Set-Valued Var. Anal (2017) 25:211–232
DOI 10.1007/s11228-016-0374-7

Monotonicity and Circular Cone Monotonicity Associated with Circular Cones

Jinchuan Zhou[1] · **Jein-Shan Chen**[2]

Received: 27 April 2015 / Accepted: 2 May 2016 / Published online: 14 May 2016

Abstract The circular cone $\mathcal{L}_\theta$ is not self-dual under the standard inner product and includes second-order cone as a special case. In this paper, we focus on the monotonicity of $f^{\mathcal{L}_\theta}$ and circular cone monotonicity of f. Their relationship is discussed as well. Our results show that the angle θ plays a different role in these two concepts.

Keywords Circular cone · Monotonicity · Circular cone monotonicity

Mathematics Subject Classification (2010) 26A27 · 26B35 · 49J52 · 65K10

1 Introduction

The circular cone [11, 32] is a pointed closed convex cone having hyperspherical sections orthogonal to its axis of revolution about which the cone is invariant to rotation. Let its half-aperture angle be θ with $\theta \in (0, 90^\circ)$. Then, the n-dimensional circular cone denoted by $\mathcal{L}_\theta$ can be expressed as

$$\mathcal{L}_\theta := \{x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \cos\theta \|x\| \le x_1\}.$$

The author's work is supported by National Natural Science Foundation of China (11101248, 11271233) and Shandong Province Natural Science Foundation (ZR2010AQ026, ZR2012AM016).

The author's work is supported by Ministry of Science and Technology, Taiwan.

✉ Jein-Shan Chen
jschen@math.ntnu.edu.tw

[1] Department of Mathematics, School of Science, Shandong University of Technology, Zibo 255049, People's Republic of China

[2] Department of Mathematics, National Taiwan Normal University, Taipei 11677, Taiwan

Note that $\mathcal{L}_{45^\circ}$ corresponds to the well-known second-order cone $\mathcal{K}^n$ (SOC, for short), which is given by

$$\mathcal{K}^n := \{x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\}.$$

There has been much study on SOC, see [5, 6, 8] and references therein; to the contrast, not much attention has been paid to circular cone at present. For optimization problems involved SOC, for example, second-order cone programming (SOCP) [1, 2, 17, 19, 21] and second-order cone complementarity problems (SOCCP) [3, 9, 14, 16, 28], the so-called SOC-functions (see [5–7])

$$f^{\mathrm{soc}}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \qquad \forall x = (x_1, x_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \tag{1}$$

play an essential role on both theory and algorithm aspects. In expression (1), $f : J \to \mathbb{R}$ with $J \subseteq \mathbb{R}$ is a real-valued function and x is decomposed as

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)} \tag{2}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $u_x^{(1)}$, $u_x^{(2)}$ are the *spectral values* and the associated *spectral vectors* of x with respect to $\mathcal{K}^n$, given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\| \quad \text{and} \quad u_x^{(i)} = \frac{1}{2}\begin{bmatrix} 1 \\ (-1)^i \bar{x}_2 \end{bmatrix} \tag{3}$$

for $i = 1, 2$ with $\bar{x}_2 := x_2/\|x_2\|$ if $x_2 \neq 0$, and $\bar{x}_2$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\bar{x}_2\| = 1$ if $x_2 = 0$. The decomposition (2) is called the spectral factorization associated with second-order cone for x. Likewise, there is a similar decomposition for x associated with circular cone case. More specifically, from [31, Theorem 3.1], the spectral factorization associated with $\mathcal{L}_\theta$ for x is in form of

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)} \tag{4}$$

where

$$\begin{cases} \lambda_1(x) := x_1 - \|x_2\|\mathrm{ctan}\theta \\ \lambda_2(x) := x_1 + \|x_2\| \tan\theta \end{cases} \tag{5}$$

and

$$\begin{cases} u_x^{(1)} := \dfrac{1}{1+\mathrm{ctan}^2\theta}\begin{bmatrix} 1 & 0 \\ 0 & \mathrm{ctan}\theta \cdot I \end{bmatrix}\begin{bmatrix} 1 \\ -\bar{x}_2 \end{bmatrix} = \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\bar{x}_2 \end{bmatrix} \\ u_x^{(2)} := \dfrac{1}{1+\tan^2\theta}\begin{bmatrix} 1 & 0 \\ 0 & \tan\theta \cdot I \end{bmatrix}\begin{bmatrix} 1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\bar{x}_2 \end{bmatrix} \end{cases} \tag{6}$$

Analogously, for any given $f : \mathbb{R} \to \mathbb{R}$, we can define the following vector-valued function for the setting of circular cone:

$$f^{\mathcal{L}_\theta}(x) := f(\lambda_1(x))\, u_x^{(1)} + f(\lambda_2(x))\, u_x^{(2)}. \tag{7}$$

For convenience, we sometime write out the explicit expression for (7) by plugging in $\lambda_i(x)$ and $u_x^{(i)}$:

$$f^{\mathcal{L}_\theta}(x) = \begin{bmatrix} \dfrac{f(x_1 - \|x_2\|\mathrm{ctan}\theta)}{1+\mathrm{ctan}^2\theta} + \dfrac{f(x_1 + \|x_2\|\tan\theta)}{1+\tan^2\theta} \\ \left(-\dfrac{f(x_1 - \|x_2\|\mathrm{ctan}\theta)\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta} + \dfrac{f(x_1 + \|x_2\|\tan\theta)\tan\theta}{1+\tan^2\theta}\right)\bar{x}_2 \end{bmatrix}. \tag{8}$$

Clearly, as $\theta = 45^\circ$, the decomposition (4)–(8) reduces to (1)–(3). Since our main target is on circular cone, in the subsequent contexts of the whole paper, λ_i and $u_x^{(i)}$ stands for (5) and (6), respectively.

Throughout this paper, we always assume that J is an open interval (finite or infinite) in $\mathbb{R}$, i.e., $J := (t, t')$ with $t, t' \in \mathbb{R} \cup \{\pm\infty\}$. Denote S the set of all $x \in \mathbb{R}^n$ whose spectral values $\lambda_i(x)$ for $i = 1, 2$ belong to J, i.e.,

$$S := \{x \in \mathbb{R}^n \mid \lambda_i(x) \in J,\ i = 1, 2\}.$$

According to [24], we know S is open if and only if J is open. In addition, as J is an interval, we know S is convex because

$$\min\{\lambda_1(x), \lambda_1(y)\} \le \lambda_1(\beta x + (1-\beta)y) \le \lambda_2(\beta x + (1-\beta)y) \le \max\{\lambda_2(x), \lambda_2(y)\}.$$

We point out that there is a close relation between $\mathcal{L}_\theta$ and $\mathcal{K}^n$ (see [31]) as below

$$\mathcal{K}^n = A\mathcal{L}_\theta \quad \text{where} \quad A := \begin{bmatrix} \tan\theta & 0 \\ 0 & I \end{bmatrix}.$$

It is well-known that $\mathcal{K}^n$ is a self-dual cone in the standard inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$. Due to $\mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta}$ by [31, Theorem 2.1], $\mathcal{L}_\theta$ is not a self-dual cone unless $\theta = 45°$. In fact, we can construct a new inner product which ensures the circular cone $\mathcal{L}_\theta$ is self-dual. More precisely, we define an inner product associated with A as $\langle x, y\rangle_A := \langle Ax, Ay\rangle$. Then

$$\begin{aligned} \mathcal{L}_\theta^* &= \{x \mid \langle x, y\rangle_A \ge 0,\ \forall y \in \mathcal{L}_\theta\} = \{x \mid \langle Ax, Ay\rangle \ge 0,\ \forall y \in A^{-1}\mathcal{K}^n\} \\ &= \{x \mid \langle Ax, y\rangle \ge 0,\ \forall y \in \mathcal{K}^n\} = \{x \mid Ax \in \mathcal{K}^n\} \\ &= A^{-1}\mathcal{K}^n = \mathcal{L}_\theta. \end{aligned}$$

However, under this new inner product the second-order cone is not self-dual, because

$$\begin{aligned} (\mathcal{K}^n)^* &= \{x \mid \langle x, y\rangle_A \ge 0,\ \forall y \in \mathcal{K}^n\} = \{x \mid \langle Ax, Ay\rangle \ge 0,\ \forall y \in \mathcal{K}^n\} \\ &= \{x \mid \langle A^2x, y\rangle \ge 0,\ \forall y \in \mathcal{K}^n\} = \{x \mid A^2x \in \mathcal{K}^n\} = A^{-2}\mathcal{K}^n. \end{aligned}$$

Since we cannot find an inner product such that the circular cone and second-order cone are both self-dual simultaneously, we must choose an inner product from the standard inner product or the new inner product associated with A. In view of the well-known properties regarding second-order cone and second-order cone programming (in which many results are based on the Jordan algebra and second-order cones are considered as self-dual cones), we adopt the standard inner product in this paper.

Our main attention in this paper is on the vector-valued function $f^{\mathcal{L}_\theta}$. It should be emphasized that the relation $\mathcal{K}^n = A\mathcal{L}_\theta$ does not guarantee that there exists a similar close relation between $f^{\mathcal{L}_\theta}$ and f^{soc}. For example, take $f(t)$ to be a simple function $\max\{t, 0\}$, which corresponds to the projection operator Π. For $x \in \mathcal{L}_\theta$, we have $Ax \in \mathcal{K}^n$ which implies

$$\Pi_{\mathcal{L}_\theta}(x) = x = A^{-1}(Ax) = A^{-1}\Pi_{\mathcal{K}^n}(Ax).$$

Unfortunately, the above relation fails to hold when $x \notin \mathcal{L}_\theta$. To see this, we let $\tan\theta = 1/4$ and $x = (-1, 1)^T$. Then, it can be verified

$$\Pi_{\mathcal{L}_\theta}(x) = 0 \quad \text{and} \quad A^{-1}\Pi_{\mathcal{K}^n}(Ax) = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{8} \end{bmatrix}$$

which says $\Pi_{\mathcal{L}_\theta}(x) \neq A^{-1}\Pi_{\mathcal{K}^n}(Ax)$. This example undoubtedly indicates that we cannot study $f^{\mathcal{L}_\theta}$ by simply resorting to f^{soc}. Hence, it is necessary to study $f^{\mathcal{L}_\theta}$ directly, and the results in this paper are neither trivial nor being taken for granted.

Much attention has been paid to symmetric cone optimizations, see [22, 23, 27] and references therein. Non-symmetric cone optimization research is much more recent; for example, the works on p-order cone [30], homogeneous cone [15, 29], matrix cone [12]; etc. Unlike the symmetric cone case in which the Euclidean Jordan algebra can unify the analysis [13], so far no unifying algebra structure has been found for non-symmetric cones. In other words, we need to study each non-symmetric cones according to their different properties involved. For circular cone, a special non-symmetric cone, and circular cone optimization, like when dealing with SOCP and SOCCP, the following studies are crucial: (i) spectral factorization associated with circular cones; (ii) smooth and nonsmooth analysis for $f^{\mathcal{L}_\theta}$ given as in (7); (iii) the so-called $\mathcal{L}_\theta$-convexity; and (iv) $\mathcal{L}_\theta$-monotonicity. The first three points have been studied in [4, 31, 32], and [33], respectively. Here, we focus on the fourth item, that is, monotonicity. The SOC-monotonicity of f have been discussed thoroughly in [5, 7, 24]; and the monotonicity of the spectral operator of symmetric cone has been studied in [18]. The main aim of this paper is studying those monotonicity properties in the framework of circular cone. Our results reveal that the angle θ plays different role in these two concepts. More precisely, the circular cone monotonicity of f depends on f and θ, whereas the monotonicity of $f^{\mathcal{L}_\theta}$ only depends on f.

To end this section, we say a few words about the notations and present the definitions of monotonicity and $\mathcal{L}_\theta$-monotonicity. A matrix $M \in \mathbb{R}^{n\times n}$ is said to be $\mathcal{L}_\theta$-invariant if $Mh \in \mathcal{L}_\theta$ for all $h \in \mathcal{L}_\theta$. We write $x \succeq_{\mathcal{L}_\theta} y$ to mean $x - y \in \mathcal{L}_\theta$ and denote $\mathcal{L}_\theta^\circ$ the polar cone of $\mathcal{L}_\theta$, i.e.,

$$\mathcal{L}_\theta^\circ := \{y \in \mathbb{R}^n \mid \langle x, y\rangle \le 0, \ \ \forall x \in \mathcal{L}_\theta\}.$$

Denote $e := (1, 0, \ldots, 0)^T$ and use $\lambda(M)$, $\lambda_{\min}(M)$, $\lambda_{\max}(M)$ for the set of all eigenvalues, the minimum, and the maximum of eigenvalues of M, respectively. Besides, $\mathbb{S}^n$ means the space of all symmetric matrices in $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n_+$ is the cone of positive semidefinite matrices. For a mapping $g : \mathbb{R}^n \to \mathbb{R}^m$, denote by D_g the set of all differentiable points of g. For convenience, we define $0/0 := 0$. Given a real-valued function $f : J \to \mathbb{R}$,

(a) f is said to be $\mathcal{L}_\theta$-monotone on J if for any $x, y \in S$,

$$x \succeq_{\mathcal{L}_\theta} y \quad \Longrightarrow \quad f^{\mathcal{L}_\theta}(x) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(y);$$

(b) $f^{\mathcal{L}_\theta}$ is said to be monotone on S if

$$\left\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\right\rangle \ge 0, \ \ \forall x, y \in S.$$

(c) $f^{\mathcal{L}_\theta}$ is said to be strictly monotone on S if

$$\left\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\right\rangle > 0, \ \ \forall x, y \in S, \ \ x \ne y.$$

(d) $f^{\mathcal{L}_\theta}$ is said to be strongly monotone on S with $\mu > 0$ if

$$\left\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\right\rangle \ge \mu\|x - y\|^2, \ \ \forall x, y \in S.$$

2 Circular Cone Monotonicity of f

This section is devoted to the study of $\mathcal{L}_\theta$-monotonicity. The main purpose is to provide characterizations of $\mathcal{L}_\theta$-monotone functions. To this end, we need a few technical lemmas.

Lemma 2.1 *Let A, B be symmetric matrices and $y^T Ay > 0$ for some y. Then, the implication $[z^T Az \ge 0 \Longrightarrow z^T Bz \ge 0]$ is valid if and only if $B \succeq_{\mathbb{S}^n_+} \lambda A$ for some $\lambda \ge 0$.*

Proof This is the well known S-Lemma, see [7, Lemma 3.1] or [25]. □

Lemma 2.2 *Given $\zeta \in \mathbb{R}$, $u \in \mathbb{R}^{n-1}$, and a symmetric matrix $\Xi \in \mathbb{R}^{n\times n}$. Denote $\mathbb{B} := \{z \in \mathbb{R}^{n-1} | \|z\| \le 1\}$. Then, the following statements hold.*

(a) *Ξ being $\mathcal{L}_\theta$-invariant is equivalent to $\Xi \begin{bmatrix} ctan\theta \\ z \end{bmatrix} \in \mathcal{L}_\theta$ for any $z \in \mathbb{B}$.*

(b) *If $\Xi = \begin{bmatrix} \zeta & u^T \\ u & H \end{bmatrix}$ with $H \in \mathbb{S}^{n-1}$, then Ξ is $\mathcal{L}_\theta$-invariant is equivalent to*

$$\zeta \ge \|u\| \tan\theta$$

and there exists $\lambda \ge 0$ such that

$$\begin{bmatrix} \zeta^2 - ctan^2\theta\|u\|^2 - \lambda & (\zeta\tan\theta)u^T - ctan\theta u^T H \\ (\zeta\tan\theta)u - ctan\theta Hu & \tan^2\theta uu^T - H^2 + \lambda I \end{bmatrix} \succeq_{\mathbb{S}^n_+} O.$$

Proof (a) The result follows from the following observation:

$$\begin{aligned} \Xi\mathcal{L}_\theta \in \mathcal{L}_\theta &\iff \Xi A^{-1}\mathcal{K}^n \in A^{-1}\mathcal{K}^n \iff A\Xi A^{-1}\mathcal{K}^n \in \mathcal{K}^n \\ &\iff A\Xi A^{-1}\begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathcal{K}^n \iff \Xi A^{-1}\begin{bmatrix} 1 \\ z \end{bmatrix} \in A^{-1}\mathcal{K}^n \\ &\iff \Xi A^{-1}\begin{bmatrix} 1 \\ z \end{bmatrix} \in \mathcal{L}_\theta \iff \Xi\begin{bmatrix} \mathrm{ctan}\theta \\ z \end{bmatrix} \in \mathcal{L}_\theta, \end{aligned} \tag{9}$$

where the third equivalence comes from [7, Lemma 3.2].
(b) From (9), we know that

$$A\,\Xi\,A^{-1}\begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} \zeta & \tan\theta u^T \\ \mathrm{ctan}\theta u & H \end{bmatrix}\begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} \zeta + \tan\theta u^T z \\ \mathrm{ctan}\theta u + Hz \end{bmatrix} \in \mathcal{K}^n,$$

which means

$$\zeta + u^T z \tan\theta \ge 0, \quad \forall z \in \mathbb{B}, \tag{10}$$

and

$$\zeta + u^T z \tan\theta \ge \|\mathrm{ctan}\theta u + Hz\|, \quad \forall z \in \mathbb{B}. \tag{11}$$

Note that condition (10) is equivalent to

$$\zeta \ge \tan\theta \max\{-u^T z | z \in \mathbb{B}\} = \tan\theta\|u\|$$

and condition (11) is equivalent to

$$\left(\zeta + \tan\theta u^T z\right)^2 \ge \|\mathrm{ctan}\theta u + Hz\|^2,$$

i.e.,

$$z^T(\tan^2\theta uu^T - H^2)z + 2\left(\zeta\tan\theta u^T - \mathrm{ctan}\theta u^T H\right)z + \zeta^2 - \mathrm{ctan}^2\theta u^T u \ge 0 \quad \forall z \in \mathbb{B},$$

which can be rewritten as

$$\begin{bmatrix} 1 & z^T \end{bmatrix}\Theta\begin{bmatrix} 1 \\ z \end{bmatrix} \ge 0 \quad \forall z \in \mathbb{B}, \tag{12}$$

with

$$\Theta := \begin{bmatrix} \zeta^2 - \mathrm{ctan}^2\theta u^T u & (\zeta\tan\theta)u^T - \mathrm{ctan}\theta u^T H \\ (\zeta\tan\theta)u - \mathrm{ctan}\theta Hu & \tan^2\theta uu^T - H^2 \end{bmatrix}.$$

We now claim that (12) is equivalent to the following implication:

$$\begin{bmatrix} k & v^T \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix}\begin{bmatrix} k \\ v \end{bmatrix} \ge 0 \implies \begin{bmatrix} k & v^T \end{bmatrix}\Theta\begin{bmatrix} k \\ v \end{bmatrix} \ge 0, \quad \forall \begin{bmatrix} k \\ v \end{bmatrix} \in \mathbb{R}^n. \tag{13}$$

First, we see that (12) corresponds to the case of $k = 1$ in (13). Hence, it only needs to show how to obtain (13) from (12). We proceed the arguments by considering the following two cases.

For $k \neq 0$, dividing by k^2 in the left side of (13) yields

$$\begin{bmatrix} 1 & (\frac{v}{k})^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 1 \\ \frac{v}{k} \end{bmatrix} \geq 0,$$

which implies $v/k \in \mathbb{B}$. Then, it follows from (12) that

$$\begin{bmatrix} 1 & (\frac{v}{k})^T \end{bmatrix} \Theta \begin{bmatrix} 1 \\ \frac{v}{k} \end{bmatrix} \geq 0.$$

Hence, the right side of (13) holds.

For $k = 0$, the left side of (13) is $\|v\| \leq 0$, which says $v = 0$, i.e., $(k, v)^T = 0$. Therefore, the right side of (13) holds clearly.

Now, applying Lemma 2.1 to Θ ensures the existence of $\lambda \geq 0$ such that

$$\begin{bmatrix} \zeta^2 - \text{ctan}^2\theta u^T u & \zeta \tan\theta u^T - \text{ctan}\theta u^T H \\ \zeta \tan\theta u - \text{ctan}\theta H u & \tan^2\theta u u^T - H^2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & -I \end{bmatrix} \succeq_{\mathbb{S}^n_+} O.$$

Thus, the proof is complete. □

Lemma 2.3 *For a matrix being in form of* $H := \begin{bmatrix} x_1 & x_2^T \\ x_2 & \alpha I + \beta \bar{x}_2 \bar{x}_2^T \end{bmatrix}$, *where* $\alpha, \beta \in \mathbb{R}$, *then*

$$\begin{aligned} & \max\{x_1 + \|x_2\|, x_1 - \beta\} + \max\{0, \alpha - x_1 + \beta\} \\ \geq\ & \lambda_{\max}(H) \geq \lambda_{\min}(H) \\ \geq\ & \min\{x_1 - \|x_2\|, x_1 - \beta\} + \min\{0, \alpha - x_1 + \beta\}. \end{aligned}$$

Proof First, we split H as sum of three special matrices, i.e.,

$$\begin{bmatrix} x_1 & x_2^T \\ x_2 & \alpha I + \beta \bar{x}_2 \bar{x}_2^T \end{bmatrix} = \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} - \beta \begin{bmatrix} 0 & 0 \\ 0 & I - \bar{x}_2 \bar{x}_2^T \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - x_1 + \beta) I \end{bmatrix}$$

and let

$$\Omega_1 := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} - \beta \begin{bmatrix} 0 & 0 \\ 0 & I - \bar{x}_2 \bar{x}_2^T \end{bmatrix}, \qquad \Omega_2 := \begin{bmatrix} 0 & 0 \\ 0 & (\alpha - x_1 + \beta) I \end{bmatrix}.$$

Then, $\lambda(\Omega_1) = \{x_1 - \|x_2\|, x_1 + \|x_2\|, x_1 - \beta\}$ by [6, Lemma 1] and $\lambda(\Omega_2) = \{0, \alpha - x_1 + \beta\}$. Thus, the desired result follows from the following facts:

$$\lambda_{\min}(\Omega_1 + \Omega_2) \geq \lambda_{\min}(\Omega_1) + \lambda_{\min}(\Omega_2) \quad \text{and} \quad \lambda_{\max}(\Omega_1 + \Omega_2) \leq \lambda_{\max}(\Omega_1) + \lambda_{\max}(\Omega_2).$$

This completes the proof. □

Next, we turn our attention to the vector-valued function $f^{\mathcal{L}_\theta}$ defined as in (7). Recall from [4, 32] that $f^{\mathcal{L}_\theta}$ is differentiable at x if and only if f is differentiable at $\lambda_i(x)$ for $i = 1, 2$ and

$$\nabla f^{\mathcal{L}_\theta}(x) = \begin{cases} f'(x_1) I & x_2 = 0; \\ \begin{bmatrix} \xi & \varrho \bar{x}_2^T \\ \varrho \bar{x}_2 & \tau I + (\eta - \tau) \bar{x}_2 \bar{x}_2^T \end{bmatrix} & x_2 \neq 0, \end{cases} \tag{14}$$

where

$$\tau := \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}, \quad \xi := \frac{f'(\lambda_1(x))}{1+\mathrm{ctan}^2\theta} + \frac{f'(\lambda_2(x))}{1+\tan^2\theta},$$
$$\varrho := -\frac{\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta} f'(\lambda_1(x)) + \frac{\tan\theta}{1+\tan^2\theta} f'(\lambda_2(x)),$$
$$\eta := \frac{\mathrm{ctan}^2\theta}{1+\mathrm{ctan}^2\theta} f'(\lambda_1(x)) + \frac{\tan^2\theta}{1+\tan^2\theta} f'(\lambda_2(x)).$$

The following result shows that if $f^{\mathcal{L}_\theta}$ is differentiable, then we can characterize the $\mathcal{L}_\theta$-monotonicity of f via the gradient $\nabla f^{\mathcal{L}_\theta}$.

Theorem 2.1 *Suppose that $f : J \to \mathbb{R}$ is differentiable. Then, f is $\mathcal{L}_\theta$-monotone on J if and only if $\nabla f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant for all $x \in S$.*

Proof "⇒" Suppose that f is $\mathcal{L}_\theta$-monotone. Take $x \in S$ and $h \in \mathcal{L}_\theta$, what we want to prove is $\nabla f^{\mathcal{L}_\theta}(x)h \in \mathcal{L}_\theta$. From the $\mathcal{L}_\theta$-monotonicity of f, we know $f^{\mathcal{L}_\theta}(x+th) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(x)$ for all $t > 0$. Note that $\mathcal{L}_\theta$ is a cone. Hence

$$\frac{f^{\mathcal{L}_\theta}(x+th) - f^{\mathcal{L}_\theta}(x)}{t} \succeq_{\mathcal{L}_\theta} 0. \tag{15}$$

Since $\mathcal{L}_\theta$ is closed, taking the limit as $t \to 0^+$ yields $\nabla f^{\mathcal{L}_\theta}(x)h \succeq_{\mathcal{L}_\theta} 0$, i.e., $\nabla f^{\mathcal{L}_\theta}(x)h \in \mathcal{L}_\theta$.
"⇐" Suppose that $\nabla f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant for all $x \in S$. Take $x, y \in S$ with $x \succeq_{\mathcal{L}_\theta} y$ (i.e., $x - y \in \mathcal{L}_\theta$). In order to show the desired result, we need to argue that $f^{\mathcal{L}_\theta}(x) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(y)$. For any $\zeta \in \mathcal{L}_\theta^\circ$, we have

$$\left\langle \zeta, f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y) \right\rangle = \int_0^1 \left\langle \zeta, \nabla f^{\mathcal{L}_\theta}\left(x + t(x-y)\right)(x-y) \right\rangle dt \le 0, \tag{16}$$

where the last step comes from $\nabla f^{\mathcal{L}_\theta}(x + t(x-y))(x-y) \in \mathcal{L}_\theta$ because $x + t(x-y) \in S$ (since S is convex) and $\nabla f^{\mathcal{L}_\theta}$ is $\mathcal{L}_\theta$-invariant over S by hypothesis. Since $\zeta \in \mathcal{L}_\theta^\circ$ is arbitrary, (16) implies $f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y) \in (\mathcal{L}_\theta^\circ)^\circ = \mathcal{L}_\theta$, where the last step is due to the fact that $\mathcal{L}_\theta$ is a closed convex cone. This means $f^{\mathcal{L}_\theta}(x) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(y)$. □

Note that f is Lipschitz continuous on J if and only if $f^{\mathcal{L}_\theta}$ is Lipschitz continuous on S, see [4, 32]. The nonsmooth version of Theorem 2.1 is given below.

Theorem 2.2 *Suppose that $f : J \to \mathbb{R}$ is Lipschitz continuous on J. Then the following statements are equivalent.*

(a) *f is $\mathcal{L}_\theta$-monotone on J;*
(b) *$\partial_B f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant for all $x \in S$;*
(c) *$\partial f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant for all $x \in S$.*

Proof "(a) ⇒ (b)" Take $V \in \partial_B f^{\mathcal{L}_\theta}(x)$, then by definition of B-subdifferential there exists $\{x_k\} \subset D_{f^{\mathcal{L}_\theta}}$ such that $x_k \to x$ and $\nabla f^{\mathcal{L}_\theta}(x_i) \to V$. According to (15), we obtain $\nabla f^{\mathcal{L}_\theta}(x_i)h \succeq_{\mathcal{L}_\theta} 0$ for $h \in \mathcal{L}_\theta$. Taking the limit yields $Vh \succeq_{\mathcal{L}_\theta} 0$. Since $V \in \partial_B f^{\mathcal{L}_\theta}(x)$ is arbitrary, this says that $\partial_B f^{\mathcal{L}_\theta}$ is $\mathcal{L}_\theta$-invariant.

"(b) ⇒ (c)" Take $V \in \partial f^{\mathcal{L}_\theta}(x)$, then by definition, there exists $V_i \in \partial_B f^{\mathcal{L}_\theta}(x)$ and $\beta_i \in [0, 1]$ such that $V = \sum_i \beta_i V_i$ and $\sum_i \beta_i = 1$. Thus, for any $h \in \mathcal{L}_\theta$, we have $Vh = \sum_i \beta_i V_i h \in \mathcal{L}_\theta$, since V_i is $\mathcal{L}_\theta$-invariant and $\mathcal{L}_\theta$ is convex. Hence $\partial f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant.
"(c) ⇒ (a)" The proof follows from Theorem 2.1 by replacing (16) with

$$\left\langle \zeta, f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y) \right\rangle = \langle \zeta, V(x-y) \rangle \leq 0,$$

for some $V \in \partial f^{\mathcal{L}_\theta}(z)$ with $z \in [x, y]$ by the mean-value theorem of Lipschitz functions [10]. □

With these preparations, we provide a sufficient condition for the $\mathcal{L}_\theta$-monotonicity.

Theorem 2.3 *Suppose that $f : J \to \mathbb{R}$ is differentiable. If for all $t_1, t_2 \in J$ with $t_1 \leq t_2$,*

$$(\tan\theta - ctan\theta)\left(f'(t_1) - f'(t_2)\right) \geq 0, \tag{17}$$

and

$$\begin{bmatrix} f'(t_1) & \dfrac{f(t_2) - f(t_1)}{t_2 - t_1} \\ \dfrac{f(t_2) - f(t_1)}{t_2 - t_1} & f'(t_2) \end{bmatrix} \succeq_{\mathbb{S}^2_+} O, \tag{18}$$

then f is $\mathcal{L}_\theta$-monotone on J.

Proof According to Theorem 2.1, it suffices to show that $\nabla f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant for all $x \in S$. We proceed by discussing the following two cases.

Case 1 For $x_2 = 0$, in this case it is clear that $\nabla f^{\mathcal{L}_\theta}(x)$ being $\mathcal{L}_\theta$-invariant, i.e., $\nabla f^{\mathcal{L}_\theta}(x)h = f'(x_1)h \in \mathcal{L}_\theta$ for all $h \in \mathcal{L}_\theta$, is equivalent to saying $f'(x_1) \geq 0$.

Case 2 For $x_2 \neq 0$, let

$$H := \tau I + (\eta - \tau)\bar{x}_2\bar{x}_2^T.$$

Then, applying Lemma 2.2 to the formula of $\nabla f^{\mathcal{L}_\theta}(x)$ in (14), $\nabla f^{\mathcal{L}_\theta}(x)$ is $\mathcal{L}_\theta$-invariant if and only if

$$\xi \geq \|\varrho\| \tan\theta \tag{19}$$

and there exists $\lambda \geq 0$ such that

$$\Upsilon := \begin{bmatrix} \xi^2 - \text{ctan}^2\theta\varrho^2 - \lambda & \xi\tan\theta\varrho\bar{x}_2^T - \text{ctan}\theta\varrho\bar{x}_2^T H \\ \xi\tan\theta\varrho\bar{x}_2 - \text{ctan}\theta\varrho H\bar{x}_2 & \tan^2\theta\varrho^2\bar{x}_2\bar{x}_2^T - H^2 + \lambda I \end{bmatrix} \succeq_{\mathbb{S}^n_+} O. \tag{20}$$

Hence, to achieve the desired result, it is equivalent to showing that the conditions (17) and (18) can guarantee the validity of the conditions (19) and (20). To check this, we first note that (19) is equivalent to

$$\begin{aligned} & -\tan\theta f'(\lambda_1(x)) - \text{ctan}\theta f'(\lambda_2(x)) \leq -\tan\theta f'(\lambda_1(x)) + \tan\theta f'(\lambda_2(x)) \\ & \leq \tan\theta f'(\lambda_1(x)) + \text{ctan}\theta f'(\lambda_2(x)) \\ \Longleftrightarrow\ & f'(\lambda_2(x)) \geq 0 \ \text{ and } \ f'(\lambda_1(x)) \geq \frac{1 - \text{ctan}^2\theta}{2} f'(\lambda_2(x)). \end{aligned} \tag{21}$$

This is ensured by (17) and (18). In fact, if $\tan\theta \geq \text{ctan}\theta$, then we know from (17) that $f'(\lambda_1(x)) \geq f'(\lambda_2(x)) \geq \left((1 - \text{ctan}^2\theta)/2\right) f'(\lambda_2(x))$ where the second inequality is due to $f'(\lambda_2(x)) \geq 0$ by (18). If $\tan\theta \leq \mathrm{ctan}\theta$, then $1 - \mathrm{ctan}^2\theta \leq 0$, and hence $f'(\lambda_1(x)) \geq 0 \geq \big((1 - \mathrm{ctan}^2\theta)/2\big) f'(\lambda_2(x))$ since $f'(\lambda_i(x)) \geq 0$ for $i = 1, 2$ by (18).

Now let us look into the entries of Υ. In the Υ_{11}-entry, we calculate

$$\begin{aligned}
&\xi^2 - \mathrm{ctan}^2\theta\varrho^2 \\
&= \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[(\tan^2\theta - \mathrm{ctan}^2\theta) f'(\lambda_1(x))^2 + 2(1 + \mathrm{ctan}^2\theta) f'(\lambda_1(x)) f'(\lambda_2(x))\Big] \\
&= \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[(\tan^2\theta - \mathrm{ctan}^2\theta) f'(\lambda_1(x))^2 + (\mathrm{ctan}^2\theta - \tan^2\theta) f'(\lambda_1(x)) f'(\lambda_2(x))\Big] \\
&\quad + \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[2 + \tan^2\theta + \mathrm{ctan}^2\theta\Big] f'(\lambda_1(x)) f'(\lambda_2(x)) \\
&= \mu + f'(\lambda_1(x)) f'(\lambda_2(x)),
\end{aligned}$$

with

$$\begin{aligned}
\mu &:= \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[(\tan^2\theta - \mathrm{ctan}^2\theta) f'(\lambda_1(x))^2 + (\mathrm{ctan}^2\theta - \tan^2\theta) f'(\lambda_1(x)) f'(\lambda_2(x))\Big] \\
&= \frac{\tan\theta - \mathrm{ctan}\theta}{\tan\theta + \mathrm{ctan}\theta} f'(\lambda_1(x)) \left[f'(\lambda_1(x)) - f'(\lambda_2(x))\right] \\
&\geq 0,
\end{aligned}$$

where the last step is due to (17). In the Υ_{12}-entry and Υ_{21}-entry, we calculate

$$\begin{aligned}
&(\xi \tan\theta)\varrho\bar{x}_2^T - \mathrm{ctan}\theta\varrho\bar{x}_2^T H \\
&= \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[\Big(-\tan^2\theta + \mathrm{ctan}^2\theta\Big) f'(\lambda_1(x))^2 + \Big(\tan^2\theta - \mathrm{ctan}^2\theta\Big) f'(\lambda_1(x)) f'(\lambda_2(x))\Big]\bar{x}_2^T \\
&= -\frac{\tan\theta - \mathrm{ctan}\theta}{\tan\theta + \mathrm{ctan}\theta} f'(\lambda_1(x)) \left[f'(\lambda_1(x)) - f'(\lambda_2(x))\right]\bar{x}_2^T \\
&= -\mu\bar{x}_2^T.
\end{aligned}$$

In the Υ_{22}-entry, we calculate

$$\begin{aligned}
\tan^2\theta\varrho^2\bar{x}_2\bar{x}_2^T - H^2 &= -\tau^2 I + \Bigg(\tau^2 + \frac{1}{(\tan\theta + \mathrm{ctan}\theta)^2}\Big[(\tan^2\theta - \mathrm{ctan}^2\theta) f'(\lambda_1(x))^2 \\
&\qquad -2(1 + \tan^2\theta) f'(\lambda_1(x)) f'(\lambda_2(x))\Big]\Bigg)\bar{x}_2\bar{x}_2^T \\
&= -\tau^2 I + \Big(\tau^2 + \mu - f'(\lambda_1(x)) f'(\lambda_2(x))\Big)\bar{x}_2\bar{x}_2^T.
\end{aligned}$$

Hence, Υ can be rewritten as

$$\begin{aligned}
\Upsilon &= \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} \\
&= \begin{bmatrix} \mu + f'(\lambda_1(x)) f'(\lambda_2(x)) - \lambda & -\mu\bar{x}_2^T \\ -\mu\bar{x}_2 & (\lambda - \tau^2)I + \big(\tau^2 + \mu - f'(\lambda_1(x)) f'(\lambda_2(x))\big)\bar{x}_2\bar{x}_2^T \end{bmatrix}.
\end{aligned}$$

Now, applying Lemma 2.3 to Υ, we have

$$\begin{aligned}
\lambda_{\min}(\Upsilon) &\geq \min\Big\{\Upsilon_{11} - |\mu|,\ \Upsilon_{11} - \Big(\tau^2 + \mu - f'(\lambda_1(x)) f'(\lambda_2(x))\Big)\Big\} \\
&\quad + \min\Big\{0,\ \Big(\lambda - \tau^2\Big) - \Upsilon_{11} + \Big(\tau^2 + \mu - f'(\lambda_1(x)) f'(\lambda_2(x))\Big)\Big\} \\
&= \min\Big\{f'(\lambda_1(x)) f'(\lambda_2(x)) - \lambda,\ 2f'(\lambda_1(x)) f'(\lambda_2(x)) - \tau^2 - \lambda\Big\} \\
&\quad + 2\min\left\{0,\ \lambda - f'(\lambda_1(x)) f'(\lambda_2(x))\right\}. \qquad (22)
\end{aligned}$$

Using $\lambda_1(x) \le \lambda_2(x)$ and condition (18) ensures

$$f'(\lambda_1(x)) \ge 0, \quad f'(\lambda_2(x)) \ge 0, \quad \text{and} \quad f'(\lambda_1(x))f'(\lambda_2(x)) \ge \left(\frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)}\right)^2,$$

which together with (15) yields

$$f'(\lambda_1(x))f'(\lambda_2(x)) \ge 0 \quad \text{and} \quad f'(\lambda_1(x))f'(\lambda_2(x)) - \tau^2 \ge 0.$$

Thus, we can plug $\lambda := f'(\lambda_1(x))f'(\lambda_2(x)) \ge 0$ into (22), which gives $\lambda_{\min}(\Upsilon) \ge 0$. Hence Υ is positive semi-definite. This completes the proof. □

Remark 2.1 The condition (17) holds automatically when $\theta = 45°$. In other case, some addition requirement needs to be imposed on f. For instance, f is required to be convex as $\theta \in (0, 45°)$ while f is required to be concave as $\theta \in (45°, 90°)$. This indicates that the angle plays an essential role in the framework of circular cone, i.e., the assumption on f is dependent on the range of the angle.

Based on the above, we can achieve a necessary and sufficient condition for $\mathcal{L}_\theta$-monotonicity in the special case of $n = 2$.

Theorem 2.4 *Suppose that $f : J \to \mathbb{R}$ is differentiable on J and $n = 2$. Then, f is $\mathcal{L}_\theta$-monotone on J if and only if $f'(t) \ge 0$ for all $t \in J$ and $(\tan\theta - ctan\theta)(f'(t_1) - f'(t_2)) \ge 0$ for all $t_1, t_2 \in J$ with $t_1 \le t_2$.*

Proof In light of the proof of Theorem 2.3, we know that f is $\mathcal{L}_\theta$-monotone if and only if for any $x \in S$,

$$f'(\lambda_2(x)) \ge 0, \quad f'(\lambda_1(x)) \ge \frac{1 - \text{ctan}^2\theta}{2} f'(\lambda_2(x)), \tag{23}$$

and there exists $\lambda \ge 0$ such that

$$\Upsilon = \begin{bmatrix} \mu + f'(\lambda_1(x))f'(\lambda_2(x)) - \lambda & \pm\mu \\ \pm\mu & \mu - f'(\lambda_1(x))f'(\lambda_2(x)) + \lambda \end{bmatrix} \succeq_{\mathbb{S}^2_+} O, \tag{24}$$

where the form of Υ comes from the fact that $\bar{x}_2 = \pm 1$ in this case. It follows from (24) that $\mu^2 - (f'(\lambda_1(x))f'(\lambda_2(x)) - \lambda)^2 - \mu^2 \ge 0$, which implies $\lambda = f'(\lambda_1(x))f'(\lambda_2(x))$. Substituting it into (24) yields

$$\Upsilon = \begin{bmatrix} \mu & \pm\mu \\ \pm\mu & \mu \end{bmatrix} = \mu \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \succeq_{\mathbb{S}^2_+} O,$$

which in turn implies $\mu \ge 0$. Hence, the conditions (23) and (24) are equivalent to

$$\begin{cases} f'(\lambda_2(x)) \ge 0, \quad f'(\lambda_1(x)) \ge \dfrac{1 - \text{ctan}^2\theta}{2} f'(\lambda_2(x)), \\ (\tan\theta - \text{ctan}\theta) f'(\lambda_1(x)) \left[f'(\lambda_1(x)) - f'(\lambda_2(x))\right] \ge 0. \end{cases}$$

Due to the arbitrariness of $\lambda_i(x) \in J$ and applying similar arguments following (21), the above conditions give $f'(t) \ge 0$ for all $t \in J$ and $(\tan\theta - \text{ctan}\theta)(f'(t_1) - f'(t_2)) \ge 0$ for all $t_1, t_2 \in J$ with $t_1 \le t_2$. Thus, the proof is complete. □

3 Monotonicity of $f^{\mathcal{L}_\theta}$

In Section 2, we have shown that the circular cone monotonicity of f depends on both the monotonicity of f and the range of the angle θ. Now the following questions arise: how about on the relationship between the monotonicity of $f^{\mathcal{L}_\theta}$ and the $\mathcal{L}_\theta$-monotonicity of f? Whether the monotonicity of $f^{\mathcal{L}_\theta}$ also depends on θ? This is the main motivation of this section. First, for a mapping $H : \mathbb{R}^n \to \mathbb{R}^n$, let us denote $\partial H(x) \succeq_{\mathbb{S}^n_+} O$ (or $\partial H(x) \succ_{\mathbb{S}^n_+} O$) to mean that each elements in $\partial H(x)$ is positive semi-definite (or positive definite), i.e.,

$$\partial H(x) \succeq_{\mathbb{S}^n_+} O \ \ (or \succ_{\mathbb{S}^n_+} O) \iff A \succeq_{\mathbb{S}^n_+} O \ \ (or \succ_{\mathbb{S}^n_+} O), \quad \forall A \in \partial H(x).$$

Taking into account of the result in [26], we readily have

Lemma 3.1 *Let f be Lipschitz continuous on J. The following statements hold:*

(a) *$f^{\mathcal{L}_\theta}$ is monotone on S if and only if $\partial f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ for all $x \in S$;*
(b) *If $\partial f^{\mathcal{L}_\theta}(x) \succ_{\mathbb{S}^n_+} O$ for all $x \in S$, then $f^{\mathcal{L}_\theta}$ is strictly monotone on S;*
(c) *$f^{\mathcal{L}_\theta}$ is strongly monotone on S if and only if there exists $\mu > 0$ such that $\partial f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} \mu I$ for all $x \in S$.*

Lemma 3.2 *Suppose that f is Lipschitz continuous. Then,*

$$\partial_B f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O \iff \partial f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O \ \ and \ \ \partial_B f^{\mathcal{L}_\theta}(x) \succ_{\mathbb{S}^n_+} O \iff \partial f^{\mathcal{L}_\theta}(x) \succ_{\mathbb{S}^n_+} O.$$

Proof The result follows immediately from the fact $\partial f^{\mathcal{L}_\theta}(x) = \mathrm{conv}\partial_B f^{\mathcal{L}_\theta}(x)$. ☐

Lemma 3.3 *The following statements hold.*

(a) *For any $h \in \mathbb{R}^n$*

$$t_1\left(h_1 - ctan\theta \bar{x}_2^T h_2\right)^2 + t_2\left(h_1 + \tan\theta \bar{x}_2^T h_2\right)^2 + t_3\left(\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right) \geq 0 \quad (25)$$

if and only if $t_i \geq 0$ for $i = 1, 2, 3$.

(b) *For any $h \in \mathbb{R}^n \backslash \{0\}$*

$$t_1\left(h_1 - ctan\theta \bar{x}_2^T h_2\right)^2 + t_2\left(h_1 + \tan\theta \bar{x}_2^T h_2\right)^2 + t_3\left(\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right) > 0 \quad (26)$$

if and only if $t_i > 0$ for $i = 1, 2, 3$.

Proof (a) The sufficiency is clear, since $|\bar{x}_2^T h_2| \leq \|\bar{x}_2\|\|h_2\| = \|h_2\|$ by Cauchy-Schwartz inequality. Now let us show the necessity. Taking $h = (1, -\mathrm{ctan}\theta\, \bar{x}_2)^T$, then (25) equal to $t_1(1+\mathrm{ctan}^2\theta)^2 \geq 0$, which implies $t_1 \geq 0$. Similarly, let $h = (1, \tan\theta\, \bar{x}_2)^T$, then (25) yields $t_2(1 + \tan^2\theta)^2 \geq 0$, implying $t_2 \geq 0$. Finally, let $h = (0, u)^T$ with u satisfying $\langle u, \bar{x}_2 \rangle = 0$ and $\|u\| = 1$, then it follows from (25) that $t_3 \geq 0$.
(b) The necessity is the same of the argument as given in part (a). For sufficiency, take a nonzero vector h. If $\|h_2\| - \bar{x}_2^T h_2 > 0$, then the result holds since $t_3 > 0$. If $\|h_2\| - \bar{x}_2^T h_2 = 0$, then $h_2 = \beta \bar{x}_2$. So the left side of (26) takes $t_1(h_1 - \beta\mathrm{ctan}\theta)^2 + t_2(h_1 + \beta\tan\theta)^2 > 0$, because $h_1 - \beta\mathrm{ctan}\theta = h_1 + \beta\tan\theta = 0$ only happened when $\beta = 0$ and $h_1 = 0$. This means $h_2 = 0$ since $h_2 = \beta\bar{x}_2$, so $h = 0$. ☐

If f is differentiable, it is known that $f^{\mathcal{L}_\theta}$ is differentiable [4, 32]. It then follows from Lemma 3.1 that $f^{\mathcal{L}_\theta}$ is monotone on S if and only if $\nabla f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ for all $x \in S$. Hence, to characterize the monotonicity of $f^{\mathcal{L}_\theta}$, the first thing is to estimate $\nabla f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ via f.

Theorem 3.1 *Given $x \in \mathbb{R}^n$ and suppose that f is differentiable at $\lambda_i(x)$ for $i = 1, 2$. Then $\nabla f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ if and only if $f'(\lambda_i(x)) \geq 0$ for $i = 1, 2$ and $f(\lambda_2(x)) \geq f(\lambda_1(x))$.*

Proof The proof is divided into the following two cases.

Case 1 For $x_2 = 0$, using $\nabla f^{\mathcal{L}_\theta}(x) = f'(x_1)e$, it is clear that $\nabla f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ is equivalent to saying $f'(x_1) \geq 0$. Then, the desired result follows.

Case 2 For $x_2 \neq 0$, denote

$$b_1 := \frac{f'(\lambda_1(x))}{1 + \mathrm{ctan}^2\theta} \quad \text{and} \quad b_2 = \frac{f'(\lambda_2(x))}{1 + \tan^2\theta}.$$

Then, $\xi = b_1 + b_2$, $\varrho = -b_1\mathrm{ctan}\theta + b_2\tan\theta$, and $\eta = b_1\mathrm{ctan}^2\theta + b_2\tan^2\theta$. Hence, for all $h = (h_1, h_2)^T \in \mathbb{R} \times \mathbb{R}^{n-1}$, we have

$$\begin{aligned}
&\langle h, \nabla f^{\mathcal{L}_\theta}(x)h\rangle \\
= &\ (b_1 + b_2)h_1^2 + 2\varrho h_1\bar{x}_2^T h_2 + \tau\|h_2\|^2 + (b_1\mathrm{ctan}^2\theta + b_2\tan^2\theta - \tau)(\bar{x}_2^T h_2)^2 \\
= &\ (b_1 + b_2)h_1^2 + 2(-b_1\mathrm{ctan}\theta + b_2\tan\theta)h_1\bar{x}_2^T h_2 + (b_1\mathrm{ctan}^2\theta + b_2\tan^2\theta)(\bar{x}_2^T h_2)^2 \\
&+\tau\left(\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right) \\
= &\ b_1\left(h_1^2 - 2\mathrm{ctan}\theta h_1\bar{x}_2^T h_2 + \mathrm{ctan}^2\theta(\bar{x}_2^T h_2)^2\right) + b_2\left(h_1^2 + 2\tan\theta h_1\bar{x}_2^T h_2 + \tan^2\theta(\bar{x}_2^T h_2)^2\right) \\
&+\tau\left(\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right) \\
= &\ b_1\left(h_1 - \mathrm{ctan}\theta\bar{x}_2^T h_2\right)^2 + b_2\left(h_1 + \tan\theta\bar{x}_2^T h_2\right)^2 + \tau\left(\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right).
\end{aligned}$$

In light of Lemma 3.3, the desired result is equivalent to

$$b_1 \geq 0, \quad b_2 \geq 0, \quad \tau = \frac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} \geq 0,$$

i.e., $f'(\lambda_1(x)) \geq 0$, $f'(\lambda_2(x)) \geq 0$, and $f(\lambda_2(x)) \geq f(\lambda_1(x))$ due to $\lambda_2(x) > \lambda_1(x)$ in this case. □

By following almost the same arguments as given in Theorem 3.1, we further obtain the following consequence.

Corollary 3.1 *Given $x \in \mathbb{R}^n$ and suppose that f is differentiable at $\lambda_i(x)$ for $i = 1, 2$. Then for $x_2 \neq 0$, $\nabla f^{\mathcal{L}_\theta}(x) \succ_{\mathbb{S}^n_+} O$ if and only if $f'(\lambda_i(x)) > 0$ for $i = 1, 2$ and $f(\lambda_2(x)) > f(\lambda_1(x))$; for $x_2 = 0$, $\nabla f^{\mathcal{L}_\theta}(x) \succ_{\mathbb{S}^n_+} O$ if and only if $f'(x_1) > 0$.*

When f is non-differentiable, we resort to the subdifferential $\partial_B(f^{\mathcal{L}_\theta})$, whose estimate is given in [32].

Lemma 3.4 *Let $f : \mathbb{R} \to \mathbb{R}$ be strictly continuous. Then, for any $x \in \mathbb{R}^n$, the B-differential $\partial_B(f^{\mathcal{L}_\theta})(x)$ is well defined and nonempty. Moreover,*

(i) *if $x_2 \neq 0$, then*

$$\partial_B(f^{\mathcal{L}_\theta})(x) = \left\{ \begin{bmatrix} \xi & \varrho x_2^T/\|x_2\| \\ \varrho x_2/\|x_2\| & \tau I + (\eta - \tau)x_2x_2^T/\|x_2\|^2 \end{bmatrix} \middle| \begin{array}{l} \tau = \dfrac{f(\lambda_2(x)) - f(\lambda_1(x))}{\lambda_2(x) - \lambda_1(x)} \\ \xi - \varrho ctan\theta \in \partial_B f(\lambda_1(x)) \\ \xi + \varrho \tan\theta \in \partial_B f(\lambda_2(x)) \\ \eta = \xi - \varrho(ctan\theta - \tan\theta) \end{array} \right\}; \tag{27}$$

(ii) *if $x_2 = 0$, then*

$$\partial_B(f^{\mathcal{L}_\theta})(x) \subset \left\{ \begin{bmatrix} \xi & \varrho w^T \\ \varrho w & \tau I + (\eta - \tau)ww^T \end{bmatrix} \middle| \begin{array}{l} \tau \in \partial f(\lambda_1(x)), \quad \|w\| = 1 \\ \xi - \varrho ctan\theta \in \partial_B f(\lambda_1(x)) \\ \xi + \varrho \tan\theta \in \partial_B f(\lambda_1(x)) \\ \eta = \xi - \varrho(ctan\theta - \tan\theta) \end{array} \right\}. \tag{28}$$

Lemma 3.4 presents an upper estimation on $\partial_B f^{\mathcal{L}_\theta}(x)$ when $x_2 = 0$. Here we give an lower estimation.

Lemma 3.5 *Suppose that f is locally Lipschitz at x with $x_2 = 0$. Then,*

$$\partial_B f(x_1)I \subseteq \partial_B f^{\mathcal{L}_\theta}(x).$$

Proof First, take $u \in \partial_B f(x_1)$. Since f is locally Lipschitz at x_1, there exists $t \in D_f$ satisfying $t \to x_1$. Note that $\lambda_i(te) = t$ for $i = 1, 2$. Then, $f^{\mathcal{L}_\theta}$ is differentiable at te, i.e., $te \in D_{f^{\mathcal{L}_\theta}}$ and $\nabla f^{\mathcal{L}_\theta}(te) = f'(t)I$. Hence,

$$uI = \lim_{t \to x_1} f'(t)I = \lim_{te \to x} \nabla f^{\mathcal{L}_\theta}(te) \in \partial_B f^{\mathcal{L}_\theta}(x).$$

Since u is an arbitrary element in $\partial_B f(x_1)$, the desired result follows. □

Using Lemmas 3.4 and 3.5, the nonsmooth version of Theorem 3.1 is given below.

Theorem 3.2 *Given $x \in \mathbb{R}^n$, then $\partial_B f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ if and only if $\partial_B f(\lambda_i(x)) \geq 0$ for $i = 1, 2$, and $f(\lambda_2(x)) \geq f(\lambda_1(x))$.*

Proof Consider the following two cases.

Case 1 For $x_2 \neq 0$, according to (27), we know for $V \in \partial_B f^{\mathcal{L}_\theta}(x)$, there exists $v_i \in \partial_B f(\lambda_i(x))$ for $i = 1, 2$ such that $\xi - \varrho \text{ctan}\theta = v_1$, $\xi + \varrho \tan\theta = v_2$, and

$$V = \begin{bmatrix} \xi & \varrho \bar{x}_2^T \\ \varrho \bar{x}_2 & \tau I + (\eta - \tau)\bar{x}_2\bar{x}_2^T \end{bmatrix}.$$

Hence,

$$\xi = \frac{v_1 \tan\theta + v_2 \text{ctan}\theta}{\tan\theta + \text{ctan}\theta}, \quad \varrho = \frac{v_2 - v_1}{\tan\theta + \text{ctan}\theta} \quad \text{and} \quad \eta = \frac{v_1 \text{ctan}\theta + v_2 \tan\theta}{\tan\theta + \text{ctan}\theta}$$

and

$$\begin{aligned}
&\langle h, Vh\rangle \\
&= \frac{v_1\tan\theta + v_2\mathrm{ctan}\theta}{\tan\theta+\mathrm{ctan}\theta}h_1^2 + 2\frac{v_2-v_1}{\tan\theta+\mathrm{ctan}\theta}\bar{x}_2^T h_2 h_1 + \frac{v_1\mathrm{ctan}\theta + v_2\tan\theta}{\tan\theta+\mathrm{ctan}\theta}(\bar{x}_2^T h_2)^2 \\
&\quad +\tau\left[\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right] \\
&= \frac{\tan\theta}{\tan\theta+\mathrm{ctan}\theta}v_1\left[h_1 - (\bar{x}_2^T h_2)\mathrm{ctan}\theta\right]^2 + \frac{\mathrm{ctan}\theta}{\tan\theta+\mathrm{ctan}\theta}v_2\left[h_1 + (\bar{x}_2^T h_2)\tan\theta\right]^2 \\
&\quad +\tau\left[\|h_2\|^2 - (\bar{x}_2^T h_2)^2\right].
\end{aligned} \tag{29}$$

Applying Lemma 3.3, we have $\langle h, Vh\rangle \geq 0$ for all $h \in \mathbb{R}^n$ if and only if $v_1, v_2 \geq 0$, and $\tau \geq 0$. In other words, $\partial_B f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$ if and only if $\partial_B f(\lambda_i(x)) \geq 0$ for $i = 1, 2$ and $f(\lambda_2(x)) \geq f(\lambda_1(x))$.

Case 2 For $x_2 = 0$, the sufficiency follows by along the same arguments as above. In fact, the every element of the left set (28) is positive semidefinite. Then, the necessity follows from Lemma 3.5. □

Likewise, we have a nonsmooth version of Corollary 3.1 which is given below.

Corollary 3.2 *Given $x \in \mathbb{R}^n$, then for $x_2 \neq 0$, $\partial_B f^{\mathcal{L}_\theta}(x) \succ 0$ if and only if $\partial_B f(\lambda_i(x)) > 0$ for $i = 1, 2$, and $f(\lambda_2(x)) > f(\lambda_1(x))$; for $x_2 = 0$, $\partial_B f^{\mathcal{L}_\theta}(x) \succ 0$ if and only if $\partial_B f(x_1) > 0$.*

One point needs to be mentioned here. Because we cannot characterize the strict monotonicity by subgradients (see Lemma 3.1), we resort to the definition of monotonicity. Indeed, inspired by [18], we obtain the following result.

Theorem 3.3 *Suppose that f is locally Lipschitz continuous on J. Then,*

(a) *$f^{\mathcal{L}_\theta}$ is monotone on S if and only if f is nondecreasing on J;*
(b) *$f^{\mathcal{L}_\theta}$ is strictly monotone on S if and only if f is strictly increasing on J;*
(c) *$f^{\mathcal{L}_\theta}$ is strongly monotone on S with $\mu > 0$ if and only if f is strongly increasing on S with $\mu > 0$.*

Proof (a) $\Longrightarrow$. Take $t_1, t_2 \in J$ with $t_1 \leq t_2$. Since $f^{\mathcal{L}_\theta}$ is monotone, we have

$$0 \leq \langle (t_1 - t_2)e, f^{\mathcal{L}_\theta}(t_1 e) - f^{\mathcal{L}_\theta}(t_2 e)\rangle = (t_1 - t_2)(f(t_1) - f(t_2)),$$

which implies $f(t_1) \leq f(t_2)$.
$\Longleftarrow$. Take $x \in S$. Since f is monotone on J, we know $\partial_B f(t) \geq 0$ for all $t \in J$, and hence $\partial f(t) \geq 0$ for $t \in J$. Because $\lambda_i(x) \in J$ for $i = 1, 2$, it implies $\partial_B f(\lambda_i(x)) \geq 0$ for $i = 1, 2$. In addition, $f(\lambda_2(x)) - f(\lambda_1(x)) = \gamma(\lambda_2(x) - \lambda_1(x))$ for some $\gamma \in \partial f(t_0)$ and $t_0 \in (\lambda_1(x), \lambda_2(x)) \subseteq J$. Hence, $f(\lambda_2(x)) \geq f(\lambda_1(x))$ due to $\gamma \in \partial f(t_0) \geq 0$. Applying Theorem 3.2 yields $\partial_B f^{\mathcal{L}_\theta}(x) \succeq_{\mathbb{S}^n_+} O$, which means $f^{\mathcal{L}_\theta}$ is monotone on S by Lemma 3.1.
(b) $\Longrightarrow$. Given $t_1, t_2 \in J$ with $t_1 > t_2$. Since $f^{\mathcal{L}_\theta}$ is strictly monotone on S, we have

$$(f(t_1) - f(t_2))(t_1 - t_2) = \langle f^{\mathcal{L}_\theta}(t_1 e) - f^{\mathcal{L}_\theta}(t_2 e), (t_1 - t_2)e\rangle > 0,$$

which together with the fact $t_1 > t_2$ yields $f(t_1) > f(t_2)$. This says f is strictly increasing on J.

$\Longleftarrow$. Let $x, y \in S$ with $x \neq y$. We shall show the strict monotonicity of $f^{\mathcal{L}_\theta}$ by checking definition. To proceed, we consider the following four cases.

Case 1 For $x_2 = y_2 = 0$, it is clear that

$$\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\rangle = (f(x_1) - f(y_1))(x_1 - y_1) > 0,$$

where the last step is due to the fact that f is strictly monotone and $x_1 \neq y_1$, since $(x_1, 0) = x \neq y = (y_1, 0)$ under this case.

Case 2 For $x_2 = 0$ and $y_2 \neq 0$, we have

$$\begin{aligned}
&\langle f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x), y - x\rangle \\
&= \left\langle \left(\begin{array}{c} \frac{1}{1+\mathrm{ctan}^2\theta}[f(\lambda_1(y)) - f(x_1)] + \frac{1}{1+\tan^2\theta}[f(\lambda_2(y)) - f(x_1)] \\ \left[\frac{\tan\theta}{1+\tan^2\theta}[f(\lambda_2(y)) - f(x_1)] - \frac{\mathrm{ctan}\theta}{1+\mathrm{ctan}^2\theta}[f(\lambda_1(y)) - f(x_1)]\right]\bar{y}_2 \end{array}\right), \begin{pmatrix} y_1 - x_1 \\ y_2 \end{pmatrix} \right\rangle \\
&= \frac{1}{1+\mathrm{ctan}^2\theta}[f(\lambda_1(y)) - f(x_1)](y_1 - x_1 - \|y_2\|\mathrm{ctan}\theta) \\
&\quad + \frac{1}{1+\tan^2\theta}[f(\lambda_2(y)) - f(x_1)](y_1 - x_1 + \|y_2\|\tan\theta) \\
&= \frac{1}{1+\mathrm{ctan}^2\theta}[f(\lambda_1(y)) - f(x_1)](\lambda_1(y) - x_1) \\
&\quad + \frac{1}{1+\tan^2\theta}[f(\lambda_2(y)) - f(x_1)](\lambda_2(y) - x_1) \\
&> 0
\end{aligned}$$

where the second step follows by taking $\bar{x}_2 = \bar{y}_2$ and the last step is due to the strict monotonicity of f.

Case 3 For $x_2 \neq 0$ and $y_2 = 0$, the argument is similar as above.

Case 4 For $x_2, y_2 \neq 0$, we further discuss two subcases. If $y_2 \notin \mathbb{R}x_2 := \{\tau x_2 | \tau \in \mathbb{R}\}$, that is, y_2 is not parallel to x_2, then $0 \notin [x_2, y_2]$. This can be verified by contradiction: if $0 \in [x_2, y_2]$, there exists $\lambda \in (0, 1)$ such that $0 = \lambda x_2 + (1-\lambda)y_2$. It implies $y_2 = -\frac{\lambda}{1-\lambda}x_2$, which is a contraction with $y_2 \notin \mathbb{R}x_2$. Hence, $z_2 \neq 0$ for all $z \in [x, y]$. We now claim that

$$\langle y - x, V(y - x)\rangle > 0, \quad \forall z \in [x, y] \text{ and } V \in \partial_B f^{\mathcal{L}_\theta}(z). \tag{30}$$

In fact, according to Theorem 3.2, $V \succeq_{\mathbb{S}^n_+} O$, and hence $\langle y-x, V(y-x)\rangle \geq 0$. However, if $\langle y - x, V(y - x)\rangle = 0$, then it follows from (29) that

$$y_2 - x_2 = \gamma\bar{z}_2 = \gamma\frac{\lambda x_2 + (1-\lambda)y_2}{\|z_2\|}$$

for some $\gamma > 0$, since $\tau = \frac{f(\lambda_2(y)) - f(\lambda_2(x))}{\lambda_2(y) - \lambda_2(x)} > 0$ by the strict increasing of f. Hence, $y_2 \in \mathbb{R}x_2$, which is a contradiction.

Therefore, applying (30) and using the fact

$$\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\rangle \in \mathrm{conv}\left\{\langle y - x, V(y - x)\rangle | V \in \bigcup_{z\in[x,y]} \partial f^{\mathcal{L}_\theta}(z)\right\},$$

we have

$$\langle f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y), x - y\rangle > 0.$$

If $y_2 \in \mathbb{R}x_2$, i.e., $y_2 = \beta x_2$ for some $\beta \in \mathbb{R}$, we proceed as follows. For $\beta > 0$, we have $\bar{y}_2 = \bar{x}_2$ and hence

$$\begin{aligned}
&\langle f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x), y - x\rangle \\
&= \frac{1}{1+\mathrm{ctan}^2\theta}\left[f(\lambda_1(y)) - f(\lambda_1(x))\right](y_1 - x_1 - (\beta-1)\|x_2\|\mathrm{ctan}\theta) \\
&\quad + \frac{1}{1+\tan^2\theta}\left[f(\lambda_2(y)) - f(\lambda_2(x))\right](y_1 - x_1 + (\beta-1)\|x_2\|\tan\theta) \\
&= \frac{1}{1+\mathrm{ctan}^2\theta}\left[f(\lambda_1(y)) - f(\lambda_1(x))\right](y_1 - x_1 - (\|y_2\| - \|x_2\|)\mathrm{ctan}\theta) \\
&\quad + \frac{1}{1+\tan^2\theta}\left[f(\lambda_2(y)) - f(\lambda_2(x))\right](y_1 - x_1 + (\|y_2\| - \|x_2\|)\tan\theta) \\
&= \frac{1}{1+\mathrm{ctan}^2\theta}\left[f(\lambda_1(y)) - f(x_1)\right](\lambda_1(y) - \lambda_1(x)) \\
&\quad + \frac{1}{1+\tan^2\theta}\left[f(\lambda_2(y)) - f(\lambda_2(x))\right](\lambda_2(y) - \lambda_2(x)) \\
&> 0.
\end{aligned}$$

For $\beta < 0$, take $z := \frac{1}{1-\beta}y + \frac{-\beta}{1-\beta}x$. Thus, $z_2 = 0$ and

$$y - x = \frac{\beta - 1}{\beta}(y - z) = (1-\beta)(z - x).$$

This yields

$$\begin{aligned}
&\langle f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(x), y - x\rangle \\
&= \langle f^{\mathcal{L}_\theta}(z) - f^{\mathcal{L}_\theta}(x), y - x\rangle + \langle f^{\mathcal{L}_\theta}(z) - f^{\mathcal{L}_\theta}(x), y - x\rangle \\
&= \frac{\beta-1}{\beta}\langle f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(z), y - z\rangle + (1-\beta)\langle f^{\mathcal{L}_\theta}(y) - f^{\mathcal{L}_\theta}(z), z - x\rangle \\
&> 0,
\end{aligned}$$

where the last step is due to $z_2 = 0$ by following the similar argument as Case 2, and coefficients $(\beta - 1)/\beta$, $(1-\beta)$ are positive.

(c) The result follows by applying part (a) to the function $f(t) - \eta t$. $\square$

Theorems 2.3 and 3.3 indicate an interesting thing: the angle θ plays different role in these two concepts. In other words, the circular cone monotonicity of f depends on f and θ, whereas the monotonicity of $f^{\mathcal{L}_\theta}$ only depends on f.

Corollary 3.3 *For two scalars $v_1, v_2 \in \mathbb{R}$ with $v_1 \le v_2$, then*

(a) *If $\partial_B f(t) \subseteq [v_1, v_2]$ for $t \in J$, then $v_1 I \preceq_{\mathbb{S}^n_+} \partial_B f^{\mathcal{L}_\theta}(x) \preceq_{\mathbb{S}^n_+} v_2 I$ for all $x \in S$;*

(b) *If $\partial_B f(t) \subseteq (v_1, v_2)$ for $t \in J$, then $v_1 I \prec_{\mathbb{S}^n_+} \partial_B f^{\mathcal{L}_\theta}(x) \prec_{\mathbb{S}^n_+} v_2 I$ for all $x \in S$.*

Proof (a) Define $g_{v_1}(t) := f(t) - v_1 t$. Then, $(g_{v_1})^{\mathcal{L}_\theta}(x) = f^{\mathcal{L}_\theta}(x) - v_1 x$ which says

$$\partial_B(g_{v_1})(t) = \partial_B f(t) - v_1 \quad \text{and} \quad \partial_B(g_{v_1})^{\mathcal{L}_\theta}(x) = \partial_B f^{\mathcal{L}_\theta}(x) - v_1 I.$$

Since $\partial_B f(t) \subseteq [v_1, v_2]$, we obtain $\partial_B g_{v_1}(t) \geq 0$ for all $t \in J$, and hence g_{v_1} is nondecreasing on J. This implies that $(g_{v_1})^{\mathcal{L}_\theta}$ is monotone by Theorem 3.3. Moreover, this together with Lemma 3.1 gives $\partial_B (g_{v_1})^{\mathcal{L}_\theta}(t) \succeq_{\mathbb{S}^n_+} O$, i.e., $\partial_B f^{\mathcal{L}_\theta}(t) \succeq_{\mathbb{S}^n_+} v_1 I$. Similarly, we can obtain $\partial_B f^{\mathcal{L}_\theta}(x) \preceq_{\mathbb{S}^n_+} v_2 I$.
(b) Note that $\partial_B f^{\mathcal{L}_\theta}$ is a closed set. Hence, if $\partial_B f(t) \subseteq (v_1, v_2)$, there exists v_1' and v_2' satisfying $v_1 < v_1', v_2' < v_2$ and $\partial_B f(t) \subseteq [v_1', v_2']$ for all $t \in J$. Applying part (a) yields $v_1' I \preceq_{\mathbb{S}^n_+} \partial_B f^{\mathcal{L}_\theta}(x) \preceq_{\mathbb{S}^n_+} v_2' I$, which in turn implies $v_1 I \prec_{\mathbb{S}^n_+} \partial_B f^{\mathcal{L}_\theta}(x) \prec_{\mathbb{S}^n_+} v_2 I$. □

Finally, let us discuss the relationship between $\mathcal{L}_\theta$-monotonicity of f in Section 2 and monotonicity of $f^{\mathcal{L}_\theta}$ in Section 3.

Theorem 3.4 *Given $f : J \to \mathbb{R}$, if f is $\mathcal{L}_\theta$-monotone, then $f^{\mathcal{L}_\theta}$ is monotone.*

Proof Given $t_1, t_2 \in J$ with $t_1 < t_2$, then $t_i e \in S$ since $\lambda_i(t_i e) = t_i \in J$ for $i = 1, 2$. Note that $t_2 e \succeq_{\mathcal{L}_\theta} t_1 e$. Since f is $\mathcal{L}_\theta$-monotone, then

$$\begin{bmatrix} f(t_2) \\ 0 \end{bmatrix} = f^{\mathcal{L}_\theta}(t_2 e) \succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(t_1 e) = \begin{bmatrix} f(t_1) \\ 0 \end{bmatrix}$$

which implies $f(t_2) \geq f(t_1)$, i.e., f is monotone on J. This, together with Theorem 3.3, means that $f^{\mathcal{L}_\theta}$ is monotone on S. □

In general, the converse of Theorem 3.4 is false. However, it holds under some particular cases.

Theorem 3.5 *When $n = 2$ and $\theta = 45°$, the following statements are equivalent.*

(a) *f is nondecreasing on J;*
(b) *f is $\mathcal{L}_\theta$-monotone on J;*
(c) *$f^{\mathcal{L}_\theta}$ is monotone on S.*

Proof If f is differentiable, then the result follows from Theorems 2.4 and 3.3. It remains to show the results hold true when f is nondifferentiable. As $n = 2$ and $\theta = 45°$, then

$$f^{\mathrm{soc}}(x) = \frac{1}{2}\begin{bmatrix} f(x_1 - |x_2|) + f(x_1 + |x_2|) \\ (f(x_1 + |x_2|) - f(x_1 - |x_2|))\,\mathrm{sign}(x_2) \end{bmatrix}.$$

Given x, y with $x \succeq_{\mathcal{K}^2} y$, i.e., $x - y \in \mathcal{K}^2$, then $x_1 - y_1 \geq |x_2 - y_2|$, which in turn implies that $\lambda_i(x) \geq \lambda_i(y)$ for $i = 1, 2$. Note that for $x_2 \neq 0$, then $\bar{x}_2 = \mathrm{sign}(x_2)$. Therefore for $x_2, y_2 \neq 0$, we have

$$\begin{aligned} f^{\mathrm{soc}}(x) - f^{\mathrm{soc}}(y) \;=\; & \frac{1}{2}\begin{bmatrix} f(x_1 - |x_2|) + f(x_1 + |x_2|) \\ (f(x_1 + |x_2|) - f(x_1 - |x_2|))\,\mathrm{sign}(x_2) \end{bmatrix} \\ & -\frac{1}{2}\begin{bmatrix} f(y_1 - |y_2|) + f(y_1 + |y_2|) \\ (f(y_1 + |y_2|) - f(y_1 - |y_2|))\,\mathrm{sign}(y_2) \end{bmatrix}. \end{aligned}$$

It needs to show that

$$\begin{aligned} & f(x_1 - |x_2|) - f(y_1 - |y_2|) + f(x_1 + |x_2|) - f(y_1 + |y_2|) \\ \geq & \left| [f(x_1 + |x_2|) - f(x_1 - |x_2|)]\,\mathrm{sign}(x_2) - [f(y_1 + |y_2|) - f(y_1 - |y_2|)]\,\mathrm{sign}(y_2) \right|. \end{aligned} \tag{31}$$

If $\mathrm{sign}(x_2) = \mathrm{sign}(y_2)$, then (31) takes

$$\begin{aligned} & f(x_1 - |x_2|) - f(y_1 - |y_2|) + f(x_1 + |x_2|) - f(y_1 + |y_2|) \\ \geq & \left| f(x_1 + |x_2|) - f(x_1 - |x_2|) - f(y_1 + |y_2|) + f(y_1 - |y_2|) \right|, \end{aligned}$$

i.e.,

$$\begin{aligned} & -f(x_1 - |x_2|) + f(y_1 - |y_2|) - f(x_1 + |x_2|) + f(y_1 + |y_2|) \\ \leq & \ f(x_1 + |x_2|) - f(x_1 - |x_2|) - f(y_1 + |y_2|) + f(y_1 - |y_2|) \\ \leq & \ f(x_1 - |x_2|) - f(y_1 - |y_2|) + f(x_1 + |x_2|) - f(y_1 + |y_2|), \end{aligned}$$

which is ensured by $f(x_1 + |x_2|) \geq f(y_1 + |y_2|)$ and $f(x_1 - |x_2|) \geq f(y_1 - |y_2|)$ since f is nondecreasing and $\lambda_i(x) \geq \lambda_i(y)$ for $i = 1, 2$.
If $\mathrm{sign}(x_2) = -\mathrm{sign}(y_2)$, then (31) takes

$$\begin{aligned} & f(x_1 - |x_2|) - f(y_1 - |y_2|) + f(x_1 + |x_2|) - f(y_1 + |y_2|) \\ \geq & \left| f(x_1 + |x_2|) - f(x_1 - |x_2|) + f(y_1 + |y_2|) - f(y_1 - |y_2|) \right|, \end{aligned}$$

i.e.,

$$\begin{aligned} & -f(x_1 - |x_2|) + f(y_1 - |y_2|) - f(x_1 + |x_2|) + f(y_1 + |y_2|) \\ \leq & \ f(x_1 + |x_2|) - f(x_1 - |x_2|) + f(y_1 + |y_2|) - f(y_1 - |y_2|) \\ \leq & \ f(x_1 - |x_2|) - f(y_1 - |y_2|) + f(x_1 + |x_2|) - f(y_1 + |y_2|), \end{aligned}$$

which is ensured by $f(x_1 + |x_2|) \geq f(y_1 - |y_2|)$ and $f(x_1 - |x_2|) \geq f(y_1 + |y_2|)$ since $x_1 - y_1 \geq |x_2 - y_2| = |x_2| + |y_2|$ where the last step is due to $\mathrm{sign}(x_2) = -\mathrm{sign}(y_2)$.

The case of either $x_2 = 0$ or $y_2 = 0$ follows by the similar arguments. This completes the proof. □

The requirements of $\theta = 45°$ and $n = 2$ in Theorem 3.5 are essential. This can be illustrated by the following examples, which indicate that the converse statement of Theorem 3.5 is false when either $\theta \neq 45°$ or $n \neq 2$.

Example 3.1 For $n = 2$, consider the function $f(t) = -1/t$ and $J := (0, \infty)$.

Then, $f'(t) = 1/t^2 \geq 0$ for all $t \in J$ and $f'(t_1) = 1/t_1^2 \geq 1/t_2^2 = f'(t_2)$ whenever $t_1, t_2 \in J$ with $t_1 \leq t_2$. Hence, according to Theorem 2.4, we know that f is $\mathcal{L}_\theta$-monotone on S (in fact $S = \mathrm{int}\mathcal{L}_\theta$ since $J = (0, \infty)$) when $\theta \geq 45^{\mathrm{o}}$. However, f is not $\mathcal{L}_\theta$-monotone on S when $\theta < 45^{\mathrm{o}}$. To see this, let $\tan\theta = 1/2$, and take $x = (16, 2)^T$, $y = (8, -2)^T$ which yields

$$x - y = \begin{bmatrix} 8 \\ 4 \end{bmatrix} \in \mathcal{L}_\theta.$$

Noting

$$f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y) = \begin{bmatrix} -\frac{13}{204} \\ \frac{1}{102} \end{bmatrix} - \begin{bmatrix} -\frac{5}{36} \\ -\frac{1}{18} \end{bmatrix} = \begin{bmatrix} \frac{23}{306} \\ \frac{10}{153} \end{bmatrix}$$

and

$$(\tan\theta)\left(f_1^{\mathcal{L}_\theta}(x) - f_1^{\mathcal{L}_\theta}(y)\right) = \frac{1}{2} \times \frac{23}{306} < \frac{20}{306} = \frac{10}{153} = \left| f_2^{\mathcal{L}_\theta}(x) - f_2^{\mathcal{L}_\theta}(y) \right|,$$

we see that $f^{\mathcal{L}_\theta}(x) \not\succeq_{\mathcal{L}_\theta} f^{\mathcal{L}_\theta}(y)$, i.e., f is not $\mathcal{L}_\theta$-monotone for $\theta < 45°$. In addition, $f^{\mathcal{L}_\theta}$ is monotone, since $f(t) = -1/t$ is monotone on $(0, \infty)$.

Whether Theorem 3.5 still holds when $n = 2$ and $\theta > 45°$? This is not true as showing by the following example.

Example 3.2 For $n = 2$, consider the function $f(t) = t^2$ and $J := (0, \infty)$.

Since $f(t) = t^2$ is increasing on $(0, \infty)$, $f^{\mathcal{L}_\theta}$ is monotone by Theorem 3.3. For $\theta > 45°$, we have $f'(t_1) - f'(t_2) = 2(t_1 - t_2) < 0$ as $t_1 < t_2$, which says f is not $\mathcal{L}_\theta$-monotone by Theorem 2.4. This can be verified by another way. Let $\tan\theta = 10$ and take $x = (2, 1)^T$, $y = (1, 9)^T$. Then, $x - y = (1, -8) \in \mathcal{L}_\theta$, i.e., $x \succeq_{\mathcal{L}_\theta} y$. Since $f^{\mathcal{L}_\theta}(x) = (5, 13.9)^T$ and $f^{\mathcal{L}_\theta}(y) = (82, 819.9)^T$, we see that $f^{\mathcal{L}_\theta}(x) - f^{\mathcal{L}_\theta}(y) \notin \mathcal{L}_\theta$ due to $f_1^{\mathcal{L}_\theta}(x) - f_1^{\mathcal{L}_\theta}(y) < 0$.

Example 3.3 For $\theta = 45°$, consider the function $f(t) := t^2$ and $J := (0, +\infty)$.

Let $n \geq 3$. Since $f(t) = t^2$ is increasing on $J = (0, +\infty)$, then f^{soc} is monotone on S. Take $x = (1, 3/10, 0, 0_{n-3})^T$ and $y = (1/2, 0, 2/5, 0_{n-3})^T$ where 0_{n-3} denotes $(0, \cdots, 0)^T \in \mathbb{R}^{n-3}$. Then

$$x - y = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{10} \\ -\frac{2}{5} \\ 0_{n-3} \end{bmatrix} \in \mathcal{K}^n,$$

i.e., $x \succeq_{\mathcal{K}^n} y$. By a simple calculation, we have

$$f^{\text{soc}}(x) = \frac{1}{200}\begin{bmatrix} 218 \\ 120 \\ 0 \\ 0_{n-3} \end{bmatrix} \quad \text{and} \quad f^{\text{soc}}(y) = \frac{1}{200}\begin{bmatrix} 82 \\ 0 \\ 80 \\ 0_{n-3} \end{bmatrix},$$

which implies

$$f^{\text{soc}}(x) - f^{\text{soc}}(y) = \frac{1}{200}\begin{bmatrix} 136 \\ 120 \\ -80 \\ 0_{n-3} \end{bmatrix} \notin \mathcal{K}^n,$$

i.e., $f^{\text{soc}}(x) \not\succeq f^{\text{soc}}(y)$. Hence f is not SOC-monotone on S.

4 Conclusions

There exist many similarities and differences between functions associated with circular cone and second-order cone. From one side, some properties of between $f^{\mathcal{L}_\theta}$ and f is analogous to that between f^{soc} and f, such as first-order differentiability, Lipschitz continuity, and nonsmoothness [4, 31]. On the other hand, as our first impression, SOC has additional algebraical property. Our results further indicate that the angle θ plays an essential property in the properties of $f^{\mathcal{L}_\theta}$, for example, $\mathcal{L}_\theta$-monotonicity. If $\theta = 45°$, then the conditions (17) and (18) reduces to [7, Theorem 3.1]. However, in the SOC setting, these conditions are necessary and sufficient conditions for f to be SOC-monotone (noting that (17) holds automatically in the SOC case). In this paper, we are only able to show that the conditions (17) and (18) are sufficient for f to be $\mathcal{L}_\theta$-monotone. How to establish the necessary and sufficient conditions for $\mathcal{L}_\theta$-monotonicity is rather important and interesting. In addition, the nonsmooth version of Theorem 2.3 merits further research.

At last, we talk about the contribution of this paper. As mentioned earlier, although it is possible to construct a new inner product which ensures the circular cone $\mathcal{L}_\theta$ is self-dual, it is not possible to make both $\mathcal{L}_\theta$ and $\mathcal{K}^n$ are self-dual under a certain inner product. The relation $\mathcal{K}^n = A\mathcal{L}_\theta$ does not guarantee that there exists a similar close relation between $f^{\mathcal{L}_\theta}$ and f^{soc}. Hence, the study on $f^{\mathcal{L}_\theta}$ is necessary. As we see, the arguments are not trivial and also pave a way to deal with non-symmetric cones.

In general, to determine whether a mapping is monotone from the original definition is not an easy thing. Fortunately, for symmetric cone programming, due to the spectral algebraic structure (called Jordan algebra) associated with symmetric cone, we can define a (vector-valued) Löwner operator by using a given simple real-valued function. The previous work has discovered that many properties of the vector-valued function are inherited by the given real-valued function, such as directional derivative, differentiability, B-subdifferentiability, semismoothness, etc.. This no doubt helps us to judge the property of vector-valued function by simply checking the real-valued functions. The monotonicity of Löwner operator in the settings of symmetric cone and second-order cone has been established in [18, 24]. We further study the monotonicity for a special non-symmetric cone, circular cone. Since checking the monotonicity of a scalar function $f : \mathbb{R} \to \mathbb{R}$ is simple, our result can help us to verify the $\mathcal{L}_\theta$-monotonicity of f and the monotonicity of $f^{\mathcal{L}_\theta}$ quickly. For example, for any fixed $\sigma \geq 0$, and let $f(t) := \frac{1}{\sigma - t}$ (or $f(t) := \sqrt{t-\sigma}$, $f(t) := \frac{t}{t+\sigma}$). Then, $f^{\mathcal{L}_\theta}$ is monotone and f is $\mathcal{L}_\theta$-monotone on $(\sigma, +\infty)$ as $\theta \in [45°, 90°)$. In particular, from the formula (3.9) in [14, Proposition 3.4], we know that for any $x \in \mathbb{R}^n$ and $w \succ_{\mathcal{K}^n} 0$, there holds

$$w^2 \succeq_{\mathcal{K}^n} x^2 \Longrightarrow w \succeq_{\mathcal{K}^n} x. \tag{32}$$

The proof for [14, Proposition 3.4] is not trivial. Here by taking $\sigma = 0$ and $f(t) = t^{1/2}$, f is $\mathcal{L}_\theta$-monotone as $\theta \in [45°, 90°)$. Hence, particularly for $\theta = 45°$, the $\mathcal{L}_\theta$-monotonicity of f ensures that $w^2 \succeq_{\mathcal{K}^n} x^2$ implies $w \succeq_{\mathcal{K}^n} |x| \succeq_{\mathcal{K}^n} x$, i.e., (32) can be proved by just checking the properties of a scalar function $f(t) = t^{1/2}$.

On the other hand, we recall that the circular cone complementarity problem is

$$x \in \mathcal{L}_\theta, \quad y \in \mathcal{L}_\theta^*, \quad \langle x, y\rangle = 0. \tag{33}$$

In a very recent work [20], the authors study how to construct the complementarity function (based on Fischer-Burmeister (FB) function and natural residual (NR) function) and the merit function for circular cone complementary problems. Here, the concept of monotonicity plays an important role to ensure the existence of solution and error bound theory, see [20] for more details. In addition, using the relation between circular cone and second-order cone, (33) can be rewritten equivalently as

$$\begin{aligned} & x \in A^{-1}\mathcal{K}^n, \quad y \in \mathcal{L}_\theta^* = \mathcal{L}_{\frac{\pi}{2}-\theta} = A\mathcal{K}^n, \quad \langle Ax, A^{-1}y\rangle = 0 \\ \Longleftrightarrow\ & u := Ax \in \mathcal{K}^n, \quad v := A^{-1}y \in \mathcal{K}^n, \quad \langle u, v\rangle = 0. \end{aligned} \tag{34}$$

Here we simply discuss the Mangasarian class of complementarity functions, defined as

$$\phi_M(a, b) := f(|a-b|) - f(a) - f(b)$$

for all $(a, b) \in \mathbb{R}^2$, where f is required to be strictly increasing and $f(0) = 0$. The corresponding vector-valued function $\Phi_M : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\Phi_M(u, v) := f^{\mathcal{L}_\theta}(|u-v|) - f^{\mathcal{L}_\theta}(u) - f^{\mathcal{L}_\theta}(v).$$

Let $\Phi_M^A(x, y) := \Phi_M(Ax, A^{-1}y)$. Then, the monotonicity of $f^{\mathcal{L}_\theta}$ (coming from the monotonicity of f) ensures that Φ_M is a complementarity function in the framework of second-order cone (as $\theta = 45°$); see the detailed discussion given in [18, Theorem 4]. Thus, we have

$$(x, y) \text{ solves } (33) \iff (u, v) \text{ solves } (34) \iff \Phi_M(u, v) = 0$$
$$\iff \Phi_M(Ax, A^{-1}y) = 0 \iff \Phi_M^A(x, y) = 0.$$

From this, we see that $\Phi_M^A(x, y)$ and $\|\Phi_M^A(x, y)\|$ serves as complementarity function and merit function of (33), respectively.

In summary, the main target of this paper is to discover the role played by the angle. In particular, our study shows that the angle is crucial for circular cone monotonicity. This is a surprising and interesting discovery.

Acknowledgments The authors are gratefully indebted to the anonymous referees and editor for their valuable suggestions and remarks that helped us improve the original presentation of the paper.

References

1. Alizadeh, F., Goldfarb, D.: Second-order cone programming. Math. Program. **95**, 3–51 (2003)
2. Bonnans, J.F., Ramirez, H.: Perturbation analysis of second-order cone programming problems. Math. Program. **104**, 205–227 (2005)
3. Chen, J.-S., Tseng, P.: An unconstrained smooth minimization reformulation of the second-order cone complementarity problem. Math. Program. **104**, 293–327 (2005)
4. Chang, Y.-L., Yang, C.-Y., Chen, J.-S.: Smooth and nonsmooth analysis of vector-valued functions associated with circular cones. Nonlinear Anal. Theory Methods Appl. **85**, 160–173 (2013)
5. Chen, J.-S.: The convex and monotone functions associated with second-order cone. Optimization **55**(4), 363–385 (2006)
6. Chen, J.-S., Chen, X., Tseng, P.: Analysis of nonsmooth vector-valued functions associated with second-order cone. Math. Program. **101**, 95–117 (2004)
7. Chen, J.-S., Chen, X., Pan, S.-H., Zhang, J.: Some characterizations for SOC-monotone and SOC-convex functions. J. Glob. Optim. **45**, 259–279 (2009)
8. Chen, J.-S., Liao, T.-K., Pan, S.-H.: Using Schur complement theorem to prove convexity of some SOC-functions. J. Nonlinear Convex Anal. **13**, 421–431 (2012)
9. Chen, J.-S., Pan, S.-H.: A survey on SOC complementarity functions and solution methods for SOCPs and SOCCPs. Pac. J. Optim. **8**, 33–74 (2012)
10. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley, New York (1983)
11. Dattorro, J.: Convex Optimization and Euclidean Distance Geometry. Meboo Publishing, California (2005)
12. Ding, C., Sun, D., Toh, K.C.: An introduction to a class of matrix cone programming. Math. Program. **144**, 141–179 (2014)
13. Faraut, U., Korányi, A.: Analysis on Symmetric Cones. Oxford Mathematical Monographs. Oxford University Press, New York (1994)
14. Fukushima, M., Luo, Z.Q., Tseng, P.: Smoothing functions for second-order-cone complementarity problems. SIAM J. Optim. **12**, 436–460 (2002)
15. Guler, O., Tuncel, L.: Characterization of the barrier parameter of homogeneous convex cones. Math. Program. **81**, 55–76 (1998)
16. Hayashi, S., Yamashita, N., Fukushima, M.: A combined smoothing and regularization method for monotone second-order cone complementarity problems. SIAM J. Optim. **15**, 593–615 (2005)
17. Kanzow, C., Ferenczi, I., Fukushima, M.: On the local convergence of semismooth Newton methods for linear and nonlinear second-order cone programs without strict complementarity. SIAM J. Optim. **20**, 297–320 (2009)
18. Kong, L.C., Tuncel, L., Xiu, N.H.: Monotonicity of Lowner operators and its applications to symmetric cone complementarity problems. Math. Program. **133**, 327–336 (2012)
19. Liu, Y.J., Zhang, L.W.: Convergence analysis of the augmented Lagrangian method for nonlinear second-order cone optimization problems. Nonlinear Anal. Theory Methods Appl. **67**, 1359–1373 (2007)
20. Miao, X.-H., Guo, S.-J., Qi, N., Chen, J.-S.: Constructions of complementarity functions and merit functions for circular cone complementarity problem. Comput. Optim. Appl. **63**, 495–522 (2016)

21. Pan, S.-H., Chen, J.-S.: A class of interior proximal-like algorithms for convex second-order cone programming. SIAM J. Optim. **19**, 883–910 (2008)
22. Pan, S.-H., Chen, J.-S.: An R-linearly convergent nonmonotone derivative-free method for symmetric cone complementarity problems. Adv. Model. Optim. **13**, 185–211 (2011)
23. Pan, S.-H., Chang, Y.-L., Chen, J.-S.: Stationary point conditions for the FB merit function associated with symmetric cones. Oper. Res. Lett. **38**, 372–377 (2010)
24. Pan, S.-H., Chiang, Y., Chen, J.-S.: SOC-Monotone and SOC-convex functions v.s. matrix-monotone and matrix-convex functions. Linear Algebra Appl. **437**(5), 1264–1284 (2012)
25. Roos, K. http://www.isa.ewi.tudelft.nl/~roos/course/WI4218/SLemma.pdf
26. Rockafellar, R.T., Wets, R.J.: Variational Analysis. Springer, New York (1998)
27. Schmieta, S.H., Alizadeh, F.: Extension of primal-dual interior point algorithms to symmetric cones. Math. Program. **96**, 409–438 (2003)
28. Sun, D., Sun, J.: Strong semismoothness of the Fischer-Burmeister SDC and SOC complementarity functions. Math. Program. **103**, 575–581 (2005)
29. Truong, V.A., Tuncel, L.: Geometry of homogeneous cones: duality mapping and optimal selfconcordant barriers. Math. Program. **100**, 295–316 (2004)
30. Xue, G., Ye, Y.: An efficient algorithm for minimizing a sum of p-norms. SIAM J. Optim. **10**, 551–579 (2000)
31. Zhou, J.-C., Chen, J.-S.: Properties of circular cone and spectral factorization associated with circular cone. J. Nonlinear Convex Anal. **14**, 807–816 (2013)
32. Zhou, J.-C., Chen, J.-S.: On the vector-valued functions associated with circular cones. Abstr. Appl. Anal. **2014**(603542), 21 (2014)
33. Zhou, J.-C., Chen, J.-S., Hung, H.-F.: Circular cone convexity and some inequalities associated with circular cones. J. Inequalities Appl. **2013**(571), 17 (2013)