

## USING SCHUR COMPLEMENT THEOREM TO PROVE CONVEXITY OF SOME SOC-FUNCTIONS

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**ABSTRACT.** In this paper, we provide an important application of the Schur Complement Theorem in establishing convexity of some functions associated with second-order cones (SOCs), called SOC-trace functions. As illustrated in the paper, these functions play a key role in the development of penalty and barrier functions methods for second-order cone programs, and establishment of some important inequalities associated with SOC.

### 1. INTRODUCTION

The second-order cone (SOC) in  $\mathbb{R}^n$ , also called Lorentz cone, is a set defined as

$$(1.1) \quad \mathcal{K}^n := \left\{ (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x_2\| \right\},$$

where  $\|\cdot\|$  denotes the Euclidean norm. When  $n = 1$ ,  $\mathcal{K}^n$  reduces to the set of nonnegative real numbers  $\mathbb{R}_+$ . As shown in [11],  $\mathcal{K}^n$  is also a set composed of the squared elements from Jordan algebra  $(\mathbb{R}^n, \circ)$ , where the Jordan product “ $\circ$ ” is a binary operation defined by

$$(1.2) \quad x \circ y := (\langle x, y \rangle, x_1 y_2 + y_1 x_2)$$

for any  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . Unless otherwise stated, in the rest of this note, we use  $e = (1, 0, \dots, 0)^T \in \mathbb{R}^n$  to denote the unit element of Jordan algebra  $(\mathbb{R}^n, \circ)$ , i.e.,  $e \circ x = x$  for any  $x \in \mathbb{R}^n$ , and for any  $x \in \mathbb{R}^n$ , use  $x_1$  to denote the first component of  $x$ , and  $x_2$  to denote the vector consisting of the rest  $n - 1$  components.

From [12, 11], we recall that each  $x \in \mathbb{R}^n$  admits a spectral factorization associated with  $\mathcal{K}^n$ , of the following form

$$(1.3) \quad x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)},$$

where  $\lambda_i(x)$  and  $u_x^{(i)}$  for  $i = 1, 2$  are the spectral values and the associated spectral vectors of  $x$ , respectively, defined by

$$(1.4) \quad \lambda_i(x) = x_1 + (-1)^i \|x_2\|, \quad u_x^{(i)} = \frac{1}{2} \left( 1, (-1)^i \bar{x}_2 \right),$$

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with  $\bar{x}_2 = \frac{x_2}{\|x_2\|}$  if  $x_2 \neq 0$ , and otherwise  $\bar{x}_2$  being any vector in  $\mathbb{R}^{n-1}$  such that  $\|\bar{x}_2\| = 1$ . When  $x_2 \neq 0$ , the spectral factorization is unique. The determinant and trace of  $x$  are defined as  $\det(x) := \lambda_1(x)\lambda_2(x)$  and  $\text{tr}(x) := \lambda_1(x) + \lambda_2(x)$ , respectively.

With the spectral factorization above, for any given scalar function  $f: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , we may define a vector-valued function  $f^{\text{soc}}: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$  by

$$(1.5) \quad f^{\text{soc}}(x) := f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}$$

where  $J$  is an interval (finite or infinite, open or closed) of  $\mathbb{R}$ , and  $S$  is the domain of  $f^{\text{soc}}$  determined by  $f$ . Obviously,  $f^{\text{soc}}$  is well-defined whether  $x_2 = 0$  or not. Assume that  $f$  is differentiable on  $\text{int}J$ . Then, by Lemma 2.2 in Section 2,  $f^{\text{soc}}$  is differentiable on  $\text{int}S$ , and moreover, for any  $x \in \text{int}S$ ,  $\nabla f^{\text{soc}}(x)$  has a structure which entails the application of the Schur Complement Theorem in establishing its positive semidefiniteness or positive definiteness. On the other hand, Lemma 2.2 shows that the SOC-trace function

$$(1.6) \quad f^{\text{tr}}(x) := f(\lambda_1(x)) + f(\lambda_2(x)) = \text{tr}(f^{\text{soc}}(x)) \quad \forall x \in S$$

is differentiable on  $\text{int}S$  with  $\nabla f^{\text{tr}}(x) = (f')^{\text{soc}}(x)$  for any  $x \in \text{int}S$ , which means that

$$(1.7) \quad \nabla^2 f^{\text{tr}}(x) = \nabla (f')^{\text{soc}}(x) \quad \forall x \in \text{int}S.$$

The two sides show that we may establish the convexity of  $f^{\text{tr}}$  with the help of the Schur Complement Theorem. In fact, the Schur Complement Theorem is frequently used in the topics related to semidefinite programming (SDP), but there is no paper to introduce its application involving SOCs, to the best of our knowledge.

Motivated by this, in this paper we provide such an application of the Schur Complement Theorem in establishing the convexity of SOC-trace functions and the compounds of SOC-trace functions and real-valued functions. As illustrated in the next section, some of these functions are the key of penalty and barrier function methods for second-order cone programs (SOCPs), as well as the establishment of some important inequalities associated with SOCs, for which the proof of convexity of these functions is a necessity. But, this requires computation of the first and second-order derivatives, which is technically much more demanding than in the linear and semidefinite cases. As will be seen, Theorem 2.1 gives a simple way to achieve this objective via the Schur Complement Theorem, by which one only needs to check the sign of the second-order derivative of a scalar function.

Throughout this note, for any  $x, y \in \mathbb{R}^n$ , we write  $x \succeq_{\mathcal{K}^n} y$  if  $x - y \in \mathcal{K}^n$ ; and write  $x \succ_{\mathcal{K}^n} y$  if  $x - y \in \text{int}\mathcal{K}^n$ . For a real symmetric matrix  $A$ , we write  $A \succeq 0$  (respectively,  $A \succ 0$ ) if  $A$  is positive semidefinite (respectively, positive definite). For any  $f: J \rightarrow \mathbb{R}$ ,  $f'(t)$  and  $f''(t)$  denote the first derivative and second-order derivative of  $f$  at the differentiable point  $t \in J$ , respectively; for any  $F: S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\nabla F(x)$  and  $\nabla^2 F(x)$  denote the gradient and the Hessian matrix of  $F$  at the differentiable point  $x \in S$ , respectively.

2. MAIN RESULTS

The Schur Complement Theorem gives a characterization for the positive semidefiniteness (definiteness) of a matrix via the positive semidefiniteness (definiteness) of the Schur-complement with respect to a block partitioning of the matrix, which is stated as below.

**Lemma 2.1** (Schur Complement Theorem [13]). *Let  $A \in \mathbb{R}^{m \times m}$  be a symmetric positive definite matrix,  $C \in \mathbb{R}^{n \times n}$  be a symmetric matrix, and  $B \in \mathbb{R}^{m \times n}$ . Then,*

$$(2.1) \quad \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \iff C - B^T A^{-1} B \succeq 0$$

and

$$(2.2) \quad \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff C - B^T A^{-1} B \succ 0.$$

In this section, we focus on on the application of the Schur Complement Theorem in establishing convexity of SOC-trace functions. To the end, we need the following lemma.

**Lemma 2.2.** *For any given  $f: J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ , let  $f^{\text{soc}}: S \rightarrow \mathbb{R}^n$  and  $f^{\text{tr}}: S \rightarrow \mathbb{R}$  be given by (1.5) and (1.6), respectively. Assume that  $J$  is open. Then, the following results hold.*

- (a) *The domain  $S$  of  $f^{\text{soc}}$  and  $f^{\text{tr}}$  is also open.*
- (b) *If  $f$  is (continuously) differentiable on  $J$ , then  $f^{\text{soc}}$  is (continuously) differentiable on  $S$ . Moreover, for any  $x \in S$ ,  $\nabla f^{\text{soc}}(x) = f'(x_1)I$  if  $x_2 = 0$ , and otherwise*

$$(2.3) \quad \nabla f^{\text{soc}}(x) = \begin{bmatrix} b(x) & c(x) \frac{x_2^T}{\|x_2\|} \\ c(x) \frac{x_2}{\|x_2\|} & a(x)I + (b(x) - a(x)) \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix},$$

where

$$\begin{aligned} b(x) &= \frac{f'(\lambda_2(x)) + f'(\lambda_1(x))}{2}, \\ c(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{2}, \\ a(x) &= \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}. \end{aligned}$$

- (c) *If  $f$  is (continuously) differentiable, then  $f^{\text{tr}}$  is (continuously) differentiable on  $S$  with  $\nabla f^{\text{tr}}(x) = (f')^{\text{soc}}(x)$ ; if  $f$  is twice (continuously) differentiable, then  $f^{\text{tr}}$  is twice (continuously) differentiable on  $S$  with  $\nabla^2 f^{\text{tr}}(x) = \nabla(f')^{\text{soc}}(x)$ .*

*Proof.* (a) Fix any  $x \in S$ . Then  $\lambda_1(x), \lambda_2(x) \in J$ . Since  $J$  is an open subset of  $\mathbb{R}$ , there exist  $\delta_1, \delta_2 > 0$  such that  $\{t \in \mathbb{R} \mid |t - \lambda_1(x)| < \delta_1\} \subseteq J$  and  $\{t \in$

$\mathbb{R}| |t - \lambda_2(x)| < \delta_2 \} \subseteq J$ . Let  $\delta := \min\{\delta_1, \delta_2\}/\sqrt{2}$ . Then, for any  $y$  satisfying  $\|y - x\| < \delta$ , we have  $|\lambda_1(y) - \lambda_1(x)| < \delta_1$  and  $|\lambda_2(y) - \lambda_2(x)| < \delta_2$  by noting that

$$\begin{aligned} & (\lambda_1(x) - \lambda_1(y))^2 + (\lambda_2(x) - \lambda_2(y))^2 \\ &= 2(x_1^2 + \|x_2\|^2) + 2(y_1^2 + \|y_2\|^2) - 4(x_1y_1 + \|x_2\|\|y_2\|) \\ &\leq 2(x_1^2 + \|x_2\|^2) + 2(y_1^2 + \|y_2\|^2) - 4(x_1y_1 + \langle x_2, y_2 \rangle) \\ &= 2(\|x\|^2 + \|y\|^2 - 2\langle x, y \rangle) = 2\|x - y\|^2, \end{aligned}$$

and consequently  $\lambda_1(y) \in J$  and  $\lambda_2(y) \in J$ . Since  $f$  is a function from  $J$  to  $\mathbb{R}$ , this means that  $\{y \in \mathbb{R}^n \mid \|y - x\| < \delta\} \subseteq S$ , and therefore the set  $S$  is open.

(b) The proof is direct by using the same arguments as those of [10, Props. 4 and 5].

(c) If  $f$  is (continuously) differentiable, then from part (b) and  $f^{\text{tr}}(x) = e^T f^{\text{soc}}(x)$  it follows that  $f^{\text{tr}}$  is (continuously) differentiable. In addition, a simple computation yields that  $\nabla f^{\text{tr}}(x) = \nabla f^{\text{soc}}(x)e = (f')^{\text{soc}}(x)$ . Similarly, by part (b), the second part follows.  $\square$

**Theorem 2.3.** *For any given  $f : J \rightarrow \mathbb{R}$ , let  $f^{\text{soc}} : S \rightarrow \mathbb{R}^n$  and  $f^{\text{tr}} : S \rightarrow \mathbb{R}$  be given by (1.5) and (1.6), respectively. Assume that  $J$  is open. If  $f$  is twice differentiable on  $J$ , then*

- (a)  $f''(t) \geq 0$  for any  $t \in J \iff \nabla(f')^{\text{soc}}(x) \succeq 0$  for any  $x \in S \iff f^{\text{tr}}$  is convex in  $S$ .
- (b)  $f''(t) > 0$  for any  $t \in J \iff \nabla(f')^{\text{soc}}(x) \succ 0 \forall x \in S \implies f^{\text{tr}}$  is strictly convex in  $S$ .

*Proof.* (a) By Lemma 2.2(c),  $\nabla^2 f^{\text{tr}}(x) = \nabla(f')^{\text{soc}}(x)$  for any  $x \in S$ , and the second equivalence follows by Prop. B.4(a) and (c) of [7]. We next come to the first equivalence. By Lemma 2.2(b), for any fixed  $x \in S$ ,  $\nabla(f')^{\text{soc}}(x) = f''(x_1)I$  if  $x_2 = 0$ , and otherwise  $\nabla(f')^{\text{soc}}(x)$  has the same expression as in (2.3) except that

$$\begin{aligned} b(x) &= \frac{f''(\lambda_2(x)) + f''(\lambda_1(x))}{2}, \\ c(x) &= \frac{f''(\lambda_2(x)) - f''(\lambda_1(x))}{2}, \\ a(x) &= \frac{f'(\lambda_2(x)) - f'(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}. \end{aligned}$$

Assume that  $\nabla(f')^{\text{soc}}(x) \succeq 0$  for any  $x \in S$ . Then, we readily have  $b(x) \geq 0$  for any  $x \in S$ . Noting that  $b(x) = f''(x_1)$  when  $x_2 = 0$ , we particularly have  $f''(x_1) \geq 0$  for all  $x_1 \in J$ , and consequently  $f''(t) \geq 0$  for all  $t \in J$ . Assume that  $f''(t) \geq 0$  for all  $t \in J$ . Fix any  $x \in S$ . Clearly,  $b(x) \geq 0$  and  $a(x) \geq 0$ . If  $b(x) = 0$ , then  $f''(\lambda_1(x)) = f''(\lambda_2(x)) = 0$ , and consequently  $c(x) = 0$ , which in turn implies that

$$(2.4) \quad \nabla(f')^{\text{soc}}(x) = \begin{bmatrix} 0 & 0 \\ 0 & a(x) \left( I - \frac{x_2 x_2^T}{\|x_2\|^2} \right) \end{bmatrix} \succeq 0.$$

If  $b(x) > 0$ , then by the first equivalence of Lemma 2.1 and the expression of  $\nabla(f')^{\text{soc}}(x)$  it suffices to argue that the following matrix

$$(2.5) \quad a(x)I + (b(x) - a(x)) \frac{x_2 x_2^T}{\|x_2\|^2} - \frac{c^2(x)}{b(x)} \frac{x_2 x_2^T}{\|x_2\|^2}$$

is positive semidefinite. Since the rank-one matrix  $x_2 x_2^T$  has only one nonzero eigenvalue  $\|x_2\|^2$ , the matrix in (2.5) has one eigenvalue  $a(x)$  of multiplicity  $n - 1$  and one eigenvalue  $\frac{b(x)^2 - c(x)^2}{b(x)}$  of multiplicity 1. Since  $a(x) \geq 0$  and  $\frac{b(x)^2 - c(x)^2}{b(x)} = f''(\lambda_1(x))f''(\lambda_2(x)) \geq 0$ , the matrix in (2.5) is positive semidefinite. By the arbitrariness of  $x$ , we have that  $\nabla(f')^{\text{soc}}(x) \succeq 0$  for all  $x \in S$ .

(b) The first equivalence is direct by using (2.2) of Lemma (2.1), noting  $\nabla(f')^{\text{soc}}(x) \succ 0$  implies  $a(x) > 0$  when  $x_2 \neq 0$ , and following the same arguments as part (a). The second part is due to [7, Prop. B.4(b)].  $\square$

**Remark 2.4.** Note that the strict convexity of  $f^{\text{tr}}$  does not necessarily imply the positive definiteness of  $\nabla^2 f^{\text{tr}}(x)$ . Consider  $f(t) = t^4$  for  $t \in \mathbb{R}$ . We next show that  $f^{\text{tr}}$  is strictly convex. Indeed,  $f^{\text{tr}}$  is convex in  $\mathbb{R}^n$  by Theorem 2.3(a) since  $f''(t) = 12t^2 \geq 0$ . Taking into account that  $f^{\text{tr}}$  is continuous, it remains to prove that

$$(2.6) \quad f^{\text{tr}}\left(\frac{x+y}{2}\right) = \frac{f^{\text{tr}}(x) + f^{\text{tr}}(y)}{2} \implies x = y.$$

Since  $h(t) = (t_0 + t)^4 + (t_0 - t)^4$  for some  $t_0 \in \mathbb{R}$  is increasing on  $[0, +\infty)$ , and the function  $f(t) = t^4$  is strictly convex in  $\mathbb{R}$ , we have that

$$\begin{aligned} f^{\text{tr}}\left(\frac{x+y}{2}\right) &= \left[\lambda_1\left(\frac{x+y}{2}\right)\right]^4 + \left[\lambda_2\left(\frac{x+y}{2}\right)\right]^4 \\ &= \left(\frac{x_1 + y_1 - \|x_2 + y_2\|}{2}\right)^4 + \left(\frac{x_1 + y_1 + \|x_2 + y_2\|}{2}\right)^4 \\ &\leq \left(\frac{x_1 + y_1 - \|x_2\| - \|y_2\|}{2}\right)^4 + \left(\frac{x_1 + y_1 + \|x_2\| + \|y_2\|}{2}\right)^4 \\ &= \left(\frac{\lambda_1(x) + \lambda_1(y)}{2}\right)^4 + \left(\frac{\lambda_2(x) + \lambda_2(y)}{2}\right)^4 \\ &\leq \frac{(\lambda_1(x))^4 + (\lambda_1(y))^4 + (\lambda_2(x))^4 + (\lambda_2(y))^4}{2} \\ &= \frac{f^{\text{tr}}(x) + f^{\text{tr}}(y)}{2}, \end{aligned}$$

and moreover, the above inequalities become the equalities if and only if

$$\|x_2 + y_2\| = \|x_2\| + \|y_2\|, \lambda_1(x) = \lambda_1(y), \lambda_2(x) = \lambda_2(y).$$

It is easy to verify that the three equalities hold if and only if  $x = y$ . So, the implication in (2.6) holds, i.e.,  $f^{\text{tr}}$  is strictly convex. However, by Theorem 2.3(b),  $\nabla(f')^{\text{soc}}(x) \succ 0$  does not hold for all  $x \in \mathbb{R}^n$  since  $f''(t) > 0$  does not hold for all  $t \in \mathbb{R}$ .

It should be mentioned that the fact that the strict convexity of  $f$  implies the strict convexity of  $f^{\text{tr}}$  was proved in [2, 5] via the definition of convex function, but here we use the Shcur Complement Theorem and the relation between  $\nabla(f')^{\text{soc}}$  and  $\nabla^2 f^{\text{tr}}$  to establish the convexity of SOC-trace functions. In addition, we also note that the necessity involved in the first equivalence of Theorem 2.3(a) was given in [12] via a different way. Next, we illustrate the application of Theorem 2.3 with some SOC-trace functions.

**Proposition 2.5.** *The following functions associated with  $\mathcal{K}^n$  are all strictly convex.*

- (a)  $F_1(x) = -\ln(\det(x))$  for  $x \in \text{int}\mathcal{K}^n$ .
- (b)  $F_2(x) = \text{tr}(x^{-1})$  for  $x \in \text{int}\mathcal{K}^n$ .
- (c)  $F_3(x) = \text{tr}(\phi(x))$  for  $x \in \text{int}\mathcal{K}^n$ , where

$$\phi(x) = \begin{cases} \frac{x^{p+1}-e}{p+1} + \frac{x^{1-q}-e}{q-1} & \text{if } p \in [0, 1], q > 1; \\ \frac{x^{p+1}-e}{p+1} - \ln x & \text{if } p \in [0, 1], q = 1. \end{cases}$$

- (d)  $F_4(x) = -\ln(\det(e-x))$  for  $x \prec_{\mathcal{K}^n} e$ .
- (e)  $F_5(x) = \text{tr}((e-x)^{-1} \circ x)$  for  $x \prec_{\mathcal{K}^n} e$ .
- (f)  $F_6(x) = \text{tr}(\exp(x))$  for  $x \in \mathbb{R}^n$ .
- (g)  $F_7(x) = \ln(\det(e + \exp(x)))$  for  $x \in \mathbb{R}^n$ .
- (h)  $F_8(x) = \text{tr}\left(\frac{x + (x^2 + 4e)^{1/2}}{2}\right)$  for  $x \in \mathbb{R}^n$ .

*Proof.* Note that  $F_1(x), F_2(x)$  and  $F_3(x)$  are the SOC-trace functions associated with  $f_1(t) = -\ln t$  ( $t > 0$ ),  $f_2(t) = t^{-1}$  ( $t > 0$ ) and  $f_3(t)$  ( $t > 0$ ), respectively, where

$$f_3(t) = \begin{cases} \frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1} & \text{if } p \in [0, 1], q > 1, \\ \frac{t^{p+1}-1}{p+1} - \ln t & \text{if } p \in [0, 1], q = 1; \end{cases}$$

$F_4(x)$  is the SOC-trace function associated with  $f_4(t) = -\ln(1-t)$  ( $t < 1$ ),  $F_5(x)$  is the SOC-trace function associated with  $f_5(t) = \frac{t}{1-t}$  ( $t < 1$ ) by noting that

$$(e-x)^{-1} \circ x = \frac{\lambda_1(x)}{\lambda_1(e-x)} u_x^{(1)} + \frac{\lambda_2(x)}{\lambda_2(e-x)} u_x^{(2)};$$

$F_6(x)$  and  $F_7(x)$  are the SOC-trace functions associated with  $f_6(t) = \exp(t)$  ( $t \in \mathbb{R}$ ) and  $f_7(t) = \ln(1+\exp(t))$  ( $t \in \mathbb{R}$ ), respectively, and  $F_8(x)$  is the SOC-trace function associated with  $f_8(t) = 2^{-1} \left(t + \sqrt{t^2 + 4}\right)$  ( $t \in \mathbb{R}$ ). It is easy to verify that the functions  $f_1$ - $f_8$  have positive second-order derivatives in their respective domain, and therefore  $F_1$ - $F_8$  are strictly convex functions by Theorem 2.3(b).  $\square$

The functions  $F_1$ ,  $F_2$  and  $F_3$  are the popular barrier functions which play a key role in the development of interior point methods for SOCPs, see, e.g., [6, 4, 14, 15, 17], where  $F_3$  covers a wide range of barrier functions, including the classical logarithmic barrier function, the self-regular functions and the non-self-regular functions; see [4] for details. The functions  $F_4$  and  $F_5$  are the popular shifted barrier functions [1, 2, 3] for SOCPs, and  $F_6$ - $F_8$  can be used as penalty functions for second-order cone programs (SOCPs), and these functions are added

to the objective of SOCPs for forcing the solution to be feasible.

Besides the application in establishing convexity for SOC-trace functions, the Schur complement theorem can be employed to establish convexity of some compound functions of SOC-trace functions and scalar-valued functions, which is usually difficult to achieve by the definition of convex function. The following proposition presents such an application.

**Proposition 2.6.** *For any  $x \in \mathcal{K}^n$ , let  $F_9(x) := -[\det(x)]^{1/p}$  with  $p > 1$ . Then,*

- (a)  $F_9$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}^n$ .
- (b)  $F_9$  is convex when  $p \geq 2$ , and moreover, it is strictly convex when  $p > 2$ .

*Proof.* (a) Note that  $-F_9(x) = \exp(p^{-1} \ln(\det(x)))$  for any  $x \in \mathbf{int}\mathcal{K}^n$ , and  $\ln(\det(x)) = f^{\text{tr}}(x)$  with  $f(t) = \ln(t)$  for  $t \in \mathbb{R}_{++}$ . By Lemma 2.2(c),  $\ln(\det(x))$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}^n$ . Hence  $-F_9(x)$  is twice continuously differentiable in  $\mathbf{int}\mathcal{K}^n$ . The result then follows.

(b) In view of the continuity of  $F_9$ , we only need to prove its convexity over  $\mathbf{int}\mathcal{K}^n$ . By part (a), we next achieve this goal by proving that the Hessian matrix  $\nabla^2 F_9(x)$  for any  $x \in \mathbf{int}\mathcal{K}^n$  is positive semidefinite when  $p \geq 2$ , and positive definite when  $p > 2$ . Fix any  $x \in \mathbf{int}\mathcal{K}^n$ . From direct computations, we obtain

$$\nabla F_9(x) = -\frac{1}{p} \begin{bmatrix} (2x_1) (x_1^2 - \|x_2\|^2)^{\frac{1}{p}-1} \\ (-2x_2) (x_1^2 - \|x_2\|^2)^{\frac{1}{p}-1} \end{bmatrix}$$

and  
(2.7)

$$\nabla^2 F_9(x) = \frac{p-1}{p^2} (\det(x))^{\frac{1}{p}-2} \begin{bmatrix} 4x_1^2 - \frac{2p(x_1^2 - \|x_2\|^2)}{p-1} & -4x_1x_2^T \\ -4x_1x_2 & 4x_2x_2^T + \frac{2p(x_1^2 - \|x_2\|^2)}{p-1}I \end{bmatrix}.$$

Since  $x \in \mathbf{int}\mathcal{K}^n$ , we have  $x_1 > 0$  and  $\det(x) = x_1^2 - \|x_2\|^2 > 0$ , and therefore

$$a_1(x) := 4x_1^2 - \frac{2p(x_1^2 - \|x_2\|^2)}{p-1} = \left(4 - \frac{2p}{p-1}\right)x_1^2 + \frac{2p}{p-1}\|x_2\|^2.$$

We next proceed the arguments by the following two cases: (1)  $a_1(x) = 0$ ; (2)  $a_1(x) > 0$ .

Case 1:  $a_1(x) = 0$ . Since  $p \geq 2$ , under this case we must have  $x_2 = 0$ , and consequently,

$$\nabla^2 F_9(x) = \frac{p-1}{p^2} (x_1)^{\frac{2}{p}-4} \begin{bmatrix} 0 & 0 \\ 0 & \frac{2p}{p-1}x_1^2I \end{bmatrix} \succeq 0.$$

Case 2:  $a_1(x) > 0$ . Under this case, we calculate that

$$\begin{aligned} & \begin{bmatrix} 4x_1^2 - \frac{2p(x_1^2 - \|x_2\|^2)}{p-1} \end{bmatrix} \begin{bmatrix} 4x_2x_2^T + \frac{2p(x_1^2 - \|x_2\|^2)}{p-1}I \end{bmatrix} - 16x_1^2x_2x_2^T \\ (2.8) \quad & = \frac{4p(x_1^2 - \|x_2\|^2)}{p-1} \left[ \frac{p-2}{p-1}x_1^2I + \frac{p}{p-1}\|x_2\|^2I - 2x_2x_2^T \right]. \end{aligned}$$

Since the rank-one matrix  $2x_2x_2^T$  has only one nonzero eigenvalue  $2\|x_2\|^2$ , the matrix in the bracket of the right hand side of (2.8) has one eigenvalue of multiplicity 1 given by

$$\frac{p-2}{p-1}x_1^2 + \frac{p}{p-1}\|x_2\|^2 - 2\|x_2\|^2 = \frac{p-2}{p-1}(x_1^2 - \|x_2\|^2) \geq 0,$$

and one eigenvalue of multiplicity  $n-1$  given by  $\frac{p-2}{p-1}x_1^2 + \frac{p}{p-1}\|x_2\|^2 \geq 0$ . Furthermore, we see that these eigenvalues must be positive when  $p > 2$  since  $x_1^2 > 0$  and  $x_1^2 - \|x_2\|^2 > 0$ . This means that the matrix on the right hand side of (2.8) is positive semidefinite, and moreover, it is positive definite when  $p > 2$ . Applying Lemma 2.1, we have that  $\nabla^2 F_9(x) \succeq 0$ , and furthermore  $\nabla^2 F_9(x) \succ 0$  when  $p > 2$ .

Since  $a_1(x) > 0$  must hold when  $p > 2$ , the arguments above show that  $F_9(x)$  is convex over  $\text{int}\mathcal{K}^n$  when  $p \geq 2$ , and strictly convex over  $\text{int}\mathcal{K}^n$  when  $p > 2$ .  $\square$

It is worthwhile to point out that  $\det(x)$  is neither convex nor concave on  $\mathcal{K}^n$ , and it is difficult to argue the convexity of those compound functions involving  $\det(x)$  by the definition of convex function. But, as shown in Proposition 2.6, the Schur Complement Theorem offers a simple way to prove their convexity.

To close section, we take a look at the application of some of convex functions above in establishing inequalities associated with SOCs. Some of these inequalities have been used to analyze the properties of SOC-function  $f^{\text{soc}}$  [9] and the convergence of interior point methods for SOCPs [2].

**Proposition 2.7.** *For any  $x \succeq_{\mathcal{K}^n} 0$  and  $y \succeq_{\mathcal{K}^n} 0$ , the following inequalities hold.*

- (a)  $\det(\alpha x + (1 - \alpha)y) \geq (\det(x))^\alpha (\det(y))^{1-\alpha}$  for any  $0 < \alpha < 1$ .
- (b)  $\det(x + y)^{1/p} \geq 2^{\frac{2}{p}-1} (\det(x)^{1/p} + \det(y)^{1/p})$  for any  $p \geq 2$ .
- (c)  $\det(x + y) \geq \det(x) + \det(y)$ .
- (d)  $\det(\alpha x + (1 - \alpha)y) \geq \alpha^2 \det(x) + (1 - \alpha)^2 \det(y)$  for any  $0 < \alpha < 1$ .
- (e)  $[\det(e + x)]^{1/2} \geq 1 + \det(x)^{1/2}$ .
- (f) If  $x \succeq_{\mathcal{K}^n} y$ , then  $\det(x) \geq \det(y)$ .
- (g)  $\det(x)^{1/2} = \inf \left\{ \frac{1}{2} \text{tr}(x \circ y) : \det(y) = 1, y \succeq_{\mathcal{K}^n} 0 \right\}$ . Furthermore, when  $x \succ_{\mathcal{K}^n} 0$ , the same relation holds with  $\inf$  replaced by  $\min$ .
- (h)  $\text{tr}(x \circ y) \geq 2\det(x)^{1/2}\det(y)^{1/2}$ .

*Proof.* (a) From Prop. 2.5(a), we know that  $\ln(\det(x))$  is strictly concave in  $\text{int}\mathcal{K}^n$ . Thus,

$$\begin{aligned} \ln(\det(\alpha x + (1 - \alpha)y)) &\geq \alpha \ln(\det(x)) + (1 - \alpha) \ln(\det(y)) \\ &= \ln(\det(x)^\alpha) + \ln(\det(y)^{1-\alpha}) \end{aligned}$$

for any  $0 < \alpha < 1$  and  $x, y \in \text{int}\mathcal{K}^n$ . This, together with the increasing of  $\ln t$  ( $t > 0$ ) and the continuity of  $\det(x)$ , implies the desired result.



(b) By Prop. 2.6(b),  $\det(x)^{1/p}$  is concave over  $\mathcal{K}^n$ . So, for any  $x, y \in \mathcal{K}^n$ , we have that

$$\begin{aligned} \det\left(\frac{x+y}{2}\right)^{1/p} &\geq \frac{1}{2} [\det(x)^{1/p} + \det(y)^{1/p}] \\ \iff 2 \left[ \left(\frac{x_1+y_1}{2}\right)^2 - \left\| \frac{x_2+y_2}{2} \right\|^2 \right]^{1/p} &\geq (x_1^2 - \|x_2\|^2)^{1/p} + (y_1^2 - \|y_2\|^2)^{1/p} \\ \iff [(x_1+y_1)^2 - \|x_2+y_2\|^2]^{1/p} &\geq \frac{4^{1/p}}{2} [(x_1^2 - \|x_2\|^2)^{1/p} + (y_1^2 - \|y_2\|^2)^{1/p}] \\ \iff \det(x+y)^{1/p} &\geq 2^{\frac{2}{p}-1} (\det(x)^{1/p} + \det(y)^{1/p}), \end{aligned}$$

which is the desired result.

(c) Using the inequality in part (b) with  $p = 2$ , we have

$$\det(x+y)^{1/2} \geq \det(x)^{1/2} + \det(y)^{1/2}.$$

Squaring both sides yields

$$\det(x+y) \geq \det(x) + \det(y) + 2\det(x)^{1/2}\det(y)^{1/2} \geq \det(x) + \det(y),$$

where the last inequality is by the nonnegativity of  $\det(x)$  and  $\det(y)$  since  $x, y \in \mathcal{K}^n$ .

(d) The inequality is direct by part (c) and the fact  $\det(\alpha x) = \alpha^2 \det(x)$ .

(e) The inequality follows from part (b) with  $p = 2$  and the fact that  $\det(e) = 1$ .

(f) Using part (c) and noting that  $x \succeq_{\mathcal{K}^n} y$ , it is easy to verify that

$$\det(x) = \det(y+x-y) \geq \det(y) + \det(x-y) \geq \det(y).$$

(g) Using the Cauchy-Schwartz inequality, it is easy to verify that

$$\text{tr}(x \circ y) \geq \lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) \quad \forall x, y \in \mathbb{R}^n.$$

For any  $x, y \in \mathcal{K}^n$ , this along with the arithmetic-geometric mean inequality implies that

$$\begin{aligned} \frac{\text{tr}(x \circ y)}{2} &\geq \frac{\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)}{2} \\ &\geq \sqrt{\lambda_1(x)\lambda_2(y)\lambda_1(y)\lambda_2(x)} \\ &= \det(x)^{1/2}\det(y)^{1/2}, \end{aligned}$$

which means that  $\inf \left\{ \frac{1}{2} \text{tr}(x \circ y) : \det(y) = 1, y \succeq_{\mathcal{K}^n} 0 \right\} = \det(x)^{1/2}$  for a fixed  $x \in \mathcal{K}^n$ . If  $x \succ_{\mathcal{K}^n} 0$ , then we can verify that the feasible point  $y^* = \frac{x^{-1}}{\sqrt{\det(x)}}$  is such that  $\frac{1}{2} \text{tr}(x \circ y^*) = \det(x)^{1/2}$ , and the second part follows.

(h) Using part (g), for any  $x \in \mathcal{K}^n$  and  $y \in \text{int}(\mathcal{K}^n)$ , we have that

$$\frac{\text{tr}(x \circ y)}{2\sqrt{\det(y)}} = \text{tr} \left( x \circ \frac{y}{\sqrt{\det(y)}} \right) \geq \sqrt{\det(x)},$$

which together with the continuity of  $\det(x)$  and  $\text{tr}(x)$  implies that

$$\text{tr}(x \circ y) \geq 2\det(x)^{1/2}\det(y)^{1/2} \quad \forall x, y \in \mathcal{K}^n.$$

Thus, we complete the proof.  $\square$

Note that some of the inequalities in Prop. 2.7 were ever established with the help of the Schwartz-inequality [9], but here we achieve the goal easily by using the convexity of SOC-functions. These inequalities all have the corresponding counterparts for matrix inequalities [8, 13, 16]. For example, Prop. 2.7(b) with  $p = 2$ , i.e.,  $p$  equal to the rank of Jordan algebra  $(\mathbb{R}^n, \circ)$ , corresponds to the Minkowski inequality of matrix case:

$$\det(A + B)^{1/n} \geq \det(A)^{1/n} + \det(B)^{1/n}$$

for any  $n \times n$  positive semidefinite matrices  $A$  and  $B$ .

### 3. CONCLUSIONS

We studied an application of the Schur complement theorem in establishing convexity of SOC-functions, especially for SOC-trace functions, which are the key of penalty and barrier function methods for SOCPs and some important inequalities associated with SOCs. One possible future direction is proving self-concordancy for such barrier/penalty functions associated with SOCs. We also believe that the results in this paper will be helpful towards establishing further properties of other SOC-functions.

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