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ScienceDirect
Nonlinear Analysis 70 (2009) 1475–1491
Nonlinear Analysis
www.elsevier.com/locate/na

A regularization method for the second-order cone complementarity problem with the Cartesian P_0-property

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Received 12 October 2007; accepted 14 February 2008

Abstract

We consider the Tikhonov regularization method for the second-order cone complementarity problem (SOCCP) with the Cartesian P_0-property. We show that many results of the regularization method for the P_0-nonlinear complementarity problem still hold for this important class of nonmonotone SOCCP. For example, under the more general setting, every regularized problem has the unique solution, and the solution trajectory generated is bounded if the original SOCCP has a nonempty and bounded solution set. We also propose an inexact regularization algorithm by solving the sequence of regularized problems approximately with the merit function approach based on Fischer–Burmeister merit function, and establish the convergence result of the algorithm. Preliminary numerical results are also reported, which verify the favorable theoretical properties of the proposed method.

Keywords: Second-order cone complementarity problem; Tikhonov regularization; Cartesian P_0-property; Fischer–Burmeister merit function

1. Introduction

We consider the second-order cone complementarity problem (SOCCP) which is to find a point $x \in \mathbb{R}^n$ such that

$$x \in \mathcal{K}, \qquad F(x) \in \mathcal{K}, \qquad \langle x, F(x) \rangle = 0, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ represents the Euclidean inner product, $F : \mathbb{R}^n \to \mathbb{R}^n$ is a mapping assumed to be continuously differentiable throughout this paper, and $\mathcal{K}$ is the Cartesian product of second-order cones (SOCs), also called Lorentz cones [9]. In other words,

$$\mathcal{K} = \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_m}, \tag{2}$$

where $m,\ n_1, \ldots, n_m \geq 1$, $n_1 + n_2 + \cdots + n_m = n$, and

$$\mathcal{K}^{n_i} := \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n_i - 1} \mid x_1 \geq \|x_2\| \right\}$$

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0362-546X/$ - see front matter
doi:10.1016/j.na.2008.02.028

with $\|\cdot\|$ denoting the Euclidean norm and $\mathcal{K}^1$ denoting the set of nonnegative reals $\mathbb{R}_+$. In what follows, we refer (1) and (2) to the SOCCP(F). An important special case of (2) is $\mathcal{K} = \mathbb{R}^n_+$, the nonnegative orthant in $\mathbb{R}^n$, which corresponds to $n_1 = \cdots = n_m = 1$ and $m = n$, and the SOCCP(F) reduces to the nonlinear complementarity problem (NCP).

There exist various methods for solving the SOCCP(F). They include the smoothing Newton method [1,12], the smoothing-regularization method [14], the merit function approaches [2,3], and the semismooth Newton method [17]. Most of these methods are proposed for the monotone SOCCP. In this paper, we will consider a particular method, i.e. the *Tikhonov regularization method*, for a class of nonmonotone SOCCP.

It is well known that the Tikhonov regularization method is designed to deal with the ill-posed problems which substitute the solution of the original problem with the solution of a sequence of well-posed problems whose solutions converge to a solution of the original problem; see [11,20] and the references therein. In the context of SOCCPs, the regularization scheme consists in solving a sequence of SOCCP(F_ε):

$$x \in \mathcal{K}, \qquad F_\varepsilon(x) \in \mathcal{K}, \qquad \langle x, F_\varepsilon(x)\rangle = 0, \tag{3}$$

where ε is a positive parameter tending to zero and $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$F_\varepsilon(x) := F(x) + \varepsilon x. \tag{4}$$

The regularization scheme was considered by [14], where it was used only to guarantee that the proposed smoothing algorithm could handle the monotone SOCCP. In this paper, we apply the regularization scheme for the SOCCP with the Cartesian P_0-property.

Specifically, paralleling to the classical results of regularization methods for convex optimization problems [6], we try to generalize as much as possible the following results to the large class of SOCCP with F having the Cartesian P_0-property:

(a) The regularized problem SOCCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$.
(b) The trajectory $x(\varepsilon)$ is continuous for $\varepsilon > 0$.
(c) For $\varepsilon \to 0$, the trajectory $x(\varepsilon)$ converges to the least l_2-norm solution of SOCCP(F) if the SOCCP(F) has a nonempty solution set, and otherwise it diverges.

In Section 3, we generalize the result (a) to the more general setting, and concentrate on the partial extension of the results (b) and (c) in Section 4. Then, we propose an inexact regularization algorithm for the SOCCP(F) in Section 5, and establish the corresponding convergence results. In Section 6, we report our numerical experience with the algorithm for solving some linear SOCPs from the DIMACS library and some SOCCPs generated randomly with the Cartesian P_0-property, and make comparisons with the merit function approach [2] to verify the favorable theoretical properties of the proposed method. Finally, we conclude this paper with several open questions.

Throughout this paper, $\mathbb{R}^n$ denotes the space of n-dimensional real column vectors, and $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ is identified with $\mathbb{R}^{n_1+\cdots+n_m}$. Thus, $(x_1, \ldots, x_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ is viewed as a column vector in $\mathbb{R}^{n_1+\cdots+n_m}$. For a differentiable mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, $F'(x) \in \mathbb{R}^{n\times n}$ denotes the Jacobian matrix of F at x while $\nabla F(x) \in \mathbb{R}^{n\times n}$ denotes the transpose Jacobian of F at x. If $\mathcal{J}$ and $\mathcal{B}$ are index sets such that $\mathcal{J}, \mathcal{B} \subseteq \{1, 2, \ldots, m\}$, we denote $M_{\mathcal{J}\mathcal{B}}$ by the block matrix consisting of the submatrices $M_{jk} \in \mathbb{R}^{n_j \times n_k}$ of M with $j \in \mathcal{J}, k \in \mathcal{B}$, and $x_{\mathcal{B}}$ by a vector consisting of subvectors $x_i \in \mathbb{R}^{n_i}$ with $i \in \mathcal{B}$. Given $x \in \mathbb{R}^n$, $[x]_+$ and $[x]_-$ denote the minimum distance projection of x onto $\mathcal{K}$ and $-\mathcal{K}$, respectively. For a set S, the notation int(S) denotes the interior of S. We write $F = (F_1, \ldots, F_m)$ with $F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ and $F_\varepsilon = (F_{\varepsilon,1}, \ldots, F_{\varepsilon,m})$ with $F_{\varepsilon,i} : \mathbb{R}^n \to \mathbb{R}^{n_i}$.

2. Preliminaries

We first review some basic concepts and properties related to the SOC $\mathcal{K}^l$ $(l > 1)$, and then introduce the concepts of Cartesian P-properties and P-properties for a matrix $M \in \mathbb{R}^{n\times n}$ and a nonlinear transformation $F : \mathbb{R}^n \to \mathbb{R}^n$, respectively.

For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{l-1}$, define their Jordan product as

$$x \circ y := (\langle x, y\rangle,\ x_1 y_2 + y_1 x_2)\,. \tag{5}$$

Write $x + y$ to mean the usual componentwise addition of vectors and x^2 to mean $x \circ x$. Then $\circ$, $+$ and $e = (1, 0, \dots, 0)^{\mathrm{T}} \in \mathbb{R}^l$ have the following basic properties [9,12]: (1) $e \circ x = x$ for all $x \in \mathbb{R}^l$. (2) $x \circ y = y \circ x$ for all $x, y \in \mathbb{R}^l$. (3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{R}^l$. (4) $(x + y) \circ z = x \circ z + y \circ z$ for all $x, y, z \in \mathbb{R}^l$. Notice that the Jordan product is not associative, but it is power associated, i.e.,$x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^l$. We stipulate $x^0 = e$. Besides, $\mathcal{K}^l$ is not closed under Jordan product.

From [9,12], any vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{l-1}$ has the spectral factorization:

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)}, \tag{6}$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \qquad u_x^{(i)} = \frac{1}{2}\left(1,\ (-1)^i \bar{x}_2\right), \tag{7}$$

with $\bar{x}_2 = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$ and otherwise $\bar{x}_2$ being any vector in $\mathbb{R}^{l-1}$ such that $\|\bar{x}_2\| = 1$. If $x_2 \neq 0$, the factorization is unique. The spectral factorizations of x, x^2 and $x^{1/2}$ have various interesting properties; see [9,12]. Here we list some that will be used later.

Property 2.1. *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{l-1}$, let $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ be the spectral values and the associated spectral vectors. Then, the following results hold:*

(a) *For any $x \in \mathbb{R}^l$, $x^2 = [\lambda_1(x)]^2\, u_x^{(1)} + [\lambda_2(x)]^2\, u_x^{(2)} \in \mathcal{K}^l$.*
(b) $x \in \mathcal{K}^l \Longleftrightarrow 0 \le \lambda_1(x) \le \lambda_2(x)$ *and* $x \in \mathrm{int}(\mathcal{K}^l) \Longleftrightarrow 0 < \lambda_1(x) \le \lambda_2(x)$.
(c) *For any $x \in \mathcal{K}^l$, $x^{1/2} = \sqrt{\lambda_1(x)}\, u_x^{(1)} + \sqrt{\lambda_2(x)}\, u_x^{(2)} \in \mathcal{K}^l$.*
(d) $x \in \mathcal{K}^l$ *if and only if the symmetric matrix* $L_x := \begin{bmatrix} x_1 & x_2^{\mathrm{T}} \\ x_2 & x_1 I \end{bmatrix}$ *is positive semidefinite, and* $x \in \mathrm{int}(\mathcal{K}^l)$ *if and only if* L_x *is positive definite.*

Next we present the definitions of Cartesian P-properties for a matrix $M \in \mathbb{R}^{n\times n}$, which are special cases of those introduced by Chen and Qi [5] for a linear transformation.

Definition 2.1. A matrix $M \in \mathbb{R}^{n\times n}$ is said to have

(a) the Cartesian P-property if for every nonzero $z = (z_1, \dots, z_m) \in \mathbb{R}^n$ with $z_i \in \mathbb{R}^{n_i}$, there exists an index $\nu \in \{1, 2, \dots, m\}$ such that $\langle z_\nu, (Mz)_\nu\rangle > 0$;
(b) the Cartesian P_0-property if for every nonzero $z = (z_1, \dots, z_m) \in \mathbb{R}^n$ with $z_i \in \mathbb{R}^{n_i}$, there exists a $\nu \in \{1, 2, \dots, m\}$ such that $z_\nu \neq 0$ and $\langle z_\nu,\ (Mz)_\nu\rangle \ge 0$.

Clearly, when $m = n$ and $n_1 = \cdots = n_m = 1$, M having the Cartesian P-property (or the Cartesian P_0-property) coincides with M being a P-matrix (or P_0-matrix) introduced in [4]. Let M be an $n \times n$ matrix with elements m_{ij}. Then, M can be denoted by

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1m} \\ M_{21} & M_{22} & \cdots & M_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ M_{m1} & M_{m2} & \cdots & M_{mm} \end{bmatrix}, \tag{8}$$

where $M_{\nu l}$ for each $\nu = 1, \dots, m$ and $l = 1, \dots, m$ is an $n_\nu \times n_l$ matrix consisting of those elements m_{kj} with $k = n_{\nu-1} + 1, \dots, n_\nu$, $j = n_{j-1} + 1, \dots, n_j$ and $n_0 = 0$. Let S be a proper subset of $\{1, 2, \dots, m\}$ and denote by $M(S)$ the matrix resulting from deleting the block matrix $M_{\nu l}$ with ν and l complementary to those indicated by S from M given as in (8). We call $M(S)$ *a principal block matrix* of M. By Definition 2.1, it is not hard to verify that every principal block matrix $M(S)$ must have the Cartesian P-property if the matrix M has the Cartesian P-property. When $m = n$ and $n_1 = \cdots = n_m = 1$, this reduces to the well-known fact that every principal submatrix of a P-matrix is again a P-matrix. Particularly, assume that the matrix M, by rearrangement, is written as

$$M = \begin{bmatrix} M_{\mathcal{J}\mathcal{J}} & M_{\mathcal{J}\mathcal{B}} \\ M_{\mathcal{B}\mathcal{J}} & M_{\mathcal{B}\mathcal{B}} \end{bmatrix}, \tag{9}$$

where $\mathcal{J}$ and $\mathcal{B}$ are index sets such that $\mathcal{J} \cup \mathcal{B} = \{1, 2, \ldots, m\}$ and $\mathcal{J} \cap \mathcal{B} = \emptyset$. Then, when M has the Cartesian P-property and $M_{\mathcal{J}\mathcal{J}}$ is nonsingular, we have the following result, which can be regarded as an extension of the fact that any Schur-complement of a P-matrix is also a P-matrix.

Proposition 2.1. *Suppose that M defined as in (9) has the Cartesian P-property and the matrix $M_{\mathcal{J}\mathcal{J}}$ is nonsingular. Then its Schur-complement in the matrix M, i.e.*

$$\widehat{M}_{\mathcal{J}\mathcal{J}} = M_{\mathcal{B}\mathcal{B}} - M_{\mathcal{B}\mathcal{J}}(M_{\mathcal{J}\mathcal{J}})^{-1}M_{\mathcal{J}\mathcal{B}}$$

also has the Cartesian P-property.

Proof. Let $y_{\mathcal{B}}$ be an arbitrary nonzero vector with the dimension same as $M_{\mathcal{B}\mathcal{B}}$. Let $x_{\mathcal{J}}$ be a vector with the dimension same as $M_{\mathcal{J}\mathcal{J}}$ such that

$$M_{\mathcal{J}\mathcal{J}}x_{\mathcal{J}} + M_{\mathcal{J}\mathcal{B}}y_{\mathcal{B}} = 0, \tag{10}$$

or equivalently,

$$x_{\mathcal{J}} = -(M_{\mathcal{J}\mathcal{J}})^{-1}M_{\mathcal{J}\mathcal{B}}y_{\mathcal{B}}. \tag{11}$$

Let $z = (x_{\mathcal{J}}, y_{\mathcal{B}}) \in \mathbb{R}^n$. Then, $z \neq 0$. From Definition 2.1(a) and the given assumption that M has the Cartesian P-property, there exists an index $i \in \{1, 2, \ldots, m\}$ such that

$$\langle z_i, (Mz)_i\rangle > 0. \tag{12}$$

Notice that the index i must belong to the set $\mathcal{B}$. If not, i.e. $i \in \mathcal{J}$, then from the definition of M we learn that inequality (12) is equivalent to

$$\langle x_i,\ [M_{\mathcal{J}\mathcal{J}}x_{\mathcal{J}} + M_{\mathcal{J}\mathcal{B}}y_{\mathcal{B}}]_i\rangle > 0,$$

which obviously contradicts equality (10). Now (12) is equivalent to

$$\langle y_i,\ [M_{\mathcal{B}\mathcal{J}}x_{\mathcal{J}} + M_{\mathcal{B}\mathcal{B}}y_{\mathcal{B}}]_i\rangle > 0.$$

Using the inequality and Eq. (11), we immediately have that

$$\begin{aligned}\langle y_i,\ [\widehat{M}_{\mathcal{J}\mathcal{J}}y_{\mathcal{B}}]_i\rangle &= \langle y_i, [M_{\mathcal{B}\mathcal{B}}y_{\mathcal{B}} - M_{\mathcal{B}\mathcal{J}}(M_{\mathcal{J}\mathcal{J}})^{-1}M_{\mathcal{J}\mathcal{B}}y_{\mathcal{B}}]_i\rangle\\ &= \langle y_i,\ [M_{\mathcal{B}\mathcal{B}}y_{\mathcal{B}} + M_{\mathcal{B}\mathcal{J}}x_{\mathcal{J}}]_i\rangle > 0.\end{aligned}$$

Thus, by Definition 2.1(a), the matrix $\widehat{M}_{\mathcal{J}\mathcal{J}}$ has the Cartesian P-property. □

Definition 2.2 (*[13]*). A matrix $M \in \mathbb{R}^{n\times n}$ is said to have

(a) the Jordan P-property (or the P_1-property) if $x \circ (Mx) \in -\mathcal{K} \Rightarrow x = 0$;
(b) the P-property if the condition that $L_{x_i}L_{(Mx)_i} = L_{(Mx)_i}L_{x_i},\ i = 1, 2, \ldots, m$ and $x \circ (Mx) \in -\mathcal{K}$ necessarily implies $x = 0$;
(c) the P_0-property if $M + \varepsilon I$ for any $\varepsilon > 0$ has the P-property.

Proposition 2.2. (a) *If a matrix $M \in \mathbb{R}^{n\times n}$ has the Cartesian P-property, then it also has the Jordan P-property and the P-property.*
(b) *If a matrix $M \in \mathbb{R}^{n\times n}$ has the Cartesian P_0-property, then it has the P_0-property.*

Proof. (a) From Definition 2.2(a) and (b), it is not hard to see that the Jordan P-property implies the P-property. Therefore, we only need to prove that the Cartesian P-property implies the Jordan P-property. Let $x = (x_1, \ldots, x_m) \in \mathbb{R}^n$ with $x_i \in \mathbb{R}^{n_i}$ be any vector such that $x \circ (Mx) \in -\mathcal{K}$. From the Cartesian structure of $\mathcal{K}$, we have

$$x_i \circ (Mx)_i \in -\mathcal{K}^{n_i} \quad \text{for } i = 1, 2, \ldots, m,$$

which, by the definition of Jordan product given by (5), means that

$$\langle x_i,\ (Mx)_i\rangle \leq 0 \quad \text{for all } i = 1, 2, \ldots, m. \tag{13}$$

Now, suppose that $x \neq 0$. Then, from Definition 2.1(a), it follows that there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that $\langle x_\nu, (Mx)_\nu \rangle > 0$, which clearly contradicts (13). Hence, M has the Jordan P-property.

(b) Observe that for any $\varepsilon > 0$, $M + \varepsilon I$ has the Cartesian P-property. By part (a) and Definition 2.2(c), M has the P_0-property. □

The Cartesian P_0-property may not imply the P-property. For example, let $m = 2$ and $n_1 = n_2 = 2$, and consider

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix} \quad \text{and} \quad x = \begin{pmatrix} -2 \\ 2 \\ -1 \\ 1 \end{pmatrix}.$$

It is easy to verify that M has the Cartesian P_0-property, $x \circ (Mx) = (0, 0, 0, 0) \in -\mathcal{K} = -(\mathcal{K}^2 \times \mathcal{K}^2)$ and $L_x L_{Mx} = L_{Mx} L_x = 0$, but $x \neq 0$, i.e., M has no P-property. Now, we are not clear whether the P-property implies the Cartesian P_0-property.

Next we introduce definitions of Cartesian P-properties for a nonlinear mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ in the setting of SOCs. The concepts of P-properties on Cartesian products in $\mathbb{R}^n$ were first established by Facchinei and Pang [10]. Recently, Chen and Qi [5] and Kong et al. [15] extended the concepts of Cartesian P-properties to the setting of positive semidefinite cones and the general Euclidean Jordan algebra, respectively.

Definition 2.3. A nonlinear mapping $F = (F_1, \ldots, F_m)$ with $F_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ is said to

(a) have the uniform Cartesian P-property if there exists a constant $\rho > 0$ such that, for any $x, y \in \mathbb{R}^n$, there is an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$\langle x_\nu - y_\nu, F_\nu(x) - F_\nu(y) \rangle \geq \rho \|x - y\|^2;$$

(b) have the Cartesian P-property if for any $x, y \in \mathbb{R}^n$ with $x \neq y$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$\langle x_\nu - y_\nu, F_\nu(x) - F_\nu(y) \rangle > 0;$$

(c) have the Cartesian P_0-property if for any $x, y \in \mathbb{R}^n$ with $x \neq y$, there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$x_\nu \neq y_\nu \text{ and } \langle x_\nu - y_\nu, F_\nu(x) - F_\nu(y) \rangle \geq 0.$$

(d) have the Cartesian R_{02}-property if for any sequence $\{x^k\}$ satisfying the condition that

$$\|x^k\| \to +\infty, \qquad \frac{[-x^k]_+}{\|x^k\|} \to 0, \qquad \frac{[-F(x^k)]_+}{\|x^k\|} \to 0, \tag{14}$$

there exists an index $\nu \in \{1, 2, \ldots, m\}$ such that

$$\liminf_{k \to +\infty} \frac{\lambda_2 \left[F_\nu(x^k) \circ x_\nu^k \right]}{\|x_\nu^k\|^2} > 0.$$

By Definition 2.3, it is not difficult to verify the following one-way implications:
Uniform Cartesian P-property $\Rightarrow$ Cartesian P-property $\Rightarrow$ Cartesian P_0-property,

$$\text{Uniform Cartesian} P\text{-property} \;\Rightarrow\; \text{Cartesian } R_{02}\text{-property}.$$

Moreover, we see that, when $m = 1$, the Cartesian P-property (or the Cartesian P_0-property) of F becomes the strict monotonicity (or monotonicity) of F. If the continuously differentiable mapping F has the Cartesian P-property (or P_0-property), then its transposed Jacobian matrix $\nabla F(x)$ at any $x \in \mathbb{R}^n$ has the corresponding Cartesian P-properties. When F degenerates into the affine function $Mx + q$, F having the uniform Cartesian P-property coincides with M having the Cartesian P-property. In addition, by Definition 2.3(b)–(c), we readily have the following result.

Proposition 2.3. *For any $\varepsilon > 0$, let $F_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be given as in (4). If F has the Cartesian P_0-property, then F_ε will have the Cartesian P-property.*

It should be pointed out that, when F has the Cartesian P-property, F_ε must not have the uniform Cartesian P-property. A counterexample is given by [11] for the case $m = 1$.

Finally, paralleling to Definition 2.2, we have the concepts of P-properties for a nonlinear mapping in the setting of SOCs, which are special cases of those given by [23].

Definition 2.4. A nonlinear mapping $F = (F_1, \ldots, F_m) : \mathbb{R}^n \to \mathbb{R}^n$ is said to have

(a) the Jordan P-property if $(x - y) \circ (F(x) - F(y)) \in -\mathcal{K} \Rightarrow x = y$;
(b) the P-property if the condition that $L_{x_i - y_i} L_{F_i(x) - F_i(y)} = L_{F_i(x) - F_i(y)} L_{x_i - y_i}$, $i = 1, 2, \ldots, m$ and $(x - y) \circ (F(x) - F(y)) \in -\mathcal{K}$ implies $x = y$;
(c) the P_0-property if $F(x) + \varepsilon x$ has the P-property for all $\varepsilon > 0$.

Proposition 2.4. (a) *If a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ has the Cartesian P-property, then it must have the Jordan P-property and the P-property.*
(b) *If a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ has the Cartesian P_0-property, then it must have the P_0-property.*

Proof. The proof is similar to that of Proposition 2.2, and we omit it. $\square$

3. Existence of regularized solutions

In this section, we show that the regularized problem SOCCP(F_ε) has a unique solution $x(\varepsilon)$ for every $\varepsilon > 0$ under the Cartesian P_0-property of F and the following condition:

Condition A. For any sequence $\{x^k\} \subseteq \mathbb{R}^n$, when there exists $i \in \{1, 2, \ldots, m\}$ such that $\lambda_2(x_i^k) \to +\infty$, $\{F_{\varepsilon,i}(x^k)\}$ is bounded below, and $\left\{\frac{\|F_i(x^k)\|}{\|x_i^k\|}\right\}$ is unbounded, there holds that

$$\limsup_{k\to+\infty} \left\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_i(x^k)}{\|F_i(x^k)\|} \right\rangle > 0.$$

The main tool to prove this result is the Fischer–Burmeister (FB) SOC complementarity function. The FB function was first introduced by Fischer [7,8], which plays a crucial role in the design of several nonsmooth Newton methods and merit function methods for the solution of NCPs. Recently, the function was extended to the setting of semidefinite complementarity problems [21,22] and SOCCPs [2], respectively.

Definition 3.1. A mapping $\phi : \mathbb{R}^l \times \mathbb{R}^l \to \mathbb{R}^l$ is called an SOC complementarity function associated with the SOC $\mathcal{K}^l$ if for any $x, y \in \mathbb{R}^l$,

$$\phi(x, y) = 0 \Longleftrightarrow \quad x \in \mathcal{K}^l,\ y \in \mathcal{K}^l,\ \langle x, y\rangle = 0. \tag{15}$$

The FB SOC complementarity function associated with $\mathcal{K}^l$ is defined as follows:

$$\phi_{\text{FB}}(x, y) := (x^2 + y^2)^{1/2} - (x + y), \quad \forall\, (x, y) \in \mathbb{R}^l \times \mathbb{R}^l. \tag{16}$$

By Property 2.1(a)–(c), clearly, the function ϕ_{FB} is well defined in $\mathbb{R}^l \times \mathbb{R}^l$. Moreover, it was shown in [12] that ϕ_{FB} satisfies the characterization (15). With the vector-valued function, Chen and Tseng [2] proposed a merit function approach for the SOCCP, and we recently developed a semismooth Newton method in [17]. In this section, we mainly employ the function as a theoretical tool. Define the operator $\Phi_{\text{FB}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_{\text{FB}}(x) := \begin{pmatrix} \phi_{\text{FB}}(x_1, F_1(x)) \\ \vdots \\ \phi_{\text{FB}}(x_m, F_m(x)) \end{pmatrix}, \tag{17}$$

which induces a natural merit function $\Psi_{\text{FB}} : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$\Psi_{\text{FB}}(x) := \frac{1}{2}\|\Phi_{\text{FB}}(x)\|^2 = \frac{1}{2}\sum_{i=1}^m \|\phi_{\text{FB}}(x_i, F_i(x))\|^2. \tag{18}$$

The following proposition summarizes some important properties of $\Psi_{\rm FB}$. Since their proofs are direct or can be found in [2,17], here we omit them.

Proposition 3.1. *Let $\Psi_{\rm FB}:\mathbb{R}^n\to\mathbb{R}_+$ be given as in (18). Then, the following results hold.*

(a) *x^* is a solution of the SOCCP(F) if and only if x^* solves the system $\Phi_{\rm FB}(x)=0$.*
(b) *$\Psi_{\rm FB}$ is continuously differentiable everywhere on $\mathbb{R}^n$.*
(c) *If F has the Cartesian P_0-property, then every stationary point of $\Psi_{\rm FB}$ is a solution of the SOCCP(F).*

Analogously, for the SOCCP(F_ε), we define the operator $\Phi_\varepsilon:\mathbb{R}^n\to\mathbb{R}^n$ by

$$\Phi_\varepsilon(x) := \begin{pmatrix}\phi_{\rm FB}(x_1, F_{\varepsilon,1}(x))\\ \vdots\\ \phi_{\rm FB}(x_m, F_{\varepsilon,m}(x))\end{pmatrix}, \tag{19}$$

where $F_{\varepsilon,i}:\mathbb{R}^n\to\mathbb{R}^{n_i}$ denotes the ith subvector of F_ε. The natural merit function $\Psi_\varepsilon:\mathbb{R}^n\to\mathbb{R}_+$ corresponding to Φ_ε is then given by

$$\Psi_\varepsilon(x) := \frac{1}{2}\|\Phi_\varepsilon(x)\|^2 = \frac{1}{2}\sum_{i=1}^m \left\|\phi_{\rm FB}(x_i, F_{\varepsilon,i}(x))\right\|^2. \tag{20}$$

The following lemma plays a crucial role in proving the main result of this section. Since the proof can be found in Lemma 5.2 of [17], here we omit it.

Lemma 3.1. *Let $\phi_{\rm FB}$ be defined as in (16). For any sequence $\{(x^k,y^k)\}\subseteq\mathbb{R}^l\times\mathbb{R}^l$, let $\lambda_1^k\le\lambda_2^k$ and $\mu_1^k\le\mu_2^k$ denote the spectral values of x^k and y^k, respectively.*

(a) *If $\lambda_1^k\to-\infty$ or $\mu_1^k\to-\infty$, then $\|\phi_{\rm FB}(x^k,y^k)\|\to+\infty$.*
(b) *If $\{\lambda_1^k\}$ and $\{\mu_1^k\}$ are bounded below, but $\lambda_2^k\to+\infty$, $\mu_2^k\to+\infty$, and $\frac{x^k}{\|x^k\|}\circ\frac{y^k}{\|y^k\|}\nrightarrow 0$, then $\|\phi_{\rm FB}(x^k,y^k)\|\to+\infty$.*

Proposition 3.2. *Suppose that $F:\mathbb{R}^n\to\mathbb{R}^n$ has the Cartesian P_0-property and satisfies Condition A. Then the function Ψ_ε given by (20) for any $\varepsilon>0$ is coercive, i.e.,*

$$\lim_{\|x^k\|\to\infty}\Psi_\varepsilon(x^k)=+\infty.$$

Proof. Suppose by contradiction that the conclusion does not hold. Then we can find an unbounded sequence $\{x^k\}\subseteq\mathbb{R}^n$ with $x^k=(x_1^k,\ldots,x_m^k)$ and $x_i^k\in\mathbb{R}^{n_i}$ such that the sequence $\{\Psi_\varepsilon(x^k)\}$ is bounded. Define the index set

$$J:=\left\{i\in\{1,2,\ldots,m\}\mid \{\|x_i^k\|\}\text{ is unbounded}\right\}.$$

Since $\{x^k\}$ is unbounded, $J\neq\emptyset$. Subsequencing if necessary, we assume without loss of generality that $\{\|x_i^k\|\}\to+\infty$ for all $i\in J$. For each $i\in J$, we define

$$J_i:=\left\{\nu\in\{1,2,\ldots,n_i\}\mid \{|x_{i\nu}^k|\}\text{ is unbounded}\right\}.$$

Let $\{y^k\}$ be a bounded sequence with $y^k=(y_1^k,\ldots,y_m^k)$ and $y_i^k\in\mathbb{R}^{n_i}$ defined as follows:

$$y_{i\nu}^k=\begin{cases}0 & \text{if } i\in J \text{ and } \nu\in J_i;\\ x_{i\nu}^k & \text{otherwise.}\end{cases}$$

From the definition of $\{y^k\}$ and the Cartesian P_0-property of F, it follows that

$$\begin{aligned}0 &\le \max_{1\le l\le m}\left\langle x_l^k-y_l^k,\ F_l(x^k)-F_l(y^k)\right\rangle\\ &=\left\langle x_i^k-y_i^k,\ F_i(x^k)-F_i(y^k)\right\rangle\end{aligned}$$

$$\le n_i \max_{\nu \in J_i} x_{i\nu}^k \left[F_{i\nu}(x^k) - F_{i\nu}(y^k)\right]$$
$$= n_i x_{ij}^k \left[F_{ij}(x^k) - F_{ij}(y^k)\right], \tag{21}$$

where i is an index from J for which the first maximum is attained, and j is an index from J_i for which the second maximum is attained. Without loss of generality, we assume that i and j are independent of k. Since $i \in J$ and $j \in J_i$,

$$|x_{ij}^k| \;\to\; +\infty. \tag{22}$$

We now consider the two cases where $x_{ij}^k \to +\infty$ and $x_{ij}^k \to -\infty$, respectively.

Case (1): $x_{ij}^k \to +\infty$. In this case, since $F_{ij}(y^k)$ is bounded by the continuity of $F_{ij}(y)$, inequality (21) implies that $F_{ij}(x^k)$ does not tend to $-\infty$. This in turn implies that

$$\left\{F_{ij}(x^k) + \varepsilon x_{ij}^k\right\} \;\to\; +\infty. \tag{23}$$

Case (2): $x_{ij}^k \to -\infty$. Now, using inequality (21) and the boundedness of $F_{ij}(y^k)$ immediately yields that $F_{ij}(x^k)$ does not tend to $+\infty$. This in turn implies that

$$\left\{F_{ij}(x^k) + \varepsilon x_{ij}^k\right\} \;\to\; -\infty. \tag{24}$$

From Eq. (22)–(24) and the definition of $F_{\varepsilon,i}(x)$, we thus obtain that

$$\|x_i^k\| \to +\infty, \quad \|F_{\varepsilon,i}(x^k)\| \to +\infty. \tag{25}$$

If $\lambda_1(x_i^k) \to -\infty$ or $\lambda_1[F_{\varepsilon,i}(x^k)] \to -\infty$, then from Lemma 3.1(a) we readily obtain that $\|\phi_{\mathrm{FB}}(x_i^k, F_{\varepsilon,i}(x^k))\| \to +\infty$. Otherwise, Eq. (25) implies that $\{x_i^k\}$ and $\{F_{\varepsilon,i}(x^k)\}$ are bounded below, but $\lambda_2(x_i^k) \to +\infty$ and $\lambda_2[F_{\varepsilon,i}(x^k)] \to +\infty$. We next prove that

$$\lim_{k\to+\infty} \frac{x_i^k}{\|x_i^k\|} \circ \frac{F_{\varepsilon,i}(x^k)}{\|F_{\varepsilon,i}(x^k)\|} \nrightarrow 0, \tag{26}$$

and consequently from Lemma 3.1(b) it follows that $\|\phi_{\mathrm{FB}}(x_i^k, F_{\varepsilon,i}(x^k))\| \to +\infty$. From the first two equations of (21) and the boundedness of $\{y^k\}$ and $\{F_i(y^k)\}$, it is not hard to verify that $\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_i(x^k)}{\|F_{\varepsilon,i}(x^k)\|}\rangle \ge 0$ for all sufficiently large k. Notice that

$$\left\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_{\varepsilon,i}(x^k)}{\|F_{\varepsilon,i}(x^k)\|}\right\rangle = \left\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_i(x^k)}{\|F_{\varepsilon,i}(x^k)\|}\right\rangle + \frac{\varepsilon\|x_i^k\|}{\|F_{\varepsilon,i}(x^k)\|}, \quad \forall\, k. \tag{27}$$

Therefore, if the sequence $\left\{\frac{\|F_i(x^k)\|}{\|x_i^k\|}\right\}$ is bounded, then equality (27) implies that

$$\limsup_{k\to+\infty} \left\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_{\varepsilon,i}(x^k)}{\|F_{\varepsilon,i}(x^k)\|}\right\rangle > 0. \tag{28}$$

If the sequence $\left\{\frac{\|F_i(x^k)\|}{\|x_i^k\|}\right\}$ is unbounded, then using Condition A and equality (27), it is easy to verify that (28) also holds. Clearly, Eq. (28) implies (26), and we thus get $\|\phi_{\mathrm{FB}}(x_i^k, F_{\varepsilon,i}(x^k))\| \to +\infty$. This contradicts the boundedness of $\{\Psi_\varepsilon(x^k)\}$. $\square$

Proposition 3.2 states that under Condition A and the Cartesian P_0-property of F the level set

$$\mathcal{L}_\gamma(x) \;:=\; \{x \in \mathbb{R}^n \mid \Psi_\varepsilon(x) \le \gamma\} \tag{29}$$

is bounded for every $\gamma \ge 0$. Now we are in a position to prove the following main result. Notice that, when $m = n$ and $n_1 = \cdots = n_m = 1$, the Cartesian P_0-property of F is equivalent to requiring that F is P_0-function;

whereas Condition A automatically holds since the assumption that $\lambda_2(x_i^k) \to +\infty$, $\{F_{\varepsilon,i}(x^k)\}$ is bounded below, and $\left\{\frac{\|F_i(x^k)\|}{\|x_i^k\|}\right\}$ is unbounded implies that there exists a subsequence $\{x_i^k\}_{k\in K}$ satisfying $x_i^k \to +\infty$ and $F_i(x^k) \to +\infty$ for $k \in K$, and consequently $\limsup_{k\to\infty}\left\langle \frac{x_i^k}{\|x_i^k\|}, \frac{F_i(x^k)}{\|F_i(x^k)\|}\right\rangle > 0$. Thus, the assertion of Proposition 3.2 reduces to that of [11, Proposition 3.4].

Theorem 3.1. *Assume that the mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ has the Cartesian P_0-property and satisfies Condition A. Then for every $\varepsilon > 0$ the problem SOCCP(F_ε) has a unique bounded solution $x(\varepsilon)$.*

Proof. Let $\varepsilon > 0$. Then the mapping F_ε has the Cartesian P-property by Proposition 2.3. This means that the regularized problem SOCCP(F_ε) has at most one solution. In fact, suppose that $x(\varepsilon)$ and $\hat{x}(\varepsilon)$ are two different solutions of the SOCCP(F_ε). From the Cartesian P-property of F_ε, it then follows that there exists an index $i \in \{1, 2, \ldots, m\}$ such that

$$\begin{aligned} 0 &< \langle x_i(\varepsilon) - \hat{x}_i(\varepsilon),\ F_{\varepsilon,i}(x(\varepsilon)) - F_{\varepsilon,i}(\hat{x}(\varepsilon))\rangle \\ &= \langle x_i(\varepsilon),\ F_{\varepsilon,i}(x(\varepsilon))\rangle - \langle x_i(\varepsilon),\ F_{\varepsilon,i}(\hat{x}(\varepsilon))\rangle \\ &\quad - \langle \hat{x}_i(\varepsilon),\ F_{\varepsilon,i}(x(\varepsilon))\rangle + \langle \hat{x}_i(\varepsilon),\ F_{\varepsilon,i}(\hat{x}(\varepsilon))\rangle \\ &= -\langle x_i(\varepsilon),\ F_{\varepsilon,i}(\hat{x}(\varepsilon))\rangle - \langle \hat{x}_i(\varepsilon),\ F_{\varepsilon,i}(x(\varepsilon))\rangle, \end{aligned} \tag{30}$$

where the last equality is due to $\langle x_i(\varepsilon), F_{\varepsilon,i}(x(\varepsilon))\rangle = 0$ and $\langle \hat{x}_i(\varepsilon), F_{\varepsilon,i}(\hat{x}(\varepsilon))\rangle = 0$. Note that the two terms on the right-hand side of (30) are nonpositive since $x_i(\varepsilon), \hat{x}_i(\varepsilon) \in \mathcal{K}^{n_i}$ and $F_{\varepsilon,i}(x(\varepsilon)), F_{\varepsilon,i}(\hat{x}(\varepsilon)) \in \mathcal{K}^{n_i}$. Thus, we obtain a contradiction with inequality (30).

To prove the existence of a solution, let $x^0 \in \mathbb{R}^n$ be an arbitrary point and define $\gamma := \Psi_\varepsilon(x^0)$. By Proposition 3.2, the corresponding level set $\mathcal{L}_\gamma(x)$ is nonempty and compact. Therefore, the continuous function $\Psi_\varepsilon(x)$ attains a global minimum $x(\varepsilon)$ on $\mathcal{L}_\gamma(x)$ which, by the definition of level sets, is also a global minimum of $\Psi_\varepsilon(x)$ on $\mathbb{R}^n$. Therefore, $x(\varepsilon)$ is a stationary point of $\Psi_\varepsilon(x)$. Since the mapping F_ε has the Cartesian P-property, we have from Proposition 3.1(c) that $x(\varepsilon)$ is a solution of the regularized problem SOCCP(F_ε). Furthermore, this solution is bounded. Combining with the discussions above, we complete the proof. □

4. Behaviour of the solution path

From Theorem 3.1, we learn that the regularized problem SOCCP(F_ε) for every $\varepsilon > 0$ has a unique solution $x(\varepsilon)$ when the mapping F has the Cartesian P_0-property and satisfies Condition A. Thus, as the parameter ε tends to 0, the solution of the regularized problem SOCCP(F_ε) generates a solution path $\mathcal{P} := \{x(\varepsilon) \mid \varepsilon > 0\}$. The aim of this section is to study the properties of the trajectory $\mathcal{P}$. Specifically, we prove that, if F has the uniform Cartesian P-property, the path $\mathcal{P}$ is bounded as $\varepsilon \to 0$ and the bound is dependent on the constant ρ involved in the uniform Cartesian P-property. We also illustrate that in this case the path $\mathcal{P}$ is not locally Lipschitz continuous as $\varepsilon \to 0$. Then, for the case that F has the Cartesian P_0-property and satisfies Condition A, we provide the condition to guarantee that $x(\varepsilon)$ remains bounded as $\varepsilon \to 0$. The reason why we are interested in the boundedness of $x(\varepsilon)$ is due to the following evident result.

Theorem 4.1. *Let $\{\varepsilon_k\}$ be a sequence of positive values converging to 0. If $\{x(\varepsilon_k)\}$ converges to a point $\bar{x}$, then $\bar{x}$ solves the SOCCP(F).*

The following proposition states that the solution $x(\varepsilon)$ of SOCCP(F_ε) is bounded for any $\varepsilon \geq 0$ if F has the uniform Cartesian P-property, but the bound is dependent on the constant ρ involved in the uniform Cartesian P-property.

Proposition 4.1. *Suppose that F has the uniform Cartesian P-property. Then, for any $\varepsilon \geq 0$, we have*

$$\|x(\varepsilon)\| \leq \rho^{-1}\|[-F(0)]_+\|, \tag{31}$$

where $\rho > 0$ is the constant involved in the uniform Cartesian P-property.

Proof. Since the uniform Cartesian P-property implies the Cartesian R_{02}-property and the P-property, from [23, Theorem 3.1] and the proof of Proposition 4.3(b) below, it follows that $x(\varepsilon)$ exists for any $\varepsilon \geq 0$. If $x(\varepsilon) \equiv 0$ for any $\varepsilon \geq 0$, then inequality (31) is direct. Suppose that $x(\varepsilon) \neq 0$ for some $\varepsilon \geq 0$. Since $x(\varepsilon)$ is the solution of the SOCCP(F_ε), it follows that

$$x_i(\varepsilon) \in \mathcal{K}^{n_i}, \quad F_{\varepsilon,i}(x(\varepsilon)) \in \mathcal{K}^{n_i} \text{ and } \left\langle x_i(\varepsilon),\ F_{\varepsilon,i}(x(\varepsilon))\right\rangle = 0, \quad i = 1, 2, \ldots, m.$$

By this and the uniform Cartesian P-property of F, we have that

$$\begin{aligned}
\rho\|x(\varepsilon)\|^2 &\leq \max_{1\leq i\leq m} \langle x_i(\varepsilon),\ F_i(x(\varepsilon)) - F_i(0)\rangle \\
&= \max_{1\leq i\leq m} \langle x_i(\varepsilon),\ -\varepsilon x_i(\varepsilon) - F_i(0)\rangle \\
&\leq \max_{1\leq i\leq m} \langle x_i(\varepsilon), -F_i(0)\rangle \\
&\leq \max_{1\leq i\leq m} \langle x_i(\varepsilon), [-F_i(0)]_+\rangle \\
&\leq \|x(\varepsilon)\|\,\|[-F_i(0)]_+\|,
\end{aligned}$$

where the third inequality is since $x_i(\varepsilon) \in \mathcal{K}^{n_i}$, $-F_i(0) = [-F_i(0)]_+ + [-F_i(0)]_-$ and $[-F_i(0)]_- \in -\mathcal{K}^{n_i}$. This leads to the desired result. □

Remark 4.1. (a) From Proposition 4.1, when F has the uniform Cartesian P-property, the SOCCP(F) has a unique bounded solution. Furthermore, if $F(0) \in \mathcal{K}$, the regularized problem SOCCP(F_ε) for every $\varepsilon \geq 0$ has the unique solution $x(\varepsilon) = 0$.
(b) When F is an affine function $Mx + q$ with $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, the assumption of Proposition 4.1 is equivalent to requiring that M has the Cartesian P-property.

Proposition 4.2. *Suppose that F has the uniform Cartesian P-property. Then, for any $\varepsilon_1, \varepsilon_2 \geq 0$, there holds that*

$$\|x(\varepsilon_1) - x(\varepsilon_2)\| \leq \rho^{-1}\|\varepsilon_1 x(\varepsilon_1) - \varepsilon_2 x(\varepsilon_2)\|, \tag{32}$$

where $\rho > 0$ is the constant same as Proposition 4.1.

Proof. Without loss of generality, we assume that $\varepsilon_1 \neq \varepsilon_2$. Let

$$y(\varepsilon_1) := F_{\varepsilon_1}(x(\varepsilon_1)), \qquad y(\varepsilon_2) := F_{\varepsilon_2}(x(\varepsilon_2)).$$

Since $x(\varepsilon_1)$ and $x(\varepsilon_2)$ are the solutions of the problem SOCCP(F_{ε_1}) and SOCCP(F_{ε_2}), respectively, we have $x_i(\varepsilon_1), y_i(\varepsilon_1) \in \mathcal{K}^{n_i}$ with $\langle x_i(\varepsilon_1), y_i(\varepsilon_1)\rangle = 0$ and $x_i(\varepsilon_2), y_i(\varepsilon_2) \in \mathcal{K}^{n_i}$ with $\langle x_i(\varepsilon_2), y_i(\varepsilon_2)\rangle = 0$ for all $i = 1, 2, \ldots, m$. From this, it then follows that

$$\begin{aligned}
&\langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ F_i(x(\varepsilon_1)) - F_i(x(\varepsilon_2))\rangle = \langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ y_i(\varepsilon_1) - \varepsilon_1 x_i(\varepsilon_1) - y_i(\varepsilon_2) + \varepsilon_2 x_i(\varepsilon_2)\rangle \\
&\quad = -\langle x_i(\varepsilon_1), y_i(\varepsilon_2)\rangle - \langle x_i(\varepsilon_2), y_i(\varepsilon_1)\rangle + \langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ \varepsilon_2 x_i(\varepsilon_2) - \varepsilon_1 x_i(\varepsilon_1)\rangle \\
&\quad \leq \langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ \varepsilon_2 x_i(\varepsilon_2) - \varepsilon_1 x_i(\varepsilon_1)\rangle,
\end{aligned}$$

where the inequality holds since $-\langle x_i(\varepsilon_1), y_i(\varepsilon_2)\rangle \leq 0$ and $-\langle x_i(\varepsilon_2), y_i(\varepsilon_1)\rangle \leq 0$. Using the last inequality and the uniform Cartesian P-property of F, we have that

$$\begin{aligned}
\rho\|x(\varepsilon_1) - x(\varepsilon_2)\|^2 &\leq \max_{1\leq i\leq m} \langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ F_i(x(\varepsilon_1)) - F_i(x(\varepsilon_2))\rangle \\
&\leq \max_{1\leq i\leq m} \langle x_i(\varepsilon_1) - x_i(\varepsilon_2),\ \varepsilon_2 x_i(\varepsilon_2) - \varepsilon_1 x_i(\varepsilon_1)\rangle, \\
&\leq \max_{1\leq i\leq m} \|x_i(\varepsilon_1) - x_i(\varepsilon_2)\|\,\|\varepsilon_2 x_i(\varepsilon_2) - \varepsilon_1 x_i(\varepsilon_1)\| \\
&\leq \|x(\varepsilon_1) - x(\varepsilon_2)\|\,\|\varepsilon_2 x(\varepsilon_2) - \varepsilon_1 x(\varepsilon_1)\|,
\end{aligned}$$

which immediately implies the desired result. Thus, we complete the proof. □

Propositions 4.1 and 4.2 characterize some properties of the path $\mathcal{P}$ as $\varepsilon \to 0$ under the uniform Cartesian P-property of F. However, these results cannot imply the locally Lipschitz continuity of $\mathcal{P}$ as $\varepsilon \to 0$. The following counterexample illustrates the fact.

Example 4.1. Let $m = 2$ and $n_1 = n_2 = 2$. Let F be given by $F(x) = Mx + q$, where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \qquad q = \begin{pmatrix} -\dfrac{1+\varepsilon}{\varepsilon} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for any given } \varepsilon > 0.$$

Since the matrix M has the Cartesian P-property, the mapping F has the uniform Cartesian P-property. For the SOCCP(F_ε), i.e., to find x such that

$$x \in \mathcal{K}^2 \times \mathcal{K}^2, \qquad F_\varepsilon(x) \in \mathcal{K}^2 \times \mathcal{K}^2, \qquad \langle x, F_\varepsilon(x)\rangle = 0,$$

we can verify that $x(\varepsilon) = (1/\varepsilon, 0, 0, 0)^{\mathrm{T}}$ is the unique solution. Obviously, $x(\varepsilon)$ is not locally Lipschitz continuous as $\varepsilon \to 0$, and furthermore, $x(\varepsilon)$ even has no bound.

Next, we concentrate on the case where F has the Cartesian P_0-property and satisfies Condition A. Under this case, we cannot prove the continuity of the mapping $\varepsilon \to x(\varepsilon)$ at any $\varepsilon > 0$ like the NCP case. The main reason is that we cannot obtain the result corresponding to Theorem 3.1 of [16] under the Cartesian P-property of F, although when $\nabla F(x)$ has the Cartesian P-property, its every principal block matrix has the Cartesian P-property, and the Schur-complement of a matrix with the Cartesian P-property also has the Cartesian P-property by Proposition 2.1. For this case, we can state the following result, whose proof will be postponed until the next section.

Theorem 4.2. *Suppose that F has the Cartesian P_0-property and satisfies Condition A. If the solution set $\mathcal{S}^*$ of the SOCCP(F) is nonempty and bounded, then the path $\mathcal{P}_{\bar{\varepsilon}} = \{x(\varepsilon) \mid \varepsilon \in (0, \bar{\varepsilon}\,]\}$ is bounded for any $\bar{\varepsilon} > 0$ and*

$$\lim_{\varepsilon \downarrow 0} \operatorname{dist}\left(x(\varepsilon) \mid \mathcal{S}^*\right) = 0.$$

As an immediate consequence of Theorem 4.2, we have the following conclusion.

Corollary 4.1. *Suppose that F has the Cartesian P_0-property and satisfies Condition A. If the SOCCP(F) has a unique solution $\bar{x}$, then $\lim_{\varepsilon\downarrow 0} x(\varepsilon) = \bar{x}$.*

As illustrated by Example 4.6 of [11], it is impossible to remove the boundedness assumption of S^* without destroying the boundedness of the path $\mathcal{P}_{\bar{\varepsilon}}$. To this end, we next provide some conditions to guarantee the nonemptyness and boundedness of S^*.

Proposition 4.3. *The SOCCP(F) has a nonempty and bounded solution set S^* under one of the following conditions:*

(a) *F is monotone, and the SOCCP(F) is strictly feasible, i.e. there is $\bar{x} \in \mathbb{R}^n$ satisfying $\bar{x}$, $F(\bar{x}) \in \operatorname{int}(\mathcal{K})$.*
(b) *The mapping F has the P_0-property and the Cartesian R_{02}-property.*

Proof. (a) Since $F(x)$ is monotone and $\nabla F(x)$ is positive semidefinite, the result is direct by Proposition 6 of [2].
(b) We prove that in this case a stronger result holds, that is, the following SOCCP(F, q)

$$x \in \mathcal{K}, \qquad F(x) + q \in \mathcal{K}, \qquad \langle x, F(x) + q\rangle = 0 \tag{33}$$

has a nonempty and bounded solution set for all $q \in \mathbb{R}^n$. By Theorem 3.1 of [23], we only need to prove that for any $\triangle > 0$, the following set

$$\{x : x \text{ solves (33) with } \|q\| \le \triangle\} \tag{34}$$

is bounded. Suppose that the set is not bounded. Then there exists a sequence $\{q^k\}$ with $\|q^k\| \le \triangle$ and a sequence $\{x^k\}$ with $\|x^k\| \to +\infty$ such that for any k,

$$x^k \in \mathcal{K}, \qquad y^k = F(x^k) + q^k \in \mathcal{K} \quad \text{and} \quad x^k \circ y^k = 0. \tag{35}$$

Without loss of generality, we assume that $\|x_i^k\| \to +\infty$. This is equivalent to saying that for any k,

$$\frac{1}{2}\sum_{i=1}^{m} \|\phi_{\mathrm{FB}}(x_i^k, y_i^k)\|^2 = 0. \tag{36}$$

Using Lemma 8 of [2] and the boundedness of q^k, we then obtain that

$$\|x^k\| \to +\infty, \qquad \lim_{k\to+\infty} \frac{[-x^k]_+}{\|x^k\|} \to 0, \qquad \lim_{k\to+\infty} \frac{[-y^k]_+}{\|x^k\|} \to 0, \quad \text{and} \quad \lim_{k\to+\infty} \frac{\|[q^k]_+\|}{\|x^k\|} \to 0. \tag{37}$$

Noting that

$$\|[q^k]_+\| = \|[y^k - F(x^k)]_+\| \geq \|[-F(x^k)]_+\|,$$

where the inequality is due to [2, Lemma 7 (c)], we have from the last term in (37) that

$$\lim_{k\to+\infty} \frac{\|[-F(x^k)]_+\|}{\|x^k\|} \to 0.$$

This, together with the first two terms in (37), shows that $\{x^k\}$ satisfies condition (14). By the Cartesian R_{02}-property of F, there exists $\nu \in \{1, 2, \ldots, m\}$ such that

$$\liminf_{k\to+\infty} \frac{\lambda_2[x_\nu^k \circ F_\nu(x^k)]}{\|x^k\|^2} > 0.$$

However, from Eq. (35) and the boundedness of q^k, we have

$$\frac{\lambda_2[x_\nu^k \circ F_\nu(x^k)]}{\|x^k\|^2} = \frac{\lambda_2[-x_\nu^k \circ q_\nu^k]}{\|x^k\|^2} \ \to\ 0.$$

This leads to a contradiction. Consequently, the set defined by (34) is bounded. □

Notice that the Cartesian R_{02}-property is implied by the R_0-property in [23]. Hence, Proposition 4.3(b) provides a weaker condition for S^* being nonempty and bounded. By Theorem 3.1 and Propositions 4.3(b) and 2.4(b), we have the following result.

Corollary 4.2. *Suppose that F has the Cartesian P_0-property and the Cartesian R_{02}-property and satisfies Condition A. Then the path $\mathcal{P}_{\bar\varepsilon} = \{x(\varepsilon) \mid \varepsilon \in (0, \bar\varepsilon\,]\}$ is bounded for any $\bar\varepsilon > 0$ and* $\lim_{\varepsilon\downarrow 0} \operatorname{dist}(x(\varepsilon) \mid \mathcal{S}^*) = 0$.

5. Inexact regularization method

The discussions from the last two sections show that the original SOCCP(F) can be solved by calculating the exact solutions of a sequence of regularized problems SOCCP(F_ε). However, in practice, it is usually not possible to solve the SOCCP(F_ε) exactly for each $\varepsilon > 0$. In this section, we propose an inexact regularization algorithm which only requires inexact solutions of these subproblems, but preserves all convergence properties of its exact counterpart. First, let us describe the specific algorithm.

Algorithm 5.1 (*Inexact Regularization Method*).

(S.0) *Choose $\varepsilon_0 > 0$ and $\tau_0 > 0$, and set $k := 0$.*

(S.1) *Compute an approximate solution x^k of SOCCP (F_ε) such that*

$$\Psi_\varepsilon(x^k) \leq \tau_k. \tag{38}$$

(S.2) *Terminate the iteration if a suitable criterion is satisfied.*

(S.3) *Choose $\varepsilon_{k+1} > 0$ and $\tau_{k+1} > 0$, set $k := k + 1$, and go to (S.1).*

Clearly, if we take $\tau_k = 0$ at each iteration, then $x^k = x(\varepsilon_k)$. In addition, the point x^k can be easily obtained by applying any effective gradient-type unconstrained optimization algorithm to the minimization problem

$$\min_{x\in\mathbb{R}^n} \Psi_\varepsilon(x), \tag{39}$$

because $\Psi_\varepsilon(x)$ is continuously differentiable everywhere and has bounded level sets for those SOCCPs with F having the Cartesian P_0-property and satisfying Condition A. In our numerical experiments, we adopt the BFGS algorithm to compute x^k.

The following well-known Mountain Pass Theorem [18] will be used in the convergence analysis of Algorithm 5.1.

Lemma 5.1. *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is smooth and coercive. Let $C \subseteq \mathbb{R}^n$ be a nonempty compact set and denote $\bar{c}$ by the least value of f on the boundary of C, i.e. $\bar{c} := \min_{x\in\partial C} f(x)$. If there are two points $a \in C$ and $b \notin C$ such that $f(a) < \bar{c}$ and $f(b) < \bar{c}$, then there exists a point $z \in \mathbb{R}^n$ such that $\nabla f(z) = 0$ and $f(z) \geq \bar{c}$.*

Now we establish the convergence results of Algorithm 5.1. To this end, assume that Algorithm 5.1 generates an infinite sequence so that the termination criterion in (S.2) is never active. The analysis technique adopted is similar to that of [11].

Theorem 5.1. *Let F be the mapping having the Cartesian P_0-property and satisfying Condition A. Assume that the solution set $\mathcal{S}^*$ of the SOCCP(F) is nonempty and bounded. If $\varepsilon_k \to 0$ and $\tau_k \to 0$, then the sequence $\{x^k\}$ generated by Algorithm 5.1 remains bounded, and every accumulation point of $\{x^k\}$ is a solution of the SOCCP(F).*

Proof. Suppose that the sequence $\{x^k\}$ is unbounded. Then, passing to a subsequence if necessary, we assume that $\{\|x^k\|\} \to +\infty$. This, together with the boundedness of $\mathcal{S}^*$, means that there exists a compact set $C \subseteq \mathbb{R}^n$ with $\mathcal{S}^* \subset \mathrm{int}C$ and $x^k \notin C$ for sufficiently large k. Let $x^* \in \mathcal{S}^*$ be a solution of the SOCCP(F). Then we have

$$\Psi_{\mathrm{FB}}(x^*) = 0 \quad \text{and} \quad \bar{c} := \min_{x\in\partial C} \Psi_{\mathrm{FB}}(x) > 0. \tag{40}$$

Let $\delta := \bar{c}/4$. Notice that $\Psi_\varepsilon(x)$ viewed as the function of x and ε is continuous on the compact set $C \times [0, \tilde{\varepsilon}]$, and so is uniformly continuous on $C \times [0, \tilde{\varepsilon}]$. Hence, there exists an $\tilde{\varepsilon} > 0$ such that for all $x \in C$ and $\varepsilon \in [0, \tilde{\varepsilon}]$

$$|\Psi_\varepsilon(x) - \Psi_{\mathrm{FB}}(x)| \leq \delta. \tag{41}$$

Combining (41) with (40), we have that for all sufficiently large k,

$$\Psi_{\varepsilon_k}(x^*) \leq \frac{1}{4}\bar{c} \tag{42}$$

and

$$c := \min_{x\in\partial C} \Psi_{\varepsilon_k}(x) \geq \frac{3}{4}\bar{c}. \tag{43}$$

On the other hand, $\Psi_{\varepsilon_k}(x^k) \leq \tau_k$ by Algorithm 5.1 and $\tau_k \to 0$, which means that

$$\Psi_{\varepsilon_k}(x^k) \leq \frac{1}{4}\bar{c} \tag{44}$$

for all k large enough. Now using (42)–(44) and setting $a = x^*$ and $b = x^k$ in Lemma 5.1, there exists a vector $\hat{x} \in \mathbb{R}^n$ such that

$$\nabla \Psi_{\varepsilon_k}(\hat{x}) = 0 \quad \text{and} \quad \Psi_{\varepsilon_k}(\hat{x}) \geq c \geq \frac{3}{4}\bar{c} > 0.$$

This says that $\hat{x}$ is a stationary point of $\Psi_{\varepsilon_k}(x)$, but not a solution of the SOCCP(F_{ε_k}). However, by Proposition 3.1(c), we know that any stationary point of $\Psi_{\varepsilon_k}(x)$ is a solution of the SOCCP(F_{ε_k}). Thus, we obtain a contradiction. □

Obviously, Theorem 4.2 follows from Theorem 5.1 by setting $\tau_k = 0$ for all k. Also Corollaries 4.1 and 4.2 can be easily extended to the inexact framework.

Corollary 5.1. *Suppose that F has the Cartesian P_0-property and satisfies Condition A. Let $\{x^k\}$ be the sequence generated by Algorithm 5.1. If $\varepsilon_k \to 0$ and $\tau_k \to 0$, and the SOCCP(F) has a unique solution $\bar{x}$, then $\lim_{\varepsilon_k\to 0} x^k = \bar{x}$.*

Corollary 5.2. *Suppose that F has the Cartesian P_0-property and the Cartesian R_{02}-property and satisfies Condition A. Let $\{x^k\}$ be the sequence generated by Algorithm 5.1. If $\varepsilon_k \to 0$ and $\tau_k \to 0$, then $\{x^k\}$ is bounded and its every accumulation point is a solution of the SOCCP(F).*

In addition, by Proposition 4.3(a), we also have the following corollary.

Corollary 5.3. *Suppose that F is monotone and satisfies Condition A and the SOCCP(F) is strictly feasible. Let $\{x^k\}$ be the sequence generated by Algorithm 5.1. If $\varepsilon_k \to 0$ and $\tau_k \to 0$, then $\{x^k\}$ is bounded and every accumulation point is a solution of the SOCCP(F).*

Finally, we stress that, as far as we know, the inexact regularization Algorithm 5.1 studied in this section is currently the only algorithm to guarantee the SOCCP(F) with the Cartesian P_0-property and a nonempty bounded solution set can actually be solved.

6. Numerical experiments

In this section, we report our preliminary numerical experience with the inexact regularization method for solving some SOCPs and SOCCPs, and make numerical comparisons with the merit function approach [2] which reformulates the SOCCP(F) as:

$$\min_{x\in\mathbb{R}^n} \Psi_{\rm FB}(x). \tag{45}$$

All experiments were done at a PC with 2.8GHz CPU and 512MB memory. The computer codes were all written in Matlab 6.5. The subproblem (39) in Algorithm 5.1 and the minimization problem (45) were both solved by the limited-memory BFGS method with 5 limited-memory vector-updates. To improve the numerical performance of the BFGS method, we replaced the monotone Armijo line search by a nonmonotone line search as described by Zhang and Hager [24]. In other words, in the BFGS method, we computed the smallest nonnegative integer m such that

$$f(x^k+\beta^m d^k)\le \mathcal{W}_k+\sigma\beta^m\nabla f(x^k)^{\rm T}d^k$$

where $f(x)=\Psi_\varepsilon(x)$ or $\Psi_{\rm FB}(x)$, d^k was the direction of the kth iterate, and

$$\mathcal{W}_k:=(\eta_{k-1}Q_{k-1}\mathcal{W}_{k-1}+f(x^k))/Q_k \quad \text{with } Q_k=\eta_{k-1}Q_{k-1}+1.$$

Throughout the experiments, we adopted $\beta=0.5$, $\sigma=10^{-4}$, $\mathcal{W}_0=f(x^0)$, $Q_0=1$ and $\eta_k\equiv 0.85$ for all k. In addition, we updated ε_k and τ_k in Algorithm 5.1 by the formula:

$$\varepsilon_k=0.1\varepsilon_{k-1} \quad \text{and} \quad \tau_k=\varepsilon_k \quad \text{for all } k,$$

where the initial regularization parameter ε_0 was given in the corresponding examples. We terminated Algorithm 5.1 and the merit function approach [2] whenever one of the following conditions was satisfied: (1) $\Psi_{\rm FB}(x^k)\le 10^{-6}$ and $|\langle x^k,F(x^k)\rangle|\le 10^{-5}$; (2) the steplength was less than 10^{-10}.

To verify the efficiency of the regularization method, we first applied the inexact regularization method for solving a class of monotone SOCCPs, which correspond to the KKT optimality conditions of the linear SOCPs from the DIMACS Implementation Challenge library [19]. The standard linear SOCPs can be described as follows:

$$\begin{aligned}&\min\ c^{\rm T}x\\ &\text{s.t. } Ax=b,\quad x\in\mathcal{K},\end{aligned} \tag{46}$$

where $A\in\mathbb{R}^{m\times n}$ has full row rank, $b\in\mathbb{R}^m$ and $c\in\mathbb{R}^n$. From [2], we know that the KKT conditions of (46) are equivalent to finding a point $\zeta\in\mathbb{R}^n$ such that

$$F(\zeta)\in\mathcal{K},\qquad G(\zeta)\in\mathcal{K},\qquad \langle F(\zeta),G(\zeta)\rangle=0 \tag{47}$$

with

$$F(\zeta):=d+(I-A^{\rm T}(AA^{\rm T})^{-1}A)\zeta,\qquad G(\zeta):=c-A^{\rm T}(AA^{\rm T})^{-1}A\zeta,$$

where d satisfies $Ax=b$. Hence, the corresponding regularized SOCCP problem is

$$F_\varepsilon(\zeta)\in\mathcal{K},\qquad G_\varepsilon(\zeta)\in\mathcal{K},\qquad \langle F_\varepsilon(\zeta),G_\varepsilon(\zeta)\rangle=0 \tag{48}$$

with

$$F_\varepsilon(\zeta):=d+(I-A^{\rm T}(AA^{\rm T})^{-1}A)\zeta+\varepsilon\zeta,\qquad G_\varepsilon(\zeta):=c-A^{\rm T}(AA^{\rm T})^{-1}A\zeta+\varepsilon\zeta,$$

and the merit functions Ψ_ε and $\Psi_{\rm FB}$ are specialized as

$$\Psi_\varepsilon(\zeta)=\frac{1}{2}\sum_{i=1}^m\|\phi_{\rm FB}(F_\varepsilon(\zeta),G_\varepsilon(\zeta))\|^2 \quad \text{and} \quad \Psi_{\rm FB}(\zeta)=\frac{1}{2}\sum_{i=1}^m\|\phi_{\rm FB}(F(\zeta),G(\zeta))\|^2.$$

Table 1
Numerical results on the four DIMACS SOCPs

Problem	Regularization method					Merit function approach			
	ε_0	$\Psi_\varepsilon(z^f)$	NF	Iter	Optval	$\Psi_{\mathrm{FB}}(z^f)$	NF	Iter	Optval
nb	1	9.85e−7	1752	1320	−0.05076515	8.28e−7	2329	1660	−0.05080010
nb_L2	0.1	6.04e−9	405	259	−1.62899997	1.10e−9	576	391	−1.62898021
nb_L2_bessel	0.1	7.69e−9	163	133	−0.10254482	3.24e−7	150	136	−0.10265116

In the experiment, the vector d in $F(\zeta)$ was computed as the solution of $\min_d \|Ad - b\|$ by Matlab's least square solver, and F and G were evaluated via the Cholesky factorization of AA^{T}. We started Algorithm 5.1 and the merit function approach with the starting point $\zeta^0 = 0$. The numerical results were reported in Table 1, in which ε_0 lists the initial regularization parameter used by every test problem, **NF** represents the number of the merit function evaluations required by the methods for solving each problem, **Iter** records the modification number of the Hessian matrix in the BFGS method, and **Optval** denotes the objective function value of the SOCP (46) at the final iteration.

From Table 1, we see that for the class of monotone SOCCPs, the inexact regularization method can generate the solutions with favorable tolerance by requiring less function evaluations and Hessian matrix modifications. For the difficult problem "nb", the efficiency of the regularization method is more remarkable.

We also applied the inexact regularization method for solving some SOCCPs with F having the Cartesian P_0-property. Since the corresponding test examples cannot be found in the literature, we considered the case where $F = Mx + q$ with the matrix $M \in \mathbb{R}^{n\times n}$ and $q = (q_1, \ldots, q_m)$ generated randomly such that M has the Cartesian P_0-property. In the experiment, the matrix M was generated by the following procedure: chose the positive semidefinite matrices $M_i \in \mathbb{R}^{n_i\times n_i}$ for $i = 1, 2, \ldots, m$, and then let M be the block diagonal matrix with $M_1, \ldots, M_m$ as block diagonals, i.e.,

$$M = \mathrm{diag}(M_1, \ldots, M_m).$$

The positive semidefinite matrices $M_i,\ i = 1, 2, \ldots, m$ were set to be

$$M_i = N_i N_i^{\mathrm{T}},$$

where $N_i \in \mathbb{R}^{n_i\times n_i}$ was a square matrix whose nonzero elements were chosen randomly from a normal distribution with mean -1, variance 4. We can verify that the matrix M generated by such a way has the Cartesian P_0-property, and furthermore, it cannot have the Cartesian P-property by controlling the nonzero density of N_i such that every block matrix M_i has at least zero eigenvalues. In the experiment, the nonzero density of N_i for $i = 1, 2, \ldots, m$ was chosen as 0.5%. The elements of every subvector q_i were chosen randomly from the interval $[-1, 1]$, and then the first element q_{i1} of q_i is set to be $\|q_{i2}\|$, where q_{i2} is a vector composed of the rest $n_i - 1$ components of q_i. In this way, the affine SOCCP can be guaranteed to have a solution. In addition, to construct SOCs of various types, we chose n_i and m such that $n_1 = n_2 = \cdots = n_m$ and $n_1 + \cdots + n_m = 1000$.

We have tested the SOCCPs with the Cartesian P_0-property for two classes of SOCs: $m = 20$ and $m = 50$. For each type of $\mathcal{K}$, we solved 10 test problems by Algorithm 5.1 and the merit function approach [2] from the starting point $\zeta^0 = (\zeta^{n_i}, \ldots, \zeta^{n_m})$, where $\bar{\zeta}^{n_i} = (10, \omega_i/\|\omega_i\|)$ for $i = 1, 2, \ldots, m$ with $\omega_i \in \mathbb{R}^{n_i-1}$ generated randomly by Matlab's **rand.m**. The numerical results were reported in Tables 2 and 3, where **NF** and **Iter** have the same meanings as those in Table 1, and **Gap** denotes the value of the function $|\langle x, F(x)\rangle|$ at the final iteration. For the case where $m = 20$, we see from Table 2 that both Algorithm 5.1 and the merit function approach [2] can solve all 10 test problems, but Algorithm 5.1 required less function evaluations and Hessian matrix modifications. For the case where $m = 50$, Table 3 shows that Algorithm 5.1 generated the favorable solutions for all test problems, whereas the merit function approach [2] failed for all test problems due to too small steplength. Observe that in the case where $m = 50$, the positive semidefinite matrices $M_i,\ i = 1, 2, \ldots, m$ have more zero eigenvalues than those in the case where $m = 20$. Hence, the numerical results in Table 3 indicate that the regularization method proposed is superior to the merit function approach [2] in dealing with the ill-posed problems.

Table 2
Numerical results for the affine Cartesian P_0-problems with $m = 20$

Problem	Regularization method					Merit function approach			
	ε_0	$\Psi_\varepsilon(z^f)$	NF	Iter	Gap	$\Psi_{FB}(z^f)$	NF	Iter	Gap
1	1	5.20e−9	266	173	4.43e−6	7.49e−8	563	418	1.56e−6
2	1	8.35e−8	655	402	2.39e−6	1.13e−8	2593	2226	1.85e−6
3	1	2.85e−10	642	371	3.28e−6	3.76e−9	1522	1191	4.96e−6
4	1	8.64e−10	570	409	6.42e−6	1.79e−8	2000	1757	7.97e−6
5	1	4.87e−8	445	271	3.28e−6	5.88e−9	1367	1056	3.01e−6
6	1	2.72e−7	420	247	9.72e−7	5.05e−9	2124	1700	6.22e−7
7	1	1.94e−8	391	242	7.67e−6	4.53e−8	1143	844	3.98e−6
8	1	1.19e−7	530	327	8.31e−6	8.21e−8	1758	1470	6.56e−6
9	1	1.15e−9	510	305	3.31−6	1.47e−9	1265	922	1.99e−6
10	1	9.25e−8	641	369	1.44−6	1.15e−8	1368	982	8.24e−6

Table 3
Numerical results for the affine Cartesian P_0-problems with $m = 50$

Problem	Regularization method					Merit function approach			
	ε_0	$\Psi_\varepsilon(z^f)$	NF	Iter	Gap	$\Psi_{FB}(z^f)$	NF	Iter	Gap
1	10	2.60e−8	270	143	2.87e−6	3.53e−6	10456	7707	2.49e−2
2	10	6.30e−8	680	418	6.71e−6	2.00e−4	12500	9640	1.58e−1
3	10	2.22e−9	534	317	6.31e−6	1.39e−5	14552	12041	3.68e−2
4	10	3.91e−8	648	296	9.96e−6	3.65e−5	13327	8817	7.75e−2
5	10	3.10e−7	617	331	4.26e−6	5.64e−5	19788	16514	6.93e−2
6	10	2.62e−7	248	130	2.20e−6	1.20e−5	8222	6535	5.19e−2
7	10	1.12e−8	548	258	3.81e−6	3.34e−4	10971	8803	1.57e−1
8	10	1.00e−7	747	400	8.66e−6	1.84e−5	12328	9642	4.03e−2
9	10	5.22e−8	490	263	5.06−6	3.76e−4	6128	4795	2.77e−1
10	10	6.89e−7	86	40	8.86−7	6.52e−7	9593	6480	7.80e−3

7. Conclusions

In this paper, we considered the Tikhonov regularization method for the SOCCP(F) with the Cartesian P_0-property. We showed that the solution path generated by the regularized subproblems possesses the favorable properties in Theorem 4.2 if the mapping F also satisfies Condition A and the SOCCP(F) has a nonempty and bounded solution set. When F has the uniform Cartesian P-property, the solution path is bounded, but the bound is related to the constant involved in the uniform Cartesian P-property. Furthermore, in this case a counterexample was given to illustrate that the solution path may not be locally Lipschitz continuous. Preliminary numerical results were reported, which verified the desirable theoretical properties of the regularization method.

There are several open questions worth investigating in our future work. The first one is, for the monotone SOCCPs, to provide appropriate conditions to guarantee the continuity of the solution path $x(\varepsilon)$, and study whether the solution trajectory $x(\varepsilon)$ converges the least l_2-norm solution of the SOCCP(F) if the SOCCP(F) has a nonempty and bounded solution set. The second one is, for SOCCPs with Cartesian P_0-property, to present appropriate conditions to guarantee the continuity of the solution path $x(\varepsilon)$, and study whether the solution trajectory $x(\varepsilon)$ converges when it is bounded.

Acknowledgements

The authors would like to thank Prof. Paul Tseng for his helpful suggestions on improving the presentation of this paper. The second author's work is partially supported by the National Science Council of Taiwan.

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