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Symmetrization of generalized natural residual function for NCP

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ABSTRACT

In contrast to the generalized Fischer–Burmeister function that is a natural extension of the popular Fischer–Burmeister function NCP-function, the generalized natural residual NCP-function based on discrete extension, recently proposed by Chen, Ko, and Wu, does not possess symmetric graph. In this paper we symmetrize the generalized natural residual NCP-function, and construct not only new NCP-functions and merit functions for the nonlinear complementarity problem, but also provide parallel functions to the generalized Fischer–Burmeister function.

1. Motivation

The nonlinear complementarity problem (NCP for short) has attracted much attention since 1970s because of its wide applications in the fields of economics, engineering, and operations research, see [11,12,18] and references therein. The mathematical format for NCP is to find a point $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ F(x) \geq 0, \quad \langle x, F(x) \rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product and $F = (F_1, \dots, F_n)^T$ is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$. For solving NCP, the so-called NCP-function $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$\phi(a, b) = 0 \Longleftrightarrow a, \quad b \geq 0,\ ab = 0,$$

plays a crucial role. More specifically, with such NCP-functions, the NCP can be recast as nonsmooth equations [23,24,29] or unconstrained minimization [13,14,17,20,21,25,28]. During the past four decades, around thirty NCP-functions are proposed, see [16] for a survey. Among them, two popular NCP-functions, the Fischer–Burmeister (denoted by FB) function [10,14,15] and the natural residual (denoted by NR) function [22,26], are frequently employed and most of the existing NCP-functions are indeed variants of these two functions. In particular, the Fischer–Burmeister function $\phi_{\rm FB}$: $\mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_{\rm FB}(a, b) = \sqrt{a^2 + b^2} - (a + b),$$

whereas the natural residual function $\phi_{\rm NR} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\phi_{\rm NR}(a, b) = a - (a - b)_+ = \min\{a, b\}.$$

Recently, a generalized Fischer–Burmeister function $\phi^p_{\rm FB}$: $\mathbb{R}^2 \to \mathbb{R}$, which includes the Fischer–Burmeister function as a special case, was considered in [1–3,7,19,27]. The function $\phi^p_{\rm FB}$ is defined as

$$\phi^p_{\rm FB}(a, b) = \|(a, b)\|_p - (a + b), \quad p > 1 \tag{1}$$

and this natural extension is based on "continuous generalization" in such a way that the 2-norm in FB function is replaced by general p-norm. In addition, its geometric view is depicted in [27] and the effect of perturbing p for different kinds of algorithms are investigated in [4,5,7–9]. More recently, a generalization of natural residual function, denoted by $\phi^p_{\rm NR}$, is proposed in [6] and defined as

$$\phi^p_{\rm NR}(a, b) = a^p - (a - b)^p_+ \quad \text{with } p > 1$$

being a positive odd integer. (2)

Notice that when $p = 1$, $\phi^p_{\rm NR}$ reduces to the natural residual function $\phi_{\rm NR}$, i.e.

$$\phi^1_{\rm NR}(a, b) = a - (a - b)_+ = \min\{a, b\} = \phi_{\rm NR}(a, b).$$

In contrast to $\phi^p_{\rm FB}$, the function $\phi^p_{\rm NR}$ is obtained by "discrete generalization" and surprisingly possesses twice differentiability, see [6]. This feature enables us that many methods such as Newton method can be employed directly for solving NCP. However,

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unlike the graph of $\phi^p_{\rm FB}$, the graph of $\phi^p_{\rm NR}$ is not symmetric which may cause some difficulty in further analysis in designing solution methods. To this end, we try to symmetrize the function $\phi^p_{\rm NR}$. More specifically, we offer two ways to obtain symmetrizations of this "generalized natural residual function", which still satisfy NCP-conditions. In other words, we construct not only new NCP-functions and merit functions for the nonlinear complementarity problem, but also provide parallel "symmetric" functions to the generalized Fischer–Burmeister function.

To close this section, we present the ideas about how we symmetrize the "generalized natural residual function". The first step is looking into the graph of $\phi^p_{\rm NR}$ given in [27]. Because we wish to symmetrize the graph of $\phi^p_{\rm NR}$, we need to consider subcases of $a \ge b$ and $a \le b$, respectively. In view of the definition of $\phi^p_{\rm NR}$, we propose the first symmetrization of $\phi^p_{\rm NR}$, denoted by $\phi^p_{\rm S\text{-}NR} : \mathbb{R}^2 \to \mathbb{R}$, which is defined by

$$\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} a^p - (a-b)^p & \text{if } a > b, \\ a^p = b^p & \text{if } a = b, \\ b^p - (b-a)^p & \text{if } a < b, \end{cases} \tag{3}$$

where $p > 1$ being a positive odd integer. We will see that $\phi^p_{\rm S\text{-}NR}$ is an NCP-function with symmetric graph in Section 2. However, $\phi^p_{\rm S\text{-}NR}$ is not differentiable in general, it is natural to ask whether there exists a symmetrization function that has not only symmetric graph but also is differentiable. To this end, we see that the induced family of merit functions $\|\phi^p_{\rm S\text{-}NR}\|^2$ will fit this purpose. Nonetheless, we can construct another simpler merit function by modifying $\phi^p_{\rm S\text{-}NR}$. In summary, we wish to construct a symmetrized function which is also differentiable. Fortunately, we figure out the second symmetrization of $\phi^p_{\rm NR}$, denoted by $\psi^p_{\rm S\text{-}NR} : \mathbb{R}^2 \to \mathbb{R}_+$, which is defined by

$$\psi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} a^p b^p - (a-b)^p b^p & \text{if } a > b, \\ a^p b^p = a^{2p} & \text{if } a = b, \\ a^p b^p - (b-a)^p a^p & \text{if } a < b, \end{cases} \tag{4}$$

where $p > 1$ being a positive odd integer. The pictures and differentiable properties of $\psi^p_{\rm S\text{-}NR}$ will be depicted in Section 3. We point it out that the value of $\psi^p_{\rm S\text{-}NR}$ is always nonnegative which indicates that $\psi^p_{\rm S\text{-}NR}$ is a merit function for NCP. Here, due to the symmetric feature, we denote these two functions as "S-NR" standing for symmetrization of NR function.

2. The first symmetrization function $\phi^p_{\rm S\text{-}NR}$

In this section, we show that the function $\phi^p_{\rm S\text{-}NR}$ defined in (3) is an NCP-function. It is not differentiable on the whole $\mathbb{R}^2$, but it is twice continuously differentiable on $\Omega := \{(a,b) \mid a \ne b\}$.

Proposition 2.1. *Let $\phi^p_{\rm S\text{-}NR}$ be defined in (3) with $p > 1$ being a positive odd integer. Then, $\phi^p_{\rm S\text{-}NR}$ is an NCP-function and is positive only on the first quadrant $\Omega = \{(a,b) \mid a > 0,\ b > 0\}$.*

Proof. It is straightforward to verify that $\phi^p_{\rm S\text{-}NR}$ is positive only on the first quadrant.

Next, we continue to show $\phi^p_{\rm S\text{-}NR}$ is an NCP-function. We will proceed it by discussing three cases. Suppose $a > b$ and $\phi^p_{\rm S\text{-}NR}(a,b) = 0$. Then, we have $a^p - (a-b)^p = 0$, which implies that $a = a - b$. Thus, we see that $a > b = 0$. Similarly, when $a < b$ and $\phi^p_{\rm S\text{-}NR}(a,b) = 0$, we have $0 = a < b$. For the third case $\phi^p_{\rm S\text{-}NR}(a,b) = 0$ and $a = b$, it is easy to see that $a = b = 0$. It is trivial to check the converse way. In summary, $\phi^p_{\rm S\text{-}NR}$ satisfies that $\phi^p_{\rm S\text{-}NR}(a,b) = 0$ if and only if $a, b \ge 0$, $ab = 0$; and hence, it is an NCP-function. □

We elaborate more about the function $\phi^p_{\rm S\text{-}NR}$ as below:

(i) For p being an even integer, $\phi^p_{\rm S\text{-}NR}$ is not an NCP-function. A counterexample is given as below:

$$\phi^2_{\rm S\text{-}NR}(-2,-4) = (-2)^2 - (-2+4)^2 = 0.$$

(ii) The function $\phi^p_{\rm S\text{-}NR}$ is neither convex nor concave function. To see this, taking $p = 3$ and using the following argument, we can verify the assertion.

$$\begin{aligned} 1 &= \phi^3_{\rm S\text{-}NR}(1,1) < \frac{1}{2}\phi^3_{\rm S\text{-}NR}(0,0) + \frac{1}{2}\phi^3_{\rm S\text{-}NR}(2,2) \\ &= \frac{0}{2} + \frac{8}{2} = 4. \\ 1 &= \phi^3_{\rm S\text{-}NR}(1,1) > \frac{1}{2}\phi^3_{\rm S\text{-}NR}(2,0) + \frac{1}{2}\phi^3_{\rm S\text{-}NR}(0,2) \\ &= \frac{0}{2} + \frac{0}{2} = 0. \end{aligned}$$

Proposition 2.2. *Let $\phi^p_{\rm S\text{-}NR}$ be defined in (3) with $p > 1$ being a positive odd integer. Then, the following holds.*

(a) *An alternative expression of $\phi^p_{\rm S\text{-}NR}$ is*

$$\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} \phi^p_{\rm NR}(a,b) & \text{if } a > b, \\ a^p = b^p & \text{if } a = b, \\ \phi^p_{\rm NR}(b,a) & \text{if } a < b. \end{cases}$$

(b) *The function $\phi^p_{\rm S\text{-}NR}$ is not differentiable. However, $\phi^p_{\rm S\text{-}NR}$ is continuously differentiable on the set $\Omega := \{(a,b) \mid a \ne b\}$ with*

$$\nabla\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} p\,[\,a^{p-1} - (a-b)^{p-1},\ (a-b)^{p-1}\,]^T & \text{if } a > b, \\ p\,[\,(b-a)^{p-1},\ b^{p-1} - (b-a)^{p-1}\,]^T & \text{if } a < b. \end{cases}$$

In a more compact form,

$$\nabla\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{\rm NR}(a,b),\ (a-b)^{p-1}\,]^T & \text{if } a > b, \\ p\,[\,(b-a)^{p-1},\ \phi^{p-1}_{\rm NR}(b,a)\,]^T & \text{if } a < b. \end{cases}$$

(c) *The function $\phi^p_{\rm S\text{-}NR}$ is twice continuously differentiable on the set $\Omega = \{(a,b) \mid a \ne b\}$ with*

$$\nabla^2\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} a^{p-2} - (a-b)^{p-2} & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} \\ \quad \text{if } a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & b^{p-2} - (b-a)^{p-2} \end{bmatrix} \\ \quad \text{if } a < b. \end{cases}$$

In a more compact form,

$$\nabla^2\phi^p_{\rm S\text{-}NR}(a,b) = \begin{cases} p(p-1)\begin{bmatrix} \phi^{p-2}_{\rm NR}(a,b) & (a-b)^{p-2} \\ (a-b)^{p-2} & -(a-b)^{p-2} \end{bmatrix} & \text{if } a > b, \\ p(p-1)\begin{bmatrix} -(b-a)^{p-2} & (b-a)^{p-2} \\ (b-a)^{p-2} & \phi^{p-2}_{\rm NR}(b,a) \end{bmatrix} & \text{if } a < b. \end{cases}$$

Proof. The arguments are just direct computations, we omit them. □

At last, we present some other variants of $\phi^p_{\rm S\text{-}NR}$. Indeed, analogous to those functions in [26], the variants of $\phi^p_{\rm S\text{-}NR}$ as below can be verified being NCP-functions.

$$\widetilde{\phi}_1(a,b) = \phi^p_{\rm S\text{-}NR}(a,b) + \alpha(a)_+(b)_+, \quad \alpha > 0.$$

$$\widetilde{\phi}_2(a,b) = \phi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \quad \alpha > 0.$$
$$\widetilde{\phi}_3(a,b) = \phi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^4, \quad \alpha > 0.$$
$$\widetilde{\phi}_4(a,b) = \phi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^2, \quad \alpha > 0.$$
$$\widetilde{\phi}_5(a,b) = \phi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+)^2((b)_+\right)^2, \quad \alpha > 0.$$

Proposition 2.3. *All the above functions $\widetilde{\phi}_i(a,b)$ for $i \in \{1,2,3,4,5\}$ are NCP-functions.*

Proof. We only show that $\widetilde{\phi}_1(a,b)$ is an NCP-function and the same argument can be applied to the other cases. First, we denote $\Omega := \{(a,b) \mid a > 0, b > 0\}$ the first quadrant and suppose that $\widetilde{\phi}_1(a,b) = 0$. If $(a,b) \in \Omega$, then $\phi^p_{\text{S-NR}}(a,b) > 0$ by Proposition 2.1; and hence, $\widetilde{\phi}_1(a,b) > 0$. This is a contradiction. Therefore, we must have $(a,b) \in \Omega^c$ which says $(a)_+(b)_+ = 0$. This further implies $\phi^p_{\text{S-NR}}(a,b) = 0$ which is equivalent to $a, b \geq 0,\ ab = 0$ by applying Proposition 2.1 again. Thus, $\widetilde{\phi}_1$ is an NCP-function. □

3. The second symmetrization function $\psi^p_{\text{S-NR}}$

In this section, we show that the function $\psi^p_{\text{S-NR}}$ defined in (4) is not only an NCP-function, but also a merit function. In particular, $\psi^p_{\text{S-NR}}$ possesses symmetric graph and is twice differentiable.

Proposition 3.1. *Let $\psi^p_{\text{S-NR}}$ be defined in (4) with $p > 1$ being a positive odd integer. Then, $\psi^p_{\text{S-NR}}$ is an NCP-function and is positive on the set*

$$\begin{aligned}\Omega &= \{(a,b) \mid ab \neq 0\} \\ &\quad \cup \{(a,b) \mid a < b = 0\} \cup \{(a,b) \mid 0 = a > b\}.\end{aligned}$$

Proof. First of all, when $a < b = 0$, we have $\psi^p_{\text{S-NR}}(a,b) = a^{2p} > 0$. Similarly, when $0 = a > b$, we have $\psi^p_{\text{S-NR}}(a,b) = b^{2p} > 0$. For $0 \neq a > b \neq 0$, suppose that $b > 0$. Then, $a > (a-b)$ which implies $a^p > (a-b)^p$ and $b^p > 0$, and hence $a^p b^p - (a-b)^p b^p > 0$. On the other hand, suppose that $b < 0$. Then, $a < (a-b)$ which implies $a^p < (a-b)^p$ and $b^p < 0$. Thus, we also have $a^p b^p - (a-b)^p b^p > 0$. For $a = b \neq 0$, it is clear that $a^p b^p = a^{2p} > 0$. For the remaining case: $0 \neq a < b \neq 0$, the proof is similar to the case of $0 \neq a > b \neq 0$. From all the above, we prove that $\psi^p_{\text{S-NR}}$ is positive on the set Ω.

Next, we go on showing that $\psi^p_{\text{S-NR}}$ is an NCP-function. Suppose that $a > b$ and $a^p b^p - (a-b)^p b^p = [a^p - (a-b)^p] b^b = 0$. If $b = 0$, then we have $a > b = 0$. Otherwise, we have $a = (a-b)$ which also yields that $a > b = 0$. Similarly, the conditions $a < b$ and $a^p b^p - (b-a)^p a^p = 0$ imply that $b > a = 0$. The remaining cases $a = b$ and $a^p b^p = 0$ give that $a = b = 0$. Thus, from all the above, $\psi^p_{\text{S-NR}}$ is an NCP-function. □

We can conclude from Proposition 3.1 that $\psi^p_{\text{S-NR}}$ is a merit function, since $\psi^p_{\text{S-NR}}$ is positive on Ω and is identically zero on the set $\{(a,b) \mid a \geq b = 0\} \cup \{(a,b) \mid 0 = a \leq b\}$. Next, we elaborate more about the function $\psi^p_{\text{S-NR}}$ as below:

(i) For p being an even integer, $\psi^p_{\text{S-NR}}$ is not an NCP-function. A counterexample is given as below.

$$\psi^2_{\text{S-NR}}(-2,-4) = (-2)^2(-4)^2 - (-2+4)^2(-4)^2 = 0.$$

(ii) The function $\psi^p_{\text{S-NR}}$ is neither convex nor concave function. To see this, taking $p = 3$ and using the following argument verify the assertion.

$$\begin{aligned}1 &= \psi^3_{\text{S-NR}}(1,1) < \frac{1}{2}\psi^3_{\text{S-NR}}(0,0) + \frac{1}{2}\psi^3_{\text{S-NR}}(2,2) \\ &= \frac{0}{2} + \frac{64}{2} = 32.\end{aligned}$$

$$\begin{aligned}1 &= \psi^3_{\text{S-NR}}(1,1) > \frac{1}{2}\psi^3_{\text{S-NR}}(2,0) + \frac{1}{2}\psi^3_{\text{S-NR}}(0,2) \\ &= \frac{0}{2} + \frac{0}{2} = 0.\end{aligned}$$

Proposition 3.2. *Let $\psi^p_{\text{S-NR}}$ be defined as in (4) with $p > 1$ being a positive odd integer. Then, the following holds.*

(a) *An alternative expression of $\phi^p_{\text{S-NR}}$ is*

$$\psi^p_{\text{S-NR}}(a,b) = \begin{cases} \phi^p_{\text{NR}}(a,b) b^p & \text{if } a > b, \\ a^p b^p = a^{2p} & \text{if } a = b, \\ \phi^p_{\text{NR}}(b,a) a^p & \text{if } a < b. \end{cases}$$

(b) *The function $\psi^p_{\text{S-NR}}$ is continuously differentiable with*

$$\nabla \psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\,[\,a^{p-1}b^p - (a-b)^{p-1}b^p,\ a^p b^{p-1} \\ \qquad - (a-b)^p b^{p-1} + (a-b)^{p-1} b^p\,]^T & \text{if } a > b, \\ p\,[\,a^{p-1}b^p, a^p b^{p-1}\,]^T = pa^{2p-1}[1\,,1]^T & \text{if } a = b, \\ p\,[\,a^{p-1}b^p - (b-a)^p a^{p-1} + (b-a)^{p-1} a^p, \\ \qquad a^p b^{p-1} - (b-a)^{p-1} a^p\,]^T & \text{if } a < b. \end{cases}$$

In a more compact form,

$$\nabla \psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\,[\,\phi^{p-1}_{\text{NR}}(a,b) b^p,\ \phi^p_{\text{NR}}(a,b) b^{p-1} \\ \qquad + (a-b)^{p-1} b^p\,]^T & \text{if } a > b, \\ p\,[\,a^{2p-1}, a^{2p-1}\,]^T & \text{if } a = b, \\ p\,[\,\phi^p_{\text{NR}}(b,a) a^{p-1} + (b-a)^{p-1} a^p, \\ \qquad \phi^{p-1}_{\text{NR}}(b,a) a^p\,]^T & \text{if } a < b. \end{cases}$$

(c) *The function $\psi^p_{\text{S-NR}}$ is twice continuously differentiable with $\nabla^2 \psi^p_{\text{S-NR}}(a,b)$ as given in Box I.*

Proof. (a) It is clear to see this part.
(b) It is easy to verify the continuous differentiability of $\psi^p_{\text{S-NR}}(a,b)$ on the set $\{(a,b) \mid a > b \text{ or } a < b\}$. We only need to check the differentiability along the line $a = b$. Suppose that $h > k$, we observe that

$$\begin{aligned}&\psi^p_{\text{S-NR}}(a+h, a+k) - \psi^p_{\text{S-NR}}(a,a) \\ &= (a+h)^p (a+k)^p - (h-k)^p b^p - a^{2p} \\ &= \left\langle pa^{2p-1}(1,1), (h,k)\right\rangle + R(a,h,k).\end{aligned}$$

Here the remainder $R(a,h,k)$ is $o(h,k)$ function of h and k, since the degree of h and k of $R(a,h,k)$ is at least 2. Similarly, from the other two cases $h = k$ and $h < k$, we can conclude that $\nabla \psi^p_{\text{S-NR}}(a,a) = pa^{2p-1}(1,1)^T$. In addition, the continuity of $\nabla \psi^p_{\text{S-NR}}(a,b)$ along the line $a = b$ is easy to verify.
(c) The arguments for this part are similar to those for part (b). We omit them. □

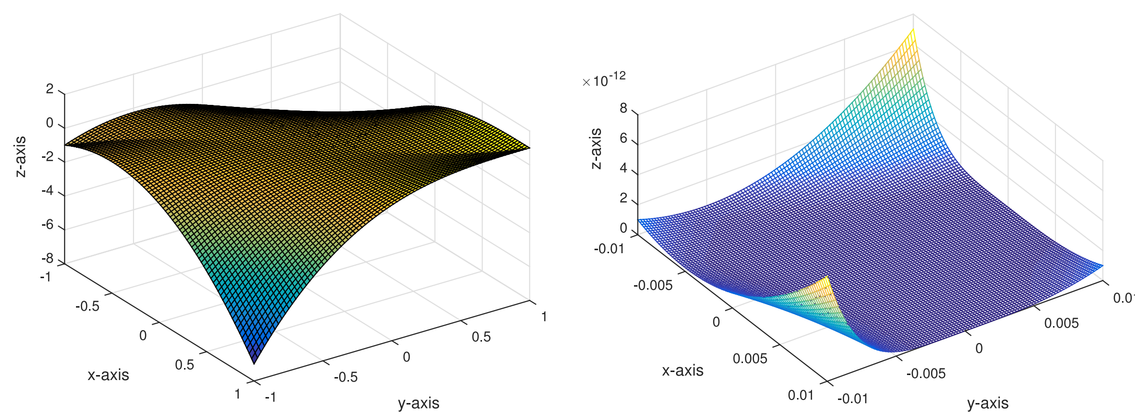
Again, we present some other variants of $\psi^p_{\text{S-NR}}$. Indeed, analogous to those functions in [26], the variants of $\psi^p_{\text{S-NR}}$ as below can be verified being NCP-functions.

$$\widetilde{\psi}_1(a,b) = \psi^p_{\text{S-NR}}(a,b) + \alpha(a)_+(b)_+, \quad \alpha > 0.$$
$$\widetilde{\psi}_2(a,b) = \psi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+(b)_+\right)^2, \quad \alpha > 0.$$
$$\widetilde{\psi}_3(a,b) = \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^4, \quad \alpha > 0.$$
$$\widetilde{\psi}_4(a,b) = \psi^p_{\text{S-NR}}(a,b) + \alpha\left((ab)_+\right)^2, \quad \alpha > 0.$$
$$\widetilde{\psi}_5(a,b) = \psi^p_{\text{S-NR}}(a,b) + \alpha\left((a)_+)^2((b)_+\right)^2, \quad \alpha > 0.$$

Proposition 3.3. *All the above functions $\widetilde{\psi}_i(a,b)$ for $i \in \{1,2,3,4,5\}$ are NCP-functions.*

$$\nabla^2 \psi^p_{\text{S-NR}}(a,b) = \begin{cases} p\begin{bmatrix} (p-1)[a^{p-2}-(a-b)^{p-2}]b^p & \begin{array}{c}(p-1)(a-b)^{p-2}b^p\\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{array} \\ \begin{array}{c}(p-1)(a-b)^{p-2}b^p\\ +p[a^{p-1}-(a-b)^{p-1}]b^{p-1}\end{array} & \begin{array}{c}(p-1)[a^p-(a-b)^p]b^{p-2}\\ +2p(a-b)^{p-1}b^{p-1}\\ -(p-1)(a-b)^{p-2}b^p\end{array} \end{bmatrix} & \text{if } a>b, \\ p\begin{bmatrix} (p-1)a^{p-2}b^p & pa^{p-1}b^{p-1} \\ pa^{p-1}b^{p-1} & (p-1)a^pb^{p-2} \end{bmatrix} & \text{if } a=b, \\ p\begin{bmatrix} \begin{array}{c}(p-1)[b^p-(b-a)^p]a^{p-2}\\ +2p(b-a)^{p-1}a^{p-1}\\ -(p-1)(b-a)^{p-2}a^p\end{array} & \begin{array}{c}(p-1)(b-a)^{p-2}a^p\\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{array} \\ \begin{array}{c}(p-1)(b-a)^{p-2}a^p\\ +p[b^{p-1}-(b-a)^{p-1}]a^{p-1}\end{array} & (p-1)[b^{p-2}-(b-a)^{p-2}]a^p \end{bmatrix} & \text{if } a<b. \end{cases}$$

Box I.



(a) The graph of $\phi^p_{\text{S-NR}}$. (b) The graph of $\psi^p_{\text{S-NR}}$.

Fig. 1. The surfaces of $\phi^p_{\text{S-NR}}$ and $\psi^p_{\text{S-NR}}$ with $p = 3$.

Proof. We only show that $\widetilde{\psi}_1$ is an NCP-function and the same argument can be applied to the other cases. Let $\Omega := \{(a,b) \mid ab \neq 0\}$ and suppose that $\widetilde{\psi}_1(a,b) = 0$. If $(a,b) \in \Omega$, then $\psi^p_{\text{S-NR}}(a,b) > 0$ by Proposition 3.1; and hence, $\widetilde{\psi}_1(a,b) > 0$. This is a contradiction. Therefore, we must have $(a,b) \in \Omega^c$ which says $(a)_+(b)_+ = 0$. This further implies $\psi^p_{\text{S-NR}}(a,b) = 0$ which is equivalent to $a, b \geq 0,\ ab = 0$ by applying Proposition 3.1 again. Thus, $\widetilde{\psi}_1$ is an NCP-function. $\square$

4. Concluding remarks

Due to space limitation, we illustrate the functions $\phi^p_{\text{S-NR}}$ and $\psi^p_{\text{S-NR}}$ for the single value $p = 3$, see Fig. 1. Nonetheless, we make some remarks about the surfaces of $\phi^p_{\text{S-NR}}$ and $\psi^p_{\text{S-NR}}$, as well as say a few words about their algebraic properties. First of all, it is clear to see that $\phi^p_{\text{S-NR}}(a,b) = \phi^p_{\text{S-NR}}(b,a)$ and $\psi^p_{\text{S-NR}}(a,b) = \psi^p_{\text{S-NR}}(b,a)$, which mean that the surfaces of $\phi^p_{\text{S-NR}}$ and $\psi^p_{\text{S-NR}}$ are both symmetric with respect to the line $a = b$. As for the algebraic structure, we can verify that

$$\psi^p_{\text{S-NR}}(a,b) = \left[\min(a,b)\right]^2 \quad \text{for } p = 1.$$

To see this, for example if $a > b$, we check that $a^1b^1 - (a-b)^1b^1 = b^2 = \min(a,b)^2$. On the other hand, for large $p = 3, 5, 7, \ldots$, the function $\psi^p_{\text{S-NR}}$ does not coincide with $\left[\min(a,b)\right]^{2p}$. Nonetheless, when we restrict $\psi^p_{\text{S-NR}}(a,b)$ on the line $a = b$ and two axes $a = 0$ and $b = 0$, we really have that

$$\psi^p_{\text{S-NR}}(a,b) = \left[\min(a,b)\right]^{2p}.$$

In summary, $\psi^p_{\text{S-NR}}$ can be viewed as a merit function relative to the original natural residual NCP-function $\phi_{\text{NR}}(a,b) = \min(a,b)$. Besides, we have to mention that $\psi^p_{\text{S-NR}}$ is twice continuously differentiable so that it is good enough to develop a lot of algorithms based on this property. However, it does not satisfy that

$$\nabla_a \psi^p_{\text{S-NR}}(a,b) \cdot \nabla_b \psi^p_{\text{S-NR}}(a,b) \geq 0.$$

(cf. Property 2.2(d) in [2]).

For example, taking $p = 3$ and $(a,b) = (0,-1)$ gives $\nabla_a \psi^3_{\text{S-NR}}(0,-1) = 3$ and $\nabla_b \psi^3_{\text{S-NR}}(0,-1) = -6$. This may cause some difficulty in analyzing the convergence rate.

As it can be seen, the surface of $\phi^p_{\text{S-NR}}$ looks like "two-wings" of an eagle and there is cusp along $x = y$. Moreover, the graph of $\phi^p_{\text{S-NR}}$ is neither convex nor concave. The surface of $\psi^p_{\text{S-NR}}$ is smooth and it is neither convex nor concave.

To sum up, we propose new NCP-functions and merit functions in this short paper. Both of them possess symmetric graphs. With our discovery of $\phi^p_{\text{S-NR}}$, $\psi^p_{\text{S-NR}}$ in this short paper, there are many future directions to be explored. We list some of them.

- Discovering benefits for such symmetrization.
- Doing numerical comparisons among ϕ^p_{FB}, ϕ^p_{NR}, $\phi^p_{\text{S-NR}}$, and $\psi^p_{\text{S-NR}}$ involved in various algorithms.
- Studying the effect when perturbing the parameter p, applying this new family of NCP-functions to suitable optimization problems.
- Extending these functions as the complementarity function associated with the second-order cone and symmetric cones.
- Developing some analytic properties on $\phi^p_{\text{S-NR}}$ such as directional differentiability, Lipschitz continuity, semismoothness.

Finally, we would like to point out that the p-th root of $\phi^p_{\text{S-NR}}$ and $\psi^p_{\text{S-NR}}$ is also NCP-functions. In other words, the functions

$$\left[\phi^p_{\text{S-NR}}\right]^{\frac{1}{p}}, \quad \left[\psi^p_{\text{S-NR}}\right]^{\frac{1}{p}}$$

are NCP-functions, too. The proof is routine, so we omit it.

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