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CONVEXITY OF SYMMETRIC CONE TRACE FUNCTIONS IN EUCLIDEAN JORDAN ALGEBRAS

YU-LIN CHANG AND JEIN-SHAN CHEN*

Abstract. In this paper, we establish convexity of some functions associated with symmetric cones, called SC trace functions. As illustrated in the paper, these functions play a key role in the development of penalty and barrier functions methods for symmtric cone programs.

1. Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called Lorentz cone, is the set defined as

$$\mathcal{K}^n := \Big\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x_2\|\Big\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. When $n = 1$, $\mathcal{K}^n$ reduces to the set of nonnegative real numbers $\mathbb{R}_+$. As shown in [13], $\mathcal{K}^n$ is also a set composed of the squared elements from Jordan algebra $(\mathbb{R}^n, \circ)$, where the Jordan product "$\circ$" is a binary operation defined by

$$x \circ y := (\langle x, y\rangle, x_1 y_2 + y_1 x_2) \tag{1.2}$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. Here for any $x \in \mathbb{R}^n$, we use x_1 to denote the first component of x, and x_2 to denote the vector consisting of the rest $n-1$ components.

From [12, 13], we recall that each $x \in \mathbb{R}^n$ admits a spectral decomposition associated with $\mathcal{K}^n$ of the following form

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1.3}$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of x, respectively, defined by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \quad u_x^{(i)} = \frac{1}{2}\Big(1, (-1)^i \bar{x}_2\Big), \tag{1.4}$$

with $\bar{x}_2 = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$, and otherwise $\bar{x}_2$ being any vector in $\mathbb{R}^{n-1}$ such that $\|\bar{x}_2\| = 1$. When $x_2 \neq 0$, the spectral factorization is unique. The determinant and trace of x are defined as $\det(x) := \lambda_1(x)\lambda_2(x)$ and $\mathrm{tr}(x) := \lambda_1(x) + \lambda_2(x)$, respectively.

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*Corresponding author. The author's work is supported by National Science Council of Taiwan.

With the spectral decomposition above, for any given scalar function $\phi : J \subseteq \mathbb{R} \to \mathbb{R}$, we may define a vector-valued function $\phi^{\rm soc} \colon S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi^{\rm soc}(x) \ := \ f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \tag{1.5}$$

where J is an interval (finite or infinite, open or closed) of $\mathbb{R}$, and S is the domain of $\phi^{\rm soc}$ determined by ϕ. Then, we can define the SOC trace function associated with ϕ

$$\phi^{\rm tr}(x) := \phi(\lambda_1(x)) + \phi(\lambda_2(x)) = {\rm tr}(\phi^{\rm soc}(x)) \quad \forall x \in S. \tag{1.6}$$

Chen, Liao and Pan [11] give the following relation between $\phi^{\rm tr}$ and $\phi^{\rm soc}$

$$\nabla\phi^{\rm tr}(x) = (\phi^{'})^{\rm soc}(x) \quad \text{and} \quad \nabla^2\phi^{\rm tr}(x) = \nabla(\phi^{'})^{\rm soc}(x) \qquad \forall x \in \mathbf{int}S. \tag{1.7}$$

By using Schur Complement Theorem, they establish the convexity of SOC trace functions and the compounds of SOC trace functions. Some of these functions are the key of penalty and barrier function methods for second-order cone programs (SOCPs), as well as the establishment of some important inequalities associated with SOCs, for which the proof of convexity of these functions is a necessity.

Some similar results associated with positive semidefinite cone are also investigated by Auslender in [1, 2]. Since both SOC and positive semidefinite cone are special cases of symmetric cone (SC for short). A natural question leads us to consider the more general case. To this end, we need to recall some concepts regarding Euclidean Jordan algebra. Let $\mathbb{A} = (\mathbb{V}, \langle\cdot,\cdot\rangle, \circ)$ be an n-dimensional Euclidean Jordan algebra (see Section 2) and $\mathcal{K}$ be the symmetric cone in $\mathbb{V}$. For any given scalar function $\phi \colon J \subseteq \mathbb{R} \to \mathbb{R}$, we define the associated function

$$\phi^{\rm sc}_{\mathbb{V}}(x) := \phi(\lambda_1(x))c_1 + \cdots + \phi(\lambda_r(x))c_r, \tag{1.8}$$

and SC trace function

$$\phi^{\rm tr}_{\mathbb{V}}(x) := \phi(\lambda_1(x)) + \cdots + \phi(\lambda_r(x)) = {\rm tr}(\phi^{\rm sc}_{\mathbb{V}}(x)) \quad \forall x \in S, \tag{1.9}$$

where $x \in \mathbb{V}$ has the spectral decomposition

$$x = \lambda_1(x)c_1 + \cdots + \lambda_r(x)c_r.$$

In this paper we extend the aforementioned results to general symmetric cone setting where we establish the convexity of SC trace functions and the compounds of SC trace functions. Throughout this note, for any $x, y \in \mathbb{V}$, we write $x \succeq_{\mathcal{K}} y$ if $x - y \in \mathcal{K}$; and write $x \succ_{\mathcal{K}} y$ if $x - y \in \mathbf{int}\mathcal{K}$. For a real symmetric matrix A, we write $A \succeq 0$ (respectively, $A \succ 0$) if A is positive semidefinite (respectively, positive definite). For any $\phi : J \to \mathbb{R}$, $\phi'(t)$ and $\phi''(t)$ denote the first derivative and second-order derivative of ϕ at the differentiable point $t \in J$, respectively. Suppose $F \colon S \subseteq \mathbb{V} \to \mathbb{R}$, $\nabla F(x)$ and $\nabla^2 F(x)$ denote the gradient and the Hessian matrix of F at the differentiable point $x \in S$, respectively.

2. Preliminaries

This section recalls some results on Euclidean Jordan algebras that will be used in subsequent analysis. More detailed expositions of Euclidean Jordan algebras can be found in Koecher's lecture notes [16] and the monograph by Faraut and Korányi [13].

Let $\mathbb{V}$ be an n-dimensional vector space over the real field $\mathbb{R}$, endowed with a bilinear mapping $(x, y) \mapsto x \circ y$ from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}$. The pair $(\mathbb{V}, \circ)$ is called a *Jordan algebra* if

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$.

Note that a Jordan algebra is not necessarily associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$ may not hold for all $x, y, z \in \mathbb{V}$. We call an element $e \in \mathbb{V}$ the *identity* element if $x \circ e = e \circ x = x$ for all $x \in \mathbb{V}$. A Jordan algebra $(\mathbb{V}, \circ)$ with an identity element e is called a *Euclidean Jordan algebra* if there is an inner product $\langle \cdot, \cdot \rangle_{\mathbb{V}}$ such that

(iii) $\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Given a Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$, we denote the set of squares as

$$\mathcal{K} := \left\{ x^2 \mid x \in \mathbb{V} \right\}.$$

From [13, Theorem III.2.1], $\mathcal{K}$ is a symmetric cone which means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior and for any two elements $x, y \in \mathbf{int}\mathcal{K}$, there exists an invertible linear transformation $\mathcal{T} : \mathbb{V} \to \mathbb{V}$ such that $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(x) = y$.

For any given $x \in \mathbb{A}$, let $\zeta(x)$ be the degree of the minimal polynomial of x, i.e.,

$$\zeta(x) := \min \Big\{ k : \{e, x, x^2, \cdots, x^k\} \text{ are linearly dependent} \Big\}.$$

Then the *rank* of $\mathbb{A}$ is defined as $\max\{\zeta(x) : x \in \mathbb{V}\}$. In this paper, we use r to denote the rank of the underlying Euclidean Jordan algebra. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$. Two idempotents c_i and c_j are said to be *orthogonal* if $c_i \circ c_j = 0$. One says that $\{c_1, c_2, \ldots, c_k\}$ is a *complete system of orthogonal idempotents* if

$$c_j^2 = c_j, \quad c_j \circ c_i = 0 \text{ if } j \neq i \text{ for all } j, i = 1, 2, \cdots, k \quad \text{and} \quad \textstyle\sum_{j=1}^k c_j = e.$$

An idempotent is *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Now we state the second version of the spectral decomposition theorem.

Theorem 2.1 ([13, Theorem III.1.2]). *Suppose that $\mathbb{A}$ is a Euclidean Jordan algebra with rank r. Then for any $x \in \mathbb{V}$, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues and $\operatorname{tr}(x) = \sum_{j=1}^{r} \lambda_j(x)$ *the trace of x.*

Since, by [13, Proposition III.1.5], a Jordan algebra $(\mathbb{V}, \circ)$ with an identity element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\operatorname{tr}(x \circ y)$ is positive definite, we may define another inner product on $\mathbb{V}$ by $\langle x, y\rangle := \operatorname{tr}(x \circ y)$ for any $x, y \in \mathbb{V}$. The inner product $\langle \cdot, \cdot\rangle$ is associative by [13, Prop. II. 4.3], i.e., $\langle x, y \circ z\rangle = \langle y, x \circ z\rangle$ for any $x, y, z \in \mathbb{V}$. For any given $x \in \mathbb{V}$, let $\mathcal{L}(x)$ be the linear operator of $\mathbb{V}$ defined by

$$\mathcal{L}(x)y := x \circ y \quad \forall y \in \mathbb{V}.$$

Then, $\mathcal{L}(x)$ is symmetric with respect to the inner product $\langle \cdot, \cdot\rangle$ in the sense that

$$\langle \mathcal{L}(x)y, z\rangle = \langle y, \mathcal{L}(x)z\rangle \quad \forall y, z \in \mathbb{V}.$$

In the sequel, we let $\|\cdot\|$ be the norm on $\mathbb{V}$ induced by the inner product, namely,

$$\|x\| := \sqrt{\langle x, x\rangle} = \left(\sum_{j=1}^{r} \lambda_j^2(x)\right)^{1/2} \quad \forall x \in \mathbb{V}. \tag{2.1}$$

A Euclidean Jordan algebra is called simple if it cannot be written as a direct sum of the other two Euclidean Jordan algebras. It is known that every Euclidean Jordan algebra is a direct sum of simple Euclidean Jordan algebras. Unless otherwise stated, in the rest of this paper, we assume that $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot\rangle)$ is a simple Euclidean Jordan algebra of rank r. Let $\{c_1, c_2, \ldots, c_r\}$ be a Jordan frame of $\mathbb{A}$. From [13, Lemma IV. 1.3], we know that the operators $\mathcal{L}(c_j)$, $j = 1, 2, \ldots, r$ commute and admit a simultaneous diagonalization. For $i, j \in \{1, 2, \ldots, r\}$, define the subspaces

$$\mathbb{V}_{ii} := \mathbb{R}c_i \ \text{ and } \ \mathbb{V}_{ij} := \left\{ x \in \mathbb{V} \mid c_i \circ x = c_j \circ x = \frac{1}{2}x \right\} \ \text{when } i \neq j.$$

Then, [13, Corollary IV.2.6] says

$$\dim(\mathbb{V}_{ij}) = \dim(\mathbb{V}_{st}) \quad \text{for any } i \neq j \in \{1, 2, \ldots, r\} \text{ and } s \neq t \in \{1, 2, \ldots, r\},$$

and $n = r + \frac{d}{2}r(r-1)$ where d denotes this common dimension. Moreover, from [13, Theorem IV.2.1], we have the following conclusion.

Theorem 2.2. *The space $\mathbb{V}$ is the orthogonal direct sum of subspaces $\mathbb{V}_{ij}$ $(1 \le i \le j \le r)$, i.e., $\mathbb{V} = \oplus_{i \le j} \mathbb{V}_{ij}$. Furthermore,*

$$\begin{array}{rcl} \mathbb{V}_{ij} \circ \mathbb{V}_{ij} & \subset & \mathbb{V}_{ii} + \mathbb{V}_{ij}, \\ \mathbb{V}_{ij} \circ \mathbb{V}_{jk} & \subset & \mathbb{V}_{ik}, \ \text{if } i \neq k, \\ \mathbb{V}_{ij} \circ \mathbb{V}_{kl} & = & \{0\}, \ \text{if } \{i, j\} \cap \{k, l\} = \emptyset. \end{array}$$

Let $x \in \mathbb{V}$ have the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x) c_j$, where $\lambda_1(x) \ge \lambda_2(x) \ge \cdots \ge \lambda_r(x)$ are the eigenvalues of x and $\{c_1, c_2, \ldots, c_r\}$ is the corresponding Jordan frame. For $i, j \in \{1, 2, \ldots, r\}$, let $\mathcal{C}_{ij}(x)$ be the orthogonal projection operator onto $\mathbb{V}_{ij}$. Then, from Theorem IV 2.1 of [13], it follows that for all $i, j = 1, 2, \ldots, r$,

$$\mathcal{C}_{jj}(x) = 2\mathcal{L}^2(c_j) - \mathcal{L}(c_j) \text{ and } C_{ij}(x) = 4\mathcal{L}(c_i)\mathcal{L}(c_j) = 4\mathcal{L}(c_j)\mathcal{L}(c_i) = \mathcal{C}_{ji}(x). \tag{2.2}$$

Moreover, the orthogonal projection operators $\{\mathcal{C}_{ij}(x) : \ i, j = 1, 2, \ldots, r\}$ satisfy

$$\mathcal{C}_{ij}(x) = \mathcal{C}_{ij}^*(x), \ \mathcal{C}_{ij}^2(x) = \mathcal{C}_{ij}(x), \ \mathcal{C}_{ij}(x)\mathcal{C}_{kl}(x) = 0 \text{ if } \{i, j\} \neq \{k, l\} \tag{2.3}$$

and

$$\sum_{1\le i\le j\le r} \mathcal{C}_{ij}(x) = \mathcal{I}. \tag{2.4}$$

Suppose $\phi : \mathbb{R} \to \mathbb{R}$ be a scalar valued function and we define the Löwner operator associated with ϕ as

$$\phi^{\mathrm{sc}}_{\mathbb{V}}(x) =: \sum_{j=1}^{r} \phi(\lambda_j(x))c_j,$$

where $x \in \mathbb{V}$ has the spectral decomposition $x = \sum_{j=1}^{r} \lambda_j(x)c_j$. Korányi [15] (or see [19]) proves the following result, which generalizes Löwner result on symmetric matrices to Euclidean Jordan algebras.

Theorem 2.3. *Let $x = \sum_{j=1}^{r} \lambda_j(x)c_j$ and (a, b) be an open interval in $\mathbb{R}$ that contains $\lambda_j(x), j = 1, 2, \dots, r$. If ϕ is continuously differentiable on (a, b), then $\phi^{\mathrm{sc}}_{\mathbb{V}}$ is differentiable at x and its derivative, for any $h \in \mathbb{V}$, is given by*

$$\left(\nabla\phi^{\mathrm{sc}}_{\mathbb{V}}\right)(x)(h) = \sum_{j=1}^{r} \left(\phi^{[1]}(\lambda(x))\right)_{jj} \mathcal{C}_{jj}(x)h + \sum_{1\le j<l\le r} \left(\phi^{[1]}(\lambda(x))\right)_{jl} \mathcal{C}_{jl}(x)h \tag{2.5}$$

where the coefficient is defined as

$$\phi^{[1]}(\lambda(x))_{jl} := \begin{cases} \phi'(\lambda_j) & \text{if } \lambda_j = \lambda_l, \\ \frac{\phi(\lambda_j)-\phi(\lambda_l)}{\lambda_j-\lambda_l} & \text{if } \lambda_j \neq \lambda_l. \end{cases} \tag{2.6}$$

Moreover, based on this theorem, Sun and Sun [19] show that $\phi^{\mathrm{sc}}_{\mathbb{V}}$ is continuously differentiable at x if and only if ϕ is continuously differentiable at $\lambda_j(x), j = 1, 2, \cdots, r$. We will exploit such property to achieve Lemma 3.1 which paves a way to our main result.

3. Main results

In this section, we present how we achieve the convexity of symmetric cone trace functions. We start with a technical lemma.

Lemma 3.1. *For any given scalar function $\phi\colon J \subseteq \mathbb{R} \to \mathbb{R}$, let $\phi^{\mathrm{sc}}_{\mathbb{V}} : S \to \mathbb{V}$ and $\phi^{\mathrm{tr}}_{\mathbb{V}} : S \to \mathbb{R}$ be given by (1.8) and (1.9), respectively. Assume that J is an open interval in $\mathbb{R}$. Then, the following results hold.*

(a) *The domain S of $\phi^{\mathrm{sc}}_{\mathbb{V}}$ and $\phi^{\mathrm{tr}}_{\mathbb{V}}$ is open and convex.*

(b) *If ϕ is (continuously) differentiable, then $\phi^{\mathrm{tr}}_{\mathbb{V}}$ is (continuously) differentiable on S with $\nabla\phi^{\mathrm{tr}}_{\mathbb{V}}(x)(h) = \langle h, (\phi')^{\mathrm{sc}}_{\mathbb{V}}(x)\rangle$ for all $h \in \mathbb{V}$.*

(c) *If ϕ is twice (continuously) differentiable, then $\phi^{\mathrm{tr}}_{\mathbb{V}}$ is twice (continuously) differentiable on S with $\nabla^2\phi^{\mathrm{tr}}_{\mathbb{V}}(x)(h, k) = \langle h, \nabla(\phi')^{\mathrm{sc}}_{\mathbb{V}}(x)k\rangle$ for all $h, k \in \mathbb{V}$.*

Proof. (a) Suppose $J = (a, b)$. Then the domain S is open because it is the intersection of two open sets S_1 and S_r, where S_1 and S_r are defined as

$$S_1 = \{x \in \mathbb{V} \colon \lambda_1(x) < b\} \text{ and } S_r = \{x \in \mathbb{V} \colon \lambda_r(x) > a\}.$$

We note here that the eigenvalue functions $\lambda_j(x)$ are continuous, see [19]. For convexity of S, we suppose $x, y \in S$ and $0 \le \lambda \le 1$. We want to verify that $\lambda x + (1-\lambda)y \in S$. First, we know that the largest eigenvalue function $\lambda_1(x)$ is a convex function [8] which implies

$$\lambda_1(\lambda x + (1-\lambda)y) \le \lambda\lambda_1(x) + (1-\lambda)\lambda_1(y) < \lambda a + (1-\lambda)a = a.$$

This means $\lambda x + (1-\lambda)y \in S_1$. Analogously, we know that the smallest eigenvalue function $\lambda_r(x)$ is concave which leads to $\lambda_r(\lambda x+(1-\lambda)y) > b$, i.e. $\lambda x+(1-\lambda)y \in S_r$.

(b) As mentioned earlier, the (continuous) differentiability is known. From the following formula

$$\phi_{\mathbb{V}}^{\mathrm{tr}}(x) := \sum_{j=1}^{r} \phi(\lambda_j(x)) = \left\langle \sum_{j=1}^{r} \phi(\lambda_j(x))c_j, e \right\rangle = \langle \phi_{\mathbb{V}}^{\mathrm{sc}}(x), e\rangle,$$

we have that, for any $h \in \mathbb{V}$,

$$\nabla\phi_{\mathbb{V}}^{\mathrm{tr}}(x)(h) = \left\langle \nabla\phi_{\mathbb{V}}^{\mathrm{sc}}(x)h, e\right\rangle = \left\langle h, \nabla\phi_{\mathbb{V}}^{\mathrm{sc}}(x)e\right\rangle$$

where we use symmetry property of $\nabla\phi_{\mathbb{V}}^{\mathrm{sc}}(x)$ in the second equation. By applying equations (2.5) and (2.6), we obtain

$$\begin{aligned} \nabla\phi_{\mathbb{V}}^{\mathrm{sc}}(x)e &= \sum_{j=1}^{r} \left(\phi^{[1]}(\lambda(x))\right)_{jj} \mathcal{C}_{jj}(x)e + \sum_{1\le j<l\le r} (\phi^{[1]}(\lambda(x)))_{jl}\mathcal{C}_{jl}(x)e \\ &= \sum_{j=1}^{r} (\phi^{[1]}(\lambda(x)))_{jj}c_j \\ &= \sum_{j=1}^{r} \phi^{'}(\lambda_j(x))c_j = (\phi^{'})_{\mathbb{V}}^{\mathrm{sc}}(x) \qquad (3.1) \end{aligned}$$

Note that $e = c_1 + \cdots + c_r$. Hence $\mathcal{C}_{jj}(x)e = c_j$ and $\mathcal{C}_{jl}(x)e = 0$ for $j \ne l$.

(c) Suppose now that ϕ is twice (continuously) differentiable. It is not hard to see that $\phi_{\mathbb{V}}^{\mathrm{tr}}$ is twice (continuously) differentiable on S with $\nabla^2\phi_{\mathbb{V}}^{\mathrm{tr}}(x)(h,k) = \langle h, \nabla(\phi^{'})_{\mathbb{V}}^{\mathrm{sc}}(x)k\rangle$ by the expression $\nabla\phi_{\mathbb{V}}^{\mathrm{tr}}(x)(h) = \langle h, (\phi^{'})_{\mathbb{V}}^{\mathrm{sc}}(x)\rangle$. □

Theorem 3.2. *For any given $f\colon J \to \mathbb{R}$, let $\phi_{\mathbb{V}}\colon S \to \mathbb{R}^n$ and $\phi_{\mathbb{V}}^{\mathrm{tr}}\colon S \to \mathbb{R}$ be given by (1.5) and (1.6), respectively. Assume that J is an open interval in $\mathbb{R}$. If ϕ is twice differentiable on J, then*

- (a) *$\phi^{''}(t) \ge 0$ for any $t \in J \iff \nabla^2\phi_{\mathbb{V}}^{\mathrm{tr}}(x) \succeq 0$ for any $x \in S \iff \phi_{\mathbb{V}}^{\mathrm{tr}}$ is convex in S.*
- (b) *$\phi^{''}(t) > 0$ for any $t \in J \iff \nabla^2\phi_{\mathbb{V}}^{\mathrm{tr}}(x) \succ 0$ $\forall x \in S \Longrightarrow \phi_{\mathbb{V}}^{\mathrm{tr}}$ is strictly convex in S.*

Proof. (a) We substitute ϕ by ϕ', then the coefficient equation (2.6) becomes

$$\phi'^{[1]}(\lambda(x))_{jl} := \begin{cases} \phi''(\lambda_j) & \text{if } \lambda_j = \lambda_l; \\ \frac{\phi'(\lambda_j)-\phi'(\lambda_l)}{\lambda_j-\lambda_l} & \text{if } \lambda_j \ne \lambda_l. \end{cases}$$

Hence the the coefficients are all nonnegative because of the assumption $\phi^{''}(t) \geq 0$. Observing that $\mathbb{V}$ is a direct sum of orthogonal spaces $\mathbb{V} = \oplus_{i\leq j}\mathbb{V}_{ij}$, we can give an orthonormal basis $\mathcal{B} = \{c_1 \ldots, c_r, c_{12}^{(1)}, \ldots, c_{12}^{(d)}, c_{13}^{(1)}, \ldots, c_{13}^{(d)}, \ldots, c_{r-1,r}^{(1)}, \ldots, c_{r-1,r}^{(d)}\}$ for $\mathbb{V}$ and $\{c_{jl}^{(1)}, \ldots, c_{jl}^{(d)}\}$ spans the space $\mathbb{V}_{jl}$, where d is the common dimension of $\mathbb{V}_{jl}, j < l$.

Let $h, k \in \mathcal{B}$. Plug in Lemma 3.1 (c), then the Hessian $\nabla^2\phi_{\mathbb{V}}^{\rm tr}(x)$ can be presented as a diagonal matrix under the basis $\mathcal{B}$

$$A = \mathrm{diag}(\phi'^{[1]}(\lambda(x))_{11}, \ldots, \phi'^{[1]}(\lambda(x))_{rr}, \overbrace{\phi'^{[1]}(\lambda(x))_{12}, \ldots, \phi'^{[1]}(\lambda(x))_{12}}^{d's},$$
$$, \ldots\ldots\ldots\ldots\ldots, \overbrace{\phi'^{[1]}(\lambda(x))_{r-1,r}, \ldots, \phi'^{[1]}(\lambda(x))_{r-1,r}}^{d's}).$$

Then, the first part equivalence follows clearly from Lemma 3.2 whereas the second part is a well-known result in analysis.

(b) The arguments are similar to those in part(a), we omit them here. $\Box$

Indeed, the fact that the strict convexity of ϕ implies the strict convexity of $\phi_{\mathbb{V}}^{\rm tr}$ was proved in [2, 8] via checking the definition of convex function. But, here our analysis is much simpler and we also give the relation between $\nabla(\phi^{'})_{\mathbb{V}}^{\rm sc}$ and $\nabla^2\phi_{\mathbb{V}}^{\rm tr}$ to achieve the convexity of SC trace functions. In addition , we note that the necessity involved in the first equivalence of Theorem 3.2(a) was given in [12] for SOC case via a different way. Next, we will illustrate the application of Theorem 3.2 with some SC trace functions.

Theorem 3.3. *The following functions associated with $\mathcal{K}$ are all strictly convex.*

(a) $F_1(x) = -\ln(\det(x))$ *for* $x \in \mathbf{int}\mathcal{K}$.
(b) $F_2(x) = \mathrm{tr}(x^{-1})$ *for* $x \in \mathbf{int}\mathcal{K}$.
(c) $F_3(x) = \mathrm{tr}(h(x))$ *for* $x \in \mathbf{int}\mathcal{K}$, *where*

$$h(x) = \begin{cases} \frac{x^{p+1}-e}{p+1} + \frac{x^{1-q}-e}{q-1} & \text{if } p \in [0,1],\ q > 1; \\ \frac{x^{p+1}-e}{p+1} - \ln x & \text{if } p \in [0,1],\ q = 1. \end{cases}$$

(d) $F_4(x) = -\ln(\det(e - x))$ *for* $x \prec_{\mathcal{K}} e$.
(e) $F_5(x) = \mathrm{tr}((e - x)^{-1} \circ x)$ *for* $x \prec_{\mathcal{K}} e$.
(f) $F_6(x) = \mathrm{tr}(\exp(x))$ *for* $x \in \mathbb{V}$.
(g) $F_7(x) = \ln(\det(e + \exp(x)))$ *for* $x \in \mathbb{V}$.
(h) $F_8(x) = \mathrm{tr}\left(\dfrac{x + (x^2 + 4e)^{1/2}}{2}\right)$ *for* $x \in \mathbb{V}$.

Proof. Note that $F_1(x), F_2(x)$ and $F_3(x)$ are the SC trace functions associated with $\phi_1(t) = -\ln t$ $(t > 0)$, $\phi_2(t) = t^{-1}$ $(t > 0)$ and $\phi_3(t)$ $(t > 0)$, respectively, where

$$\phi_3(t) = \begin{cases} \frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1} & \text{if } p \in [0,1],\ q > 1, \\ \frac{t^{p+1}-1}{p+1} - \ln t & \text{if } p \in [0,1],\ q = 1, \end{cases}$$

$F_4(x)$ is the SC trace function associated with $\phi_4(t) = -\ln(1-t)$ $(t < 1)$, $F_5(x)$ is the SC trace function associated with $\phi_5(t) = \frac{t}{1-t}$ $(t < 1)$ by noting that

$$(e-x)^{-1} \circ x = \frac{\lambda_1(x)}{1-\lambda_1(x)} c_1(x) + \cdots + \frac{\lambda_r(x)}{1-\lambda_r(x)} c_r(x);$$

$F_6(x)$ and $F_7(x)$ are the SC trace functions associated with $\phi_6(t) = \exp(t)$ $(t \in \mathbb{R})$ and $\phi_7(t) = \ln(1+\exp(t))$ $(t \in \mathbb{R})$, respectively, and $F_8(x)$ is the SC trace function associated with $\phi_8(t) = 2^{-1}\left(t + \sqrt{t^2+4}\right)$ $(t \in \mathbb{R})$. It is easy to verify that the functions ϕ_1-ϕ_8 have positive second-order derivatives in their respective domain, and therefore F_1-F_8 are strictly convex functions by Theorem 3.2(b). $\square$

Analogous to SOC case, e.g., [6, 7, 17, 18, 20], the functions F_1, F_2 and F_3 can be served as barrier functions for symmetric cone programming (SCP) which also play a key role in the development of interior point methods for SCPs. The function F_3 covers a wide range of barrier functions for SCPs, including the classical logarithmic barrier function, the self-regular functions and the non-self-regular functions; see [7] for details. The functions F_4 and F_5 are called shifted barrier functions [1, 2, 3] for SOCPs, and F_6-F_8 can be used as penalty functions for SCPs.

Besides the application in establishing convexity for SC trace functions, our establishment of convexity of some compound functions of SC trace functions and scalar-valued functions is much simpler, which is usually difficult to achieve by the definition of convex function.

4. Conclusions

We establish convexity of SC-functions, especially for SC trace functions, which are the key of penalty and barrier function methods for symmetric cone programming and some important inequalities associated with symmetric cones. We believe that the results in this paper will be helpful towards establishing further properties of other SC functions.

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YU-LIN CHANG
Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `ylchang@math.ntnu.edu.tw`

JEIN-SHAN CHEN
Member of Mathematics Division, National Center for Theoretical Sciences, Taipei Office
Department of Mathematics , National Taiwan Normal University, Taipei 11677, Taiwan
E-mail address: `jschen@math.ntnu.edu.tw`