

## ALTERNATIVE PROOFS FOR SOME RESULTS OF VECTOR-VALUED FUNCTIONS ASSOCIATED WITH SECOND-ORDER CONE

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ABSTRACT. Let  $\mathcal{K}^n$  be the Lorentz/second-order cone in  $\mathbb{R}^n$ . For any function  $f$  from  $\mathbb{R}$  to  $\mathbb{R}$ , one can define a corresponding vector-valued function  $f^{\text{soc}}(x)$  on  $\mathbb{R}^n$  by applying  $f$  to the spectral values of the spectral decomposition of  $x \in \mathbb{R}^n$  with respect to  $\mathcal{K}^n$ . It was shown by J.-S. Chen, X. Chen and P. Tseng that this vector-valued function inherits from  $f$  the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. It was also proved by D. Sun and J. Sun that the vector-valued Fischer-Burmeister function associated with second-order cone is strongly semismooth. All proofs for the above results are based on a special relation between the vector-valued function and the matrix-valued function over symmetric matrices. In this paper, we provide a straightforward and intuitive way to prove all the above results by using the simple structure of second-order cone and spectral decomposition.

### 1. INTRODUCTION

The second-order cone (SOC) in  $\mathbb{R}^n$ , also called the Lorentz cone, is defined to be

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\},$$

where  $\|\cdot\|$  denotes the Euclidean norm. If  $n = 1$ ,  $\mathcal{K}^1$  is the set of nonnegative reals  $\mathbb{R}_+$ . Recently, there have been much study on second-order cone in optimization, particularly in the context of applications and solution methods for second-order cone program (SOCP) [1, 2, 9, 12, 14, 18, 22]. For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , we can decompose  $x$  as

$$(1) \quad x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)},$$

where  $\lambda_1, \lambda_2$  and  $u^{(1)}, u^{(2)}$  are the *spectral values* and the associated *spectral vectors* of  $x$ , with respect to  $\mathcal{K}^n$ , given by

$$(2) \quad \lambda_i = x_1 + (-1)^i \|x_2\|,$$
$$(3) \quad u^{(i)} = \begin{cases} \frac{1}{2} \left( 1, (-1)^i \frac{x_2}{\|x_2\|} \right), & \text{if } x_2 \neq 0, \\ \frac{1}{2} \left( 1, (-1)^i w \right), & \text{if } x_2 = 0, \end{cases}$$

for  $i = 1, 2$ , with  $w$  being any vector in  $\mathbb{R}^{n-1}$  satisfying  $\|w\| = 1$ . If  $x_2 \neq 0$ , the decomposition (1) is unique. With this spectral decomposition, for any function

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$f : \mathbb{R} \rightarrow \mathbb{R}$ , the following vector-valued function associated with  $\mathcal{K}^n$  ( $n \geq 1$ ) was considered (see [9]):

$$(4) \quad f^{\text{soc}}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)} \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

If  $f$  is defined only on a subset of  $\mathbb{R}$ , then  $f^{\text{soc}}$  is defined on the corresponding subset of  $\mathbb{R}^n$ . The definition (4) is unambiguous whether  $x_2 \neq 0$  or  $x_2 = 0$ . The above definition (4) is analogous to one associated with the semidefinite cone  $\mathcal{S}^n$ , see [20, 21].

The study of this function is motivated by second-order cone complementarity problem (SOCCP), see [3, 4] and references therein. In fact, in the paper [4], it studied the continuity and differentiability properties of the vector-valued function  $f^{\text{soc}}$ . In particular, it showed that the properties of continuity, strict continuity, Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, and ( $\rho$ -order) semismoothness are each inherited by  $f^{\text{soc}}$  from  $f$ . These results parallel those obtained recently in [5] for matrix-valued functions and are useful in the design and analysis of smoothing and nonsmooth methods for solving SOCP and SOCCP. The proofs are based on an elegant relation between the vector-valued function  $f^{\text{soc}}$  and its matrix-valued counterpart (see Lemma 4.1 of [4]). This relation enables applying the results from [5] for matrix-valued functions to the vector-valued function  $f^{\text{soc}}$ . In this paper, we study an intuitive way to prove all the aforementioned results without using the relation as will be seen in Sec. 3.

A popular approach to solving SOCCP is to reformulate it as an unconstrained minimization problem. Specifically, it is to find a smooth (continuously differentiable) function  $\psi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$  such that

$$(5) \quad \psi(x, y) = 0 \iff \langle x, y \rangle = 0, \quad x \in \mathcal{K}^n, \quad y \in \mathcal{K}^n,$$

which yields that the SOCCP can be expressed as an unconstrained smooth (continuously differentiable) minimization problem:  $\min_{\zeta \in \mathbb{R}^n} f(\zeta) := \psi(F(\zeta), G(\zeta))$ , for some  $F$  and  $G$ . For detailed reformulation, please refer to [3]. Such a  $\psi$  is usually called a *merit function*. A popular choice of  $\psi$  is

$$(6) \quad \psi(x, y) = \frac{1}{2} \|\phi(x, y)\|^2 \quad x, y \in \mathbb{R}^n,$$

where

$$(7) \quad \phi(x, y) := (x^2 + y^2)^{1/2} - x - y.$$

Here  $(\cdot)^2$  and  $(\cdot)^{1/2}$  are well-defined via the *Jordan product* as will be explained in Sec. 2. The function  $\phi$  is called Fischer-Burmeister function. It is the natural extension of Fischer-Burmeister function over  $\mathbb{R}^n$  to  $\mathcal{K}^n$ . D. Sun and J. Sun proved that  $\phi$  is strongly semismooth in [19], while  $\psi$  was proved smooth (continuously differentiable) everywhere by J.-S. Chen and P. Tseng in [3]. In this paper, we also provide an alternative proof for property of strong semismoothness of  $\phi$  in Sec. 4.

In what follows, for any differentiable (in the Fréchet sense) mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we denote its Jacobian(not transposed) at  $x \in \mathbb{R}^n$  by  $\nabla F(x) \in \mathbb{R}^{m \times n}$ , i.e.,  $(F(x+u) - F(x) - \nabla F(x)u)/\|u\| \rightarrow 0$  as  $u \rightarrow 0$ . “:=” means “define”. We write  $z = O(\alpha)$  (respectively,  $z = o(\alpha)$ ), with  $\alpha \in \mathbb{R}$  and  $z \in \mathbb{R}^n$ , to mean  $\|z\|/|\alpha|$  is uniformly bounded (respectively, tends to zero) as  $\alpha \rightarrow 0$ .

## 2. BASIC CONCEPTS AND KNOWN RESULTS

In this section, we review some basic materials regarding vector-valued functions. These contain continuity, (local) Lipschitz continuity, directional differentiability, differentiability, continuous differentiability, as well as ( $\rho$ -order) semismoothness. We also recall some known results for vector-valued functions for which we will provide alternative proofs later.

Let the mapping  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . Then  $F$  is continuous at  $x \in \mathbb{R}^n$  if  $F(y) \rightarrow F(x)$  as  $y \rightarrow x$ ; and  $F$  is continuous if  $F$  is continuous at every  $x \in \mathbb{R}^n$ . We say  $F$  is strictly continuous (also called ‘locally Lipschitz continuous’) at  $x \in \mathbb{R}^n$  if there exist scalars  $\kappa > 0$  and  $\delta > 0$  such that

$$\|F(y) - F(z)\| \leq \kappa \|y - z\| \quad \forall y, z \in \mathbb{R}^n \text{ with } \|y - x\| \leq \delta, \|z - x\| \leq \delta;$$

and  $F$  is strictly continuous if  $F$  is strictly continuous at every  $x \in \mathbb{R}^n$ . We say  $F$  is directionally differentiable at  $x \in \mathbb{R}^n$  if

$$F'(x; h) := \lim_{t \rightarrow 0^+} \frac{F(x + th) - F(x)}{t} \quad \text{exists} \quad \forall h \in \mathbb{R}^n;$$

and  $F$  is directionally differentiable if  $F$  is directionally differentiable at every  $x \in \mathbb{R}^n$ .  $F$  is differentiable (in the Fréchet sense) at  $x \in \mathbb{R}^n$  if there exists a linear mapping  $\nabla F(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  such that

$$F(x + h) - F(x) - \nabla F(x)h = o(\|h\|).$$

If  $F$  is differentiable at every  $x \in \mathbb{R}^n$  and  $\nabla F$  is continuous, then  $F$  is continuously differentiable. We notice that, in the above expression about strict continuity of  $F$ , if  $\delta$  can be taken to be  $\infty$ , then  $F$  is called Lipschitz continuous with Lipschitz constant  $\kappa$ .

It is well-known that if  $F$  is strictly continuous, then  $F$  is almost everywhere differentiable by Rademacher’s Theorem—see [6] and [17, Sec. 9J]. In this case, the generalized Jacobian  $\partial F(x)$  of  $F$  at  $x$  (in the Clarke sense) can be defined as the convex hull of the generalized Jacobian  $\partial_B F(x)$ , where

$$\partial_B F(x) := \left\{ \lim_{x^j \rightarrow x} \nabla F(x^j) \mid F \text{ is differentiable at } x^j \in \mathbb{R}^n \right\}.$$

The notation  $\partial_B$  is adopted from [15]. In [17, Chap. 9], the case of  $m = 1$  is considered and the notations “ $\bar{\nabla}$ ” and “ $\bar{\partial}$ ” are used instead of, respectively, “ $\partial_B$ ” and “ $\partial$ ”. Assume  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is strictly continuous, then  $F$  is said to be semismooth at  $x$  if  $F$  is directionally differentiable at  $x$  and, for any  $V \in \partial F(x + h)$ , we have

$$F(x + h) - F(x) - Vh = o(\|h\|).$$

Moreover,  $F$  is called  $\rho$ -order semismooth at  $x$  ( $0 < \rho < \infty$ ) if  $F$  is semismooth at  $x$  and, for any  $V \in \partial F(x + h)$ , we have

$$F(x + h) - F(x) - Vh = O(\|h\|^{1+\rho}).$$

The following lemma, proven by D. Sun and J. Sun [20, Thm. 3.6] using the definition of generalized Jacobian, enables one to study the semismooth property of  $f^{\text{soc}}$  by examining only those points  $x \in \mathbb{R}^n$  where  $f^{\text{soc}}$  is differentiable and thus

work only with the Jacobian of  $f^{\text{soc}}$ , rather than the generalized Jacobian. It is a very useful working lemma for verifying semismoothness property.

**Lemma 2.1.** *Suppose  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is strictly continuous and directionally differentiable in a neighborhood of  $x \in \mathbb{R}^n$ . Then, for any  $0 < \rho < \infty$ , the following two statements are equivalent:*

(a) *For any  $v \in \partial F(x + h)$  and  $h \rightarrow 0$ ,*

$$F(x + h) - F(x) - vh = o(\|h\|) \quad (\text{respectively, } O(\|h\|)^{1+\rho}).$$

(b) *For any  $h \rightarrow 0$  such that  $F$  is differentiable at  $x + h$ ,*

$$F(x + h) - F(x) - \nabla F(x + h)h = o(\|h\|) \quad (\text{respectively, } O(\|h\|)^{1+\rho}).$$

We say  $F$  is semismooth (respectively,  $\rho$ -order semismooth) if  $F$  is semismooth (respectively,  $\rho$ -order semismooth) at every  $x \in \mathbb{R}^n$ . We say  $F$  is *strongly semismooth* if it is 1-order semismooth. Convex functions and piecewise continuously differentiable functions are examples of semismooth functions. The composition of two (respectively,  $\rho$ -order) semismooth functions is also a (respectively,  $\rho$ -order) semismooth function. The property of semismoothness, as introduced by Mifflin [13] for functionals and scalar-valued functions and further extended by L. Qi and J. Sun [16] for vector-valued functions, is of particular interest due to the key role it plays in the superlinear convergence analysis of certain generalized Newton methods [10, 11, 15, 16, 23]. For extensive discussions of semismooth functions, see [8, 13, 16].

For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , we define their *Jordan product* as

$$(8) \quad x \circ y = (x^T y, y_1 x_2 + x_1 y_2).$$

We write  $x^2$  to mean  $x \circ x$  and write  $x + y$  to mean the usual componentwise addition of vectors. Then,  $\circ$ ,  $+$ , together with  $e = (1, 0, \dots, 0) \in \mathbb{R}^n$ , give rise to a Jordan algebra associated with  $\mathcal{K}^n$  [7, Chap. II]. If  $x \in \mathcal{K}^n$ , then there exists a unique vector in  $\mathcal{K}^n$ , which we denote by  $x^{1/2}$ , such that  $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$ . For any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , we also define the symmetric matrix

$$(9) \quad L_x = \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix},$$

viewed as a linear mapping from  $\mathbb{R}^n$  to  $\mathbb{R}^{n \times n}$ . The matrix  $L_x$  has various interesting properties that were studied in [9]. Especially, we have  $L_x \cdot y = x \circ y$  for any  $x, y \in \mathbb{R}^n$ .

Now, we summarize the results shown in [4] for which we will provide alternative proofs that are straightforward and intuitive in the subsequent sections.

**Proposition 2.2.** *For any  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the following results hold:*

- (a)  $f^{\text{soc}}$  is continuous at an  $x \in \mathbb{R}^n$  with eigenvalues  $\lambda_1, \lambda_2$  if and only if  $f$  is continuous at  $\lambda_1, \lambda_2$ .
- (b)  $f^{\text{soc}}$  is directionally differentiable at an  $x \in \mathbb{R}^n$  with eigenvalues  $\lambda_1, \lambda_2$  if and only if  $f$  is directionally differentiable at  $\lambda_1, \lambda_2$ .
- (c)  $f^{\text{soc}}$  is differentiable at an  $x \in \mathbb{R}^n$  with eigenvalues  $\lambda_1, \lambda_2$  if and only if  $f$  is differentiable at  $\lambda_1, \lambda_2$ .

- (d)  $f^{\text{soc}}$  is continuously differentiable at an  $x \in \mathbb{R}^n$  with eigenvalues  $\lambda_1, \lambda_2$  if and only if  $f$  is continuously differentiable at  $\lambda_1, \lambda_2$ .
- (e)  $f^{\text{soc}}$  is strictly continuous at an  $x \in \mathbb{R}^n$  with eigenvalues  $\lambda_1, \lambda_2$  if and only if  $f$  is strictly continuous at  $\lambda_1, \lambda_2$ .
- (f)  $f^{\text{soc}}$  is Lipschitz continuous (with respect to  $\|\cdot\|$ ) with constant  $\kappa$  if and only if  $f$  is Lipschitz continuous with constant  $\kappa$ .
- (g)  $f^{\text{soc}}$  is semismooth if and only if  $f$  is semismooth.

**Proposition 2.3.** *The vector-valued Fischer-Burmeister function associated with second-order cone defined as (7) is strongly semismooth.*

### 3. ALTERNATIVE PROOFS OF CONTINUITY AND DIFFERENTIABILITY

In this section, we present alternative proofs for Prop. 2.2 of Sec. 2 which is one of the main purposes of this paper. Unlike the existing proofs which employed an elegant lemma ([4, Lem. 4.1]), our arguments come from an intuitive way only using the simple structure of second-order cone and basic definitions. We need some technical lemmas before starting the alternative proofs.

**Lemma 3.1.** *Let  $\lambda_1 \leq \lambda_2$  be the spectral values of  $x \in \mathbb{R}^n$  and  $m_1 \leq m_2$  be the spectral values of  $y \in \mathbb{R}^n$ . Then we have*

$$(10) \quad |\lambda_1 - m_1|^2 + |\lambda_2 - m_2|^2 \leq 2 \|x - y\|^2,$$

and hence,  $|\lambda_i - m_i| \leq \sqrt{2} \|x - y\|, \forall i = 1, 2$ .

*Proof.* The proof follows from a direct computation. □

**Lemma 3.2.** *Let  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ .*

- (a) *If  $x_2 \neq 0, y_2 \neq 0$ , then we have*

$$(11) \quad \|u^{(i)} - v^{(i)}\| \leq \frac{1}{\|x_2\|} \|x - y\|, \quad \forall i = 1, 2,$$

where  $u^{(i)}, v^{(i)}$  are the unique spectral vectors of  $x$  and  $y$ , respectively.

- (b) *If either  $x_2 = 0$  or  $y_2 = 0$ , then we can choose  $u^{(i)}, v^{(i)}$  such that the left hand side of inequality (11) is zero.*

*Proof.* (a) From the spectral factorization (1)-(3), we know that

$$u^{(i)} = \frac{1}{2} \left( 1, (-1)^i \frac{x_2}{\|x_2\|} \right), \quad v^{(i)} = \frac{1}{2} \left( 1, (-1)^i \frac{y_2}{\|y_2\|} \right),$$

where  $u^{(i)}, v^{(i)}$  are unique. Thus, we have  $u^{(i)} - v^{(i)} = \frac{1}{2} \left( 0, (-1)^i \left( \frac{x_2}{\|x_2\|} - \frac{y_2}{\|y_2\|} \right) \right)$ .

Then

$$\begin{aligned} \|u^{(i)} - v^{(i)}\| &= \frac{1}{2} \left\| \frac{x_2}{\|x_2\|} - \frac{y_2}{\|y_2\|} \right\| = \frac{1}{2} \left\| \frac{x_2 - y_2}{\|x_2\|} + \frac{(\|y_2\| - \|x_2\|)y_2}{\|x_2\| \cdot \|y_2\|} \right\| \\ &\leq \frac{1}{2} \left( \frac{1}{\|x_2\|} \|x_2 - y_2\| + \frac{1}{\|x_2\|} \left| \|y_2\| - \|x_2\| \right| \right) \\ &\leq \frac{1}{2} \left( \frac{1}{\|x_2\|} \|x_2 - y_2\| + \frac{1}{\|x_2\|} \|x_2 - y_2\| \right) \leq \frac{1}{\|x_2\|} \|x - y\|, \end{aligned}$$

where the first inequality follows from the triangle inequality.

(b) We can choose the same spectral vectors for  $x$  and  $y$  by the spectral factorization (1)-(3) since either  $x_2 = 0$  or  $y_2 = 0$ . Then, it is obvious.  $\square$

**Lemma 3.3.** For any  $w \neq 0 \in \mathbb{R}^n$ , we have  $\nabla_w \left( \frac{w}{\|w\|} \right) = \frac{1}{\|w\|} \left( I - \frac{ww^T}{\|w\|^2} \right)$ .

*Proof.* The verification is routine, so we omit it.  $\square$

Now, we are ready to present our alternative proofs for Prop. 2.2. As mentioned, all of our proofs are from intuitive definitions as well as the structure of second-order cone. Some portion of the proofs are similar to the original ones, we omit them when there is the case.

*Proof.* (a) “ $\Leftarrow$ ” Suppose  $f$  is continuous at  $\lambda_1, \lambda_2$ . For any fixed  $x \in \mathbb{R}^n$  and  $y \rightarrow x$ , let the spectral factorizations of  $x, y$  be  $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$  and  $y = m_1 v^{(1)} + m_2 v^{(2)}$ . Then, we discuss two cases.

Case (i): If  $x_2 \neq 0$ , then we have

$$(12) \quad f^{\text{soc}}(y) - f^{\text{soc}}(x) = f(m_1) \left( v^{(1)} - u^{(1)} \right) + \left( f(m_1) - f(\lambda_1) \right) u^{(1)} \\ + f(m_2) \left( v^{(2)} - u^{(2)} \right) + \left( f(m_2) - f(\lambda_2) \right) u^{(2)}.$$

Since  $f$  is continuous at  $\lambda_1, \lambda_2$ , and from Lemma 3.1,  $|m_i - \lambda_i| \leq \sqrt{2} \|y - x\|$ , we obtain  $f(m_i) \rightarrow f(\lambda_i)$  as  $y \rightarrow x$ . Also by Lemma 3.2, we know that  $\|v^{(i)} - u^{(i)}\| \rightarrow 0$  as  $y \rightarrow x$ . Thus, equation (12) yields  $f^{\text{soc}}(y) \rightarrow f^{\text{soc}}(x)$  as  $y \rightarrow x$ , since both  $f(m_i)$  and  $\|u^{(i)}\|$  are bounded. Hence,  $f^{\text{soc}}$  is continuous at  $x \in \mathbb{R}^n$ .

Case (ii): If  $x_2 = 0$ , no matter  $y_2$  is zero or not, we can arrange that  $x, y$  have the same spectral vectors. Thus,  $f^{\text{soc}}(y) - f^{\text{soc}}(x) = \left( f(m_1) - f(\lambda_1) \right) u^{(1)} + \left( f(m_2) - f(\lambda_2) \right) u^{(2)}$ . Then,  $f^{\text{soc}}$  is continuous at  $x \in \mathbb{R}^n$  by similar arguments.

“ $\Rightarrow$ ” The proof for this direction is straightforward and similar to the arguments in [4, Prop. 5.2].  $\square$

*Proof.* (b) “ $\Leftarrow$ ” Suppose  $f$  is directionally differentiable at  $\lambda_1, \lambda_2$ . Fix any  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , then we discuss two cases as below.

Case (i): If  $x_2 \neq 0$ , then we have  $f^{\text{soc}}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}$  where  $\lambda_i = x_1 + (-1)^i \|x_2\|$  and  $u^{(i)} = \frac{1}{2} \left( 1, (-1)^i \frac{x_2}{\|x_2\|} \right)$  for all  $i = 1, 2$ . From Lemma 3.3, we know that  $u^{(i)}$  is Fréchet-differentiable with respect to  $x$ , with

$$(13) \quad \nabla_x u^{(i)} = \frac{(-1)^i}{2\|x_2\|} \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}, \quad \forall i = 1, 2.$$

Also by the expression of  $\lambda_i$ , we know that  $\lambda_i$  is Fréchet-differentiable with respect to  $x$ , with

$$(14) \quad \nabla_x \lambda_i = \left( 1, (-1)^i \frac{x_2}{\|x_2\|} \right) = 2u^{(i)}, \quad \forall i = 1, 2.$$

Since  $f$  is directionally differentiable at  $\lambda_1, \lambda_2$ , then the chain rule and product rule for directional differentiation give

$$\begin{aligned} (f^{\text{soc}})'(x; h) &= f(\lambda_1) \nabla_x u^{(1)} h + u^{(1)} f'(\lambda_1; h) (\nabla_x \lambda_1)^T \\ &\quad + f(\lambda_2) \nabla_x u^{(2)} h + u^{(2)} f'(\lambda_2; h) (\nabla_x \lambda_2)^T \\ &= \frac{f(\lambda_2) - f(\lambda_1)}{2\|x_2\|} \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} h \\ &\quad + 2f'(\lambda_1; h) u^{(1)} (u^{(1)})^T + 2f'(\lambda_2; h) u^{(2)} (u^{(2)})^T \\ &= \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} h \\ &\quad + 2f'(\lambda_1; h) u^{(1)} (u^{(1)})^T + 2f'(\lambda_2; h) u^{(2)} (u^{(2)})^T, \end{aligned}$$

where the second equality uses equations (13) and (14), and the last equality employs the fact that  $\lambda_2 - \lambda_1 = 2\|x_2\|$ . Notice that we can obtain  $u^{(i)}(u^{(i)})^T$  as follows by direct computation :

$$u^{(i)}(u^{(i)})^T = \frac{1}{4} \begin{bmatrix} 1 & (-1)^i \frac{x_2^T}{\|x_2\|} \\ (-1)^i \frac{x_2}{\|x_2\|} & \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}, \quad \forall i = 1, 2.$$

Now let

$$(15) \quad \tilde{a} = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} h, \quad \tilde{b} = \frac{f'(\lambda_2; h) + f'(\lambda_1; h)}{2}, \quad \tilde{c} = \frac{f'(\lambda_2; h) - f'(\lambda_1; h)}{2}.$$

Then, we can rewrite the previous expression of  $(f^{\text{soc}})'(x; h)$  as

$$\begin{aligned} (16) \quad (f^{\text{soc}})'(x; h) &= \tilde{a} \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} + \begin{bmatrix} \tilde{b} & \frac{\tilde{c} x_2^T}{\|x_2\|} \\ \frac{\tilde{c} x_2}{\|x_2\|} & \frac{b x_2 x_2^T}{\|x_2\|^2} \end{bmatrix} \\ &= \begin{bmatrix} \tilde{b} & \frac{\tilde{c} x_2^T}{\|x_2\|} \\ \frac{\tilde{c} x_2}{\|x_2\|} & \tilde{a} I + (\tilde{b} - \tilde{a}) \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix}. \end{aligned}$$

This enables that  $f^{\text{soc}}$  is directionally differentiable at  $x$  when  $x_2 \neq 0$  with  $(f^{\text{soc}})'(x; h)$  being in form of (16).

Case (ii): If  $x_2 = 0$ , we compute the directional derivative  $(f^{\text{soc}})'(x; h)$  at  $x$  for any direction  $h$  by definition. Let  $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . We have two subcases.

First, consider the subcase of  $h_2 \neq 0$ . From the spectral factorization, we can choose  $u^{(i)} = \frac{1}{2} \left( 1, (-1)^i \frac{h_2}{\|h_2\|} \right)$  for all  $i = 1, 2$ , such that

$$\begin{cases} f^{\text{soc}}(x + th) &= f(\lambda + \Delta\lambda_1)u^{(1)} + f(\lambda + \Delta\lambda_2)u^{(2)} \\ f^{\text{soc}}(x) &= f(\lambda)u^{(1)} + f(\lambda)u^{(2)}, \end{cases}$$

where  $\lambda = x_1$  and  $\Delta\lambda_i = t \left( h_1 + (-1)^i \|h_2\| \right)$  for all  $i = 1, 2$ . Thus, we obtain

$$f^{\text{soc}}(x + th) - f^{\text{soc}}(x) = \left( f(\lambda + \Delta\lambda_1) - f(\lambda) \right) u^{(1)} + \left( f(\lambda + \Delta\lambda_2) - f(\lambda) \right) u^{(2)}.$$

The fact that

$$\lim_{t \rightarrow 0^+} \frac{f(\lambda + \Delta\lambda_1) - f(\lambda)}{t} = \lim_{t \rightarrow 0^+} \frac{f(\lambda + t(h_1 - \|h_2\|)) - f(\lambda)}{t} = f'(\lambda; h_1 - \|h_2\|),$$

and

$$\lim_{t \rightarrow 0^+} \frac{f(\lambda + \Delta\lambda_2) - f(\lambda)}{t} = \lim_{t \rightarrow 0^+} \frac{f(\lambda + t(h_1 + \|h_2\|)) - f(\lambda)}{t} = f'(\lambda; h_1 + \|h_2\|),$$

yields

$$\begin{aligned} (17) \quad & \lim_{t \rightarrow 0^+} \frac{f^{\text{soc}}(x + th) - f^{\text{soc}}(x)}{t} \\ &= \lim_{t \rightarrow 0^+} \frac{f(\lambda + \Delta\lambda_1) - f(\lambda)}{t} u^{(1)} + \lim_{t \rightarrow 0^+} \frac{f(\lambda + \Delta\lambda_2) - f(\lambda)}{t} u^{(2)} \\ &= f'(\lambda; h_1 - \|h_2\|) u^{(1)} + f'(\lambda; h_1 + \|h_2\|) u^{(2)}, \end{aligned}$$

where  $u^{(1)} = \frac{1}{2} \left( 1, -\frac{h_2}{\|h_2\|} \right)$ ,  $u^{(2)} = \frac{1}{2} \left( 1, \frac{h_2}{\|h_2\|} \right)$ . Hence,  $(f^{\text{soc}})'(x; h)$  exists with form of (17).

Secondly, for the subcase of  $h_2 = 0$ , the same argument applies except  $h_2/\|h_2\|$  is replaced by any  $w \in \mathbb{R}^{n-1}$  with  $\|w\| = 1$ , i.e., choosing  $u^{(i)} = \frac{1}{2}(1, (-1)^i w)$ , for all  $i = 1, 2$ . Analogously,

$$(18) \quad \lim_{t \rightarrow 0^+} \frac{f^{\text{soc}}(x + th) - f^{\text{soc}}(x)}{t} = f'(\lambda; h_1) u^{(1)} + f'(\lambda; h_1) u^{(2)}.$$

Hence,  $(f^{\text{soc}})'(x; h)$  exists with form of (18). From the above, it shows that  $f^{\text{soc}}$  is directionally differentiable at  $x$  when  $x_2 = 0$  and its directional derivative  $(f^{\text{soc}})'(x; h)$  is either in form of (17) or (18).

" $\Rightarrow$ " Suppose  $f^{\text{soc}}$  is directionally differentiable at  $x \in \mathbb{R}^n$  with spectral values  $\lambda_1, \lambda_2$ , we will prove that  $f$  is directionally differentiable at  $\lambda_1, \lambda_2$ . For  $\lambda_1 \in \mathbb{R}$  and any direction  $d_1 \in \mathbb{R}$ , let  $h := d_1 u^{(1)} + 0u^{(2)}$  where  $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$ . Then,  $x + th = (\lambda_1 + td_1)u^{(1)} + \lambda_2 u^{(2)}$ , and

$$\frac{f^{\text{soc}}(x + th) - f^{\text{soc}}(x)}{t} = \frac{f(\lambda_1 + td_1) - f(\lambda_1)}{t} u^{(1)}.$$



Since  $f^{\text{soc}}$  is directionally differentiable at  $x$ , the above equation yields that

$$f'(\lambda_1; d_1) = \lim_{t \rightarrow 0^+} \frac{f(\lambda_1 + td_1) - f(\lambda_1)}{t} \text{ exists.}$$

This means  $f$  is directionally differentiable at  $\lambda_1$ . Similarly,  $f$  is also directionally differentiable at  $\lambda_2$ .  $\square$

*Proof.* (c) “ $\Leftarrow$ ” The proof of this direction is identical to the proof shown as in (b), but with “directionally differentiable” replaced by “differentiable”. We omit the proof and only present the formula of  $f^{\text{soc}}(x)$  as below. For  $x_2 \neq 0$ , we have

$$(19) \quad \nabla f^{\text{soc}}(x) = \begin{bmatrix} b & \frac{cx_2^T}{\|x_2\|} \\ \frac{cx_2}{\|x_2\|} & aI + (b-a) \frac{x_2 x_2^T}{\|x_2\|^2} \end{bmatrix},$$

where

$$(20) \quad a = \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1}, \quad b = \frac{f'(\lambda_2) + f'(\lambda_1)}{2}, \quad c = \frac{f'(\lambda_2) - f'(\lambda_1)}{2}.$$

If  $x_2 = 0$ , then

$$(21) \quad \nabla f^{\text{soc}}(x) = f'(\lambda)I.$$

“ $\Rightarrow$ ” Let  $f^{\text{soc}}$  be Fréchet-differentiable at  $x \in \mathbb{R}^n$  with spectral eigenvalues  $\lambda_1, \lambda_2$ , we will show that  $f$  is Fréchet-differentiable at  $\lambda_1, \lambda_2$ . Suppose not, then  $f$  is not Fréchet-differentiable at  $\lambda_i$  for some  $i \in \{1, 2\}$ . Thus, either  $f$  is not directionally differentiable at  $\lambda_i$  or, if it is, the right- and left-directional derivatives of  $f$  at  $\lambda_i$  are unequal. In either case, this implies that there exist two sequences of non-zero scalars  $t^\nu$  and  $\tau^\nu$ ,  $\nu = 1, 2, \dots$ , converging to zero, such that the limits

$$\lim_{\nu \rightarrow \infty} \frac{f(\lambda_i + t^\nu) - f(\lambda_i)}{t^\nu}, \quad \lim_{\nu \rightarrow \infty} \frac{f(\lambda_i + \tau^\nu) - f(\lambda_i)}{\tau^\nu}$$

exist (possible  $\infty$  or  $-\infty$ ) and either are unequal or both equal to  $\infty$  or are both equal to  $-\infty$ . Now for any  $x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}$ , let  $h := 1 \cdot u^{(1)} + 0 \cdot u^{(2)} = u^{(1)}$ . Then,  $x + th = (\lambda_1 + t)u^{(1)} + \lambda_2 u^{(2)}$  and  $f^{\text{soc}}(x + th) = f(\lambda_1 + t)u^{(1)} + f(\lambda_2)u^{(2)}$ . Thus,

$$\begin{aligned} \lim_{\nu \rightarrow \infty} \frac{f^{\text{soc}}(x + t^\nu h) - f^{\text{soc}}(x)}{t^\nu} &= \lim_{\nu \rightarrow \infty} \frac{f^{\text{soc}}(\lambda_1 + t^\nu) - f^{\text{soc}}(\lambda_1)}{t^\nu} u^{(1)} \\ \lim_{\nu \rightarrow \infty} \frac{f^{\text{soc}}(x + \tau^\nu h) - f^{\text{soc}}(x)}{\tau^\nu} &= \lim_{\nu \rightarrow \infty} \frac{f^{\text{soc}}(\lambda_1 + \tau^\nu) - f^{\text{soc}}(\lambda_1)}{\tau^\nu} u^{(1)}. \end{aligned}$$

It follows that these two limits either are unequal or are both non-finite. This implies that  $f^{\text{soc}}$  is not Fréchet-differentiable at  $x$  where is a contradiction.  $\square$

*Proof.* (d) “ $\Leftarrow$ ” Suppose  $f$  is continuously differentiable. From equation (19), it can be seen that  $\nabla f^{\text{soc}}$  is continuous at every  $x$  with  $x_2 \neq 0$ . It remains to show that  $\nabla f^{\text{soc}}$  is continuous at every  $x$  with  $x_2 = 0$ . Fix any  $x = (x_1, 0) \in \mathbb{R}^n$ , so  $\lambda_1 = \lambda_2 = x_1$ . Then, from equation (20), we have

$$\lim_{y \rightarrow x} a = \lim_{y \rightarrow x} \frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} = f'(x_1)$$

$$\begin{aligned} \lim_{y \rightarrow x} b &= \lim_{y \rightarrow x} \frac{1}{2}(f'(\lambda_2) + f'(\lambda_1)) = f'(x_1) \\ \lim_{y \rightarrow x} c &= \lim_{y \rightarrow x} \frac{1}{2}(f'(\lambda_2) - f'(\lambda_1)) = 0. \end{aligned}$$

Taking the limit in equation (19) as  $y \rightarrow x$  yields  $\lim_{y \rightarrow x} \nabla f^{\text{soc}}(y) = f'(x_1)I = \nabla f^{\text{soc}}(x)$ , which says  $\nabla f^{\text{soc}}$  is continuous at every  $x \in \mathbb{R}^n$ .

“ $\Rightarrow$ ” The proof for this direction is similar to the original proof of [4, Prop. 5.4], so we omit it. □

*Proof. (e) and (f)* The original proofs in [4] use the working Lemma 2.1 directly which is the same idea as the one used for the whole paper, so the proofs for part(e) and (f) are identical to theirs. We therefore omit them. □

*Proof. (g)* “ $\Rightarrow$ ” Suppose  $f^{\text{soc}}$  is semismooth, then  $f^{\text{soc}}$  is strictly continuous and directionally differentiable. By part (b) and (e),  $f$  is strictly continuous and directionally differentiable. Now, for any  $\alpha \in \mathbb{R}$  and any  $\eta \in \mathbb{R}$  such that  $f$  is differentiable at  $\alpha + \eta$ , part (c) yields that  $f^{\text{soc}}$  is differentiable at  $x + h$ , where  $x := (\alpha, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $h := (\eta, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . Hence, we can choose the same spectral vectors for  $x + h$  and  $x$  such that

$$\begin{cases} f^{\text{soc}}(x + h) = f(\alpha + \eta)u^{(1)} + f(\alpha + \eta)u^{(2)}, \\ f^{\text{soc}}(x) = f(\alpha)u^{(1)} + f(\alpha)u^{(2)}. \end{cases}$$

Since  $f^{\text{soc}}$  is semismooth, by Lemma 2.1, we have

$$(22) \quad f^{\text{soc}}(x + h) - f^{\text{soc}}(x) - \nabla f^{\text{soc}}(x + h)h = o(\|h\|).$$

On the other hand, (21) says  $\nabla f^{\text{soc}}(x + h)h = f'(\alpha + \eta)Ih = (f'(\alpha + \eta)\eta, 0)$ . Plugging this into equation (22), it yields that  $f(\alpha + \eta) - f(\alpha) - f'(\alpha + \eta)\eta = o(|\eta|)$ . Thus, by Lemma 2.1 again, it says that  $f$  is semismooth at  $\alpha$ . Since  $\alpha$  is arbitrary,  $f$  is semismooth.

“ $\Leftarrow$ ” Suppose  $f$  is semismooth, then  $f$  is strictly continuous and directionally differentiable. By part (b) and (c),  $f^{\text{soc}}$  is strictly continuous and directionally differentiable. For any  $x \in \mathbb{R}^n$  and  $h \in \mathbb{R}^n$  such that  $f^{\text{soc}}$  is differentiable at  $x + h$ , we will verify that

$$f^{\text{soc}}(x + h) - f^{\text{soc}}(x) - \nabla f^{\text{soc}}(x + h)h = o(\|h\|).$$

Case (i): If  $x_2 \neq 0$ , let  $\lambda_i$  be the spectral eigenvalues of  $x$  and  $u^{(i)}$  be the associated vectors. We denote  $x + h$  by  $z$  for convenience, i.e.,  $z := x + h$  and let  $m_i$  be the spectral values of  $z$  with the associated vectors  $v^{(i)}$ . Hence, we have

$$\begin{cases} f^{\text{soc}}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}, \\ f^{\text{soc}}(x + h) = f(m_1)v^{(1)} + f(m_2)v^{(2)}. \end{cases}$$

In addition, from (19), we know

$$\nabla f^{\text{soc}}(x+h) = \begin{bmatrix} \hat{b} & \frac{\hat{c}z_2^T}{\|z_2\|} \\ \frac{\hat{c}z_2}{\|z_2\|} & \hat{a}I - (\hat{b} - \hat{a})\frac{z_2z_2^T}{\|z_2\|^2} \end{bmatrix},$$

where

$$\hat{a} = \frac{f(m_2) - f(m_1)}{m_2 - m_1}, \quad \hat{b} = \frac{f'(m_2) + f'(m_1)}{2}, \quad \hat{c} = \frac{f'(m_2) - f'(m_1)}{2}.$$

With this, we can write out  $f^{\text{soc}}(x+h) - f^{\text{soc}}(x) - \nabla f^{\text{soc}}(x+h)h := (\Xi_1, \Xi_2)$  where  $\Xi_1 \in \mathbb{R}$  and  $\Xi_2 \in \mathbb{R}^{n-1}$ . Since the expansion is very long, for simplicity, we denote  $\Xi_1$  be the first component and  $\Xi_2$  be the second component of the expansion. We will show that  $\Xi_1$  and  $\Xi_2$  are both  $o(\|h\|)$ .

First, we compute the first component  $\Xi_1$ :

$$\begin{aligned} \Xi_1 &= \frac{1}{2} \left\{ f(m_1) - f(\lambda_1) - f'(m_1) \left( h_1 - \frac{z_2^T h_2}{\|z_2\|} \right) \right\} \\ &\quad + \frac{1}{2} \left\{ f(m_2) - f(\lambda_2) - f'(m_2) \left( h_1 + \frac{z_2^T h_2}{\|z_2\|} \right) \right\} \\ &= \frac{1}{2} \left\{ f(m_1) - f(\lambda_1) - f'(m_1) \left( h_1 - (\|z_2\| - \|x_2\|) \right) + o(\|h\|) \right\} \\ &\quad + \frac{1}{2} \left\{ f(m_2) - f(\lambda_2) - f'(m_2) \left( h_1 + (\|z_2\| - \|x_2\|) \right) + o(\|h\|) \right\} \\ &= o\left( h_1 - (\|z_2\| - \|x_2\|) \right) + o(\|h\|) + o\left( h_1 + (\|z_2\| - \|x_2\|) \right) + o(\|h\|). \end{aligned}$$

In the above expression of  $\Xi_1$ , the third equality holds since the following:

$$\begin{aligned} \frac{z_2^T h_2}{\|z_2\|} &= \frac{z_2^T (z_2 - x_2)}{\|z_2\|} = \|z_2\| - \frac{\|z_2\|\|x_2\|}{\|z_2\|} \cos \theta \\ &= \|z_2\| - \|x_2\| \left( 1 + O(\theta^2) \right) = \|z_2\| - \|x_2\| \left( 1 + O(\|h\|^2) \right) \\ &= \|z_2\| - \|x_2\| \left( 1 + o(\|h\|) \right), \end{aligned}$$

where  $\theta$  is the angle between  $x_2$  and  $z_2$  and note that  $z_2 - x_2 = h_2$  gives  $O(\theta^2) = O(\|h\|^2)$ . Also the last equality in expression of  $\Xi_1$  holds since  $f$  is semismooth and

$$m_i - \lambda_i = h_1 + (-1)^i (\|z_2\| - \|x_2\|).$$

On the other hand, due to  $h_1 + (-1)^i (\|z_2\| - \|x_2\|) \leq h_1 + \|z_2 - x_2\| = h_1 + \|h_2\|$ , we observe that when  $\|h\| \rightarrow 0$  then  $h_1 + (-1)^i (\|z_2\| - \|x_2\|) \rightarrow 0$  and  $h_1 + (-1)^i (\|z_2\| - \|x_2\|) = O(\|h\|)$ . Thus,  $o\left( h_1 + (-1)^i (\|z_2\| - \|x_2\|) \right) = o(\|h\|)$ , which yields the first component  $\Xi_1$  is  $o(\|h\|)$ .

Now consider the second component  $\Xi_2$ :

$$\begin{aligned}
\Xi_2 &= -\frac{1}{2}f(m_1)\frac{z_2}{\|z_2\|} + \frac{1}{2}f(m_2)\frac{z_2}{\|z_2\|} + \frac{1}{2}f(\lambda_1)\frac{x_2}{\|x_2\|} - \frac{1}{2}f(\lambda_2)\frac{x_2}{\|x_2\|} \\
&\quad - \frac{1}{2}\left(f'(m_2) - f'(m_1)\right)\frac{z_2 h_1}{\|z_2\|} - \frac{f(m_2) - f(m_1)}{m_2 - m_1}h_2 \\
&\quad - \left\{\frac{1}{2}\left(f'(m_2) - f'(m_1)\right) - \frac{f(m_2) - f(m_1)}{m_2 - m_1}\right\}\frac{z_2 z_2^T h_2}{\|z_2\|^2} \\
&= -\frac{1}{2}\left\{f(m_1)\frac{z_2}{\|z_2\|} - f(\lambda_1)\frac{x_2}{\|x_2\|} - f'(m_1)\frac{z_2 h_1}{\|z_2\|} - \frac{2f(m_1)}{m_2 - m_1}h_2\right. \\
&\quad \left.+ \left(f'(m_1) + \frac{2f(m_1)}{m_2 - m_1}\right)\frac{z_2 z_2^T h_2}{\|z_2\|^2}\right\} \\
&\quad + \frac{1}{2}\left\{f(m_2)\frac{z_2}{\|z_2\|} - f(\lambda_2)\frac{x_2}{\|x_2\|} - f'(m_2)\frac{z_2 h_1}{\|z_2\|} - \frac{2f(m_2)}{m_2 - m_1}h_2\right. \\
&\quad \left.- \left(f'(m_2) + \frac{2f(m_2)}{m_2 - m_1}\right)\frac{z_2 z_2^T h_2}{\|z_2\|^2}\right\} \\
&:= \Xi_2^{(1)} + \Xi_2^{(2)},
\end{aligned}$$

where  $\Xi_2^{(1)}$  denotes the first half part while  $\Xi_2^{(2)}$  denotes the second half part. We will show that both  $\Xi_2^{(1)}$  and  $\Xi_2^{(2)}$  are  $o(\|h\|)$ . For symmetry, it is enough to show that  $\Xi_2^{(1)}$  is  $o(\|h\|)$ . From the observations that

$$\begin{cases} z_2 &= x_2 + h_2, \\ m_2 - m_1 &= 2\|z_2\|, \\ m_i - \lambda_i &= h_1 + (-1)^i(\|z_2\| - \|x_2\|) = O(\|h\|), \end{cases}$$

we have the following:

$$\begin{aligned}
\Xi_2^{(1)} &= -\frac{1}{2}\left\{f(m_1)\frac{z_2}{\|z_2\|} - f(\lambda_1)\frac{x_2}{\|x_2\|} - f'(m_1)\frac{z_2 h_1}{\|z_2\|} - \frac{2f(m_1)}{m_2 - m_1}h_2\right. \\
&\quad \left.+ \left(f'(m_1) + \frac{2f(m_1)}{m_2 - m_1}\right)\frac{z_2 z_2^T h_2}{\|z_2\|^2}\right\} \\
&= -\frac{1}{2}\frac{z_2}{\|z_2\|}\left\{f(m_1) - f(\lambda_1) - f'(m_1)\left(h_1 - \frac{z_2^T h_2}{\|z_2\|}\right)\right\} \\
&\quad + \frac{1}{2}\left(f(m_1) - f(\lambda_1)\right)\left(\frac{-h_2}{\|z_2\|} + \frac{z_2 z_2^T h_2}{\|z_2\|^3}\right) \\
&\quad - \frac{1}{2}f(\lambda_1)\left(\frac{z_2}{\|z_2\|} - \frac{x_2}{\|x_2\|} - \frac{h_2}{\|z_2\|} + \frac{z_2 z_2^T h_2}{\|z_2\|^3}\right).
\end{aligned}$$

Following the same arguments as in the first component  $\Xi_1$ , it can be seen that

$$f(m_1) - f(\lambda_1) - f'(m_1)\left(h_1 - \frac{z_2^T h_2}{\|z_2\|}\right) = o(\|h\|).$$

Since  $m_1 - \lambda_1 = h_1 - (\|z_2\| - \|x_2\|) = O(\|h\|)$  and  $f$  is strictly continuous, then  $f(m_1) - f(\lambda_1) = O(\|h\|)$ . In addition,  $-h_2/\|z_2\| + z_2 z_2^T h_2/\|z_2\|^3 = O(\|h\|)$ . Hence,

$$\left(f(m_1) - f(\lambda_1)\right) \left(\frac{-h_2}{\|z_2\|} + \frac{z_2 z_2^T h_2}{\|z_2\|^3}\right) = O(\|h\|^2) = o(\|h\|).$$

Therefore, it remains to show that the last part of  $\Xi_2^{(1)}$  is  $o(\|h\|)$ . Now, note that

$$\frac{z_2}{\|z_2\|} - \frac{x_2}{\|x_2\|} - \frac{h_2}{\|z_2\|} + \frac{z_2 z_2^T h_2}{\|z_2\|^3} = x_2 \left( \frac{1}{\|z_2\|} - \frac{1}{\|x_2\|} + \frac{z_2^T h_2}{\|z_2\|^3} \right) + O(\|h\|^2).$$

Let  $\theta(z_2) := -1/\|z_2\|$ , then  $\nabla\theta(z_2) = -\frac{-1}{\|z_2\|^2} \frac{z_2}{\|z_2\|} = \frac{z_2}{\|z_2\|^3}$ . This implies that

$$\frac{1}{\|z_2\|} - \frac{1}{\|x_2\|} + \frac{z_2^T h_2}{\|z_2\|^3} = \theta(x_2) - \theta(z_2) - \nabla\theta(z_2)(x_2 - z_2) = O(\|h\|^2),$$

where the last equality is from first Taylor approximation. Thus, we obtain

$$f(\lambda_1) \left( \frac{z_2}{\|z_2\|} - \frac{x_2}{\|x_2\|} - \frac{h_2}{\|z_2\|} + \frac{z_2 z_2^T h_2}{\|z_2\|^3} \right) = o(\|h\|).$$

From all the above, we therefore verified that (22) is satisfied which says  $f^{\text{soc}}$  is semismooth under the case (i).

Case (ii): If  $x_2 = 0$ , we need to discuss two subcases. First subcase, consider  $h_2 \neq 0$ .

Then  $x = (x_1, 0)$  and  $x + h = (x_1 + h_1, h_2)$ . We can choose  $u^{(i)} = \frac{1}{2} \left( 1, (-1)^i \frac{h_2}{\|h_2\|} \right)$  such that  $x = \lambda u^{(1)} + \lambda u^{(2)}$  and  $x + h = m_1 u^{(1)} + m_2 u^{(2)}$  where  $\lambda = x_1$  and  $m_i = x_1 + h_1 + (-1)^i \|h_2\|$ ,  $i = 1, 2$ . Hence,

$$\begin{cases} f^{\text{soc}}(x) & = f(x_1)u^{(1)} + f(x_1)u^{(2)}, \\ f^{\text{soc}}(x + h) & = f(m_1)u^{(1)} + f(m_2)u^{(2)}. \end{cases}$$

Also from part (c), we know

$$\begin{aligned} \nabla f^{\text{soc}}(x + h) &= \frac{f(m_2) - f(m_1)}{m_2 - m_1} \begin{bmatrix} 0 & 0 \\ 0 & I - \frac{h_2 h_2^T}{\|h_2\|^2} \end{bmatrix} \\ &\quad + 2f'(m_1)u^{(1)}(u^{(1)})^T + 2f'(m_2)u^{(2)}(u^{(2)})^T, \end{aligned}$$

where

$$u^{(i)}(u^{(i)})^T = \frac{1}{4} \begin{bmatrix} 1 & (-1)^i \frac{h_2^T}{\|h_2\|} \\ (-1)^i \frac{h_2}{\|h_2\|} & \frac{h_2 h_2^T}{\|h_2\|^2} \end{bmatrix}, \quad \forall i = 1, 2.$$

Therefore, by direct computations, we have

$$\begin{aligned} \nabla f^{\text{soc}}(x + h)h &= \left( \frac{1}{2}f'(m_1)(h_1 - \|h_2\|), \left(1 - \frac{h_1}{\|h_2\|}\right)h_2 \right) \\ &\quad + \left( \frac{1}{2}f'(m_2)(h_1 + \|h_2\|), \left(1 + \frac{h_1}{\|h_2\|}\right)h_2 \right). \end{aligned}$$

Combining all of these, we obtain that

$$\begin{aligned}
& f^{\text{soc}}(x+h) - f^{\text{soc}}(x) - \nabla f^{\text{soc}}(x+h)h \\
&= \left\{ f(m_1)u^{(1)} + f(m_2)u^{(2)} \right\} - \left\{ f(x_1)u^{(1)} + f(x_1)u^{(2)} \right\} \\
&\quad - \left\{ \left( \frac{1}{2}f'(m_1)(h_1 - \|h_2\|), \left(1 - \frac{h_1}{\|h_2\|}\right)h_2 \right) \right. \\
&\quad \left. + \left( \frac{1}{2}f'(m_2)(h_1 + \|h_2\|), \left(1 + \frac{h_1}{\|h_2\|}\right)h_2 \right) \right\} \\
&= \left( \frac{1}{2} \{ f(m_1) - f(x_1) - f'(m_1)(h_1 - \|h_2\|) \} \right. \\
&\quad \left. + \frac{1}{2} \{ f(m_2) - f(x_1) - f'(m_2)(h_1 + \|h_2\|) \} \right), \\
&\quad \frac{1}{2} \left\{ f(m_1) - f(x_1) - f'(m_1)(h_1 - \|h_2\|) \right\} \left( -\frac{h_2}{\|h_2\|} \right) \\
&\quad + \frac{1}{2} \left\{ f(m_2) - f(x_1) - f'(m_2)(h_1 + \|h_2\|) \right\} \left( \frac{h_2}{\|h_2\|} \right) \\
&= o(h_1 - \|h_2\|)u^{(1)} + o(h_1 + \|h_2\|)u^{(2)}.
\end{aligned}$$

The third equality holds since  $f$  is semismooth and Lemma 2.1. When  $h$  goes to zero, both  $h_1 - \|h_2\|$  and  $h_1 + \|h_2\|$  go to zero, so the above expression yields that (22) is satisfied. Hence,  $f^{\text{soc}}$  is semismooth under this subcase.

Secondly, for the subcase of  $h_2 = 0$ , then  $x = (x_1, 0)$  and  $x+h = (x_1+h_1, 0)$ . We can choose  $u^{(i)} = \frac{1}{2} \left( 1, (-1)^i w \right)$  with  $\|w\| = 1$  such that  $x = \lambda u^{(1)} + \lambda u^{(2)}$  and  $x+h = m u^{(1)} + m u^{(2)}$ , where  $\lambda = x_1$  and  $m = x_1 + h_1$ . Hence,

$$\begin{cases} f^{\text{soc}}(x) &= f(x_1)u^{(1)} + f(x_1)u^{(2)} \\ f^{\text{soc}}(x+h) &= f(x_1+h_1)u^{(1)} + f(x_1+h_1)u^{(2)}. \end{cases}$$

Also (21) says  $\nabla f^{\text{soc}}(x+h) = f'(x_1+h_1)I$ . Therefore,  $\nabla f^{\text{soc}}(x+h)h = (f'(x_1+h_1)h_1, 0)$ . Combining all of these, we obtain that

$$\begin{aligned}
& f^{\text{soc}}(x+h) - f^{\text{soc}}(x) - \nabla f^{\text{soc}}(x+h)h \\
&= \left\{ f(x_1+h_1)u^{(1)} + f(x_1+h_1)u^{(2)} \right\} \\
&\quad - \left\{ f(x_1)u^{(1)} + f(x_1)u^{(2)} \right\} - \left( f'(x_1+h_1)h_1, 0 \right) \\
&= \left( f(x_1+h_1) - f(x_1) - f'(x_1+h_1)h_1, 0 \right) \\
&= \left( o(|h_1|), 0 \right),
\end{aligned}$$

where the third equality holds since  $f$  is semismooth and Lemma 2.1. When  $h$  goes to zero, it implies  $h_1$  goes to zero, so the above expression implies that (22) is satisfied which says  $f^{\text{soc}}$  is semismooth in this subcase.

From all the above, we proved that if  $f$  is semismooth then  $f^{\text{soc}}$  is semismooth.  $\square$

#### 4. SEMISMOOTHNESS PROPERTY OF $\phi$

It was shown by L. Qi and J. Sun in [16] that for NCP case a key to superlinear convergence is a certain semismoothness property of  $\phi$ . Indeed, the property of semismoothness of  $\phi$  is of some interest since it leads to investigation of nonsmooth methods. In this section, we will also give an alternative proof for this property. The idea is straightforward though it involves more algebraic computations. This kind of nonlinear analysis would help understanding and analyzing other merit functions. Let  $\rho : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be defined by

$$(23) \quad \rho(x, y) := (x^2 + y^2)^{1/2}.$$

To prove  $\phi$  is strongly semismooth, it is enough to show that  $\rho$  is strongly semismooth. In other words, we will show that  $\rho$  is strictly continuous, directionally differentiable and satisfies conditions of Lemma 2.1, hence it is strongly semismooth. Before the long proof, we need some technical lemmas that will be used very often for the analysis of semismoothness of vector-valued Fischer-Burmeister function associated with second-order cone.

**Lemma 4.1.** [3, Lem. 3.2] *For any  $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  with  $x^2 + y^2$  on the boundary of  $\mathcal{K}^n$ , we have  $x_1^2 = \|x_2\|^2, y_1^2 = \|y_2\|^2, x_1 y_1 = x_2^T y_2, x_1 y_2 = y_1 x_2$ .*

**Lemma 4.2.** [3, Lem. 3.3] *For any  $x = (x_1, x_2)$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  with  $x_1 x_2 + y_1 y_2 \neq 0$ , we have*

$$\begin{aligned} \left( x_1 - \frac{(x_1 x_2 + y_1 y_2)^T x_2}{\|x_1 x_2 + y_1 y_2\|} \right)^2 &\leq \left\| x_2 - x_1 \frac{x_1 x_2 + y_1 y_2}{\|x_1 x_2 + y_1 y_2\|} \right\|^2 \\ &\leq \|x\|^2 + \|y\|^2 - 2\|x_1 x_2 + y_1 y_2\|. \end{aligned}$$

**Lemma 4.3.** *If  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  with  $x^2 + y^2$  on the boundary of  $\mathcal{K}^n$ , then for any  $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $k = (k_1, k_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  we have*

$$\begin{aligned} (a) \quad \langle x_1 x_2 + y_1 y_2, x_1 h_2 + h_1 x_2 + y_1 k_2 + k_1 y_2 \rangle &= (x_1^2 + y_1^2) \left( \langle x, h \rangle + \langle y, k \rangle \right), \\ (b) \quad \langle x \circ h + y \circ k, x^2 + y^2 \rangle &= 4(x_1^2 + y_1^2) \left( \langle x, h \rangle + \langle y, k \rangle \right). \end{aligned}$$

*Proof.* (a) By direct computation and applying Lemma 4.1 in the first equality, we obtain

$$\begin{aligned} &\langle x_1 x_2 + y_1 y_2, x_1 h_2 + h_1 x_2 + y_1 k_2 + k_1 y_2 \rangle \\ &= x_1^2 \langle x_2, h_2 \rangle + y_1^2 \langle x_2, h_2 \rangle + x_1^3 h_1 + y_1^2 x_1 h_1 \\ &\quad + x_1^2 y_1 k_1 + y_1^3 k_1 + x_1^2 \langle y_2, k_2 \rangle + y_1^2 \langle y_2, k_2 \rangle \\ &= (x_1^2 + y_1^2) \left( \langle x, h \rangle + \langle y, k \rangle \right). \end{aligned}$$

(b) This is a consequence of part (a), since

$$\begin{aligned} & \langle x \circ h + y \circ k, x^2 + y^2 \rangle \\ &= 2(x_1^2 + y_1^2) \left( \langle x, h \rangle + \langle y, k \rangle \right) + 2\langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle \\ &= 4(x_1^2 + y_1^2) \left( \langle x, h \rangle + \langle y, k \rangle \right), \end{aligned}$$

where the first equality holds because  $x^2 + y^2 = (2x_1^2 + 2y_1^2, 2x_1x_2 + 2y_1y_2)$ .  $\square$

**Lemma 4.4.** For  $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  with  $\|x\|^2 + \|y\|^2 = 2\|x_1x_2 + y_1y_2\| > 0$ , let  $u := x^2 + y^2$  and  $z := (x+h)^2 + (y+k)^2$  where  $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $k = (k_1, k_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ . If we denote  $m_1, m_2$  as the spectral values of  $z$  while  $\lambda_1, \lambda_2$  the spectral values of  $u$ , then we have

$$m_1 - \lambda_1 = O(\|(h, k)\|^2), \quad m_2 - \lambda_2 = O(\|(h, k)\|).$$

*Proof.* Since  $m_1, m_2$  are the spectral values of  $z$ , we know that

$$\begin{aligned} m_1 &= \|x+h\|^2 + \|y+k\|^2 - 2\|(x+h_1)(x_2+h_2) + (y_1+k_1)(y_2+k_2)\|, \\ m_2 &= \|x+h\|^2 + \|y+k\|^2 + 2\|(x+h_1)(x_2+h_2) + (y_1+k_1)(y_2+k_2)\|. \end{aligned}$$

Also  $\lambda_1, \lambda_2$  are spectral values of  $u$ , we have

$$\begin{aligned} \lambda_1 &= \|x\|^2 + \|y\|^2 - 2\|x_1x_2 + y_1y_2\|, \\ \lambda_2 &= \|x\|^2 + \|y\|^2 + 2\|x_1x_2 + y_1y_2\|. \end{aligned}$$

We denote  $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$  and  $u = (u_1, u_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ , then we obtain

$$\begin{aligned} & m_1 - \lambda_1 \\ &= \left( \|x+h\|^2 + \|y+k\|^2 - \|x\|^2 - \|y\|^2 \right) - \left( \|z_2\| - \|u_2\| \right) \\ &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) + \left( \|h\|^2 + \|k\|^2 \right) - \left( \|z_2\| - \|u_2\| \right) \\ &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) \\ &\quad - \frac{4\|(x_1+h_1)(x_2+h_2) + (y_1+k_1)(y_2+k_2)\|^2 - 4\|x_1x_2 + y_1y_2\|^2}{\|z_2\| + \|u_2\|} \\ &\quad + O(\|(h, k)\|^2) \\ &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) - \frac{8}{\|z_2\| + \|u_2\|} \left( \langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle \right. \\ &\quad \left. + O(\|(h, k)\|^2) \right) + O(\|(h, k)\|^2) \\ &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) - \frac{4\|u_2\|}{\|z_2\| + \|u_2\|} \left( \langle x, h \rangle + \langle y, k \rangle \right) + O(\|(h, k)\|^2) \\ &= \frac{2(\langle x, h \rangle + \langle y, k \rangle)}{\|z_2\| + \|u_2\|} \left( \|z_2\| - \|u_2\| \right) + O(\|(h, k)\|^2) \end{aligned}$$



$$\begin{aligned}
 &= \frac{2(\langle x, h \rangle + \langle y, k \rangle)}{(\|z_2\| + \|u_2\|)^2} \left( \|z_2\|^2 - \|u_2\|^2 \right) + O(\|(h, k)\|^2) \\
 &= \frac{2(\langle x, h \rangle + \langle y, k \rangle)}{(\|z_2\| + \|u_2\|)^2} \left( 8\langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle + O(\|(h, k)\|^2) \right) \\
 &\quad + O(\|(h, k)\|^2) \\
 &= \frac{16\|u_2\|}{(\|z_2\| + \|u_2\|)^2} \left( \langle x, h \rangle + \langle y, k \rangle \right)^2 + O(\|(h, k)\|^2) \\
 &= O(\|(h, k)\|^2),
 \end{aligned}$$

where the fifth and ninth equalities hold due to the following (by applying Lemma 4.3):

$$(24) \quad \langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle = \frac{\|u_2\|}{2} \left( \langle x, h \rangle + \langle y, k \rangle \right).$$

Using the same ideas as above, we also obtain

$$\begin{aligned}
 &m_2 - \lambda_2 \\
 &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) \\
 &\quad + \frac{4\|(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)\|^2 - 4\|x_1x_2 + y_1y_2\|^2}{\|z_2\| + \|u_2\|} \\
 &\quad + O(\|(h, k)\|^2) \\
 &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) + \frac{8}{\|z_2\| + \|u_2\|} \left( \langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle \right. \\
 &\quad \left. + O(\|(h, k)\|^2) \right) + O(\|(h, k)\|^2) \\
 &= 2 \left( \langle x, h \rangle + \langle y, k \rangle \right) + \frac{4\|u_2\|}{\|z_2\| + \|u_2\|} \left( \langle x, h \rangle + \langle y, k \rangle \right) + O(\|(h, k)\|^2) \\
 &= O(\|(h, k)\|).
 \end{aligned}$$

Thus, we complete the proof.  $\square$

It can be verified that the vector-valued function  $\rho$  given as (23) is strictly continuous and directionally differentiable by following the similar arguments as in the original proof in [19] by D. Sun and J. Sun. Therefore, to prove  $\rho$  is strongly semismooth, it is enough to verify that  $\rho$  satisfies the condition in Lemma 2.1(b). We show the detailed verifications in the following proposition which is another main work of this paper.

**Proposition 4.5.** *The vector-valued function  $\rho$  given as (23) is strongly semismooth. Hence, the vector-valued function  $\phi$  defined as (7) is strongly semismooth.*

*Proof.* We will have to discuss three cases to complete the proof.

Case (1): If  $(x, y) = (0, 0)$ , to show  $\rho$  is strongly semismooth at  $(0, 0)$ , we need to verify the condition of Lemma 2.1, that is,

$$(25) \quad \rho(h, k) - \rho(0, 0) - \nabla\rho(h, k) \cdot \begin{pmatrix} h \\ k \end{pmatrix} = O(\|(h, k)\|^2),$$

for any  $(h, k) \rightarrow (0, 0)$  such that  $\rho$  is differentiable at  $(h, k)$ . Since  $\rho$  is differentiable at  $(h, k)$  and  $\nabla\rho(h, k) = [\nabla_x\rho(h, k) \quad \nabla_y\rho(h, k)] = \left[ L_{(h^2+k^2)^{1/2}}^{-1} L_h \quad L_{(h^2+k^2)^{1/2}}^{-1} L_k \right]$  (see [3, Prop. 3.1(b)] or [9, Cor. 5.2]), we have

$$\begin{aligned} & \rho(h, k) - \rho(0, 0) - \nabla\rho(h, k) \cdot \begin{pmatrix} h \\ k \end{pmatrix} \\ &= (h^2 + k^2)^{1/2} - L_{(h^2+k^2)^{1/2}}^{-1} L_h \cdot h - L_{(h^2+k^2)^{1/2}}^{-1} L_k \cdot k \\ &= L_{(h^2+k^2)^{1/2}}^{-1} L_{(h^2+k^2)^{1/2}} \cdot (h^2 + k^2)^{1/2} - L_{(h^2+k^2)^{1/2}}^{-1} \cdot h^2 - L_{(h^2+k^2)^{1/2}}^{-1} \cdot k^2 \\ &= L_{(h^2+k^2)^{1/2}}^{-1} \left( (h^2 + k^2)^{1/2} \circ (h^2 + k^2)^{1/2} \right) - L_{(h^2+k^2)^{1/2}}^{-1} \cdot (h^2 + k^2) \\ &= L_{(h^2+k^2)^{1/2}}^{-1} \cdot 0 \\ &= O(\|(h, k)\|^2). \end{aligned}$$

Hence (25) is satisfied, which means  $\rho$  is semismooth at  $(0, 0)$

Case (2): If  $(x, y) \neq (0, 0)$  with  $x^2 + y^2$  lying in the interior of  $\mathcal{K}^n$ , it was already known (see [9]) that  $\rho$  is differentiable at such  $(x, y)$ . Hence, it is strongly semismooth at such  $(x, y)$ .

Case (3): If  $(x, y) \neq (0, 0)$  with  $x^2 + y^2$  on the boundary of  $\mathcal{K}^n$ , that is,  $\|x\|^2 + \|y\|^2 = 2\|x_1x_2 + y_1y_2\| > 0$  (Note that  $x^2 + y^2 = (\|x\|^2 + \|y\|^2, 2x_1x_2 + 2y_1y_2)$ ). Let  $u := x^2 + y^2$  with spectral values  $\lambda_1, \lambda_2$ , then we have

$$\begin{aligned} u^{1/2} &= \rho(x, y) = (x^2 + y^2)^{1/2} \\ &= \left( \sqrt{(\|x\|^2 + \|y\|^2)/2}, \frac{x_1x_2 + y_1y_2}{\sqrt{(\|x\|^2 + \|y\|^2)/2}} \right) \\ &= \left( \sqrt{x_1^2 + y_1^2}, \frac{x_1x_2 + y_1y_2}{\sqrt{x_1^2 + y_1^2}} \right), \end{aligned}$$

where the third and fourth equalities is true due to Lemma 4.1. Now, by Lemma 2.1 again, we need to verify, for any  $(h, k) \rightarrow (0, 0)$  such that  $\rho$  is differentiable at  $(x + h, y + k)$ , that

$$(26) \quad \rho(x + h, y + k) - \rho(x, y) - \nabla\rho(x + h, y + k) \cdot \begin{pmatrix} h \\ k \end{pmatrix} = O(\|(h, k)\|^2).$$

Since  $\rho$  is differentiable at  $(x + h, y + k)$ , we know

$$\begin{aligned} \nabla\rho(x + h, y + k) &= [\nabla_x\rho(x + h, y + k) \quad \nabla_y\rho(x + h, y + k)] \\ &= \left[ L_{((x+h)^2+(y+k)^2)^{1/2}}^{-1} L_{x+h} \quad L_{((x+h)^2+(y+k)^2)^{1/2}}^{-1} L_{y+k} \right]. \end{aligned}$$

Let  $z := (x + h)^2 + (y + k)^2$  with spectral values  $m_1, m_2$ , we have

$$\begin{aligned} z^{1/2} &= \rho(x + h, y + k) \\ &= \left( \frac{\sqrt{m_1} + \sqrt{m_2}}{2}, \frac{\sqrt{m_2} - \sqrt{m_1}}{2} \frac{(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)}{\|(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)\|} \right) \\ &= \left( \frac{\sqrt{m_1} + \sqrt{m_2}}{2}, \frac{2}{\sqrt{m_1} + \sqrt{m_2}} [(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)] \right), \end{aligned}$$

where

$$\begin{aligned} m_1 &:= \|x + h\|^2 + \|y + k\|^2 - 2\|(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)\|, \\ m_2 &:= \|x + h\|^2 + \|y + k\|^2 + 2\|(x_1 + h_1)(x_2 + h_2) + (y_1 + k_1)(y_2 + k_2)\|. \end{aligned}$$

Now let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  denote the function  $f(\cdot) = \sqrt{(\cdot)}$ , we have  $\nabla f^{\text{soc}}(z) = \frac{1}{2}L_{z^{1/2}}^{-1}$  (This is a result in the proof of [9, Cor. 5.2]). It together with part (c) of Prop. 2.2 gives

$$L_{z^{1/2}}^{-1} = 2\nabla f^{\text{soc}}(z) = \begin{bmatrix} 2b & \frac{2cz_2^T}{\|z_2\|} \\ \frac{2cz_2}{\|z_2\|} & 2aI + 2(b - a) \frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix},$$

where

$$a = \frac{f(m_2) - f(m_1)}{m_2 - m_1}, \quad b = \frac{f'(m_2) + f'(m_1)}{2}, \quad c = \frac{f'(m_2) - f'(m_1)}{2}.$$

In summary, up to here, we have

$$\begin{aligned} z^{1/2} &= \left( \frac{f(m_2) + f(m_1)}{2}, \frac{f(m_2) - f(m_1)}{2} \frac{z_2}{\|z_2\|} \right), \\ u^{1/2} &= \left( \frac{f(\lambda_2) + f(\lambda_1)}{2}, \frac{f(\lambda_2) - f(\lambda_1)}{2} \frac{u_2}{\|u_2\|} \right), \\ \nabla \rho(x + h, y + k) &= \begin{bmatrix} L_{z^{1/2}}^{-1} L_{x+h} & L_{z^{1/2}}^{-1} L_{y+k} \end{bmatrix}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} &\rho(x + h, y + k) - \rho(x, y) - \nabla \rho(x + h, y + k) \cdot \begin{pmatrix} h \\ k \end{pmatrix} \\ &= z^{1/2} - u^{1/2} - L_{z^{1/2}}^{-1} L_{x+h} \cdot h - L_{z^{1/2}}^{-1} L_{y+k} \cdot k \\ &= z^{1/2} - u^{1/2} - L_{z^{1/2}}^{-1} \left( x \circ h + y \circ k + h^2 + k^2 \right) \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{f(m_2) + f(m_1)}{2}, \frac{f(m_2) - f(m_1)}{2} \frac{z_2}{\|z_2\|} \right) \\
&\quad - \left( \frac{f(\lambda_2) + f(\lambda_1)}{2}, \frac{f(\lambda_2) - f(\lambda_1)}{2} \frac{u_2}{\|u_2\|} \right) \\
&\quad - \begin{bmatrix} 2b & \frac{2cz_2^T}{\|z_2\|} \\ \frac{2cz_2}{\|z_2\|} & 2aI + 2(b-a) \frac{z_2 z_2^T}{\|z_2\|^2} \end{bmatrix} \\
&\quad \cdot \begin{bmatrix} \langle x, h \rangle + \langle y, k \rangle + \|h\|^2 + \|k\|^2 \\ x_1 h_2 + h_1 x_2 + y_1 k_2 + k_1 y_2 + 2h_1 h_2 + 2k_1 k_2 \end{bmatrix} \\
&:= (\Xi_1, \Xi_2),
\end{aligned}$$

where  $\Xi_1 \in \mathbb{R}$  denotes the first component while  $\Xi_2 \in \mathbb{R}^{n-1}$  represents the second component. We will show that  $\Xi_1$  and  $\Xi_2$  are both  $O(\|(h, k)\|^2)$ . First of all, we compute the first component  $\Xi_1$ :

$$\begin{aligned}
\Xi_1 &= \frac{1}{2}f(m_2) + \frac{1}{2}f(m_1) - \frac{1}{2}f(\lambda_2) - \frac{1}{2}f(\lambda_1) \\
&\quad - \left( f'(m_2) + f'(m_1) \right) \left( \langle x, h \rangle + \langle y, k \rangle + \|h\|^2 + \|k\|^2 \right) \\
&\quad - \left( f'(m_2) - f'(m_1) \right) \frac{1}{\|z_2\|} \langle z_2, x_1 h_2 + h_1 x_2 + y_1 k_2 + k_1 y_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \\
&= \frac{1}{2} \left[ f(m_1) - f(\lambda_1) - f'(m_1) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \right. \\
&\quad \left. + f'(m_1) \frac{1}{\|z_2\|} \langle z_2, 2x_1 h_2 + 2h_1 x_2 + 2y_1 k_2 + 2k_1 y_2 + 4h_1 h_2 + 4k_1 k_2 \rangle \right] \\
&\quad + \frac{1}{2} \left[ f(m_2) - f(\lambda_2) - f'(m_2) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \right. \\
&\quad \left. - f'(m_2) \frac{1}{\|z_2\|} \langle z_2, 2x_1 h_2 + 2h_1 x_2 + 2y_1 k_2 + 2k_1 y_2 + 4h_1 h_2 + 4k_1 k_2 \rangle \right] \\
&= \frac{1}{2} \left[ f(m_1) - f(\lambda_1) - f'(m_1) \Delta_1 \right] + \frac{1}{2} \left[ f(m_2) - f(\lambda_2) - f'(m_2) \Delta_2 \right],
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \\
&\quad - \frac{1}{\|z_2\|} \langle z_2, 2x_1 h_2 + 2h_1 x_2 + 2y_1 k_2 + 2k_1 y_2 + 4h_1 h_2 + 4k_1 k_2 \rangle, \\
\Delta_2 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \\
&\quad + \frac{1}{\|z_2\|} \langle z_2, 2x_1 h_2 + 2h_1 x_2 + 2y_1 k_2 + 2k_1 y_2 + 4h_1 h_2 + 4k_1 k_2 \rangle.
\end{aligned}$$

We further observe that

$$\begin{aligned}
 \Delta_2 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 + \frac{1}{\|z_2\|} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \\
 &\quad + \frac{1}{\|z_2\|} \left( \|z_2\|^2 - \langle z_2, u_2 \rangle + \langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \right) \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 + \|z_2\| - \frac{\|z_2\|\|u_2\| \cos \theta}{\|z_2\|} + O(\|(h, k)\|^2) \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 + \left( \|z_2\| - \|u_2\|(1 + O(\theta^2)) \right) + O(\|(h, k)\|^2) \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 + \left( \|z_2\| - \|u_2\| \right) + O(\|(h, k)\|^2) \\
 &= (m_2 - \lambda_2) + O(\|(h, k)\|^2),
 \end{aligned}$$

where  $\theta$  is the angle between  $z_2$  and  $u_2$  and  $O(\theta^2) = O(\|(h, k)\|^2)$  due to  $\|z_2 - u_2\| = O(\|(h, k)\|)$ . In addition, the last equality holds by the following equation in proof of Lemma 4.4:

$$(27) \quad m_2 - \lambda_2 = 2\langle x, h \rangle + 2\langle y, k \rangle + \|h\|^2 + \|k\|^2 + \|z_2\| - \|u_2\|.$$

Hence, we have

$$\begin{aligned}
 &\frac{1}{2} \left[ f(m_2) - f(\lambda_2) - f'(m_2)\Delta_2 \right] \\
 &= \frac{1}{2} \left[ f(m_2) - f(\lambda_2) - f'(m_2) \left( (m_2 - \lambda_2) + O(\|(h, k)\|^2) \right) \right] \\
 &= O(|m_2 - \lambda_2|^2) + O(\|(h, k)\|^2) \\
 &= O(\|(h, k)\|^2),
 \end{aligned}$$

where the second equality is true since  $f$  is strongly semismooth at  $\lambda_2$  and  $f'(\lambda_2)$  is bounded, while the last equality holds due to Lemma 4.4. Therefore, it remains to show that the other term about  $\lambda_1$  in the expression of  $\Xi_1$  is  $O(\|(h, k)\|^2)$ . However, we can not use the same idea as above to prove it since  $f$  is not strongly semismooth at  $\lambda_1 = 0$ . Our approach way is as below. We rewrite  $\Delta_1$  as

$$\begin{aligned}
 \Delta_1 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 - \frac{1}{\|z_2\|} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \\
 &\quad - \frac{1}{\|z_2\|} \left( \|z_2\|^2 - \langle z_2, u_2 \rangle + \langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \right) \\
 &= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \\
 &\quad - \|z_2\| + \frac{1}{\|z_2\|} \langle z_2, u_2 \rangle - \frac{1}{\|z_2\|} \langle z_2, 2h_1h_2 + 2k_1k_2 \rangle
 \end{aligned}$$

$$\begin{aligned}
&= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 - \|z_2\| + \frac{\|z_2\|\|u_2\|\cos\theta}{\|z_2\|} \\
&\quad - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \\
&= 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 - \left( \|z_2\| - \|u_2\|(1 - \theta^2/2 + O(\theta^4)) \right) \\
&\quad - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \\
&= 2m_1 + \left( \|z_2\| - \|u_2\| \right) - 2\left( \langle x, h \rangle + \langle y, k \rangle \right) + \|u_2\|(-\theta^2/2) \\
&\quad - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle + O(\theta^4) \\
&:= 2m_1 + \widetilde{\Delta}_1 + O(\theta^4),
\end{aligned}$$

where  $\theta$  is the angle between  $z_2$  and  $u_2$  and the last equality holds due to Lemma 4.4. We will show that  $\widetilde{\Delta}_1$  is actually  $O(\|(h, k)\|^3)$  in the following.

$$\begin{aligned}
\widetilde{\Delta}_1 &= \left( \|z_2\| - \|u_2\| \right) - 2\left( \langle x, h \rangle + \langle y, k \rangle \right) \\
&\quad + \|u_2\|(-\theta^2/2) - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \\
&= \frac{\|z_2\|^2 - \|u_2\|^2}{\|z_2\| + \|u_2\|} - 2\left( \langle x, h \rangle + \langle y, k \rangle \right) \\
&\quad + \|u_2\|(-\theta^2/2) - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \\
&= \frac{4}{\|z_2\| + \|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 + \langle u_2, h_1h_2 + k_1k_2 \rangle \right) \\
&\quad + \|u_2\|(\langle x, h \rangle + \langle y, k \rangle) + O(\|(h, k)\|^3) - 2\left( \langle x, h \rangle + \langle y, k \rangle \right) \\
&\quad + \|u_2\|(-\theta^2/2) - \frac{1}{\|z_2\|}\langle z_2, 2h_1h_2 + 2k_1k_2 \rangle \\
&= \frac{4\|u_2\|}{\|z_2\| + \|u_2\|} \left( \langle x, h \rangle + \langle y, k \rangle \right) - 2\left( \langle x, h \rangle + \langle y, k \rangle \right) \\
&\quad + \frac{4\|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2}{\|z_2\| + \|u_2\|} + \|u_2\|(-\theta^2/2) \\
&\quad + \left( \frac{4}{\|z_2\| + \|u_2\|} - \frac{2}{\|z_2\|} \right) \langle u_2, h_1h_2 + k_1k_2 \rangle + O(\|(h, k)\|^3) \\
&= \frac{\langle x, h \rangle + \langle y, k \rangle}{\|z_2\| + \|u_2\|} \left( 4\|u_2\| - 2(\|z_2\| + \|u_2\|) \right) \\
&\quad + \frac{4\|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2}{\|z_2\| + \|u_2\|} + \|u_2\|(-\theta^2/2) + O(\|(h, k)\|^3)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{2(\langle x, h \rangle + \langle y, k \rangle)}{(\|z_2\| + \|u_2\|)^2} \left( \|u_2\|^2 - \|z_2\|^2 \right) \\
 &\quad + \frac{4\|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2}{\|z_2\| + \|u_2\|} + \|u_2\|(-\theta^2/2) + O(\|(h, k)\|^3) \\
 &= \frac{-8(\langle x, h \rangle + \langle y, k \rangle)}{(\|z_2\| + \|u_2\|)^2} \left( \|u_2\|(\langle x, h \rangle + \langle y, k \rangle) \right) \\
 &\quad + \frac{4\|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2}{\|z_2\| + \|u_2\|} + \|u_2\|(-\theta^2/2) + O(\|(h, k)\|^3) \\
 &= \frac{-8\|u_2\|}{(\|z_2\| + \|u_2\|)^2} \left( (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \frac{8\|u_2\| \cdot \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2}{(\|z_2\| + \|u_2\|)^2} + \|u_2\|(-\theta^2/2) + O(\|(h, k)\|^3) \\
 &= \frac{8\|u_2\|}{(\|z_2\| + \|u_2\|)^2} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \|u_2\|(-\theta^2/2) + O(\|(h, k)\|^3),
 \end{aligned}$$

where the fifth equality is true since the following equation:

$$\left( \frac{4}{\|z_2\| + \|u_2\|} - \frac{2}{\|z_2\|} \right) \langle u_2, h_1h_2 + k_1k_2 \rangle = O(\|(h, k)\|^3).$$

On the other hand, the first two terms in the last equality of expression of  $\widetilde{\Delta}_1$  could be cancelled as shown below:

$$\begin{aligned}
 &\frac{8\|u_2\|}{(\|z_2\| + \|u_2\|)^2} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) + \|u_2\|(-\theta^2/2) \\
 &= \frac{8\|u_2\|}{(2\|u_2\|)^2} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \frac{\langle z_2, u_2 \rangle - \|u_2\| \cdot \|z_2\|}{\|z_2\|} + O(\|(h, k)\|^3) \\
 &= \frac{2}{\|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \frac{\langle u_2 + d_2, u_2 \rangle - \|u_2\| \cdot \|u_2 + d_2\|}{\|z_2\|} + O(\|(h, k)\|^3) \\
 &= \frac{2}{\|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \frac{(\langle u_2, d_2 \rangle)^2 - (\|u_2\| \cdot \|d_2\|)^2}{2\|u_2\|^3} + O(\|(h, k)\|^3) \\
 &= \frac{2}{\|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
 &\quad + \frac{1}{2\|u_2\|^3} \left( \left[ 4\langle x_1x_2 + y_1y_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 \rangle \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + 4\langle x_1x_2 + y_1y_2, h_1h_2 + k_1k_2 \rangle \Big]^2 \\
& - \|u_2\|^2 \cdot \left[ 4\|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 + O(\|(h, k)\|^3) \right] \Big) + O(\|(h, k)\|^3) \\
& = \frac{2}{\|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
& + \frac{4}{2\|u_2\|^3} \left( \|u_2\| \cdot \left[ (\langle x, h \rangle + \langle y, k \rangle) + \langle u_2, h_1h_2 + k_1k_2 \rangle \right]^2 \right. \\
& \left. - \|u_2\|^2 \cdot \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 \right) + O(\|(h, k)\|^3) \\
& = \frac{2}{\|u_2\|} \left( \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 - (\langle x, h \rangle + \langle y, k \rangle)^2 \right) \\
& + \frac{2}{\|u_2\|} \left( (\langle x, h \rangle + \langle y, k \rangle)^2 - \|x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2\|^2 \right) + O(\|(h, k)\|^3) \\
& = O(\|(h, k)\|^3),
\end{aligned}$$

where we denote  $\underline{d}_2 := z_2 - u_2$  and equation (24) is applied in the sixth equality. Thus, we proved  $\underline{\Delta}_1$  is  $O(\|(h, k)\|^3)$ , which implies that  $\Delta_1 = 2m_1 + \underline{\Delta}_1$  is  $2m_1 + O(\|(h, k)\|^3)$ . Then, we obtain

$$\begin{aligned}
& \frac{1}{2} \left[ f(m_1) - f(\lambda_1) - f'(m_1)(\Delta_1) \right] \\
& = \frac{1}{2} \left[ f(m_1) - f(\lambda_1) - f'(m_1) \left( 2m_1 + O(\|(h, k)\|^3) \right) \right] \\
& = f'(m_1) \cdot O(\|(h, k)\|^3) \\
& = O(\|(h, k)\|^2),
\end{aligned}$$

where the second equality holds since  $f'(m_1) = \frac{1}{2\sqrt{m_1}}$ ,  $f(\lambda_1) = 0$  and the third equality is true because  $f'(m_1) = O\left(\frac{1}{\|(h, k)\|}\right)$  (by applying Lemma 4.2 and 4.4).

Therefore, we completed proving that the first component  $\Xi_1$  is  $O(\|(h, k)\|^2)$ . Now, we move onto the second component  $\Xi_2$ . Recall that we have  $z_2 = u_2 + d_2$  and  $m_2 - m_1 = 2\|z_2\|$ . Thus, we can simplify  $\Xi_2$  as below:

$$\begin{aligned}
\Xi_2 & = \frac{1}{2} \left( f(m_2) - f(m_1) \right) \frac{z_2}{\|z_2\|} - \frac{1}{2} \left( f(\lambda_2) - f(\lambda_1) \right) \frac{u_2}{\|u_2\|} \\
& - \left( f'(m_2) - f'(m_1) \right) \left( \langle x, h \rangle + \langle y, k \rangle + \|h\|^2 + \|k\|^2 \right) \frac{z_2}{\|z_2\|} \\
& - 2 \left( \frac{f(m_2) - f(m_1)}{m_2 - m_1} \right) \left( x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 + 2h_1h_2 + 2k_1k_2 \right) \\
& - \left( f'(m_2) + f'(m_1) \right) \frac{z_2}{\|z_2\|^2} \langle z_2, x_1h_2 + h_1x_2 + y_1k_2 + k_1y_2 + 2h_1h_2 + 2k_1k_2 \rangle
\end{aligned}$$



$$\begin{aligned}
 & + 2 \left( \frac{f(m_2) - f(m_1)}{m_2 - m_1} \right) \frac{z_2}{\|z_2\|^2} \langle z_2, x_1 h_2 + h_1 x_2 + y_1 k_2 + k_1 y_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \\
 = & - \frac{1}{2} \left[ f(m_1) \frac{z_2}{\|z_2\|} - f(\lambda_1) \frac{u_2}{\|u_2\|} \right. \\
 & - f'(m_1) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \frac{z_2}{\|z_2\|} \\
 & - f(m_1) \frac{1}{\|z_2\|} \left( z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \right) \\
 & + f'(m_1) \frac{z_2}{\|z_2\|^2} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \\
 & \left. + f(m_1) \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \right] \\
 & + \frac{1}{2} \left[ f(m_2) \frac{z_2}{\|z_2\|} - f(\lambda_2) \frac{u_2}{\|u_2\|} \right. \\
 & - f'(m_2) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \frac{z_2}{\|z_2\|} \\
 & - f(m_2) \frac{1}{\|z_2\|} \left( z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \right) \\
 & - f'(m_2) \frac{z_2}{\|z_2\|^2} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \\
 & \left. + f(m_2) \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \right] \\
 := & \Xi_2^{(1)} + \Xi_2^{(2)},
 \end{aligned}$$

where  $\Xi_2^{(1)}$  denotes the first half part of the above expression while  $\Xi_2^{(2)}$  denotes the second part. We will show that both  $\Xi_2^{(1)}$  and  $\Xi_2^{(2)}$  are  $O(\|(h, k)\|^2)$ . We look at  $\Xi_2^{(2)}$  first:

$$\begin{aligned}
 \Xi_2^{(2)} & = \frac{1}{2} \left\{ f(m_2) \frac{z_2}{\|z_2\|} - f(\lambda_2) \frac{u_2}{\|u_2\|} \right. \\
 & - f'(m_2) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \frac{z_2}{\|z_2\|} \\
 & - f(m_2) \frac{1}{\|z_2\|} \left( z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \right) \\
 & - f'(m_2) \frac{z_2}{\|z_2\|^2} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \\
 & \left. + f(m_2) \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \right\} \\
 & = \frac{1}{2} \left\{ \frac{z_2}{\|z_2\|} \cdot \left[ f(m_2) - f(\lambda_2) - f'(m_2) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \right. \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\|z_2\|} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \Big] + f(\lambda_2) \left( \frac{z_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} \right) \\
& + f(m_2) \left( - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right. \\
& \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) \Big\} \\
= & \frac{1}{2} \left\{ \frac{z_2}{\|z_2\|} \cdot \left[ f(m_2) - f(\lambda_2) - f'(m_2) \left( m_2 - \lambda_2 + O(\|(h, k)\|^2) \right) \right] \right. \\
& + f(\lambda_2) \left( \frac{z_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right. \\
& \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) \\
& \left. + \left( f(m_2) - f(\lambda_2) \right) \left( - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right. \right. \\
& \left. \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) \right\},
\end{aligned}$$

where we add and subtract  $f(\lambda_2) \frac{z_2}{\|z_2\|}$  in the second equality; we also add and subtract

$$\frac{1}{2} f(\lambda_2) \left( - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right)$$

in the third equality. In addition, the equation (27) is also used in the last equality.

Since  $f$  is strongly semismooth at  $\lambda_2$ , by the same arguments as in the first component  $\Xi_1$ , we have

$$f(m_2) - f(\lambda_2) - f'(m_2) \left( m_2 - \lambda_2 + O(\|(h, k)\|^2) \right) = O(\|(h, k)\|^2).$$

Note that  $f(m_2) - f(\lambda_2) = O(\|(h, k)\|)$  because  $m_2 - \lambda_2 = O(\|(h, k)\|)$  as well as  $f$  is strictly continuous at  $\lambda_2$ . On the other hand, it is not hard to verify that

$$\left( - \frac{(z_2 - u_2 + 2h_2h_2 + 2k_1k_2)}{\|z_2\|} + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) = O(\|(h, k)\|).$$

Hence, we obtain that the third term of  $\Xi_2^{(2)}$  is  $O(\|(h, k)\|)$ , i.e.,

$$\begin{aligned}
& \left( f(m_2) - f(\lambda_2) \right) \left( - \frac{(z_2 - u_2 + 2h_2h_2 + 2k_1k_2)}{\|z_2\|} \right. \\
& \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) = O(\|(h, k)\|^2).
\end{aligned}$$

Thus, it remains to show that the second term of  $\Xi_2^{(2)}$  is  $O(\|(h, k)\|^2)$ . Note that

$$\begin{aligned}
& \frac{z_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \\
= & \frac{u_2 + d_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} - \frac{(d_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} + \frac{u_2 + d_2}{\|z_2\|^3} \langle z_2, d_2 + 2h_1h_2 + 2k_1k_2 \rangle
\end{aligned}$$

$$\begin{aligned}
 &= \frac{u_2 + d_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} - \frac{d_2}{\|z_2\|} + \frac{u_2 + d_2}{\|z_2\|^3} \langle z_2, d_2 \rangle + O(\|(h, k)\|^2) \\
 &= u_2 \left( \frac{1}{\|z_2\|} - \frac{1}{u_2} + \frac{\langle z_2, d_2 \rangle}{\|z_2\|^3} \right) + O(\|(h, k)\|^2).
 \end{aligned}$$

If we let  $\theta(z_2) := -1/\|z_2\|$ , then a previous technique leads to

$$\frac{1}{\|z_2\|} - \frac{1}{\|u_2\|} + \frac{\langle z_2, d_2 \rangle}{\|z_2\|^3} = \theta(u_2) - \theta(z_2) - \nabla\theta(z_2)(u_2 - z_2) = O(\|(h, k)\|^2),$$

where the last equality is from Taylor approximation. Thus, we obtain

$$\begin{aligned}
 f(\lambda_2) \left( \frac{z_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} - \frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right. \\
 \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) = O(\|(h, k)\|^2).
 \end{aligned}$$

So far, we therefore proved that  $\Xi_2^{(2)}$  is  $O(\|(h, k)\|^2)$ .

Finally, we will go back to show  $\Xi_2^{(1)}$  is  $O(\|(h, k)\|^2)$ . Again, the idea used for  $\Xi_2^{(2)}$  can not be applied to  $\Xi_2^{(1)}$  since  $f$  is not semismooth at  $\lambda_1 = 0$ . However,  $\Xi_2^{(1)}$  can be rewritten as below.

$$\begin{aligned}
 \Xi_2^{(1)} &= -\frac{1}{2} \left\{ f(m_1) \frac{z_2}{\|z_2\|} - f(\lambda_1) \frac{u_2}{\|u_2\|} \right. \\
 &\quad - f'(m_1) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \frac{z_2}{\|z_2\|} \\
 &\quad - f(m_1) \frac{1}{\|z_2\|} \left( z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \right) \\
 &\quad + f'(m_1) \frac{z_2}{\|z_2\|^2} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \\
 &\quad \left. + f(m_1) \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right\} \\
 &= -\frac{1}{2} \left\{ \frac{z_2}{\|z_2\|} \cdot \left[ f(m_1) - f(\lambda_1) - f'(m_1) \left( 2\langle x, h \rangle + 2\langle y, k \rangle + 2\|h\|^2 + 2\|k\|^2 \right) \right. \right. \\
 &\quad \left. \left. - \frac{1}{\|z_2\|} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right] + f(\lambda_1) \left( \frac{z_2}{\|z_2\|} - \frac{u_2}{\|u_2\|} \right) \right. \\
 &\quad + f(m_1) \left( -\frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right. \\
 &\quad \left. \left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1h_2 + 2k_1k_2 \rangle \right) \right\} \\
 &= -\frac{1}{2} \left\{ \frac{z_2}{\|z_2\|} \cdot \left[ f(m_1) - f(\lambda_1) - f'(m_1) \left( \Delta_1 + O(\|(h, k)\|^3) \right) \right] \right. \\
 &\quad \left. + f(m_1) \left( -\frac{(z_2 - u_2 + 2h_1h_2 + 2k_1k_2)}{\|z_2\|} \right) \right\}
 \end{aligned}$$

$$\left. + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \right\}.$$

Then follow the same arguments as in the first component  $\Xi_1$ , we can obtain

$$f(m_1) - f(\lambda_1) - f' \left( \Delta_1 + O(\|(h, k)\|^3) \right) = O(\|(h, k)\|^2).$$

On the other hand, it is not hard to show that

$$\left( -\frac{(z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2)}{\|z_2\|} + \frac{z_2}{\|z_2\|^3} \langle z_2, z_2 - u_2 + 2h_1 h_2 + 2k_1 k_2 \rangle \right) = O(\|(h, k)\|),$$

and  $f(m_1) = O(\|(h, k)\|)$  by Lemma 4.4. Thus, we obtain that the second term of expression of  $\Xi_2^{(1)}$  is  $O(\|(h, k)\|^2)$ . With this, we therefore complete that  $\Xi_2^{(1)}$  is  $O(\|(h, k)\|^2)$ . From all the above, we proved that  $(\Xi_1, \Xi_2)$  is  $O(\|(h, k)\|^2)$  which implies  $\rho$  is strongly semismooth in Case (3).  $\square$

## 5. CONCLUSION

We have provided alternative proofs for some results of vector-valued functions associated with second-order cone, which are useful for designing and analyzing smoothing and nonsmooth methods for solving SOCP and SOCCP. Our proofs involve more algebraic computations than existing proofs do, in general. Nonetheless, our proofs come from the straightforward, intuitive thinking and basic definitions as well as the simple structure of second-order cone. We believe that the intuitive way we presented here would be helpful for analysis of other merit functions used for solving SOCP and SOCCP that is one of our future research interests.

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