_j(x) - \frac{\lambda_j}{c}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j}{c}\right\|^2\right] \\
&\leq p(0) + \|\mu\| \cdot \|h(x)\| + \frac{\|\lambda\|^2}{2c} \\
&\leq p(0) + \varepsilon,
\end{aligned}
$$

where the last inequality is due to $\|h(x)\| \leq \alpha$ since $x \in \Gamma(\alpha)$ by (31). Then, from the definition of (29), the desired result follows. □

With the above, we now state the existence of global saddle points of NSOCP as below.

Theorem 3.1 *Suppose that S^* is nonempty and $\Gamma(\varepsilon_0) \cap \Upsilon(\varepsilon_0)$ is bounded for some $\varepsilon_0 > 0$. Suppose also that there exists $(\lambda^*, \mu^*) \in \mathbb{R}^p \times \mathbb{R}^l$ such that for each $x^* \in S^*$, the triple (x^*, λ^*, μ^*) is a local saddle point of $\mathcal{L}_c$ for some $c > 0$. Then, (x, λ^*, μ^*) with $x \in S^*$ is a global saddle point of $\mathcal{L}_c$ as $c > 0$ sufficiently large.*

Proof Choose $\bar{x} \in S^*$ arbitrarily. Since $(\bar{x}, \lambda^*, \mu^*)$ is a local saddle point by hypothesis, there exists $c_1 > 0$ and $\delta_1 > 0$ such that

$$\mathcal{L}_{c_1}(\bar{x}, \lambda, \mu) \leq \mathcal{L}_{c_1}(\bar{x}, \lambda^*, \mu^*) \leq \mathcal{L}_{c_1}(x, \lambda^*, \mu^*), \ \ \forall (x, \lambda, \mu) \in \mathbb{B}(\bar{x}, \delta_1) \times \mathbb{R}^p \times \mathbb{R}^l.$$

To complete the proof, we only need to show that the second inequality holds for all $x \notin \mathbb{B}(\bar{x}, \delta_1)$ as $c > 0$ sufficiently large because the first inequality holds due to $\mathcal{L}_c(\bar{x}, \lambda, \mu) \leq f(\bar{x}) = \mathcal{L}_c(\bar{x}, \lambda^*, \mu^*)$ for all $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$ and $c > 0$ by Theorem 2.1.

Suppose on the contrary that we can find $c_k \to \infty$ and $x^k \notin \mathbb{B}(\bar{x}, \delta_1)$ such that

$$\mathcal{L}_{c_k}(x^k, \lambda^*, \mu^*) < \mathcal{L}_{c_k}(\bar{x}, \lambda^*, \mu^*) = f(\bar{x}) = p(0). \tag{32}$$

Take $\varepsilon \in (0, \varepsilon_0)$. According to Lemmas 3.1-3.2, we obtain from (32) that

$$x^k \in \Gamma(\varepsilon) \cap \Upsilon(\varepsilon) \subseteq \Gamma(\varepsilon_0) \cap \Upsilon(\varepsilon_0)$$

as k large enough. Hence, $\{x^k\}$ is bounded. We assume without loss of generality that $x^k \to \tilde{x}$, i.e., $\tilde{x} \in \Gamma(\varepsilon) \cap \Upsilon(\varepsilon)$ since $\Gamma(\varepsilon)$ and $\Upsilon(\varepsilon)$ are both closed (due to the continuity of f, h, and g). Since $\varepsilon > 0$ is taken arbitrarily, it further implies that $\tilde{x} \in \Gamma(0) \cap \Upsilon(0) = S^*$ (which can be also obtained by using the upper semi-continuity of the set-valued mapping $\Gamma(\cdot) \cap \Upsilon(\cdot)$). Thus, we obtain from the hypothesis that $(\tilde{x}, \lambda^*, \mu^*)$ is also a local saddle point of $\mathcal{L}_c$ for some c, say c_2, i.e., there exists $\delta_2 > 0$ such that

$$\mathcal{L}_{c_2}(\tilde{x}, \lambda, \mu) \leq \mathcal{L}_{c_2}(\tilde{x}, \lambda^*, \mu^*) \leq \mathcal{L}_{c_2}(x, \lambda^*, \mu^*), \quad (x, \lambda, \mu) \in \mathbb{B}(\tilde{x}, \delta_2) \times \mathbb{R}^p \times \mathbb{R}^l.$$

Since $x^k \to \tilde{x}$, we know $x^k \in \mathbb{B}(\tilde{x}, \delta_2)$ as k sufficiently large. Therefore,

$$\mathcal{L}_{c_k}(x^k, \lambda^*, \mu^*) \geq \mathcal{L}_{c_2}(x^k, \lambda^*, \mu^*) \geq \mathcal{L}_{c_2}(\tilde{x}, \lambda^*, \mu^*) = f(\tilde{x}) = f(\bar{x}),$$

where the first inequality is due to the monotonicity of $\mathcal{L}_c$ in c by Lemma 2.1, the first equality follows from Theorem 2.1, and the second equality comes from the fact that $\bar{x}, \tilde{x} \in S^*$ (i.e., $f(\bar{x}) = f(\tilde{x}) = p(0)$). This contradicts (32). □

Remark 3.1 In [16], the authors develop the existence of global saddle points by requiring the solution set to be unique. This is a condition imposed on the solution set, while our assumption is imposed on the perturbation set. Indeed, these two assumptions are independent. This is illustrated by the following two examples. The first one shows that the set $\Upsilon(\varepsilon) \cap \Gamma(\varepsilon)$ may be unbounded even if the solution to NSOCP (1) is unique.

Example 3.1 Consider the following NSOCP:

$$\begin{aligned} \min \quad & f(x) = x_2 \\ \text{s.t.} \quad & g(x) = x \succeq_{\mathcal{K}_2} 0 \\ & h(x) = x_1(x_1 x_2 - 1) = 0. \end{aligned}$$

The optimal solution is unique, i.e., $x^* = (0, 0)$. But, for any $\varepsilon > 0$, the set $\Upsilon(\varepsilon) \cap \Gamma(\varepsilon)$ is unbounded because $(n, 1/n) \in \Upsilon(\varepsilon) \cap \Gamma(\varepsilon)$ whenever $n > [1/\varepsilon]$.

The second example shows that the solution set is not necessary unique even if the perturbation set $\Upsilon(\varepsilon) \cap \Gamma(\varepsilon)$ is bounded. Moreover, it also shows that our result is applicable for multiple solutions.

Example 3.2 Consider the following NSOCP:

$$\begin{aligned} \min \quad & f(x) = e^{x_1^3 - x_2^2} \\ \text{s.t.} \quad & g(x) = \begin{bmatrix} x_1 \\ x_2^2 \end{bmatrix} \succeq_{\mathcal{K}_2} 0 \\ & h(x) = x_2^4 - 1 = 0. \end{aligned}$$

The optimal solution are $x^* = (1, 1)$ and $(1, -1)$. Note that, for all $\varepsilon > 0$, the sets $\Upsilon(\varepsilon)$ and $\Gamma(\varepsilon)$ are both unbounded, but their intersection $\Upsilon(\varepsilon) \cap \Gamma(\varepsilon)$ is bounded. According to the KKT condition (10), we know $\lambda^* = (3, -3)$ and $\mu^* = -1$ is a common Lagrangian multipliers at $(1, 1)$ and $(1, -1)$. However, for either $x^* = (1, 1)$ or $x^* = (1, -1)$, we always have

$$\nabla^2_{xx} L(x^*, \lambda^*, \mu^*) = \begin{bmatrix} 15 & -6x_2^* \\ -6x_2^* & -4 \end{bmatrix}$$

and

$$\mathcal{H}(x^*, \lambda^*) = -\frac{\lambda_1^*}{x_1^*} \begin{bmatrix} 1 & 0 \\ 0 & 2x_2^* \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2x_2^* \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 12 \end{bmatrix}.$$

Even though $\nabla^2_{xx} L(x^*, \lambda^*, \mu^*)$ is indefinite,

$$\nabla^2_{xx} L(x^*, \lambda^*, \mu^*) + \mathcal{H}(x^*, \lambda^*) = \begin{bmatrix} 12 & -6x_2^* \\ -6x_2^* & 8 \end{bmatrix}$$

is positive definite and hence (x^*, λ^*, μ^*) is a local saddle point of $\mathcal{L}_c$ for some $c > 0$ by Theorem 2.3.

This example again clarifies the importance of the sigma-term involved in second-order conditions in NSOCP.

4 Exact penalty representation

In this section, we further study another important concept, exact penalty representation. We show that this concept has close relationship to global saddle points. First, we introduce what exact penalty representation means.

Definition 4.1 A pair $(\lambda^*, \mu^*) \in \mathbb{R}^p \times \mathbb{R}^l$ is said to support an exact penalty representation in the framework of $\mathcal{L}_c$ if there exists $c^* > 0$ such that

$$p(0) = \inf_{x \in \mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) \qquad \forall c \geq c^* \tag{33}$$

$$S^* = \arg\min_{x \in \mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) \qquad \forall c \geq c^*. \tag{34}$$

Proposition 4.1 *A pair (λ^*, μ^*) supports an exact penalty representation in the framework of $\mathcal{L}_c$ if and only if there exists $c^* > 0$ such that*

$$p(0) = \inf_{x \in \mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) \tag{35}$$

$$S^* = \arg\min_{x \in \mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*). \tag{36}$$

Proof We only need to show the sufficiency because the necessity is trivial. Let $c > c^*$. For any $\varepsilon > 0$, there must exist a feasible point x_0 such that $f(x_0) \leq p(0) + \varepsilon$. Hence,

$$\begin{aligned} p(0) = \inf_{x \in \mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) &\leq \inf_{x \in \mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) \\ &\leq \mathcal{L}_c(x_0, \lambda^*, \mu^*) \leq f(x_0) \leq p(0) + \varepsilon, \end{aligned} \tag{37}$$

where the first equality follows from (35), the first inequality comes from the monotonicity of $\mathcal{L}_c$ with respect to c, and the third inequality is due to the feasibility of x_0. Since $\varepsilon > 0$ is arbitrary, we obtain from (37) that $p(0) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*)$ for all $c \geq c^*$, which establishes (33).

To show the validity of (34), we consider the following two cases.

Case 1: $S^* = \emptyset$. Suppose on the contrary that there exists $\bar{c} > c^*$ such that

$$\arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\bar{c}}(x, \lambda^*, \mu^*) \neq \emptyset.$$

Then, pick $\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\bar{c}}(x, \lambda^*, \mu^*)$ which leads to

$$\mathcal{L}_{c^*}(\bar{x}, \lambda^*, \mu^*) \leq \mathcal{L}_{\bar{c}}(\bar{x}, \lambda^*, \mu^*) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_{\bar{c}}(x, \lambda^*, \mu^*) = p(0) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*), \tag{38}$$

where the last two steps are due to (33) just shown above. Hence,

$$\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) = S^*$$

by applying (36). This means that S^* is not empty which is a contradiction.

Case 2: $S^* \neq \emptyset$. Take $\bar{x} \in S^*$. For any $c > c^*$, we know

$$\mathcal{L}_c(\bar{x}, \lambda^*, \mu^*) \leq f(\bar{x}) = p(0) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*),$$

where the first inequality is due to the feasibility of $\bar{x}$ and the last step comes from (33). Hence, we obtain

$$\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*)$$

which says $S^* \subset \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*)$.
On the other hand, take $\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*)$. Similar to (38) we obtain

$$\mathcal{L}_{c^*}(\bar{x}, \lambda^*, \mu^*) \leq \mathcal{L}_c(\bar{x}, \lambda^*, \mu^*) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) = p(0) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*),$$

i.e., $\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) = S^*$, where the equality is due to (36). Hence, $\arg\min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) \subset S^*$. This completes the proof. □

Remark 4.1 In Definition 4.1, in order to clarify a pair (λ^*, μ^*) to be an exact penalty representation, we have to check (33) and (34) for all $c \geq c^*$. At the first glance, this task is more difficult and impossible in applications. However, our result shows that it is enough to only check at some c^* not all other $c > c^*$.

The close relationship between global saddle points and exact penalty representations is described as below.

Theorem 4.1 *Suppose that S^* is nonempty. Then a triple (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_c$ if and only if $x^* \in S^*$ and (λ^*, μ^*) supports an exact penalty representation in the framework of $\mathcal{L}_c$.*

Proof (a) Sufficiency. Since $x^* \in S^*$, it follows from (34) that $x^* \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*)$ and from (33) that

$$\mathcal{L}_{c^*}(x^*, \lambda, \mu) \le f(x^*) = \mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*) = \min_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) \le \mathcal{L}_{c^*}(x, \lambda^*, \mu^*)$$

whenever $x \in \mathbb{R}^n$ and $(\lambda, \mu) \in \mathbb{R}^p \times \mathbb{R}^l$. That is, (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_{c^*}$.

(b) Necessity. Following almost the same argument given in Theorem 2.1, we know that if (x^*, λ^*, μ^*) is a global saddle point of $\mathcal{L}_c$ for some $c^* > 0$, then x^* is a global optimal solution of (NSOCP) (i.e., $x^* \in S^*$) and $\mathcal{L}_c(x^*, \lambda^*, \mu^*) = f(x^*)$ for all $c > 0$. Hence for all $c \ge c^*$, we have

$$\begin{aligned} p(0) &= f(x^*) = \mathcal{L}_{c^*}(x^*, \lambda^*, \mu^*) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) \\ &\le \min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*) \le \mathcal{L}_c(x^*, \lambda^*, \mu^*) \le f(x^*) \\ &= p(0), \end{aligned}$$

where the third equality comes from (4) because (x^*, λ^*, μ^*) is a global saddle point by hypothesis. Thus, $p(0) = \min_{x\in\mathbb{R}^n} \mathcal{L}_c(x, \lambda^*, \mu^*)$ for all $c \ge c^*$. This establishes (33).

Next, let $\bar{x} \in S^*$. For all $\tilde{c} \ge c^*$, we have

$$\mathcal{L}_{\tilde{c}}(\bar{x}, \lambda^*, \mu^*) \le f(\bar{x}) = p(0) = \inf_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*),$$

where the first inequality is due to the feasibility of $\bar{x}$. Hence, $\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*)$ which says $S^* \subset \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*)$.

On the other hand, let $\bar{x} \in \arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*)$ with $\tilde{c} > c^*$. Then,

$$0 = \min_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*) - \min_{x\in\mathbb{R}^n} \mathcal{L}_{c^*}(x, \lambda^*, \mu^*) \ge \mathcal{L}_{\tilde{c}}(\bar{x}, \lambda^*, \mu^*) - \mathcal{L}_{c^*}(\bar{x}, \lambda^*, \mu^*) \ge 0,$$

where the first equality comes from (33) and the last inequality is due to the monotonicity of $\mathcal{L}_c$ with respect to c. This further implies

$$\mathcal{L}_c(\bar{x}, \lambda^*, \mu^*) = \mathcal{L}_{c^*}(\bar{x}, \lambda^*, \mu^*) \qquad \forall c \in [c^*, \tilde{c}].$$

Therefore, $\nabla_c \mathcal{L}_c(\bar{x}, \lambda^*, \mu^*) = 0$ for $c \in (c^*, \tilde{c})$. Note that

$$\begin{aligned} \nabla_c \mathcal{L}_c(\bar{x}, \lambda^*, \mu^*) &= \frac{1}{2}\|h(\bar{x})\|^2 + \frac{1}{2}\sum_{j=1}^{J}\left[\operatorname{dist}^2\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}, \mathcal{K}_j\right) - \left\|\frac{\lambda_j^*}{c}\right\|^2\right] \\ &\quad + \frac{c}{2}\sum_{j=1}^{J}\left[2\left\langle g_j(\bar{x}) - \frac{\lambda_j^*}{c} - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right), \frac{\lambda_j^*}{c^2}\right\rangle + 2\frac{\|\lambda_j^*\|^2}{c^3}\right] \\ &= \frac{1}{2}\|h(\bar{x})\|^2 + \frac{1}{2}\sum_{j=1}^{J}\left[\left\|g_j(\bar{x}) - \frac{\lambda_j^*}{c} - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right)\right\|^2\right. \\ &\quad \left. + 2\left\langle g_j(\bar{x}) - \frac{\lambda_j^*}{c} - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right), \frac{\lambda_j^*}{c}\right\rangle + \left\|\frac{\lambda_j^*}{c}\right\|^2\right] \end{aligned}$$

$$= \frac{1}{2}\|h(\bar{x})\|^2 + \frac{1}{2}\sum_{j=1}^{J}\left\|\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c} - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right)\right) + \frac{\lambda_j^*}{c}\right\|^2$$

$$= \frac{1}{2}\|h(\bar{x})\|^2 + \frac{1}{2}\sum_{j=1}^{J}\left\|g_j(\bar{x}) - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right)\right\|^2.$$

Thus $h(\bar{x}) = 0$ and $g_j(\bar{x}) = \Pi_{\mathcal{K}_j}\big(g_j(\bar{x}) - (\lambda_j^*/c)\big) \in \mathcal{K}_j$. This means that $\bar{x}$ is feasible and

$$\operatorname{dist}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}, \mathcal{K}_j\right) = \left\|g_j(\bar{x}) - \frac{\lambda_j^*}{c} - \Pi_{\mathcal{K}_j}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{c}\right)\right\| = \left\|\frac{\lambda_j^*}{c}\right\|,$$

from which (by letting $c \to \tilde{c}$) we have

$$\operatorname{dist}\left(g_j(\bar{x}) - \frac{\lambda_j^*}{\tilde{c}}, \mathcal{K}_j\right) = \left\|\frac{\lambda_j^*}{\tilde{c}}\right\|.$$

Thus, $f(\bar{x}) = \mathcal{L}_{\tilde{c}}(\bar{x}, \lambda^*, \mu^*) = p(0)$ where the last step is due to (33). Since $\bar{x}$ is feasible as shown above, it ensures that $\bar{x} \in S^*$ which says $\arg\min_{x\in\mathbb{R}^n} \mathcal{L}_{\tilde{c}}(x, \lambda^*, \mu^*) \subset S^*$. $\square$

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