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Analysis of nonsmooth vector-valued functions associated with infinite-dimensional second-order cones

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ABSTRACT

Given a Hilbert space $\mathcal{H}$, the infinite-dimensional Lorentz/second-order cone $\mathbb{K}$ is introduced. For any $x \in \mathcal{H}$, a spectral decomposition is introduced, and for any function $f : \mathbb{R} \to \mathbb{R}$, we define a corresponding vector-valued function $f^{\mathcal{H}}(x)$ on Hilbert space $\mathcal{H}$ by applying f to the spectral values of the spectral decomposition of $x \in \mathcal{H}$ with respect to $\mathbb{K}$. We show that this vector-valued function inherits from f the properties of continuity, Lipschitz continuity, differentiability, smoothness, as well as s-semismoothness. These results can be helpful for designing and analyzing solution methods for solving infinite-dimensional second-order cone programs and complementarity problems.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$, and we write the norm induced by $\langle\cdot,\cdot\rangle$ as $\|\cdot\|$. For any given closed convex cone $K \subseteq \mathcal{H}$,

$$K^* := \{x \in \mathcal{H} \mid \langle x, y\rangle \geq 0,\ \forall y \in K\}$$

is the dual cone of K. A closed convex cone K in $\mathcal{H}$ is called *self-dual* if K coincides with its dual cone K^*; for example, the non-negative orthant cone $\mathbb{R}^n_+$ and the second-order cone (also called Lorentz cone) $\mathbb{K}^n := \{(r, x') \in \mathbb{R} \times \mathbb{R}^{n-1} \mid r \geq \|x'\|\}$. As discussed in [1], this Lorentz cone $\mathbb{K}^n$ can be rewritten as

$$\mathbb{K}^n := \left\{x \in \mathbb{R}^n \mid \langle x, e\rangle \geq \frac{1}{\sqrt{2}}\|x\|\right\} \quad \text{with } e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

This motivates us to consider the following closed convex cone in the Hilbert space $\mathcal{H}$:

$$K(e, r) := \{x \in \mathcal{H} \mid \langle x, e\rangle \geq r\|x\|\}$$

where $e \in \mathcal{H}$ with $\|e\| = 1$ and $r \in \mathbb{R}$ with $0 < r < 1$. It can be seen that $K(e, r)$ is pointed, i.e., $K(e, r) \cap (-K(e, r)) = \{0\}$. Moreover, by denoting

$$\langle e\rangle^{\perp} := \{x \in \mathcal{H} \mid \langle x, e\rangle = 0\},$$

we may express the closed convex cone $K(e, r)$ as

$$K(e, r) = \left\{x' + \lambda e \in \mathcal{H} \mid x' \in \langle e\rangle^{\perp} \text{ and } \lambda \geq \frac{r}{\sqrt{1 - r^2}}\|x'\|\right\}.$$

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When $\mathcal{H} = \mathbb{R}^n$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $K(e, \frac{1}{\sqrt{2}})$ coincides with $\mathbb{K}^n$. In view of this, we shall call $K(e, \frac{1}{\sqrt{2}})$ the infinite-dimensional second-order cone (or infinite-dimensional Lorentz cone) in $\mathcal{H}$ determined by e. In the rest of this paper, we shall only consider any fixed unit vector $e \in \mathcal{H}$, and denote

$$\mathbb{K} = K\left(e, \frac{1}{\sqrt{2}}\right)$$

since two infinite-dimensional second-order cones $\mathbb{K}(e_1)$ and $\mathbb{K}(e_2)$ associated with different unit elements e_1 and e_2 in $\mathcal{H}$ are isometric. This means there exists a bijective isometry P which maps $\mathbb{K}(e_1)$ onto $\mathbb{K}(e_2)$ such that $\|Px\| = \|x\|$ for any $x \in \mathbb{K}(e_1)$. For example, let $e_1 = (1, 0, 0)$ and $e_2 = (0, 0, 1)$. Then, for any $x \in \mathbb{K}(e_1)$ and $y \in \mathbb{K}(e_2)$, we have the following relation:

$$y = Px = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} x.$$

Moreover, this mapping preserves the Jordan algebra structure, i.e., $P(x \circ y) = Px \circ Py$. In the infinite-dimensional Hilbert space, P is indeed a unitary operator. In light of this fact, we can consider the infinite-dimensional second-order cone associated with a fixed arbitrary unit element in $\mathcal{H}$.

Unless specifically stated otherwise, we shall alternatively write any point $x \in \mathcal{H}$ as $x = x' + \lambda e$ with $x' \in \langle e \rangle^{\perp}$ and $\lambda = \langle x, e \rangle$. In addition, for any $x, y \in \mathcal{H}$, we shall write $x \succ_{\mathbb{K}} y$ (respectively, $x \succeq_{\mathbb{K}} y$) if $x - y \in \mathrm{int}\mathbb{K}$ (respectively, $x - y \in \mathbb{K}$). Now, we introduce the spectral decomposition for any element $x \in \mathcal{H}$. For any $x = x' + \lambda e \in \mathcal{H}$, we can decompose x as

$$x = \alpha_1(x) \cdot v_x^{(1)} + \alpha_2(x) \cdot v_x^{(2)}, \tag{1}$$

where $\alpha_1(x), \alpha_2(x)$ and $v_x^{(1)}, v_x^{(2)}$ are the spectral values and the associated spectral vectors of x, with respect to $\mathbb{K}$, given by

$$\alpha_i(x) = \lambda + (-1)^i \|x'\|, \tag{2}$$

$$v_x^{(i)} = \begin{cases} \frac{1}{2}\left(e + (-1)^i \frac{x'}{\|x'\|}\right), & x' \neq 0 \\ \frac{1}{2}(e + (-1)^i w), & x' = 0 \end{cases} \tag{3}$$

for $i = 1, 2$ with w being any vector in $\mathcal{H}$ satisfying $\|w\| = 1$. With this spectral decomposition, for any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathbb{K}$ is defined:

$$f^{\mathcal{H}}(x) = f(\alpha_1(x))v_x^{(1)} + f(\alpha_2(x))v_x^{(2)} \quad \forall x \in \mathcal{H}. \tag{4}$$

The above definition is analogous to the one in finite-dimensional second-order cone case [2,3].

The motivation of studying $f^{\mathcal{H}}$ defined as in (4) is from concerning with the complementarity problem associated with infinite-dimensional second-order cone $\mathbb{K}$, i.e., to find an $x \in \mathcal{H}$ such that

$$x \in \mathbb{K}, \quad T(x) \in \mathbb{K} \quad \text{and} \quad \langle x, T(x) \rangle = 0, \tag{5}$$

where T is a mapping from $\mathcal{H}$ to $\mathcal{H}$. We denote this problem (5) as $\mathrm{CP}(\mathbb{K}, T)$. More specifically, when dealing with such complementarity problem by nonsmooth function approach, i.e., recasting it as a nonsmooth system of equations, we need to check what kind of properties of f can be inherited by $f^{\mathcal{H}}$ so that we can know to what extent the convergence analysis of solutions methods based on such nonsmooth system can be obtained. Indeed, the format of the aforementioned complementarity problem $\mathrm{CP}(\mathbb{K}, T)$ indeed follows the direction of complementarity problems associated with symmetric cones in Euclidean Jordan algebra. Recently, nonlinear symmetric cone optimization and complementarity problems in finite-dimensional spaces such as semidefinite cone optimization and complementarity problems, second-order cone optimization and complementarity problems, and general symmetric cone optimization and complementarity problems, become an active research field of mathematical programming. Taking second-order cone optimization and complementarity problems for example, there have proposed many effective solution methods, including the interior-point methods [4–7], the smoothing Newton methods [8,3,9], the semismooth Newton methods [10,11], and the merit function method [12,13]. However, there are very limited works about nonlinear symmetric cone optimization and complementarity problems in infinite-dimensional spaces, for instance [14], in which with the JB algebras of finite rank primal–dual interior-point methods are presented for some special type of infinite-dimensional cone optimization problems.

It is our intention to extend the above methods for infinite-dimensional complementarity problem $\mathrm{CP}(\mathbb{K}, T)$, in which the vector-valued function $f^{\mathcal{H}}$ will play a key role. In this paper, we study the continuity and differential properties of the vector-valued function $f^{\mathcal{H}}$ in general. In particular, we show that the properties of continuity, strict continuity (locally Lipschitz continuity), Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, and s-semismoothness are each inherited by $f^{\mathcal{H}}$ from f. These results can give some concept in designing solutions methods for solving infinite-dimensional second-order cone programs and infinite-dimensional second-order cone complementarity problems.

2. Preliminaries

For any $x = x' + \lambda e \in \mathcal{H}$ and $y = y' + \mu e \in \mathcal{H}$, we define the Jordan product of x and y by

$$x \circ y := (\mu x' + \lambda y') + \langle x, y\rangle e, \tag{6}$$

and write $x^2 = x \circ x$. Clearly, when $\mathcal{H} = \mathbb{R}^n$ and $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$, this definition coincides with the one given in [15, Chapter II] which is the case of finite-dimensional second-order cone associated with Euclidean Jordan algebra. The following technical lemmas will be frequently used in the subsequent analysis.

Lemma 2.1. *Let $\alpha_1(x), \alpha_2(x)$ be the spectral values of $x \in \mathcal{H}$ and $\alpha_1(y), \alpha_2(y)$ be the spectral values of $y \in \mathcal{H}$. Then we have*

$$|\alpha_1(x) - \alpha_1(y)|^2 + |\alpha_2(x) - \alpha_2(y)|^2 \le 2\|x - y\|^2, \tag{7}$$

and hence, $|\alpha_i(x) - \alpha_i(y)| \le \sqrt{2}\|x - y\|$, $\forall i = 1, 2$.

Proof. The proof can be obtained by direct computation like in [2, Lemma 2]. □

Lemma 2.2. *Let $x = x' + \lambda e \in \mathcal{H}$ and $y = y' + \mu e \in \mathcal{H}$.*

(a) *If $x' \neq 0$ and $y' \neq 0$, then we have*

$$\|v_x^{(i)} - v_y^{(i)}\| \le \frac{1}{\|x'\|}\|x - y\| \quad \forall i = 1, 2, \tag{8}$$

where $v_x^{(i)}, v_y^{(i)}$ are the spectral vectors of x and y, respectively.

(b) *If either $x' = 0$ or $y' = 0$, then we can choose $v_x^{(i)}, v_y^{(i)}$ such that the left hand side of inequality (8) is zero.*

Proof. The proof is similar to [16, Lemma 3.2], so we omit it here. □

Lemma 2.3. *For any $x \neq 0 \in \mathcal{H}$, the following hold.*

(a) *If $g(x) = \|x\|$, we have $g'(x)h = \frac{\langle x,h\rangle}{\|x\|}$.*

(b) *If $g(x) = \frac{x}{\|x\|}$, we have $g'(x)h = \frac{h}{\|x\|} - \frac{\langle x,h\rangle}{\|x\|^3}x$.*

Proof. (a) See Example 3.1(V) of [1].

(b) First, we compute that

$$\begin{aligned} g(x+h) - g(x) &= \frac{x+h}{\|x+h\|} - \frac{x}{\|x\|} \\ &= \frac{h}{\|x+h\|} - \left(\frac{1}{\|x\|} - \frac{1}{\|x+h\|}\right) \cdot x \\ &= \frac{h}{\|x+h\|} - \frac{\sqrt{\langle x+h, x+h\rangle} - \sqrt{\langle x, x\rangle}}{\sqrt{\langle x, x\rangle} \cdot \sqrt{\langle x+h, x+h\rangle}} \cdot x \\ &= \frac{h}{\|x+h\|} - \frac{2\langle x, h\rangle + \langle h, h\rangle}{\sqrt{\langle x, x\rangle} \cdot \sqrt{\langle x+h, x+h\rangle}(\sqrt{\langle x+h, x+h\rangle} + \sqrt{\langle x, x\rangle})} \cdot x \\ &= \frac{h}{\|x\|} - \frac{\langle x, h\rangle}{\|x\|^3}x + o(\|h\|). \end{aligned}$$

From the above, it is clear that $g'(x)h = \frac{h}{\|x\|} - \frac{\langle x,h\rangle}{\|x\|^3}x$. □

Semismooth function, as introduced by Mifflin [17] for functionals and further extended by Qi and Sun [18] for vector-valued functions, is of particular interest due to the central role it plays in the superlinear convergence analysis of certain generalized Newton methods, see [18,19] and references therein. Given a mapping $F : \mathbb{R}^n \to \mathbb{R}^m$, it is well known that if F is strictly continuous (locally Lipschitz continuous), then F is almost everywhere differentiable by Rademacher's Theorem — see [20] and [21, Sec. 9J]. In this case, the generalized Jacobian $\partial F(x)$ of F at x (in the Clarke sense) can be defined as the convex hull of the B-subdifferential $\partial_B F(x)$, where

$$\partial_B F(x) := \left\{ \lim_{x^j \to x} \nabla F(x^j) | F \text{ is differentiable at } x^j \in \mathbb{R}^n \right\}.$$

The notation ∂_B is adopted from [19]. In [21, Chap. 9], the case of $m = 1$ is considered and the notations "$\bar{\nabla}$" and "$\bar{\partial}$" are used instead of, respectively, "∂_B" and "∂". Assume $F : \mathbb{R}^n \to \mathbb{R}^m$ is strictly continuous, then F is said to be semismooth at x if F is directionally differentiable at x and, for any $V \in \partial F(x + h)$ and $h \to 0$, we have

$$F(x+h) - F(x) - Vh = o(\|h\|). \tag{9}$$

Moreover, F is called ρ-order semismooth at x $(0 < \rho < \infty)$ if F is semismooth at x and, for any $V \in \partial F(x+h)$ and $h \to 0$, we have

$$F(x+h) - F(x) - Vh = O(\|h\|^{1+\rho}).$$

The Rademacher theorem does not hold in function spaces, see [22]. Hence, the aforementioned definitions of generalized Jacobian and semismoothness cannot be used in infinite-dimensional spaces. To overcome this difficulty, in the paper [22], so-called slanting functions and slant differentiability of operators in Banach spaces are proposed and used to formulate a concept of semismoothness in infinite-dimensional spaces. We shall introduce them as below. Let $X, Y \subset \mathcal{H}$. A function $F : X \to Y$ is said to be directionally differentiable at x if the limit

$$\delta^+ F(x; h) := \lim_{t \to 0^+} \frac{F(x+th) - F(x)}{t} \tag{10}$$

exists, where $\delta^+ F(x; h)$ is called the directional derivative of F at x with respect to the direction h. A function $F : X \to Y$ is said to be B-differentiable at x if it is directionally differentiable at x and

$$\lim_{h \to 0} \frac{F(x+h) - F(x) - \delta^+ F(x; h)}{\|h\|} = 0 \tag{11}$$

in which we call $\delta^+ F(x; \cdot)$ the B-derivative of F at x. In finite-dimensional Euclidean spaces, Shapiro [23] shows that a locally Lipschitz continuous function F is B-differentiable at x if and only if it is directionally differentiable at x. From (9) and (11) (also see [18]), it can be seen that F is semismooth at x if and only if F is B-differentiable (hence directionally differentiable) at x and, for each $V \in \partial F(x+h)$, there has

$$\delta^+ F(x; h) - Vh = o(\|h\|).$$

As mentioned earlier, these results do not hold in infinite-dimensional spaces. Therefore, the slant differentiability is introduced to circumvent this hurdle. In what follows, we state its definition.

Definition 2.1. Let D be an open domain in X and $L(X, Y)$ denote the set of all bounded linear operators from X onto Y.

(a) A function $F : D \subset X \to Y$ is said to be slantly differentiable at $x \in D$ if there exists a mapping $f^\circ : D \to L(X, Y)$ such that the family $\{f^\circ(x+h)\}$ of bounded linear operators is uniformly bounded in the operator norm for h sufficiently small and

$$\lim_{h \to 0} \frac{F(x+h) - F(x) - f^\circ(x+h)h}{\|h\|} = 0. \tag{12}$$

The function f° is called a slanting function for F at x.

(b) A function $F : D \subset X \to Y$ is said to be slantly differentiable in an open domain $D_0 \subset D$ if there exists a mapping $f^\circ : D \to L(X, Y)$ such that f° is a slanting function for F at every $x \in D_0$. In this case, f° is called a slanting function for F in D_0.

Definition 2.2. Suppose that $f^\circ : D \to L(X, Y)$ is a slanting function for F at $x \in D$ We denote the set

$$\partial_S F(x) := \left\{ \lim_{x_k \to x} f^\circ(x_k) \right\} \tag{13}$$

and call it the slant derivative of F associated with f° at $x \in D$. Note that $f^\circ(x) \in \partial_S F(x)$ which says $\partial_S F(x)$ is always nonempty.

A function F may be slantly differentiable at all points of D, but there is no common slanting function of F at all points of D. Moreover, a slantly differentiable function F at x can have infinitely many slanting functions at x. A slanting function f° for F at x is a single-valued function, but not continuous in general. In addition, a continuous function is not necessarily slantly differentiable. For more details about slanting functions and slantly differentiability, please refer to [22].

Definition 2.3. A mapping $F : X \to Y$ is said to be s-semismooth at x if there is a slanting function f° for F in a neighborhood $\mathcal{N}_x$ of x such that f° and the associated slant derivative satisfy the following two conditions.

(a) $\lim_{t \to 0^+} f^\circ(x+th)h$ exists for every $h \in X$ and

$$\lim_{\|h\| \to 0} \frac{\lim_{t \to 0^+} f^\circ(x+th)h - f^\circ(x+h)h}{\|h\|} = 0.$$

(b) $f^\circ(x+h)h - Vh = o(\|h\|)$ for all $V \in \partial_S F(x+h)$.

We point it out that the function F defined in Definition 2.3 was called semismooth in [22]. However, we here rename it as "s-semismooth" because when X, Y are both finite-dimensional spaces it does not reduce to the original definition introduced by Qi and Sun [18] in finite-dimensional spaces. The main key causing this is the limits in $\partial_S F(x)$ and $\partial_B F(x)$ are approached by different ways. In order to distinguish such difference, we hence use the term "s-semismooth" to convey concept of semismoothness in infinite-dimensional spaces.

3. Continuous properties of $f^{\mathcal{H}}$

In this section, we show properties of continuity and (local) Lipschitz continuity of $f^{\mathcal{H}}$. The arguments are straightforward by checking their definitions.

Proposition 3.1. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x), \alpha_2(x)$ and spectral vectors $v_x^{(1)}, v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, $f^{\mathcal{H}}$ is continuous at $x \in \mathcal{H}$ if and only if f is continuous at $\alpha_1(x), \alpha_2(x)$.*

Proof. ($\Rightarrow$) This part of proof is similar to the argument of [2, Proposition 2(a)].
($\Leftarrow$) This direction of proof is also similar to [16, Proposition 2.2(a)], we omit it. □

Proposition 3.2. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x), \alpha_2(x)$ and spectral vectors $v_x^{(1)}, v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, the following hold.*

(a) *$f^{\mathcal{H}}$ is strictly continuous at $x \in \mathcal{H}$ if and only if f is strictly continuous at $\alpha_1(x), \alpha_2(x)$.*
(b) *$f^{\mathcal{H}}$ is Lipschitz continuous (with respect to $\|\cdot\|$) if and only if f is Lipschitz continuous.*

Proof. (a) ($\Leftarrow$) Suppose f is strictly continuous at $\alpha_1(x), \alpha_2(x)$. Then, there exist $\kappa_i > 0$ and $\delta_i > 0$ for $i = 1, 2$, such that

$$|f(\xi) - f(\zeta)| \le \kappa_i|\xi - \zeta| \quad \forall \xi, \zeta \in [\alpha_i(x) - \delta_i, \alpha_i(x) + \delta_i] \; i = 1, 2.$$

Let $\delta = \frac{1}{\sqrt{2}} \min\{\delta_1, \delta_2\}$ and for any $y, z \in \mathcal{B}(x, \delta)$, we have

$$\begin{aligned} f^{\mathcal{H}}(y) - f^{\mathcal{H}}(z) &= (f(\alpha_1(y))v_y^{(1)} + f(\alpha_2(y))v_y^{(2)}) - (f(\alpha_1(z))v_z^{(1)} + f(\alpha_2(z))v_z^{(2)}) \\ &= f(\alpha_1(y))(v_y^{(1)} - v_z^{(1)}) + (f(\alpha_1(y)) - f(\alpha_1(z)))v_z^{(1)} \\ &\quad + f(\alpha_2(y))(v_y^{(2)} - v_z^{(2)}) + (f(\alpha_2(y)) - f(\alpha_2(z)))v_z^{(2)} \end{aligned} \tag{14}$$

where $y = \alpha_1(y)v_y^{(1)} + \alpha_2(y)v_y^{(2)}$ and $z = \alpha_1(z)v_z^{(1)} + \alpha_2(z)v_z^{(2)}$. By Lemmas 2.1 and 2.2 and the similar argument in [2, Proposition 6(a)], the proof can be obtained.
($\Rightarrow$) This part of proof is quite simple and similar to [2, Proposition 6(a)], we omit it here.
(b) The argument of proof is similar to [2, Proposition 6(c)]. □

4. Differential properties of $f^{\mathcal{H}}$

In this section, we show properties of directional differentiability, differentiability, continuous differentiability and B-differentiability of $f^{\mathcal{H}}$. For simplicity, in the arguments we sometimes abbreviate $\alpha_i(x)$ as α_i when there is no ambiguity in the context. Note that, unlike in finite-dimensional second-order cone case [2], Propositions 4.1 and 4.2 are proved by different approaches since the chain rule for directional differentiability in infinite-dimensional space does not hold in general, see [23].

Proposition 4.1. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x), \alpha_2(x)$ and spectral vectors $v_x^{(1)}, v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, $f^{\mathcal{H}}$ is directionally differentiable at $x \in \mathcal{H}$ if and only if f is directionally differentiable at $\alpha_1(x), \alpha_2(x)$.*

Proof. ($\Leftarrow$) Suppose f is directionally differentiable at $\alpha_1(x), \alpha_2(x)$. Fix $x = x' + \lambda e \in \mathcal{H}$ and any direction $h = h' + le \in \mathcal{H}$, we discuss two cases as below.
Case (i). If $x' \neq 0$, then we have $f^{\mathcal{H}}(x) = f(\alpha_1(x))v_x^{(1)} + f(\alpha_2(x))v_x^{(2)}$ where $\alpha_i(x) = \lambda + (-1)^i\|x'\|$ and $v_x^{(i)} = \frac{1}{2}(e + (-1)^i\frac{x'}{\|x'\|})$ for $i = 1, 2$. Now $x + th = (x' + th') + (\lambda + tl)e$ with spectral values $\alpha_i(x + th) = \lambda + tl + (-1)^i\|x' + th'\|$ and spectral vectors $v_{x+th}^{(i)} = \frac{1}{2}(e + (-1)^i\frac{x'+th'}{\|x'+th'\|})$ for $i = 1, 2$. We consider Eq. (14) again in which replacing y with $x + th$, then we have

$$\begin{aligned} f^{\mathcal{H}}(x + th) - f^{\mathcal{H}}(x) = f(\alpha_1(x + th))(v_{x+th}^{(1)} - v_x^{(1)}) + (f(\alpha_1(x + th)) - f(\alpha_1(x)))v_x^{(1)} \\ + f(\alpha_2(x + th))(v_{x+th}^{(2)} - v_x^{(2)}) + (f(\alpha_2(x + th)) - f(\alpha_2(x)))v_x^{(2)}. \end{aligned} \tag{15}$$

Because the process of checking argument is similar to [2, Proposition 3], we only present the result here.
By denoting

$$\begin{aligned} \tilde{a} &= \frac{f(\alpha_2(x)) - f(\alpha_1(x))}{\alpha_2(x) - \alpha_1(x)}, \\ \tilde{b} &= \frac{\delta^+f(\alpha_2(x); k_2) + \delta^+f(\alpha_1(x); k_1)}{2}, \\ \tilde{c} &= \frac{\delta^+f(\alpha_2(x); k_2) - \delta^+f(\alpha_1(x); k_1)}{2}, \end{aligned} \tag{16}$$

where $k_i = \langle h, e\rangle + (-1)^i \frac{\langle x', h\rangle}{\|x'\|}$ for $i = 1, 2$, we can write the expression of $\delta^+ f^{\mathcal{H}}(x; h)$ as

$$\delta^+ f^{\mathcal{H}}(x; h) = \tilde{a}\left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2}x'\right) + \tilde{b}e + \tilde{c}\frac{x'}{\|x'\|}. \tag{17}$$

Case (ii). If $x' = 0$, we compute the directional derivative $\delta^+ f^{\mathcal{H}}(x; h)$ at $x \in \mathcal{H}$ for any direction h by definition. Let $h = h' + le \in \mathcal{H}$ with $h' \in \langle e\rangle^\perp$ and $l \in \mathbb{R}$. We discuss two subcases.

Subcase (a). If $h' \neq 0$, from the spectral decomposition, we choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i \frac{h'}{\|h'\|})$ for $i = 1, 2$ such that

$$f^{\mathcal{H}}(x + th) = f(\lambda + th_1)v_x^{(1)} + f(\lambda + th_2)v_x^{(2)}$$
$$f^{\mathcal{H}}(x) = f(\lambda)v_x^{(1)} + f(\lambda)v_x^{(2)}$$

where $h_i = l + (-1)^i\|h'\|$ for $i = 1, 2$. Now, we compute

$$\begin{aligned}\lim_{t\to 0^+}\frac{f^{\mathcal{H}}(x + th) - f^{\mathcal{H}}(x)}{t} &= \lim_{t\to 0^+}\frac{f(\lambda + th_1) - f(\lambda)}{t}v_x^{(1)} + \lim_{t\to 0^+}\frac{f(\lambda + th_2) - f(\lambda)}{t}v_x^{(2)}\\ &= \delta^+ f(\lambda; l - \|h'\|)v_x^{(1)} + \delta^+ f(\lambda; l + \|h'\|)v_x^{(2)}.\end{aligned} \tag{18}$$

This shows that $\delta^+ f^{\mathcal{H}}(x; h)$ exists under this subcase.

Subcase (b). If $h' = 0$, we choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i w)$ for any $w \in \mathcal{H}$ with $\|w\| = 1$. Analogous to (18), we have

$$\begin{aligned}\lim_{t\to 0^+}\frac{f^{\mathcal{H}}(x + th) - f^{\mathcal{H}}(x)}{t} &= \lim_{t\to 0^+}\frac{f(\lambda + tl) - f(\lambda)}{t}v_x^{(1)} + \lim_{t\to 0^+}\frac{f(\lambda + tl) - f(\lambda)}{t}v_x^{(2)}\\ &= \delta^+ f(\lambda; l)v_x^{(1)} + \delta^+ f(\lambda; l)v_x^{(2)}.\end{aligned} \tag{19}$$

Hence, $\delta^+ f^{\mathcal{H}}(x; h)$ exists under this subcase.

From all the above, we have proved that $f^{\mathcal{H}}$ is directionally differentiable at $x \in \mathcal{H}$ when $x' = 0$ and its directional derivative $\delta^+ f^{\mathcal{H}}(x; h)$ is either in form of (18) or (19).

($\Rightarrow$) Suppose $f^{\mathcal{H}}$ is directionally differentiable at $x \in \mathcal{H}$, we will prove that f is directionally differentiable at α_1, α_2. For $\alpha_1 \in \mathbb{R}$ and any direction $d_1 \in \mathbb{R}$, let $h = d_1 v_x^{(1)} + 0v_x^{(2)}$ where $x = \alpha_1 v_x^{(1)} + \alpha_2 v_x^{(2)}$. Then, $x + th = (\alpha_1 + td_1)v_x^{(1)} + \alpha_2 v_x^{(2)}$ and

$$\frac{f^{\mathcal{H}}(x + th) - f^{\mathcal{H}}(x)}{t} = \frac{f(\alpha_1 + td_1) - f(\alpha_1)}{t}v_x^{(1)}.$$

Since $f^{\mathcal{H}}$ is directionally differentiable at x, the above equation implies that

$$\delta^+ f(\alpha_1; d_1) = \lim_{t\to 0^+}\frac{f(\alpha_1 + td_1) - f(\alpha_1)}{t} \quad \text{exists.}$$

This means f is directionally differentiable at α_1. Similarly, it can be verified that f is also directionally differentiable at α_2. □

Proposition 4.2. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x), \alpha_2(x)$ and spectral vectors $v_x^{(1)}, v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, $f^{\mathcal{H}}$ is differentiable at $x \in \mathcal{H}$ if and only if f is differentiable at $\alpha_1(x), \alpha_2(x)$.*

Proof. ($\Leftarrow$) Suppose f is differentiable at α_1, α_2. Fix $x = x' + \lambda e \in \mathcal{H}$ and $h = h' + le \in \mathcal{H}$, we discuss two cases as below.

Case (i). If $x' \neq 0$, then we have $f^{\mathcal{H}}(x) = f(\alpha_1)v_x^{(1)} + f(\alpha_2)v_x^{(2)}$ where $\alpha_i = \lambda + (-1)^i\|x'\|$ and $v_x^{(i)} = \frac{1}{2}(e + (-1)^i\frac{x'}{\|x'\|})$ for $i = 1, 2$. By using Lemma 2.3 and the chain rule and product rule for differentiation, the argument is similar to [2, Proposition 4] so we omit the process and present the result as following. Denoting

$$a = \frac{f(\alpha_2) - f(\alpha_1)}{\alpha_2 - \alpha_1}, \qquad b = \frac{f'(\alpha_2) + f'(\alpha_1)}{2}, \qquad c = \frac{f'(\alpha_2) - f'(\alpha_1)}{2}. \tag{20}$$

We can write the expression of $(f^{\mathcal{H}})'(x)h$ as

$$(f^{\mathcal{H}})'(x)h = ah + (b - a)\left(\langle h, e\rangle e + \frac{\langle x', h\rangle}{\|x'\|^2}x'\right) + \frac{c}{\|x'\|}(\langle x', h\rangle e + \langle h, e\rangle x'). \tag{21}$$

Case (ii). The proof is identical to that of Case (ii) in Proposition 4.1, but with th replaced by h. We omit it and only present the formula of $(f^{\mathcal{H}})'(x)h$ as below. If $x' = 0$, then

$$(f^{\mathcal{H}})'(x)h = f'(\lambda)h. \tag{22}$$

($\Rightarrow$) This part of proof is similar to [16, Proposition 2.2(c)]. □

Proposition 4.3. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x)$, $\alpha_2(x)$ and spectral vectors $v_x^{(1)}$, $v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, $f^{\mathcal{H}}$ is continuously differentiable (smooth) at $x \in \mathcal{H}$ if and only if f is continuously differentiable at $\alpha_1(x)$, $\alpha_2(x)$.*

Proof. ($\Leftarrow$) This part of proof is similar to [16, Proposition 2.2(d)], so we omit it.

($\Rightarrow$) This direction of proof is some variant of argument in [2, Proposition 5], we also skip it here. □

Proposition 4.4. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x)$, $\alpha_2(x)$ and spectral vectors $v_x^{(1)}$, $v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, $f^{\mathcal{H}}$ is B-differentiable at $x \in \mathcal{H}$ if and only if f is B-differentiable at $\alpha_1(x)$, $\alpha_2(x)$.*

Proof. ($\Leftarrow$) If f is B-differentiable at $\alpha_1(x)$, $\alpha_2(x)$, f is directionally differentiable at $\alpha_1(x)$, $\alpha_2(x)$. By Proposition 4.1, $f^{\mathcal{H}}$ is directionally differentiable at x. It remains to verify that

$$\lim_{h\to 0} \frac{f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x;h)}{\|h\|} = 0.$$

We write $x = x' + \lambda e$ and $h = h' + le \in \mathcal{H}$. Again, two cases will be discussed.

Case (i). If $x' \neq 0$, considering Eq. (15) in which we replace $x + th$ with $x + h$, it yields

$$\begin{aligned} f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) &= f(\alpha_1(x+h))(v_{x+h}^{(1)} - v_x^{(1)}) + (f(\alpha_1(x+h)) - f(\alpha_1(x)))v_x^{(1)} \\ &\quad + f(\alpha_2(x+h))(v_{x+h}^{(2)} - v_x^{(2)}) + (f(\alpha_2(x+h)) - f(\alpha_2(x)))v_x^{(2)}. \end{aligned} \tag{23}$$

Indeed, sum of the first and third can be simplified as

$$\begin{aligned} & f(\alpha_1(x+h))(v_{x+h}^{(1)} - v_x^{(1)}) + f(\alpha_2(x+h))(v_{x+h}^{(2)} - v_x^{(2)}) \\ &\quad = (f(\alpha_2(x+h)) - f(\alpha_1(x+h))) \cdot \frac{1}{2} \cdot \left(\frac{x'+h'}{\|x'+h'\|} - \frac{x'}{\|x'\|} \right) \\ &\quad = \frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{2\|x'\|} \left(h' - \frac{\langle x', h'\rangle}{\|x'\|^2} x' + o(\|h'\|) \right) \\ &\quad = \frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{\alpha_2(x) - \alpha_1(x)} \left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x' + o(\|h'\|) \right), \end{aligned} \tag{24}$$

where the second equality is due to Lemma 2.3(b) and the last equality uses the fact that $\alpha_2(x) - \alpha_1(x) = 2\|x'\|$. From (17), we know that

$$\begin{aligned} \delta^+ f^{\mathcal{H}}(x;h) &= \frac{f(\alpha_2(x)) - f(\alpha_1(x))}{\alpha_2(x) - \alpha_1(x)} \left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x' \right) \\ &\quad + \delta^+ f(\alpha_1(x); k_1) v_x^{(1)} + \delta^+ f(\alpha_2(x); k_2) v_x^{(2)} \end{aligned} \tag{25}$$

where $k_i = \langle h, e\rangle + (-1)^i \frac{\langle x', h\rangle}{\|x'\|}$ for $i = 1, 2$. Since $\lim_{h\to 0}(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x') = 0$, following almost the same arguments as in Proposition 4.1 gives

$$\begin{aligned} \alpha_i(x+h) - \alpha_i(x) &= l + (-1)^i (\|x'+h'\| - \|x'\|) \\ &= \langle h, e\rangle + (-1)^i \left(\frac{\langle x', h'\rangle}{\|x'\|} + o(\|h'\|) \right) \\ &= k_i + (-1)^i o(\|h'\|) \quad \forall i = 1, 2. \end{aligned}$$

Let $T_i := k_i + (-1)^i o(\|h'\|) = \alpha_i(x+h) - \alpha_i(x)$ for $i = 1, 2$, we obtain

$$\begin{aligned} \lim_{h\to 0} \frac{f(\alpha_i(x+h)) - f(\alpha_i(x))}{\|h\|} &= \lim_{h\to 0} \frac{f(\alpha_i(x) + T_i \cdot 1) - f(\alpha_i(x))}{T_i} \cdot \frac{k_i + (-1)^i o(\|h'\|)}{\|h\|} \\ &= \delta^+ f(\alpha_i(x); 1) \cdot \widetilde{k_i} \\ &= \delta^+ f(\alpha_i(x); \widetilde{k_i}), \end{aligned}$$

where the last equality uses the positive homogeneity property of $\delta^+ f(\alpha_i(x); \cdot)$ again and $\widetilde{k_i} := \lim_{h\to 0} \frac{k_i}{\|h\|}$. We notice that $0 < \|\widetilde{k_i}\| \leq 2$ and $\widetilde{k_i}$ can be viewed as a directional vector here. By the above discussion, we have

$$\begin{aligned} \lim_{h\to 0} \frac{1}{\|h\|} \Bigg(& f(\alpha_1(x+h))(v_{x+h}^{(1)} - v_x^{(1)}) + f(\alpha_2(x+h))(v_{x+h}^{(2)} - v_x^{(2)}) \\ & - \frac{f(\alpha_2(x)) - f(\alpha_1(x))}{\alpha_2(x) - \alpha_1(x)} \left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x' \right) \Bigg) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{(f(\alpha_2(x+h)) - f(\alpha_2(x))) - (f(\alpha_1(x+h)) - f(\alpha_1(x)))}{\|h\| \cdot (\alpha_2(x) - \alpha_1(x))} \cdot \left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x' \right) \\
&\quad + \lim_{h\to 0} \frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{\alpha_2(x) - \alpha_1(x)} \cdot \frac{o(\|h'\|)}{\|h\|} \\
&= \frac{\delta^+ f(\alpha_2(x); \widetilde{k_2}) - \delta^+ f(\alpha_1(x); \widetilde{k_1})}{\alpha_2(x) - \alpha_1(x)} \cdot 0 + 0 \\
&= 0.
\end{aligned} \tag{26}
$$

By assumption, f is B-differentiable at $\alpha_1(x), \alpha_2(x)$ and employ almost the same arguments, we compute

$$
\begin{aligned}
&\lim_{h\to 0} \frac{f(\alpha_i(x+h)) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); k_i)}{\|h\|} \cdot v_x^{(i)} \\
&= \lim_{h\to 0} \left(\frac{f(\alpha_i(x) + T_i) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); T_i)}{\|T_i\|} \cdot \frac{\|k_i + (-1)^i o(\|h'\|)\|}{\|h\|} \right. \\
&\quad \left. + \ \frac{\delta^+ f(\alpha_i(x); T_i) - \delta^+ f(\alpha_i(x); k_i)}{\|h\|} \right) \cdot v_x^{(i)} \\
&= \left(0 \cdot \lim_{h\to 0} \|\widetilde{k_i}\| + \lim_{h\to 0} \delta^+ f \left(\alpha_i(x); (-1)^i \frac{o(\|h'\|)}{\|h\|} \right) \right) \cdot v_x^{(i)} = 0 \quad \forall i = 1, 2.
\end{aligned} \tag{27}
$$

Now from Eqs. (23) and (25)–(27), we see that

$$
\lim_{h\to 0} \frac{f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x; h)}{\|h\|} = 0
$$

which says that $f^{\mathcal{H}}$ is B-differentiable at x.

Case (ii). If $x' = 0$, we need to further consider the following two subcases:

Subcase (a). If $h' \neq 0$, we choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i \frac{h'}{\|h'\|})$ for $i = 1, 2$ such that $v_{x+h}^{(i)} = v_x^{(i)}$. Then,

$$
f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) = (f(\alpha_1(x+h)) - f(\alpha_1(x)))v_x^{(1)} + (f(\alpha_2(x+h)) - f(\alpha_2(x)))v_x^{(2)},
$$

and from Case (ii)(a) of Proposition 4.1, we have

$$
\delta^+ f^{\mathcal{H}}(x; h) = \delta^+ f(\lambda; l - \|h'\|)v_x^{(1)} + \delta^+ f(\lambda; l + \|h'\|)v_x^{(2)},
$$

where $\lambda = \alpha_1(x) = \alpha_2(x)$. Again by the B-differentiability of f at $\alpha_1(x)$ and $\alpha_2(x)$, we have

$$
\begin{aligned}
\lim_{h\to 0} \frac{f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x; h)}{\|h\|} &= \lim_{h\to 0} \frac{f(\alpha_1(x+h)) - f(\alpha_1(x)) - \delta^+ f(\alpha_1(x); l - \|h'\|)}{\|h\|} \cdot v_x^{(1)} \\
&\quad + \lim_{h\to 0} \frac{f(\alpha_2(x+h)) - f(\alpha_2(x)) - \delta^+ f(\alpha_2(x); l + \|h'\|)}{\|h\|} \cdot v_x^{(2)} \\
&= 0,
\end{aligned}
$$

which implies the B-differentiability of $f^{\mathcal{H}}$ at x.

Subcase (b). If $h' = 0$, we choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i w)$ with any $w \in \mathcal{H}$ with $\|w\| = 1$. With almost the same arguments as in Case (ii)-(b) of Proposition 4.1, the B-differentiability of $f^{\mathcal{H}}$ can be verified, we omit the detail here.

($\Rightarrow$) If $f^{\mathcal{H}}$ is B-differentiable at x, then $f^{\mathcal{H}}$ is directionally differentiable at x by definition. Then, f is also directionally differentiable at $\alpha_i(x), \alpha_2(x)$ by Proposition 4.1. In order to prove the B-differentiability of f at $\alpha_i(x), \alpha_2(x)$, all we have to do is proving the following condition:

$$
\lim_{t\to 0} \frac{f(\alpha_i(x) + t) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); t)}{|t|} = 0 \quad \forall i = 1, 2.
$$

Since $f^{\mathcal{H}}$ is B-differentiable at x, the following condition is true:

$$
\lim_{h\to 0} \frac{f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x; h)}{\|h\|} = 0.
$$

Again, we write $x = x' + \lambda e$ and $h = h' + le \in \mathcal{H}$ and discuss two cases.

Case (i). If $x' \neq 0$, from the proof of first part, we know

$$\begin{aligned}
&\lim_{h\to 0} \frac{f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x;h)}{\|h\|} \\
&\quad = \lim_{h\to 0} \left[\frac{(f(\alpha_2(x+h)) - f(\alpha_2(x))) - (f(\alpha_1(x+h)) - f(\alpha_1(x)))}{\|h\| \cdot (\alpha_2(x) - \alpha_1(x))} \cdot \left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x' \right) \right. \\
&\qquad \left. + \frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{\alpha_2(x) - \alpha_1(x)} \cdot \frac{o(\|h'\|)}{\|h\|} + \sum_{i=1}^{2} \frac{f(\alpha_i(x+h)) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); k_i)}{\|h\|} \cdot v_x^{(i)} \right]
\end{aligned}$$

where $k_i = \langle h, e\rangle + (-1)^i \frac{\langle x', h\rangle}{\|x'\|}$ for $i = 1, 2$.

Because $\lim_{h\to 0}(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2} x') = 0$ and $v_x^{(1)} \perp v_x^{(2)}$, we must have

$$\lim_{h\to 0} \frac{f(\alpha_i(x+h)) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); k_i)}{\|h\|} = 0 \quad \forall i = 1, 2. \tag{28}$$

Note that $h \in \mathcal{H}$ is arbitrary, we can choose $h = te$ where $t \in \mathbb{R}$ is also arbitrary. Then, we have

$$k_i = \alpha_i(x+h) - \alpha_i(x) = t \quad \forall i = 1, 2.$$

This together with the fact that $t \to 0$ as $h \to 0$ gives

$$\lim_{t\to 0} \frac{f(\alpha_i(x) + t) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); t)}{|t|} = 0 \quad \forall i = 1, 2,$$

which means that f is B-differentiable at $\alpha_i(x)$ for $i = 1, 2$.

Case (ii). If $x' = 0$, we consider the two subcases of $h' = 0$ or $h' \neq 0$. The proof is routine check as earlier verifications, so we omit it. □

5. *S*-semismooth properties of $f^{\mathcal{H}}$

In this section, we show s-semismooth properties of $f^{\mathcal{H}}$. To this end, we first present some equivalent criteria for s-semismooth functions in infinite-dimensional spaces. In fact, we immediate obtain the following criteria from the very basic definition and combining some known results in [22].

Proposition 5.1. *Suppose that $F : X \to Y$ is slantly differentiable on a neighborhood $\mathcal{N}_x$ of x. Let f° be a slanting function for F in $\mathcal{N}_x$ and $\partial_S F$ be the slant derivative associated with f° in $\mathcal{N}_x$. Then, F is s-semismooth at x if and only if one of the following holds:*

(a) $\lim_{t\to 0^+} f^\circ(x+th)h$ *exists for every* $h \in X$,

$$\lim_{\|h\|\to 0} \frac{\lim_{t\to 0^+} f^\circ(x+th)h - f^\circ(x+h)h}{\|h\|} = 0, \tag{29}$$

and

$$f^\circ(x+h)h - Vh = o(\|h\|) \quad \forall V \in \partial_S F(x+h). \tag{30}$$

(b) *F is B-differentiable at x, and*

$$\delta^+ F(x; h) - Vh = o(\|h\|) \quad \forall V \in \partial_S F(x+h). \tag{31}$$

(c) *F is B-differentiable at x, and*

$$F(x+h) - F(x) - Vh = o(\|h\|) \quad \forall V \in \partial_S F(x+h). \tag{32}$$

Proof. (a) This is clear from the original definition of s-semismooth function given as in Definition 2.3.
(b) This is result of [22, Theorem 3.3].
(c) Using part(a) and [22, Theorem 2.9] yield F being B-differentiable at x, and

$$\delta^+ F(x; h) - f^\circ(x+h)h = o(\|h\|). \tag{33}$$

Then, by definition of F being B-differentiable, condition (31) holds. □

The conditions in Proposition 5.1 are indeed hard to be verified since it is difficult to write out the set $\partial_S F(x+h)$. Hence, we further establish some equivalent conditions which are useful in subsequent analysis regarding s-semismooth property which is the main contribution of this paper. We also want to point out the following observation. Suppose that $F : X \to Y$ is slantly differentiable on a neighborhood $\mathcal{N}_x$ of x. Let f° be a slanting function for F with uniform bound $\|f^\circ\| \le L$ in $\mathcal{N}_x$. It is easy to derive that $\|F(y) - F(z)\| \le 2L\|y - z\|$ for any $y, z \in \mathcal{N}_x$. However, we have no idea whether it is true or not for the opposite direction.

Proposition 5.2. *Suppose that $F : X \to Y$ is slantly differentiable on a neighborhood $\mathcal{N}_x$ of x. Let f° be a slanting function for F in $\mathcal{N}_x$ and $\partial_S F$ be the slant derivative associated with f° in $\mathcal{N}_x$. Then, the following hold.*

(a) *If F is s-semismooth at x, then F is B-differentiable at x, and*

$$F(x+h) - F(x) - \delta^+ F(x+h; h) = o(\|h\|) \tag{34}$$

for all $x+h$ at which F is B-differentiable.

(b) *If F is B-differentiable on a neighborhood $\mathcal{N}_x$ of x and (34) holds for all $x+h$ at which F is B-differentiable, then F is s-semismooth at x.*

Proof. (a) The B-differentiability of F at x is clear by Proposition 5.1. It remains to claim that when F is B-differentiable at $x+h$, there has

$$\frac{\|F(x+h) - F(x) - \delta^+ F(x+h; h)\|}{\|h\|} \to 0 \quad \text{as } h \to 0. \tag{35}$$

If not, there exist a $\delta > 0$ and a sequence $h_i \to 0$ such that F is B-differentiable at $x + h_i$ for each $i = 1, 2, \ldots,$ and

$$\frac{\|F(x+h_i) - F(x) - \delta^+ F(x+h_i; h_i)\|}{\|h_i\|} \ge \delta. \tag{36}$$

By assumption, F is s-semismooth at x, then for each $i \ge 1$ there exist $V_i \in \partial_S F(x+h_i)$ and $y_i \in \mathcal{N}_{x+h}$ such that

$$\|V_i - f^\circ(y_i)\| \le \|h_i\|, \qquad \|y_i - (x+h_i)\| \le \|h_i\|^2 \tag{37}$$

and

$$\frac{\|F(x+h_i) - F(x) - V_i h_i\|}{\|h_i\|} \to 0 \quad \text{as } h_i \to 0. \tag{38}$$

By [22, Proposition 2.8], for each h_i there exist $t_i > 0$ with $0 < t_i \le \|h_i\|$ such that

$$\|f^\circ(x+h_i+t_ih_i)h_i - \delta^+ F(x+h_i; h_i)\| \le \|h_i\|^2. \tag{39}$$

Now we compute

$$\begin{aligned} f^\circ(y_i)h_i - f^\circ(x+h_i+t_ih_i)h_i = {} & (F(x+h_i+t_ih_i) - F(x) - f^\circ(x+h_i+t_ih_i)(h_i+t_ih_i)) \\ & - (F(y_i) - F(x) - f^\circ(y_i)(y_i - x)) + (F(y_i) - F(x+h_i+t_ih_i)) \\ & + f^\circ(y_i)(x+h_i-y_i) + f^\circ(x+h_i+t_ih_i)(t_ih_i). \end{aligned} \tag{40}$$

Because F is slantly differentiable at x, the first and second term of (40) implies

$$\frac{F(x+h_i+t_ih_i) - F(x) - f^\circ(x+h_i+t_ih_i)(h_i+t_ih_i)}{\|h_i+t_ih_i\|} \to 0 \quad \text{as } i \to \infty$$

and

$$\frac{F(y_i) - F(x) - f^\circ(y_i)(y_i - x)}{\|y_i - x\|} \to 0 \quad \text{as } i \to \infty$$

which lead to

$$\begin{aligned} & \frac{F(x+h_i+t_ih_i) - F(x) - f^\circ(x+h_i+t_ih_i)(h_i+t_ih_i)}{\|h_i\|} \\ & \quad = \frac{F(x+h_i+t_ih_i) - F(x) - f^\circ(x+h_i+t_ih_i)(h_i+t_ih_i)}{\|h_i+t_ih_i\|} \cdot \frac{\|h_i+t_ih_i\|}{\|h_i\|} \\ & \quad \to 0 \quad \text{as } i \to \infty \end{aligned} \tag{41}$$

and

$$\frac{F(y_i)-F(x)-f^\circ(y_i)(y_i-x)}{\|h_i\|} = \frac{F(y_i)-F(x)-f^\circ(y_i)(y_i-x)}{\|y_i-x\|}\cdot\frac{\|y_i-x\|}{\|h_i\|} \to 0 \quad \text{as } i\to\infty. \tag{42}$$

Here we use that fact that $\|h_i+t_ih_i\|=(1+t_i)\|h_i\|$ and $\|y_i-x\|=\|y_i-x-h_i+h_i\|\le\|y_i-x-h_i\|+\|h_i\|\le\|h_i\|^2+\|h_i\|$. Besides, for the third, fourth and fifth term of (40), since F is slantly differentiable in a neighborhood $\mathcal{N}_x$ of x, $\|f^\circ(x)\|$ is uniformly bounded in $\mathcal{N}_x$, say $\|f^\circ(x)\|\le M$ in $\mathcal{N}_x$. Hence we have

$$\begin{aligned}\|F(y_i)-F(x+h_i+t_ih_i)\| &\le M\|y_i-(x+h_i+t_ih_i)\|\\ &\le M(\|h_i\|^2+t_i\|h_i\|),\end{aligned}$$

$$\|f^\circ(y_i)(x+h_i-y_i)\|\le M\|x+h_i-y_i\|\le M\|h_i\|^2$$

and

$$\|f^\circ(x+h_i+t_ih_i)(t_ih_i)\|\le M\|t_ih_i\|\le M\|h_i\|^2$$

which implies

$$\frac{\|F(y_i)-F(x+h_i+t_ih_i)\|}{\|h_i\|}\to 0 \quad \text{as } i\to\infty, \tag{43}$$

$$\frac{\|f^\circ(y_i)(x+h_i-y_i)\|}{\|h_i\|}\to 0 \quad \text{as } i\to\infty, \tag{44}$$

$$\frac{\|f^\circ(x+h_i+t_ih_i)(t_ih_i)\|}{\|h_i\|}\to 0 \quad \text{as } i\to\infty. \tag{45}$$

Combining (41)–(45) all together, we have

$$\frac{\|f^\circ(y_i)h_i-f^\circ(x+h_i+t_ih_i)h_i\|}{\|h_i\|}\to 0 \quad \text{as } i\to\infty. \tag{46}$$

Now consider

$$\begin{aligned}F(x+h_i)-F(x)-\delta^+F(x+h_i;h_i) = {}&[F(x+h_i)-F(x)-V_ih_i]+[V_ih_i-f^\circ(y_i)h_i]\\ &+[f^\circ(y_i)h_i-f^\circ(x+h_i+t_ih_i)h_i]\\ &+[f^\circ(x+h_i+t_ih_i)h_i-\delta^+F(x+h_i;h_i)].\end{aligned}$$

From (38), (37), (46) and (39), we have

$$\frac{\|F(x+h_i)-F(x)-\delta^+F(x+h_i;h_i)\|}{\|h_i\|}\to 0 \quad \text{as } h_i\to 0.$$

This is a contradiction to Eq. (36), hence (35) holds for all $x+h$ at which F is B-differentiable.

(b) By Proposition 5.1(c), it suffice to show that for each $V\in\partial_SF(x+h)$, there has

$$\frac{\|F(x+h)-F(x)-Vh\|}{\|h\|}\to 0 \quad \text{as } \|h\|\to 0.$$

If not, there exist $\delta>0$ and a sequence $h_i\to 0$, $V_i\in\partial_SF(x+h_i)$ and $y_i\in\mathcal{N}_{x+h_i}$ such that $\|y_i-(x+h_i)\|\le\|h_i\|^2$, $\|V_i-f^\circ(y_i)\|\le\|h_i\|$ and

$$\frac{\|F(x+h_i)-F(x)-V_ih_i\|}{\|h_i\|}\ge\delta.$$

By assumption, F is B-differentiable in a neighborhood of x and satisfies (34) which yields

$$\frac{\|F(x+h_i)-F(x)-\delta^+F(x+h_i;h_i)\|}{\|h_i\|}\to 0 \quad \text{as } \|h_i\|\to 0.$$

Then, we consider

$$\begin{aligned}F(x+h_i)-F(x)-V_ih_i = {}&[F(x+h_i)-F(x)-\delta^+F(x+h_i;h_i)]+[f^\circ(y_i)h_i-V_ih_i]\\ &+[f^\circ(x+h_i+t_ih_i)h_i-f^\circ(y_i)h_i]+[\delta^+F(x+h_i;h_i)-f^\circ(x+h_i+t_ih_i)h_i].\end{aligned}$$

With similar argument and choice of t_i in part (a), we have

$$\frac{\|F(x+h_i)-F(x)-V_ih_i\|}{\|h_i\|}\to 0 \quad \text{as } i\to\infty.$$

This leads to a contradiction. Thus, the proof is complete. □

Lemma 5.1. *$f^{\mathcal{H}}$ has the continuity or differential properties in a neighborhood $\mathcal{N}_x$ of x with spectral values $\alpha_1(x)$, $\alpha_2(x)$ if and only if f has the continuity or differential properties in neighborhoods $\mathcal{N}_{\alpha_i(x)}$ of $\alpha_i(x)$ for all $i=1,2$.*

Proof. (⇐) Suppose f has the continuity or differential properties in neighborhoods $\mathcal{B}(\alpha_i(x),\delta_i)$ of $\alpha_i(x)$. By taking $\delta=\min\{\delta_i\}$, we may assume that f has the continuity or differential properties in neighborhoods $\mathcal{B}(\alpha_i(x),\delta)$ of $\alpha_i(x)$. Then, for any $y\in\mathcal{B}(x,\frac{\delta}{\sqrt{2}})$ with $\|y-x\|\le\frac{\delta}{\sqrt{2}}$, applying Lemma 2.1 gives

$$|\alpha_i(y)-\alpha_i(x)|\le\sqrt{2}\|y-x\|\le\delta,\quad \forall i=1,2,$$

which means $\alpha_i(y)\in\mathcal{B}(\alpha_i(x),\delta)$. From assumption we know that f has the continuity or differential properties at $\alpha_i(y)$. Then by previous propositions of this article, $f^{\mathcal{H}}$ has the continuity or differential properties at $y\in\mathcal{B}(x,\frac{\delta}{\sqrt{2}})$.

(⇒) Suppose $f^{\mathcal{H}}$ has the continuity or differential properties in a neighborhood $\mathcal{B}(x,\delta)$ of x. For any $s\in\mathcal{B}(\alpha_i(x),\delta)$, say $s=\alpha_i(x)+k$, $0\le|k|<\delta$, let $y=x+ke$ with $\|y-x\|=|k|<\delta$. That is, $y\in\mathcal{B}(x,\delta)$, by assumption and previous propositions of this article, we know that f has the continuity or differential properties at $\alpha_i(y)=\alpha_i(x+ke)=s$. □

Lemma 5.2. *Let $x=x'+\lambda e\in\mathcal{H}$ with spectral values $\alpha_1(x)$, $\alpha_2(x)$ and spectral vectors $v_x^{(1)}$, $v_x^{(2)}$. If $x'\neq 0$, then the following hold.*

(a) *$(\alpha_i(x))'$ is a slanting function for $\alpha_i(x)$ in a neighborhood $\mathcal{N}_x$ of x for all $i=1,2$.*
(b) *$(v_x^{(i)})'$ is a slanting function for $v_x^{(i)}$ in a neighborhood $\mathcal{N}_x$ of x for all $i=1,2$.*

Proof. As usual, we write $y=y'+\mu e\in\mathcal{N}_x$.
(a) If $x'\neq 0$, for any nonzero $(y-x)\in\mathcal{H}$, since $\alpha_i(y+y-x)=(2\mu-\lambda)+(-1)^i\|2y'-x'\|$ and $\alpha_i(y)=\mu+(-1)^i\|y'\|$, there has

$$\begin{aligned}\alpha_i(y+y-x)-\alpha_i(y)&=(\mu-\lambda)+(-1)^i(\|2y'-x'\|-\|y'\|)\\&=(\mu-\lambda)+(-1)^i\frac{\langle x'+2(y'-x'),x'+2(y'-x')\rangle-\langle x'+(y'-x'),x'+(y'-x')\rangle}{\|x'+2(y'-x')\|+\|x'+(y'-x')\|}\\&=(\mu-\lambda)+(-1)^i\frac{2\langle x',y'-x'\rangle+3\langle y'-x',y'-x'\rangle}{\|x'+2(y'-x')\|+\|x'+(y'-x')\|}\\&=\langle y-x,e\rangle+(-1)^i\frac{\langle x',y'-x'\rangle}{\|x'\|}+o(\|y'-x'\|).\end{aligned}$$

This implies

$$(\alpha_i(y))'(y-x)=\langle y-x,e\rangle+(-1)^i\frac{\langle x',y'-x'\rangle}{\|x'\|}\quad\forall i=1,2. \tag{47}$$

On the other hand,

$$\begin{aligned}\alpha_i(y)-\alpha_i(x)&=(\mu-\lambda)+(-1)^i(\|y'\|-\|x'\|)\\&=\langle y-x,e\rangle+(-1)^i\frac{\langle x'+(y'-x'),x'+(y'-x')\rangle-\langle x',x'\rangle}{\|x'+(y'-x')\|+\|x'\|}\\&=\langle y-x,e\rangle+(-1)^i\frac{\langle x',y'-x'\rangle}{\|x'\|}+o(\|y'-x'\|)\quad\forall i=1,2.\end{aligned} \tag{48}$$

Then, the fact that $\|y'-x'\|\le\|y-x\|$ together with Eqs. (47)–(48) yields

$$\alpha_i(y)-\alpha_i(x)-(\alpha_i(y))'(y-x)=o(\|y-x\|)\quad\forall i=1,2,$$

which says the condition (12) in Definition 2.1(a) is satisfied.
Now, it remains to show that $\{(\alpha_i(x))'\}$ is uniformly bounded. To see this, for any $z\in\mathcal{H}$, we estimate it as following.
If $x'\neq 0$, then

$$(\alpha_i(x))'z=\langle z,e\rangle+(-1)^i\frac{\langle x',z\rangle}{\|x'\|}\quad\forall i=1,2.$$

Hence,

$$\|(\alpha_i(x))'\| = \sup_{z\neq 0}\frac{\|(\alpha_i(x))'z\|}{\|z\|} = \left\|\left\langle \frac{z}{\|z\|}, e\right\rangle + (-1)^i\left\langle \frac{x'}{\|x'\|}, \frac{z}{\|z\|}\right\rangle\right\| \le 2 \quad \forall i = 1, 2.$$

From all the above, we prove $(\alpha_i(x))'$ is a slanting function for $\alpha_i(x)$ in a neighborhood $\mathcal{N}_x$ of x for all $i = 1, 2$.

(b) The verification is routine, we only list the key steps here.

$$v_y^{(i)} - v_x^{(i)} = \frac{(-1)^i}{2\|x'\|}\left((y'-x') - \frac{\langle x', y'-x'\rangle}{\|x'\|^2}x'\right) + o(\|y'-x'\|) \quad \forall i = 1, 2.$$

$$(v_y^{(i)})'(y-x) = \frac{(-1)^i}{2\|x'\|}\left((y'-x') - \frac{\langle x', y'-x'\rangle}{\|x'\|^2}x'\right) \quad \forall i = 1, 2.$$

For any $z = z' + \xi e \in \mathcal{H}$, there hold

$$\sup_{z\neq 0}\frac{\|(v_y^{(i)})'z\|}{\|z\|} = \frac{1}{2\|y'\|}\left\|\frac{z'}{\|z\|} - \left\langle \frac{y'}{\|y'\|}, \frac{z}{\|z\|}\right\rangle \frac{y'}{\|y'\|}\right\| \le \frac{1}{\|y'\|}. \quad \square$$

As in finite-dimensional case, we were hoping to establish that f is s-semismooth at $\alpha_1(x), \alpha_2(x)$ if and only if $f^{\mathcal{H}}$ is s-semismooth at $x \in \mathcal{H}$. However, it is not possible to achieve this due to some essential difference between concepts of s-semismoothness and semismoothness. As shown in the following proposition, we need some additional condition to carry it.

Proposition 5.3. *Suppose $x = x' + \lambda e \in \mathcal{H}$ with spectral values $\alpha_1(x), \alpha_2(x)$ and spectral vectors $v_x^{(1)}, v_x^{(2)}$. Let $f^{\mathcal{H}}$ be defined as in (4). Then, the following hold.*

(a) *If f is s-semismooth at $\alpha_1(x), \alpha_2(x)$ and f is B-differentiable on neighborhood of $\alpha_1(x), \alpha_2(x)$, then $f^{\mathcal{H}}$ is s-semismooth at $x \in \mathcal{H}$.*

(b) *If $f^{\mathcal{H}}$ is s-semismooth at $x \in \mathcal{H}$ and $f^{\mathcal{H}}$ is B-differentiable on neighborhood of x, then f is s-semismooth at $\alpha_1(x), \alpha_2(x)$.*

Proof. (a) Since f is s-semismooth at $\alpha_1(x), \alpha_2(x)$, by Definition 2.3, there exists slanting functions f_i° for f in neighborhood $\mathcal{N}_{\alpha_i(x)}$ of $\alpha_i(x)$ for $i = 1, 2$. Denote $f^\circ(z) = f_1^\circ(z)$ if $z \in \mathcal{N}_{\alpha_1(x)}$ and $f^\circ(z) = f_2^\circ(z)$ if $z \in \mathcal{N}_{\alpha_2(x)}$, then f° is a slanting function for f in $\mathcal{N}_{\alpha_1(x)} \bigcup \mathcal{N}_{\alpha_2(x)}$. For any $y \in \mathcal{N}_x$, since $|\alpha_i(y) - \alpha_i(x)| \le \sqrt{2}\|y-x\|$ by Lemma 2.1, we have $\alpha_i(y) \in \mathcal{N}_{\alpha_i(x)}$ and

$$f(\alpha_i(y)) - f(\alpha_i(x)) - f^\circ(\alpha_i(y))(\alpha_i(y) - \alpha_i(x)) = o(|\alpha_i(y) - \alpha_i(x)|) = o(\|y-x\|), \tag{49}$$

where $\alpha_i(y) \in \mathcal{N}_{\alpha_i(x)}$ for $i = 1, 2$. From definition of slanting function, we know

$$\|f^\circ(\alpha_i(y))\| \le L, \tag{50}$$

where L is a positive number and $\alpha_i(y) \in \mathcal{N}_{\alpha_i(x)}$ for $i = 1, 2$. In addition, by Lemma 5.2(a),

$$\alpha_i(y) - \alpha_i(x) - (\alpha_i(y))'(y-x) = o(\|y-x\|) \quad \forall i = 1, 2,$$

and hence Eq. (49) turns into

$$f(\alpha_i(y)) - f(\alpha_i(x)) - f^\circ(\alpha_i(y))(\alpha_i(y))'(y-x) = o(\|y-x\|), \quad \forall i = 1, 2. \tag{51}$$

Now for any $y \in \mathcal{N}_x$, we define

$$(f^{\mathcal{H}})^\circ(y)(h) = \begin{cases} \displaystyle\sum_{i=1}^{2}(f^\circ(\alpha_i(y))(\alpha_i(y))'(h)v_y^{(i)} + f(\alpha_i(y))(v_y^{(i)})'(h)), & \text{if } x' \neq 0,\\ \displaystyle\sum_{i=1}^{2} f^\circ(\alpha_i(y))\alpha_i(y-x)v_y^{(i)}, & \text{if } x' = 0, \end{cases} \tag{52}$$

we will show that $(f^{\mathcal{H}})^\circ$ is a slanting function for $f^{\mathcal{H}}$ in a neighborhood $\mathcal{N}_x$ of x. We write $y = y' + \mu e \in \mathcal{N}_x$ and discuss two cases.

Case (i). If $x' \neq 0$, considering Eq. (23) in which we replace $x+h$ with y gives

$$\begin{aligned} f^{\mathcal{H}}(y) - f^{\mathcal{H}}(x) &= f(\alpha_1(y))(v_y^{(1)} - v_x^{(1)}) + (f(\alpha_1(y)) - f(\alpha_1(x)))v_x^{(1)} \\ &\quad + f(\alpha_2(y))(v_y^{(2)} - v_x^{(2)}) + (f(\alpha_2(y)) - f(\alpha_2(x)))v_x^{(2)}. \end{aligned} \tag{53}$$

In fact, the sum of first and third terms in (53) can be simplified as

$$f(\alpha_1(y))(v_y^{(1)} - v_x^{(1)}) + f(\alpha_2(y))(v_y^{(2)} - v_x^{(2)}) = (f(\alpha_2(y)) - f(\alpha_1(y))) \cdot \frac{1}{2} \cdot \left(\frac{y'}{\|y'\|} - \frac{x'}{\|x'\|}\right)$$
$$= \frac{f(\alpha_2(y)) - f(\alpha_1(y))}{2\|x'\|}\left((y' - x') - \frac{\langle x', y' - x'\rangle}{\|x'\|^2}x' + o(\|y' - x'\|)\right). \tag{54}$$

Now, we compute

$$(f^{\mathcal{H}})^{\circ}(y)(y - x) = f^{\circ}(\alpha_1(y))(\alpha_1(y))'(y - x)v_y^{(1)} + f(\alpha_1(y))(v_y^{(1)})'(y - x)$$
$$= f^{\circ}(\alpha_2(y))(\alpha_2(y))'(y - x)v_y^{(2)} + f(\alpha_2(y))(v_y^{(2)})'(y - x). \tag{55}$$

By Lemma 5.2(b), the sum of second and fourth terms in (55) becomes

$$f(\alpha_1(y))(v_y^{(1)})'(y - x) + f(\alpha_2(y))(v_y^{(2)})'(y - x) = (f(\alpha_2(y)) - f(\alpha_1(y)))(v_y^{(2)})'(y - x)$$
$$= (f(\alpha_2(y)) - f(\alpha_1(y))) \cdot \frac{1}{2\|x'\|} \cdot \left((y' - x') - \frac{\langle x', y' - x'\rangle}{\|x'\|^2}x'\right). \tag{56}$$

Next subtracting the first/third term of (55) from the second/fourth term of (53), we obtain

$$\|(f(\alpha_i(y)) - f(\alpha_i(x)))v_x^{(i)} - f^{\circ}(\alpha_i(y))(\alpha_i(y))'(y - x)v_y^{(i)}\|$$
$$\le \|f(\alpha_i(y)) - f(\alpha_i(x)) - f^{\circ}(\alpha_i(y))(\alpha_i(y))'(y - x)\|\|v_x^{(i)}\| + \|f^{\circ}(\alpha_i(y))(\alpha_i(y))'(y - x)\| \cdot \|v_y^{(i)} - v_x^{(i)}\| \tag{57}$$
$$\le \frac{1}{\sqrt{2}} \cdot o(\|y - x\|) + L \cdot 2\|y - x\| \cdot \frac{1}{\|x'\|} \cdot \|y - x\|$$
$$= o(\|y - x\|) \quad \forall i = 1, 2, \tag{58}$$

where the second inequality comes from (50)–(51), Lemma 2.2 and Lemma 5.2(a). The third inequality is due to the fact that $x' \neq 0$. Now putting (53)–(57) all together yields

$$f^{\mathcal{H}}(y) - f^{\mathcal{H}}(x) - (f^{\mathcal{H}})^{\circ}(y)(y - x) = o(\|y - x\|), \tag{59}$$

where $y \in \mathcal{N}_x$.

Case (ii). If $x' = 0$, and if $y' \neq 0$, we choose $\frac{y'}{\|y'\|} \neq 0$ such that $v_x^{(i)} = \frac{1}{2}(e + (-1)^i \frac{y'}{\|y'\|}) = v_y^{(i)}$. If $y' = 0$, we choose a same unit vector w with $\|w\| = 1$ such that $v_x^{(i)} = \frac{1}{2}(e + (-1)^i w) = v_y^{(i)}$, for all $i = 1, 2$. In this case, Eq. (54) becomes zero because of $v_x^{(i)} = v_y^{(i)}$, $\forall i = 1, 2$, when $x' = 0$. From (49) and the fact that $\alpha_i(y - x) = \alpha_i(y) - \alpha_i(x)$ when $x' = 0$, we have

$$f^{\mathcal{H}}(y) - f^{\mathcal{H}}(x) - (f^{\mathcal{H}})^{\circ}(y)(y - x) = \sum_{i=1}^{2}[f(\alpha_i(y)) - f(\alpha_i(x)) - f^{\circ}(\alpha_i(y))(\alpha_i(y) - \alpha_i(x))]v_x^{(i)}$$
$$= o(\|y - x\|).$$

Now since f°, $(\alpha_i(y))'$ and $(v_y^{(i)})'$ are slanting functions of f, $\alpha_i(y)$ and $v_y^{(i)}$ for all $i = 1, 2$, we can easily check that $(f^{\mathcal{H}})^{\circ}$ is uniformly bounded in a neighborhood $\mathcal{N}_x$ of x. From above discussion, $(f^{\mathcal{H}})^{\circ}$ is a slanting function for $f^{\mathcal{H}}$ in a neighborhood $\mathcal{N}_x$ of x. In addition, because of the assumption that f is B-differentiable on neighborhood of $\alpha_1(x), \alpha_2(x)$, we obtain the result that $f^{\mathcal{H}}$ is B-differentiable on neighborhood of x by Lemma 5.1.

In order to prove that $f^{\mathcal{H}}$ is s-semismooth at x, by Proposition 5.2(b), all we need to do is to identify (34) holds for all $x + h$ at which $f^{\mathcal{H}}$ is B-differentiable. Let $h = h' + le$.

Case (i). If $x' \neq 0$, From Eqs. (23) and (24), we have

$$f^{\mathcal{H}}(x + h) - f^{\mathcal{H}}(x) = \frac{f(\alpha_2(x + h)) - f(\alpha_1(x + h))}{\alpha_2(x) - \alpha_1(x)}\left(h - \langle h, e\rangle e - \frac{\langle x', h\rangle}{\|x'\|^2}x' + o(\|h'\|)\right)$$
$$+ \sum_{i=1}^{2}(f(\alpha_i(x + h) - f(\alpha_i(x))))v_x^{(i)}. \tag{60}$$

From (25), we further have

$$\delta^+ f^{\mathcal{H}}(x + h; h) = \frac{f(\alpha_2(x + h)) - f(\alpha_1(x + h))}{\alpha_2(x + h) - \alpha_1(x + h)}\left(h - \langle h, e\rangle e - \frac{\langle x' + h', h'\rangle}{\|x' + h'\|^2}(x' + h')\right)$$
$$+ \sum_{i=1}^{2}\delta^+ f(\alpha_i(x + h); \overline{k_i})v_{x+h}^{(i)}, \tag{61}$$

where $\overline{k_i} = \langle h, e\rangle + (-1)^i \frac{\langle x'+h', h'\rangle}{\|x'+h'\|}$, for all $i = 1, 2$. Since $\alpha_2(x) - \alpha_1(x) = 2\|x'\|$ and $\alpha_2(x+h) - \alpha_1(x+h) = 2\|x'+h'\|$, the first term of (60) becomes

$$\frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{2}\left(\frac{h'}{\|x'\|} - \frac{\langle x', h'\rangle}{\|x'\|^3}x' + o(\|h'\|)\right)$$

and the first term of (61) becomes

$$\begin{aligned}&\frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{2}\left(\frac{h'}{\|x'+h'\|} - \frac{\langle x', h'\rangle + \langle h', h'\rangle}{\|x'+h'\|^3}(x'+h')\right)\\ &\quad = \frac{f(\alpha_2(x+h)) - f(\alpha_1(x+h))}{2}\left(\frac{h'}{\|x'\|} - \frac{\langle x', h'\rangle}{\|x'\|^3}x' + o(\|h'\|)\right).\end{aligned}$$

Hence the first terms of (60) and (61) are equal. Now we consider the second/third terms of (60) and (61). For all $i = 1, 2$,

$$\begin{aligned}&f(\alpha_i(x+h) - f(\alpha_i(x)))v_x^{(i)} - \delta^+ f(\alpha_i(x+h); \overline{k_i})v_{x+h}^{(i)}\\ &\quad = (f(\alpha_i(x)+T_i) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); T_i))v_x^{(i)} + (\delta^+ f(\alpha_i(x); T_i)v_x^{(i)} - \delta^+ f(\alpha_i(x+h); \overline{k_i})v_{x+h}^{(i)}),\end{aligned} \tag{62}$$

where $T_i = k_i + (-1)^i o(\|h'\|) = \langle h, e\rangle + (-1)^i \frac{\langle x', h\rangle}{\|x'\|} + (-1)^i o(\|h'\|)$. Since f is B-differentiable at $\alpha_i(x)$, the first term of (62) is equal to $o(\|h\|)$. Let us consider the second term of (62), we separate it into three parts as following:

$$\begin{aligned}&\delta^+ f(\alpha_i(x); T_i)v_x^{(i)} - \delta^+ f(\alpha_i(x+h); \overline{k_i})v_{x+h}^{(i)} = (\delta^+ f(\alpha_i(x); T_i) - \delta^+ f(\alpha_i(x+h); T_i))v_x^{(i)}\\ &\quad + (\delta^+ f(\alpha_i(x+h); T_i) - \delta^+ f(\alpha_i(x+h); \overline{k_i}))v_x^{(i)} + \delta^+ f(\alpha_i(x+h); \overline{k_i})(v_x^{(i)} - v_{x+h}^{(i)}).\end{aligned} \tag{63}$$

From the assumption that f is B-differentiable at $\alpha(x) + T_i$ and Eq. (34), we have

$$f(\alpha_i(x)+T_i) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x)+T_i; T_i) = o(\|T_i\|).$$

That is,

$$f(\alpha_i(x+h)) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x+h); T_i) = o(\|T_i\|).$$

Now, the first part of (63) turns into

$$\begin{aligned}(\delta^+ f(\alpha_i(x); T_i) - \delta^+ f(\alpha_i(x+h); T_i))v_x^{(i)} &= -(f(\alpha_i(x)+T_i) - f(\alpha_i(x)) - \delta^+ f(\alpha_i(x); T_i) - o(\|T_i\|))v_x^{(i)}\\ &= -(o(\|T_i\|) - o(\|T_i\|))v_x^{(i)}\\ &= o(\|h\|).\end{aligned}$$

Moreover, since $\lim_{h\to 0}\frac{T_i}{\|h\|} = \lim_{h\to 0}\frac{\overline{k_i}}{\|h\|}$, the second part of (63) becomes

$$\begin{aligned}\lim_{h\to 0}\frac{\delta^+ f(\alpha_i(x+h); T_i) - \delta^+ f(\alpha_i(x+h); \overline{k_i})v_x^{(i)}}{\|h\|} &= \lim_{h\to 0}\left(\delta^+ f\left(\alpha_i(x+h); \frac{T_i}{\|h\|}\right) - \delta^+ f\left(\alpha_i(x+h); \frac{\overline{k_i}}{\|h\|}\right)\right)v_x^{(i)}\\ &= 0,\end{aligned}$$

while the third part of (63) becomes

$$\begin{aligned}&\lim_{h\to 0}\frac{\delta^+ f(\alpha_i(x+h); \overline{k_i})(v_x^{(i)} - v_{x+h}^{(i)})}{\|h\|} = \lim_{h\to 0}\delta^+ f(\alpha_i(x+h); \overline{k_i})\cdot\frac{1}{\|h\|}\cdot\left(-\frac{(-1)^i}{2}\right)\left(\frac{h'}{\|x'\|} - \frac{\langle x', h'\rangle}{\|x'\|^3} + o(\|h'\|)\right)\\ &\quad = \lim_{h\to 0}\delta^+ f(\alpha_i(x+h); \overline{k_i})\cdot\left(-\frac{(-1)^i}{2}\right)\left(\frac{1}{\|x'\|}\frac{h'}{\|h\|} - \left\langle\frac{x'}{\|x'\|}, \frac{h'}{\|h\|}\right\rangle\frac{x'}{\|x'\|^2} + \frac{o(\|h'\|)}{\|h\|}\right)\\ &\quad = 0,\end{aligned}$$

due to $\lim_{h\to 0}\overline{k_i} = \lim_{h\to 0}(\langle h, e\rangle + (-1)^i\frac{\langle x'+h', h'\rangle}{\|x'+h'\|}) = 0$.
From above discussion, we can obtain the result that

$$f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x+h; h) = o(\|h\|).$$

Case (ii). If $x' = 0$, we consider the following two subcases:
Subcase (a). If $h' \neq 0$, we can choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i \frac{h'}{\|h'\|})$ for all $i = 1, 2$ such that $v_{x+h}^{(i)} = v_x^{(i)}$, $\alpha_i(x) = \lambda$ and $\alpha_i(x+h) = \lambda + l + (-1)^i\|h'\| = \lambda + h_i$ where $h_i = l + (-1)^i\|h'\|$. From Eq. (23), we have

$$\begin{aligned} f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) &= \sum_{i=1}^{2}[f(\alpha_i(x+h)) - f(\alpha_i(x))]v_x^{(i)} \\ &= \sum_{i=1}^{2}[f(\lambda+h_i) - f(\lambda)]v_x^{(i)} \end{aligned} \tag{64}$$

and

$$\begin{aligned} \delta^+ f^{\mathcal{H}}(x+h;h) &= \lim_{t\to 0^+}\frac{1}{t}(f^{\mathcal{H}}(x+h+th) - f^{\mathcal{H}}(x+h)) \\ &= \lim_{t\to 0^+}\sum_{i=1}^{2}\frac{1}{t}(f(\lambda+h_i+th_i) - f(\lambda+h_i)) \\ &= \sum_{i=1}^{2}\delta^+ f(\lambda+h_i;h_i)v_x^{(i)}. \end{aligned} \tag{65}$$

Combining Eqs. (64), (65) and the fact that f satisfies Eq. (34), we have

$$\begin{aligned} f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) - \delta^+ f^{\mathcal{H}}(x+h;h) &= \sum_{i=1}^{2}(f(\lambda+h_i) - f(\lambda) - \delta^+ f(\lambda+h_i;h_i))v_x^{(i)} \\ &= o(\|h\|). \end{aligned}$$

Subcase (b). If $h' = 0$, we can choose $v_x^{(i)} = \frac{1}{2}(e + (-1)^i\omega)$ by any $\omega \in \mathcal{H}$ with $\|\omega\| = 1$. With almost the same argument, we only list the result as following:

$$f^{\mathcal{H}}(x+h) - f^{\mathcal{H}}(x) = \sum_{i=1}^{2}(f(\lambda+l) - f(\lambda))v_x^{(i)}$$

$$\delta^+ f^{\mathcal{H}}(x+h;h) = \sum_{i=1}^{2}\delta^+ f(\lambda+l;l)v_x^{(i)}.$$

From above discussion, $f^{\mathcal{H}}$ satisfies condition (34) for all $x+h$ at which $f^{\mathcal{H}}$ is B-differentiable. Hence, $f^{\mathcal{H}}$ is s-semismooth at x by Proposition 5.2(b).
(b) Since $f^{\mathcal{H}}$ is s-semismooth at x, by Definition 2.3, there is a slanting function $(f^{\mathcal{H}})^\circ$ for $f^{\mathcal{H}}$ in a neighborhood $\mathcal{N}_x$ of x. Now we define a function $f^\circ : \mathbb{R} \to L(\mathbb{R}, \mathbb{R})$ by

$$f^\circ(\alpha_i(x))t = 2\langle (f^{\mathcal{H}})^\circ(x)(te), v_x^{(i)}\rangle.$$

We will argue that f° is a slanting function for f in a neighborhood $\mathcal{N}_{\alpha_i(x)}$ of $\alpha_i(x)$, $i = 1, 2$. To see this, we fix some $i = 1, 2$ and any $\alpha_i(y) \in \mathcal{N}_{\alpha_i(x)}$ with $\alpha_i(y) - \alpha_i(x) = t \in \mathbb{R}$. Without loss of generality, we can choose $y = x + te \in \mathcal{H}$. Since $(f^{\mathcal{H}})^\circ$ is a slanting function for $f^{\mathcal{H}}$ at $y \in \mathcal{N}_x$, we know that $\{(f^{\mathcal{H}})^\circ(x+te)\}$ is uniformly bounded in the operator norm. With the following calculation

$$\frac{|f^\circ(\alpha_i(x)+t)\bar{t}|}{|\bar{t}|} = \frac{|2\langle (f^{\mathcal{H}})^\circ(y)(\bar{t}e), v_y^{(i)}\rangle|}{|\bar{t}|} \le \frac{2\|(f^{\mathcal{H}})^\circ(y)(\bar{t}e)\|}{\|\bar{t}e\|} = \frac{2\|(f^{\mathcal{H}})^\circ(x+te)(\bar{t}e)\|}{\|\bar{t}e\|},$$

it implies that $\{f^\circ(\alpha_i(x)+t)\}$ is uniformly bounded in the operator norm for t. Now from the definition of slanting function, we have

$$f^{\mathcal{H}}(y) - f^{\mathcal{H}}(x) - (f^{\mathcal{H}})^\circ(y)(y-x) = o(\|y-x\|).$$

That is,

$$f^{\mathcal{H}}(x+te) - f^{\mathcal{H}}(x) - (f^{\mathcal{H}})^\circ(x+te)(te) = o(|t|). \tag{66}$$

Due to the fact that $v_y^{(i)} = v_{x+te}^{(i)} = v_x^{(i)}$ and from Eq. (15), we have

$$f^{\mathcal{H}}(x+te) - f^{\mathcal{H}}(x) = \sum_{i=1}^{2}(f(\alpha_i(x+te)) - f(\alpha_i(x)))v_x^{(i)}. \tag{67}$$

After combining (66)–(67), there hold

$$(f^{\mathcal{H}})^{\circ}(x+te)(te)=\sum_{i=1}^{2}(f(\alpha_i(x+te))-f(\alpha_i(x)))v_x^{(i)}-o(|t|). \tag{68}$$

Now, recalling the definition of f°,

$$f^{\circ}(\alpha_i(x+te))t=f^{\circ}(\alpha_i(y))t=2\langle (f^{\mathcal{H}})^{\circ}(y)(te),v_y^{(i)}\rangle=2\langle (f^{\mathcal{H}})^{\circ}(x+te)(te),v_y^{(i)}\rangle. \tag{69}$$

With the fact $v_x^{(1)}\perp v_x^{(2)}$ and apply (68) into (69), we have

$$\begin{aligned} f^{\circ}(\alpha_i(y))t &= 2(f(\alpha_i(x+te))-f(\alpha_i(x)))\|v_x^{(i)}\|^2-\langle o(|t|),v_y^{(i)}\rangle \\ &= f(\alpha_i(y))-f(\alpha_i(x))-o(|t|), \end{aligned}$$

which says

$$f(\alpha_i(y))-f(\alpha_i(x))-f^{\circ}(\alpha_i(y))(\alpha_i(y)-\alpha_i(x))=o(|\alpha_i(y)-\alpha_i(x)|).$$

That means f° is a slanting function for f in a neighborhood $\mathcal{N}_{\alpha_i(x)}$ of $\alpha_i(x)$ for $i=1,2$.
Because of the assumption that $f^{\mathcal{H}}$ is B-differentiable at neighborhood of x, we know that f is also B-differentiable at neighborhood of $\alpha_i(x)$, $i=1,2$ by Lemma 5.1. Now for any $t\in\mathbb{R}$, $te\in\mathcal{H}$, after replacing h with te in Eq. (61) and the fact that $v_{x+te}^{(i)}=v_x^{(i)}$ and $\alpha_i(x+te)=\alpha_i(x)+t$, we obtain

$$\delta^+f^{\mathcal{H}}(x+te;te)=\sum_{i=1}^{2}\delta^+f(\alpha_i(x+te);t)v_{x+te}^{(i)}=\sum_{i=1}^{2}\delta^+f((\alpha_i(x)+t);t)v_x^{(i)}. \tag{70}$$

Since $f^{\mathcal{H}}$ is s-semismooth at x, by Proposition 5.2(a), $f^{\mathcal{H}}$ must satisfy Eq. (34) for $te\in\mathcal{H}$. That is,

$$f^{\mathcal{H}}(x+te)-f^{\mathcal{H}}(x)-\delta^+f^{\mathcal{H}}(x+te;te)=o(\|te\|)=o(|t|).$$

This together with (67) and (70) gives

$$\sum_{i=1}^{2}(f(\alpha_i(x+te))-f(\alpha_i(x))-\delta^+f((\alpha_i(x)+t);t))v_x^{(i)}=o(|t|).$$

Since we know that $v_x^{(1)}\perp v_x^{(2)}$, it implies

$$f(\alpha_i(x)+t)-f(\alpha_i(x))-\delta^+f((\alpha_i(x)+t);t)=o(|t|)$$

for all $i=1,2$. Thus, by Proposition 5.2(b), f is s-semismooth at $\alpha_i(x)$, $i=1,2$. □

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References

[1] Y. Chiang, S.-H. Pan, J.-S. Chen, A merit function method for infinite-dimensional SOCCPs, Journal of Mathematical Analysis and Applications (2011), in press (doi:10.1016/j.jmaa.2011.05.019).
[2] J.-S. Chen, X. Chen, P. Tseng, Analysis of nonsmooth vector-valued functions associated with second-order cones, Mathematical Programming 101 (2004) 95–117.
[3] M. Fukushima, Z.-Q. Luo, P. Tseng, Smoothing functions for second-order cone complementarity problems, SIAM Journal on Optimization 12 (2002) 436–460.
[4] E.D. Andersen, C. Roos, T. Terlaky, On implementing a primal–dual interior-point method for conic quadratic optimization, Mathematical Programming 95 (2003) 249–277.
[5] R.D.C. Monteiro, T. Tsuchiya, Polynomial convergence of primal–dual algorithms for the second-order cone programs based on the MZ-family of directions, Mathematical Programming 88 (2000) 61–83.
[6] T. Tsuchiya, A convergence analysis of the scaling-invariant primal–dual path-following algorithms for second-order cone programming, Optimization Methods and Software 11 (1999) 141–182.
[7] A. Yoshise, Interior point trajectories and a homogenous model for nonlinear complementarity problems over symmetric cones, SIAM Journal on Optimization 17 (2006) 1129–1153.
[8] X.-D. Chen, D. Sun, J. Sun, Complementarity functions and numerical experiments on some smoothing Newton methods for second-order cone complementarity problems, Computational Optimization and Applications 25 (2003) 39–56.
[9] S. Hayashi, N. Yamashita, M. Fukushima, A combined smoothing and regularization method for monotone second-order cone complementarity problems, SIAM Journal of Optimization 15 (2005) 593–615.
[10] C. Kanzow, I. Ferenczi, M. Fukushima, On the local convergence of semismooth Newton methods for linear and nonlinear second-order cone programs without strict complementarity, SIAM Journal on Optimization 20 (2009) 297–320.

[11] S.-H. Pan, J.-S. Chen, A damped Gauss–Newton method for the second-order cone complementarity problem, Applied Mathematics and Optimization 59 (2009) 293–318.
[12] J.-S. Chen, Two classes of merit functions for second-order cone complementarity problem, Mathematical Methods of Operations Research 64 (2006) 495–519.
[13] J.-S. Chen, P. Tseng, An unconstrained smooth minimization reformulation of the second-order cone complementarity problem, Mathematical Programming 104 (2005) 293–327.
[14] L. Faybusovich, T. Tsuchiya, Primal–dual algorithms and infinite-dimensional Jordan algebras of finite rank, Mathematical Programming 97 (2003) 471–493.
[15] U. Faraut, A. Korányi, Analysis on Symmetric Cones, in: Oxford Mathematical Monographs, Oxford University Press, New York, 1994.
[16] J.-S. Chen, Alternative proofs for some results of vector-valued functions associated with second-order cone, Journal of Nonlinear Analysis and Convex Analysis 6 (2005) 297–325.
[17] R. Mifflin, Semismooth and semiconvex function in constrained optimization, SIAM Journal on Control and Optimization 15 (1977) 959–972.
[18] L. Qi, J. Sun, A nonsmooth version of Newton's method, Mathematical Programming 58 (1993) 353–367.
[19] L. Qi, Convergence analysis of some algorithms for solving nonsmooth equations, Mathematics of Operations Research 18 (1993) 227–244.
[20] F. Clarke, Optimization and Nonsmooth Analysis, Wiley, New York, 1983.
[21] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, Springer-Verlag, Berlin, 1998.
[22] X. Chen, Z. Nashed, L. Qi, Smoothing method and semismooth methods for nondifferentiable operator equations, SIAM Journal on Numerical Analysis 38 (2000) 1200–1216.
[23] A. Shapiro, On concepts of directional differentiability, Journal of Optimization Theory and Applications 66 (1990) 477–487.