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Conditions for Error Bounds and Bounded Level Sets of Some Merit Functions for the Second-Order Cone Complementarity Problem

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Abstract Recently this author studied several merit functions systematically for the second-order cone complementarity problem. These merit functions were shown to enjoy some favorable properties, to provide error bounds under the condition of strong monotonicity, and to have bounded level sets under the conditions of monotonicity as well as strict feasibility. In this paper, we weaken the condition of strong monotonicity to the so-called uniform P^*-property, which is a new concept recently developed for linear and nonlinear transformations on Euclidean Jordan algebra. Moreover, we replace the monotonicity and strict feasibility by the so-called R_{01} or R_{02}-functions to keep the property of bounded level sets.

Keywords Error bounds · Jordan products · Level sets · Merit functions · Second-order cones · Spectral factorization

1 Introduction

The second-order cone complementarity problem (SOCCP), which is a natural extension of nonlinear complementarity problem (NCP), is to find $\zeta \in \mathbb{R}^n$ satisfying

$$\langle F(\zeta), \zeta \rangle = 0, \quad F(\zeta) \in \mathcal{K}, \ \zeta \in \mathcal{K}, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous mapping, and $\mathcal{K}$ is the Cartesian product of second-order cones (SOC), also called Lorentz

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cones [1]. In other words,

$$\mathcal{K} = \mathcal{K}^{n_1} \times \cdots \times \mathcal{K}^{n_m}, \tag{2}$$

where $m, n_1, \dots, n_m \geq 1$, $n_1 + \cdots + n_m = n$, and

$$\mathcal{K}^{n_i} := \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid \|x_2\| \leq x_1\}, \tag{3}$$

with $\|\cdot\|$ denoting the Euclidean norm and $\mathcal{K}^1$ denoting the set of nonnegative reals $\mathbb{R}_+$. A special case of (2) is $\mathcal{K} = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$, which corresponds to $m = n$ and $n_1 = \cdots = n_m = 1$. If $\mathcal{K} = \mathbb{R}^n_+$, then (1) reduces to the nonlinear complementarity problem. Throughout this paper, we assume $\mathcal{K} = \mathcal{K}^n$ for simplicity, i.e., $\mathcal{K}$ is a single second-order cone (all the analysis can be easily carried over to the general case where $\mathcal{K}$ has the direct product structure (2)).

Second-order cone programs (SOCP) are convex optimization problems in which a linear function is minimized over the intersection of an affine linear manifold with Cartesian product of SOCs. Linear programs, convex quadratic programs and quadratically constrained convex quadratic programs can all be formulated as SOCP problems. Many other problems from engineering, control, finance, and robust optimization can also be recast as SOCP problems [2, 3]. It is well-known that the KKT optimality conditions of SOCP forms a SOCCP which is also a natural extension of nonlinear complementarity problems (NCP). Thus studying the SOCCP is very important from the above points of view.

There have been various methods proposed for solving SOCCP. They include interior-point methods [3–9], non-interior smoothing Newton methods [10–12]. Recently in the papers [13–15], the author studied an alternative approach based on reformulating SOCCP as an unconstrained smooth minimization problem. For this approach, it aims to find a smooth function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\psi(x, y) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{K}^n, \;\; y \in \mathcal{K}^n, \;\; \langle x, y \rangle = 0. \tag{4}$$

Then SOCCP can be expressed as an unconstrained smooth (global) minimization problem:

$$\min_{\zeta \in \mathbb{R}^n} f(\zeta) := \psi(F(\zeta), \zeta). \tag{5}$$

We call such a f a *merit function* for the SOCCP.

A popular choice of ψ is the squared norm of Fischer-Burmeister function, i.e., $\psi_{\mathrm{FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ associated with second-order cone given by

$$\psi_{\mathrm{FB}}(x, y) = \frac{1}{2}\|\phi_{\mathrm{FB}}(x, y)\|^2, \tag{6}$$

where $\phi_{\mathrm{FB}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is the well-known Fischer-Burmeister function (originally proposed for NCP, see [16, 17]) defined by

$$\phi_{\mathrm{FB}}(x, y) = (x^2 + y^2)^{1/2} - x - y. \tag{7}$$

More specifically, for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* associated with $\mathcal{K}^n$ as

$$x \circ y := (\langle x, y \rangle, y_1 x_2 + x_1 y_2). \tag{8}$$

The Jordan product $\circ$, unlike scalar or matrix multiplication, is not associative, which is a main source on complication in the analysis of SOCCP. The identity element under this product is $e := (1, 0, \dots, 0)^T \in \mathbb{R}^n$. We write x^2 to mean $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. It is known that $x^2 \in \mathcal{K}^n$ for all $x \in \mathbb{R}^n$. Moreover, if $x \in \mathcal{K}^n$, then there exists a unique vector in $\mathcal{K}^n$, denoted by $x^{1/2}$, such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Thus, ϕ_{FB} defined as (7) is well-defined for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ and maps $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$. It was shown in [11] that $\phi_{\mathrm{FB}}(x, y) = 0$ if and only if (x, y) satisfies (4). Therefore, ψ_{FB} defined as (6) induces a merit function $f_{\mathrm{FB}} := \psi_{\mathrm{FB}}(F(\zeta), \zeta))$ for the SOCCP.

The function ψ_{FB} given as in (6) was proved smooth with computable gradient formulas and enjoys several favorable properties, nonetheless, it does not have additional bounded level-set and error bound properties (see [15]). To conquer this, four other functions associated with second-order cone were considered in [13–15]. The first one is $\psi_{\mathrm{YF}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi_{\mathrm{YF}}(x, y) := \psi_0(\langle x, y \rangle) + \psi_{\mathrm{FB}}(x, y), \tag{9}$$

where $\psi_0 : \mathbb{R} \to \mathbb{R}_+$ is any smooth function satisfying

$$\psi_0(t) = 0 \quad \forall t \le 0 \quad \text{and} \quad \psi_0'(t) > 0 \quad \forall t > 0. \tag{10}$$

The function ψ_{YF} was studied by Yamashita and Fukushima in [18] for SDCP (semidefinite complementarity problems) case and was extended to SOCCP case in [15]. An example of $\psi_0(t)$ is $\psi_0(t) = \frac{1}{4}(\max\{0, t\})^4$. A slight modification of ψ_{YF} yields $\widehat{\psi_{\mathrm{YF}}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\widehat{\psi_{\mathrm{YF}}}(x, y) := \frac{1}{2}\|(x \circ y)_+\|^2 + \psi_{\mathrm{FB}}(x, y), \tag{11}$$

where $(\cdot)_+$ means the orthogonal projection onto the second-order cone $\mathcal{K}^n$. The third function is $\psi_{\mathrm{LT}} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi_{\mathrm{LT}}(x, y) := \psi_0(\langle x, y \rangle) + \tilde{\psi}(x, y), \tag{12}$$

where $\tilde{\psi} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ satisfies

$$\tilde{\psi}(x, y) = 0,\ \langle x, y \rangle \le 0 \quad \Longleftrightarrow \quad x \in \mathcal{K}^n,\ y \in \mathcal{K}^n,\ \langle x, y \rangle = 0. \tag{13}$$

The function ψ_0 is the same as the above (namely, it satisfies (10)) and examples of $\tilde{\psi}$ are

$$\tilde{\psi}_1(x, y) := \frac{1}{2}\big(\|(-x)_+\|^2 + \|(-y)_+\|^2\big) \quad \text{and} \quad \tilde{\psi}_2(x, y) := \frac{1}{2}\|\phi_{\mathrm{FB}}(x, y)_+\|^2 \tag{14}$$

which were recently investigated in [14]. The function ψ_{LT} was proposed by Luo and Tseng for NCP case in [19] and was extended to the SDCP case by Tseng in [20]. The last function $\widehat{\psi_{\mathrm{LT}}}: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, a slight variant of ψ_{LT}, is defined by

$$\widehat{\psi_{\mathrm{LT}}}(x, y) := \frac{1}{2}\|(x \circ y)_+\|^2 + \tilde{\psi}(x, y), \tag{15}$$

where $\tilde{\psi}$ is given as in (13).

Each of the above functions naturally induces a merit function as follows:

$$\begin{aligned} f_{\mathrm{YF}}(\zeta) &:= \psi_{\mathrm{YF}}(F(\zeta), \zeta), \\ \widehat{f_{\mathrm{YF}}}(\zeta) &:= \widehat{\psi_{\mathrm{YF}}}(F(\zeta), \zeta), \\ f_{\mathrm{LT}}(\zeta) &:= \psi_{\mathrm{LT}}(F(\zeta), \zeta), \\ \widehat{f_{\mathrm{LT}}}(\zeta) &:= \widehat{\psi_{\mathrm{LT}}}(F(\zeta), \zeta). \end{aligned} \tag{16}$$

It was shown that f_{YF} provides error bound [15, Prop. 5] if F is strongly monotone and f_{YF} has bounded level set [15, Prop. 6] if F is monotone as well as SOCCP is strictly feasible. The same results hold for $\widehat{f_{\mathrm{YF}}}$ [13, Prop. 4.1 and Prop. 4.2], for f_{LT} [14, Prop. 4.1 and Prop. 4.3], and for $\widehat{f_{\mathrm{LT}}}$ [14, Prop. 4.2 and Prop. 4.4]. The main purpose of this paper is to weaken the condition of strong monotonicity to so-called uniform P^*-property (will be introduced in Sect. 2) which is a new concept recently developed for linear and nonlinear transformations on Euclidean Jordan Algebra [21, 22]. Moreover, we replace the monotonicity and strict feasibility by the so-called R_{01} (or R_{02})-functions (will be introduced in Sect. 2) to ensure that the level sets for $f_{\mathrm{YF}}, \widehat{f_{\mathrm{YF}}}, f_{\mathrm{LT}}, \widehat{f_{\mathrm{LT}}}$ are still bounded.

2 Preliminaries

In this section, we review some definitions and preliminary materials that will be used in the subsequent analysis. First, we recall from [11] that each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ admits a spectral factorization, associated with $\mathcal{K}^n$, of the form

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)}, \tag{17}$$

where $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\begin{aligned} \lambda_i(x) &= x_1 + (-1)^i \|x_2\|, \\ u_x^{(i)} &= \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0; \\ \frac{1}{2}\big(1, (-1)^i w_2\big), & \text{if } x_2 = 0, \end{cases} \end{aligned} \tag{18}$$

for $i = 1, 2$, with w_2 being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w_2\| = 1$. If $x_2 \neq 0$, the factorization is unique. The set $\{u_x^{(1)}, u_x^{(2)}\}$ is called a *Jordan frame* and possesses the following properties.

Property 2.1 For any $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$ with the spectral values $\lambda_1(x),\lambda_2(x)$ and spectral vectors $u_x^{(1)},u_x^{(2)}$ given as in (18), we have

(a) $u_x^{(1)}$ and $u_x^{(2)}$ are orthogonal under Jordan product and have length $1/\sqrt{2}$, i.e.,

$$u_x^{(1)}\circ u_x^{(2)}=0,\qquad \|u_x^{(1)}\|=\|u_x^{(2)}\|=\frac{1}{\sqrt{2}}.$$

(b) $u_x^{(1)}$ and $u_x^{(2)}$ are *idempotent* under Jordan product, i.e.,

$$u_x^{(i)}\circ u_x^{(i)}=u_x^{(i)},\quad i=1,2.$$

The above spectral factorization of x, as well as x^2 and $x^{1/2}$ have various interesting properties; see [11]. For instances, for any $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, with spectral values $\lambda_1(x),\lambda_2(x)$ and spectral vectors $u_x^{(1)},u_x^{(2)}$, the following results hold: (1) $x^2=\lambda_1(x)^2\,u_x^{(1)}+\lambda_2(x)^2\,u_x^{(2)}\in\mathcal{K}^n$. (2) If $x\in\mathcal{K}^n$, then $0\le\lambda_1(x)\le\lambda_2(x)$ and $x^{1/2}=\sqrt{\lambda_1(x)}\,u_x^{(1)}+\sqrt{\lambda_2(x)}\,u_x^{(2)}$. It is also well-known that for any $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, we have $x\in\mathcal{K}^n$ if and only if

$$L_x:=\begin{bmatrix} x_1 & x_2^T\\ x_2 & x_1 I\end{bmatrix}$$

is positive semi-definite (see [11, p. 437] and [23]). If $x\in\operatorname{int}(\mathcal{K}^n)$, then $0<\lambda_1(x)\le\lambda_2(x)$, and L_x is invertible with

$$L_x^{-1}=\frac{1}{x_1^2-\|x_2\|^2}\begin{bmatrix} x_1 & -x_2^T\\ -x_2 & \frac{x_1^2-\|x_2\|^2}{x_1}I+\frac{1}{x_1}x_2x_2^T\end{bmatrix}.$$

In general, we have $x\circ y=L_xy$ for all $y\in\mathbb{R}^n$, and $L_x\succ 0$ if and only if $x\in\operatorname{int}(\mathcal{K}^n)$. We say that x,y *operator commute* if L_x and L_y commute, i.e., $L_xL_y=L_yL_x$. From [1, Lemma X.2.2], we know that x and y operator commute if and only if x and y share a common Jordan frame in their spectral factorizations.

We now recall definitions of various monotonicities and P-properties of a continuous mapping which are needed for the assumptions of our main results later. To this end, we denote

$$x\sqcap y:=x-(x-y)_+,\qquad x\sqcup y:=y+(x-y)_+, \tag{19}$$

which will be used in the definitions of P-properties. As below, we state the definitions of various P-properties associated with SOC. Indeed, such definitions are borrowed from [21, 22, 24], and are generalization of the familiar P-properties for matrices. Some of them may look slightly different from the original ones given in [21, 22, 24]; this is because ours are in SOCCP style.

Definition 2.1 Let $x\preceq_{\mathcal{K}^n}y$ denote $y-x\in\mathcal{K}^n$ for any $x,y\in\mathbb{R}^n$. Then, for a continuous mapping $F:\mathbb{R}^n\to\mathbb{R}^n$,

(a) F is monotone if

$$\langle F(\zeta) - F(\xi), \zeta - \xi\rangle \geq 0 \quad \forall \zeta, \xi \in \mathbb{R}^n;$$

(b) F is strictly monotone if

$$\langle F(\zeta) - F(\xi), \zeta - \xi\rangle > 0 \quad \forall \zeta \neq \xi \in \mathbb{R}^n;$$

(c) F is strongly monotone if there exists $\rho > 0$ such that

$$\langle F(\zeta) - F(\xi), \zeta - \xi\rangle \geq \rho\|\zeta - \xi\|^2 \quad \forall \zeta,\ \xi \in \mathbb{R}^n;$$

(d) F has Order P-property if

$$(\zeta - \xi) \sqcap (F(\zeta) - F(\xi)) \preceq_{\mathcal{K}^n} 0 \preceq_{\mathcal{K}^n} (\zeta - \xi) \sqcup (F(\zeta) - F(\xi)) \Longrightarrow \zeta = \xi;$$

(e) F has Jordan P-property if

$$(\zeta - \xi) \circ (F(\zeta) - F(\xi)) \preceq_{\mathcal{K}^n} 0 \Longrightarrow \zeta = \xi,$$

or equivalently,

$$\zeta \neq \xi \Longrightarrow \lambda_2[(\zeta - \xi) \circ (F(\zeta) - F(\xi))] > 0;$$

(f) F has P-property if

$$\left.\begin{array}{r} \zeta - \xi \quad \text{and} \quad F(\zeta) - F(\xi) \quad \text{operator commute} \\ (\zeta - \xi) \circ (F(\zeta) - F(\xi)) \preceq_{\mathcal{K}^n} 0 \end{array}\right\} \Longrightarrow \zeta = \xi;$$

(g) F has uniform P^*-property if there exists $\rho > 0$ such that

$$\max_{i=1,2} \langle (\zeta - \xi) \circ (F(\zeta) - F(\xi))\ ,\ u_\xi^{(i)}\rangle \geq \rho\|\zeta - \xi\|^2 \quad \forall \zeta,\ \xi \in \mathbb{R}^n,$$

where $u_\xi^{(i)}$, $i = 1, 2$, are the spectral vectors of ξ;

(h) F has uniform Jordan P-property if there exists $\rho > 0$ such that

$$\lambda_2[(\zeta - \xi) \circ (F(\zeta) - F(\xi))] \geq \rho\|\zeta - \xi\|^2 \quad \forall \zeta,\ \xi \in \mathbb{R}^n;$$

(i) F has uniform P-property if there exists $\rho > 0$ such that for any $\zeta, \xi \in \mathbb{R}^n$ with $\zeta - \xi$ operator commuting with $F(\zeta) - F(\xi)$, we have

$$\lambda_2[(\zeta - \xi) \circ (F(\zeta) - F(\xi))] \geq \rho\|\zeta - \xi\|^2;$$

(j) F has P_0-property if $F(\zeta) + \varepsilon\zeta$ has the P-property for all $\varepsilon > 0$.

As remarked in [22, Remark 3.1], when F is linear, strong monotonicity and strict monotonicity coincide; and uniform (Jordan) P-property and (Jordan) P-property also coincide. In addition, there have been established some inter-connections between the above concepts, for instances, the following implications hold (see [22, 24]). For more details about P-properties, please refer to [21, 22] and [25].

Property 2.2 For a continuous mapping $F:\mathbb{R}^n\to\mathbb{R}^n$,

(a) strong monotonicity $\Rightarrow$ strict monotonicity $\Rightarrow$ Order P-property $\Rightarrow$ Jordan P-property $\Rightarrow$ P-property $\Rightarrow$ P_0-property;
(b) strong monotonicity $\Rightarrow$ uniform P^*-property $\Rightarrow$ uniform Jordan P-property $\Rightarrow$ uniform P-property $\Rightarrow$ P-property;
(c) monotonicity $\Rightarrow$ P_0-property.

It is also worthy to point out that, when F is linear and self-adjoint, there have strongly monotonicity = Order P-property = Jordan P-property = P-property (see [21, Theorem 21]). Therefore, from Property 2.2(a) and (b), strongly monotonicity, strictly monotonicity, Order P-property, uniform P^*-property, Jordan P-property, uniform Jordan P-property, uniform P-property, and P-property all coincide when F is linear and self-adjoint. This gives a rough direction to construct a counterexample that F has uniform P^*-property but is not strongly monotone function.

To close this section, we want to introduce some other concepts which will be used in analysis of boundedness of level sets. In fact, they are extensions of R_0-property for NCP case. It is known that R_0-property is used to prove the existence of solutions for P_0-NCP. Such properties were recently studied for the following complementarity problems (see [22, Sects. 2, 3]): find $x\in V$ such that

$$x\in K,\qquad F(x)+q\in K,\quad \text{and}\quad \langle x, F(x)+q\rangle=0,$$

where V is a Euclidean Jordan algebra with the associated cone K and $q\in V$. We employ their definitions to prove the properties of bounded level sets for $f_{\mathrm{YF}}, \widehat{f_{\mathrm{YF}}}, f_{\mathrm{LT}}, \widehat{f_{\mathrm{LT}}}$.

Definition 2.2 For a mapping $F:\mathbb{R}^n\to\mathbb{R}^n$, it is called

(a) R_{01}-function if, for any sequence $\{\zeta^k\}$ such that

$$\|\zeta^k\|\to\infty,\qquad \frac{(-\zeta^k)_+}{\|\zeta^k\|}\to 0,\qquad \frac{(-F(\zeta^k))_+}{\|\zeta^k\|}\to 0, \tag{20}$$

we have

$$\liminf_{k\to\infty}\frac{\langle\zeta^k, F(\zeta^k)\rangle}{\|\zeta^k\|^2}>0; \tag{21}$$

(b) R_{02}-function if, for any sequence $\{\zeta^k\}$ such that (20) hold, we have

$$\liminf_{k\to\infty}\frac{\lambda_2(\zeta^k\circ F(\zeta^k))}{\|\zeta^k\|^2}>0. \tag{22}$$

The above concepts are taken from [24] and are extensions of the ones defined for NCP and SDCP settings. In particular, R_{02}-property is equivalent to R_0-property defined in Definition 3.2 of [22]; and hence is equivalent to R_0-matrix when F is linear (note counterexample of R_0-matrix but not monotone matrix can be found in Chap. 3 of [26]). It is easy to see that every R_{01}-function is R_{02}-function [24, Lemma 4]. Also, from Lemma 5 of [24] or Proposition 3.2 of [22], if F has the uniform Jordan P-property then F is R_{02}-function (see [24, Lemma 5]). Where are R_{01}-function and R_{02}-function located in Property 2.2(a) and (b)? There is no answer yet, to the author's best knowledge. However, when F is linear, a chart describes the relation between P-properties and R_0-property is given in [27].

3 Conditions for Error Bounds

The error bound is an important concept that indicates how close an arbitrary point is to the solution set of SOCCP. Thus, an error bound may be used to provide stopping criterion for an iterative method. In this section, we study conditions under which the merit functions $\widehat{f_{\mathrm{YF}}}$, $\widehat{f_{\mathrm{LT}}}$ defined as in(16) provide error bounds for SOCCP. In fact, there have existing results: Proposition 5 of [15], Proposition 4.1 of [13], Proposition 4.1 of [14], and Proposition 4.2 of [14], which indicate that f_{YF}, $\widehat{f_{\mathrm{YF}}}$, f_{LT}, $\widehat{f_{\mathrm{LT}}}$ provide error bounds for SOCCP, respectively, when F is strongly monotone. Our main work is to substitute the condition of strong monotonicity by a weaker condition, uniform P^*-property. We notice that this replacement can be done only for $\widehat{f_{\mathrm{YF}}}$, $\widehat{f_{\mathrm{LT}}}$, and it is not clear yet whether it is true for f_{YF}, f_{LT} or not. The reasons for it will be explained in the section of final remarks (Sect. 5). We begin with the following technical lemmas to reach our claims.

Lemma 3.1 *For any $\zeta \in \mathbb{R}^n$ and $\xi \in \mathcal{K}^n$, we have $\langle \zeta, \xi\rangle \le \langle (\zeta)_+, \xi\rangle$.*

Proof For any $\zeta \in \mathbb{R}^n$, we can write $\zeta = (\zeta)_+ + (\zeta)_-$ where $(\cdot)_+, (\cdot)_-$ represent the projection onto $\mathcal{K}^n$ and $-\mathcal{K}^n$, respectively. Since $\xi \in \mathcal{K}^n$ and $(\zeta)_- \in -\mathcal{K}^n$, we have $\langle (\zeta)_-, \xi\rangle \le 0$. Thus, $\langle \zeta, \xi\rangle = \langle (\zeta)_+, \xi\rangle + \langle (\zeta)_-, \xi\rangle \le \langle (\zeta)_+, \xi\rangle$. In fact, the result is true for any closed convex cone. □

Lemma 3.2 ([15, Lemma 5.2]) *Let $\psi_{\mathrm{FB}}, \phi_{\mathrm{FB}}$ be given by (6) and (7), respectively. Then, for any $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have*

$$4\psi_{\mathrm{FB}}(x,y) \ge 2\left\|\phi_{\mathrm{FB}}(x,y)_+\right\|^2 \ge \left\|(-x)_+\right\|^2 + \left\|(-y)_+\right\|^2.$$

Lemma 3.3 *Let $\tilde{\psi}_i, i = 1, 2,$ be given as in (14). Then, $\tilde{\psi}_i$ satisfies the following inequality*:

$$\tilde{\psi}_i(x,y) \ge \alpha\left(\|(-x)_+\|^2 + \|(-y)_+\|^2\right) \quad \forall (x,y) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{23}$$

for some positive constant α and $i = 1, 2$.

Proof For $\tilde{\psi}_1$, it is clear by definition (14) where $\alpha = \frac{1}{2}$. For ψ_2, the inequality is still true, where $\alpha = \frac{1}{4}$, due to Lemma 3.2. □

Proposition 3.1 *Let $\widehat{f_{\mathrm{LT}}}$ be given as in (15) and (16) with $\tilde{\psi}$ satisfying (23). Suppose that F has uniform P^*-property and SOCCP has a solution ζ^*. Then, there exists a* *scalar* $\tau > 0$ *such that*

$$\tau\|\zeta - \zeta^*\|^2 \le \|(F(\zeta)\circ\zeta)_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\|, \quad \forall\zeta \in \mathbb{R}^n. \tag{24}$$

Moreover,

$$\tau\|\zeta - \zeta^*\|^2 \le \left(\sqrt{2} + \frac{\sqrt{2}}{\sqrt{\alpha}}\right)\widehat{f_{\mathrm{LT}}}(\zeta)^{1/2}, \quad \forall\zeta \in \mathbb{R}^n, \tag{25}$$

where α *is a positive constant.*

Proof From the assumption of uniform P^*-property, there exists $\rho > 0$ such that

$$\rho\|\zeta - \zeta^*\|^2 \le \max_{i=1,2}\left\langle(\zeta - \zeta^*)\circ(F(\zeta) - F(\zeta^*)), u_{\zeta^*}^{(i)}\right\rangle, \tag{26}$$

where $u_{\zeta^*}^{(i)}$, $i = 1, 2$, are the spectral vectors of ζ^*. On the other hand, since ζ^* is a solution of SOCCP, we have $\zeta^* \in \mathcal{K}^n$, $F(\zeta^*) \in \mathcal{K}^n$, $\zeta^*\circ F(\zeta^*) = 0$. Thus,

$$\begin{aligned}
&(\zeta - \zeta^*)\circ(F(\zeta) - F(\zeta^*))\\
&\quad = \zeta\circ F(\zeta) - \zeta^*\circ F(\zeta) - \zeta\circ F(\zeta^*) + \zeta^*\circ F(\zeta^*)\\
&\quad = \zeta\circ F(\zeta) - \zeta^*\circ F(\zeta) - \zeta\circ F(\zeta^*).
\end{aligned}$$

Now, we express the spectral factorizations of ζ and $F(\zeta)$ as below:

$$\begin{aligned}
\zeta &= \lambda_1(\zeta)\cdot u_\zeta^{(1)} + \lambda_2(\zeta)\cdot u_\zeta^{(2)},\\
F(\zeta) &= \lambda_1(F(\zeta))\cdot u_{F(\zeta)}^{(1)} + \lambda_2(F(\zeta))\cdot u_{F(\zeta)}^{(2)}.
\end{aligned}$$

We notice that ζ^* and $F(\zeta^*)$ operator commute due to $\zeta^*\circ F(\zeta^*) = F(\zeta^*)\circ\zeta^* = 0$, $\zeta^* \in \mathcal{K}^n$, and $F(\zeta^*) \in \mathcal{K}^n$. Hence, they share the same Jordan frame; indeed, we can express them as

$$\zeta^* = \lambda_1(\zeta^*)\cdot u_{\zeta^*}^{(1)} + \lambda_2(\zeta^*)\cdot u_{\zeta^*}^{(2)} = \sum_{j=1}^{2}\lambda_j(\zeta^*)\cdot u_{\zeta^*}^{(i)},$$

$$F(\zeta^*) = \lambda_2(F(\zeta^*))\cdot u_{\zeta^*}^{(1)} + \lambda_1(F(\zeta^*))\cdot u_{\zeta^*}^{(2)} = \sum_{j=1}^{2}\lambda_{j^*}(F(\zeta^*))\cdot u_{\zeta^*}^{(i)},$$

where j^* denotes $j^* = 2$ for $j = 1$ and $j^* = 1$ for $j = 2$. It needs to point out that, when x and y share the same Jordan frame, it does not necessarily hold $u_x^{(i)} = u_y^{(i)}$ for $i = 1, 2$. In general, it holds that $u_x^{(1)} = u_y^{(2)}$ and $u_x^{(2)} = u_y^{(1)}$. Then, for $i = 1, 2$. we have

$$\begin{aligned}
&\left\langle(\zeta - \zeta^*)\circ(F(\zeta) - F(\zeta^*)), u_{\zeta^*}^{(i)}\right\rangle\\
&\quad = \left\langle\zeta\circ F(\zeta) - \zeta^*\circ F(\zeta) - \zeta\circ F(\zeta^*), u_{\zeta^*}^{(i)}\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle \zeta \circ F(\zeta), u_{\zeta^*}^{(i)} \rangle + \langle -\zeta^* \circ F(\zeta), u_{\zeta^*}^{(i)} \rangle + \langle -\zeta \circ F(\zeta^*), u_{\zeta^*}^{(i)} \rangle \\
&= \langle \zeta \circ F(\zeta), u_{\zeta^*}^{(i)} \rangle + \langle -F(\zeta), \zeta^* \circ u_{\zeta^*}^{(i)} \rangle + \langle -\zeta, F(\zeta^*) \circ u_{\zeta^*}^{(i)} \rangle \\
&= \langle \zeta \circ F(\zeta), u_{\zeta^*}^{(i)} \rangle + \left\langle -F(\zeta), \sum_{j=1}^{2} \lambda_j(\zeta^*) u_{\zeta^*}^{(j)} \circ u_{\zeta^*}^{(i)} \right\rangle \\
&\quad + \left\langle -\zeta, \sum_{j=1}^{2} \lambda_{j^*}(F(\zeta^*)) u_{\zeta^*}^{(j)} \circ u_{\zeta^*}^{(i)} \right\rangle \\
&= \langle (\zeta \circ F(\zeta)), u_{\zeta^*}^{(i)} \rangle + \lambda_i(\zeta^*) \langle -F(\zeta), u_{\zeta^*}^{(i)} \rangle + \lambda_{i^*}(F(\zeta^*)) \langle -\zeta, u_{\zeta^*}^{(i)} \rangle \\
&\le \langle (\zeta \circ F(\zeta))_+, u_{\zeta^*}^{(i)} \rangle + \lambda_i(\zeta^*) \langle (-F(\zeta))_+, u_{\zeta^*}^{(i)} \rangle + \lambda_{i^*}(F(\zeta^*)) \langle (-\zeta)_+, u_{\zeta^*}^{(i)} \rangle \\
&\le \|(\zeta \circ F(\zeta))_+\| \cdot \|u_{\zeta^*}^{(i)}\| + \lambda_i(\zeta^*) \|(-F(\zeta))_+\| \\
&\quad \times \|u_{\zeta^*}^{(i)}\| + \lambda_{i^*}(F(\zeta^*)) \|(-\zeta)_+\| \cdot \|u_{\zeta^*}^{(i)}\| \\
&= \frac{1}{\sqrt{2}} \left[\|(\zeta \circ F(\zeta))_+\| + \lambda_i(\zeta^*) \|(-F(\zeta))_+\| + \lambda_{i^*}(F(\zeta^*)) \|(-\zeta)_+\| \right] \\
&\le \max\left\{ \frac{1}{\sqrt{2}}, \frac{\lambda_i(\zeta^*)}{\sqrt{2}}, \frac{\lambda_{i^*}(F(\zeta^*))}{\sqrt{2}} \right\} \\
&\quad \times \left[\|(\zeta \circ F(\zeta))_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\| \right] \\
&\le \max\left\{ \frac{1}{\sqrt{2}}, \frac{\lambda_2(\zeta^*)}{\sqrt{2}}, \frac{\lambda_2(F(\zeta^*))}{\sqrt{2}} \right\} \\
&\quad \times \left[\|(\zeta \circ F(\zeta))_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\| \right],
\end{aligned}
$$

where the last equality uses Property 2.1, the first inequality is from Lemma 3.1, and the second inequality uses the fact that $\lambda_i(\zeta^*) \ge 0, \lambda_{i^*}(F(\zeta^*)) \ge 0$ since $\zeta^* \in \mathcal{K}^n, F(\zeta^*) \in \mathcal{K}^n$. Also, note that i^* denotes $i^* = 2$ for $i = 1$ and $i^* = 1$ for $i = 2$. Now, let

$$\tau := \frac{\rho}{\max\{\frac{1}{\sqrt{2}}, \frac{\lambda_2(\zeta^*)}{\sqrt{2}}, \frac{\lambda_2(F(\zeta^*))}{\sqrt{2}}\}} > 0;$$

then the above and (26) give

$$\tau \|\zeta - \zeta^*\|^2 \le \|(F(\zeta) \circ \zeta)_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\|, \quad \forall \zeta \in \mathbb{R}^n.$$

Next, we come to the second part of the proposition. By $\widehat{f_{\mathrm{LT}}}(\zeta) = \frac{1}{2}\|(F(\zeta) \circ \zeta)_+\|^2 + \tilde{\psi}(F(\zeta), \zeta)$, we have

$$\|(F(\zeta) \circ \zeta)_+\| \le \sqrt{2}\widehat{f_{\mathrm{LT}}}(\zeta)^{1/2}.$$

In addition, we know that

$$\|(-F(\zeta))_+\| + \|(-\zeta)_+\| \le \sqrt{2}(\|(-F(\zeta))_+\|^2 + \|(-\zeta)_+\|^2)^{1/2}$$

$$\leq \frac{\sqrt{2}}{\sqrt{\alpha}}\tilde{\psi}(F(\zeta),\zeta)^{1/2}$$
$$\leq \frac{\sqrt{2}}{\sqrt{\alpha}}\widehat{f_{\mathrm{LT}}}(\zeta)^{1/2},$$

where the second inequality is true by Lemma 3.3. Thus,

$$\|(F(\zeta)\circ\zeta)_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\| \leq \left(\sqrt{2}+\frac{\sqrt{2}}{\sqrt{\alpha}}\right)\widehat{f_{\mathrm{LT}}}(\zeta)^{1/2}.$$

This together with (24) yields (25). □

Proposition 3.2 *Let $\widehat{f_{\mathrm{YF}}}$ be given as in* (11) *and* (16). *Suppose that F has uniform P^*-property and SOCCP has a solution ζ^*. Then, there exists a scalar $\tau > 0$ such that*

$$\tau\|\zeta-\zeta^*\|^2 \leq \|(F(\zeta)\circ\zeta)_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\|, \quad \forall\zeta\in\mathbb{R}^n. \tag{27}$$

Moreover,

$$\tau\|\zeta-\zeta^*\|^2 \leq 3\sqrt{2}\widehat{f_{\mathrm{YF}}}(\zeta)^{1/2}, \quad \forall\zeta\in\mathbb{R}^n. \tag{28}$$

Proof It follows totally the same arguments as in the proof for Proposition 3.1 to obtain (27). It remains to show the second part. Since by $\widehat{f_{\mathrm{YF}}}(\zeta) = \frac{1}{2}\|(F(\zeta)\circ\zeta)_+\|^2 + \psi_{\mathrm{FB}}(F(\zeta),\zeta)$, we have

$$\|(F(\zeta)\circ\zeta)_+\| \leq \sqrt{2}\widehat{f_{\mathrm{YF}}}(\zeta)^{1/2}.$$

In addition, we know that

$$\begin{aligned}\|(-F(\zeta))_+\| + \|(-\zeta)_+\| &\leq \sqrt{2}\big(\|(-F(\zeta))_+\|^2 + \|(-\zeta)_+\|^2\big)^{1/2}\\ &\leq 2\sqrt{2}\psi_{\mathrm{FB}}(F(\zeta),\zeta)^{1/2}\\ &\leq 2\sqrt{2}\widehat{f_{\mathrm{YF}}}(\zeta)^{1/2},\end{aligned}$$

where the second inequality is true by Lemma 3.2. Thus,

$$\|(F(\zeta)\circ\zeta)_+\| + \|(-F(\zeta))_+\| + \|(-\zeta)_+\| \leq 3\sqrt{2}\widehat{f_{\mathrm{YF}}}(\zeta)^{1/2}.$$

This together with (27) yield (28). □

4 Conditions for Bounded Level Sets

The boundedness of level sets of a merit function is also important since it ensures that the sequences generated by a descent method has at least one accumulation point. In particular, there have existing results of bounded level sets for f_{YF}, $\widehat{f_{\mathrm{YF}}}$, f_{LT}, $\widehat{f_{\mathrm{LT}}}$, respectively, for instances, Proposition 6 of [15], Proposition 4.2 of [13], Proposition 4.3 of [14] and Proposition 4.4 of [14], which require that F is monotone and SOCCP is strict feasible. We note that the strict feasibility is necessary. For example, when $F(\zeta) \equiv 0$ every $\zeta \in \mathcal{K}^n$ is a solution of SOCCP and hence the solution set is unbounded. In this section, we study another condition to replace this kind of "*strict*" condition by F being R_{01}-function for cases of f_{YF}, f_{LT}, while by F being R_{02}-functions for cases of $\widehat{f_{\mathrm{LT}}}$, $\widehat{f_{\mathrm{YF}}}$.

We want to point it out that for f_{LT} and $\widehat{f_{\mathrm{LT}}}$ to possess property of bounded level sets, an additional condition is required (see Lemma 4.1). In fact, the examples of $\tilde{\psi}_1$ and $\tilde{\psi}_2$ given in (14) both satisfy this additional condition (Lemma 4.1). It was also proved in Lemma 9 of [15] that this additional condition is satisfied with ψ_{FB} as well.

Lemma 4.1 ([14, Lemma 4.4]) *For any $\{(x^k, y^k)\}_{k=1}^{\infty} \subseteq \mathbb{R}^n \times \mathbb{R}^n$, let $\lambda_1(x)^k \leq \lambda_2(x)^k$ and $\mu_1(y)^k \leq \mu_2(y)^k$ denote the spectral values of x^k and y^k, respectively. Then, if $\lambda_1(x)^k \to -\infty$ or $\mu_1(y)^k \to -\infty$, we have $\tilde{\psi}_i(x^k, y^k) \to \infty$, for $i = 1, 2$.*

Now, we come to another main work of this paper that is to claim the monotonicity of F plus the strict feasibility of SOCCP can be replaced by R_{01} (or R_{02})-functions to ensure property of bounded level sets.

Proposition 4.1 (a) *Let f_{YF} be given as in (9) and (16). Suppose that F is a R_{01}-function. Then, the level set*

$$\mathcal{L}(\gamma) := \{\zeta \in \mathbb{R}^n \mid f_{\mathrm{YF}}(\zeta) \leq \gamma\}$$

is bounded for all $\gamma \geq 0$.

(b) *Let f_{LT} be given as in (12) and (16) with $\tilde{\psi}$ satisfying Lemma 4.1. Suppose that F is a R_{01}-function. Then, the level set*

$$\mathcal{L}(\gamma) := \{\zeta \in \mathbb{R}^n \mid f_{\mathrm{LT}}(\zeta) \leq \gamma\}$$

is bounded for all $\gamma \geq 0$.

Proof (a) We will prove this result by contradiction. Suppose there exists an unbounded sequence $\{\zeta^k\} \subset \mathcal{L}(\gamma)$ for some $\gamma \geq 0$. It can be seen that the sequence of the smaller spectral values of $\{\zeta^k\}$ and $\{F(\zeta^k)\}$ are bounded below. In fact, if not, it follows from Lemma 4.1 (note ψ_{FB} also satisfies this lemma, see Lemma 9 of [15]) that $f_{\mathrm{YF}}(\zeta^k) \to \infty$, which contradicts $\{\zeta^k\} \subset \mathcal{L}(\gamma)$.

Therefore, $\{(-\zeta^k)_+\}$ and $\{(-F(\zeta^k))_+\}$ are bounded above, which says condition (20) is held. Then, by the assumption of R_{01}-function, we have

$$\liminf_{k\to\infty} \frac{\langle \zeta^k, F(\zeta^k)\rangle}{\|\zeta^k\|^2} > 0.$$

This yields $\langle \zeta^k, F(\zeta^k)\rangle \to \infty$, and hence $f_{\mathrm{YF}}(\zeta^k) \to \infty$ by definition of f_{YF} given as in (9) and (10). Thus, it is a contradiction to $\{\zeta^k\} \subset \mathcal{L}(\gamma)$.

(b) Same arguments as part (a). □

Proposition 4.2 (a) *Let $\widehat{f_{\rm LT}}$ be given as in* (15) *and* (16) *with $\tilde{\psi}$ satisfying Lemma* 4.1. *Suppose that F is a R_{02}-function. Then, the level set*

$$\mathcal{L}(\gamma) := \{\zeta \in \mathbb{R}^n \mid \widehat{f_{\rm LT}}(\zeta) \le \gamma\}$$

is bounded for all $\gamma \ge 0$.

(b) *Let $\widehat{f_{\rm YF}}$ be given as in* (11) *and* (16). *Suppose that F is a R_{02}-function. Then, the level set*

$$\mathcal{L}(\gamma) := \{\zeta \in \mathbb{R}^n \mid \widehat{f_{\rm YF}}(\zeta) \le \gamma\}$$

is bounded for all $\gamma \ge 0$.

Proof (a) Again, we will prove this result by contradiction. Suppose there exists an unbounded sequence $\{\zeta^k\} \subset \mathcal{L}(\gamma)$ for some $\gamma \ge 0$. It can be seen that the sequence of the smaller spectral values of $\{\zeta^k\}$ and $\{F(\zeta^k)\}$ are bounded below. In fact, if not, it follows from Lemma 4.1 (note we assume $\tilde{\psi}$ satisfies this lemma) that $\widehat{f_{\rm LT}}(\zeta^k) \to \infty$, which contradicts $\{\zeta^k\} \subset \mathcal{L}(\gamma)$.

Thus, $\{(-\zeta^k)_+\}$ and $\{(-F(\zeta^k))_+\}$ are bounded above, which says condition (20) is held. Then, by the assumption of R_{02}-function, we have

$$\liminf_{k\to\infty} \frac{\lambda_2(\zeta^k \circ F(\zeta^k))}{\|\zeta^k\|^2} > 0.$$

This yields $\lambda_2(\zeta^k \circ F(\zeta^k)) \to \infty$, and hence $\|(\zeta^k \circ F(\zeta^k))_+\| \to \infty$. This together with definition of $\widehat{f_{\rm LT}}$ given as in (15) and (10) imply $\widehat{f_{\rm LT}}(\zeta^k) \to \infty$. But, this contradicts $\{\zeta^k\} \subset \mathcal{L}(\gamma)$. Therefore, we complete the proof.

(b) Same arguments as part (a). □

5 Final Remarks

In this paper, we have studied conditions for error bounds and bounded level sets of some merit functions, $f_{\rm YF}$, $\widehat{f_{\rm YF}}$, $f_{\rm LT}$, $\widehat{f_{\rm LT}}$ given as in (16) for SOCCP. For property of bounded level sets, we propose a new condition, F being R_{01}-function, to replace the traditional condition of monotonicity of F and strict feasibility of SOCCP in the cases of $f_{\rm YF}$, $f_{\rm LT}$. In the contrast, we propose another condition, F being R_{02}-function, to replace the traditional condition of monotonicity of F and strict feasibility of SOCCP in the cases of $\widehat{f_{\rm LT}}$, $\widehat{f_{\rm YF}}$. We notice that the condition of R_{02}-function is even weaker than R_{01}-function, which means we need a bit stronger condition in cases of $f_{\rm YF}$, $f_{\rm LT}$ to obtain property of bounded level sets than in cases of $\widehat{f_{\rm LT}}$, $\widehat{f_{\rm YF}}$ to do. This observation seems true for property of error bounds. More specifically, we have established the new condition of uniform P^*-property to ensure that $\widehat{f_{\rm LT}}$, $\widehat{f_{\rm YF}}$ provide error bounds (see Propositions 3.1 and 3.2). Thus, due to this observation, we suspect that there needs a condition between strongly monotonicity and uniform P^*-property (see the implications in Property 2.2(b)) to ensure $f_{\rm YF}$, $f_{\rm LT}$ to provide error bounds for SOCCP. However, we still don't know whether there is a condition between strongly monotonicity and uniform P^*-property (see Property 2.2(b)). That is the reason we don't have similar results of error bounds for f_{YF}, f_{LT} yet.

We can elaborate more to explain the above reason from the other aspect. In fact, the existing results of error bounds in Proposition 5 of [15] and Proposition 4.1 of [14] (for f_{YF} and f_{LT}, respectively) say that there exists a scalar $\tau > 0$ such that

$$\tau\|\zeta - \zeta^*\|^2 \le \max\{0, \langle F(\zeta), \zeta\rangle\} + \|(-F(\zeta))_+\| + \|(-\zeta)_+\|, \quad \forall \zeta \in \mathbb{R}^n. \tag{29}$$

Now, by the fact from Lemma 4.1 of [13],

$$\langle x, y\rangle \le \sqrt{2}\|(x \circ y)_+\|, \quad \forall x, y \in \mathbb{R}^n,$$

we can see that

$$\max\{0, \langle F(\zeta), \zeta\rangle\} \le \sqrt{2}\|(F(\zeta) \circ \zeta)_+\|.$$

In other words, if (29) is true then (24) is also held. But the converse is not guaranteed. If we follow the same arguments as in Propositions 3.1 and 3.2, we can obtain (24). Nonetheless, (24) does not imply (29) as explained above. Thus, the uniform P^*-property does not guarantee the property of error bounds for f_{YF}, f_{LT}. Therefore, it is still worth of watching up on the issue of finding a weaker condition than strong monotonicity for f_{YF}, f_{LT} to provide error bounds for SOCCP.

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