Journal of Nonlinear and Convex Analysis
Volume 10, Number 3, 2009, 429–435

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A NOTE ON CONVEXITY OF TWO SIGNOMIAL FUNCTIONS

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ABSTRACT. In this note, we provide correct proofs for showing the convexity of two signomial functions which are frequently used in some recent papers [4, 6, 7, 8, 9] by Tsai et al.. Their arguments contain repeated flaws that motivate our work of this note.

1. MOTIVATION AND BASIC CONCEPTS

In this note, we consider two signomial functions whose convexity play important roles in some recent papers [4, 6, 7, 8, 9] dealing with geometric programming problems. However, the verifications therein contain some certain flaws and those incorrect arguments are repeatedly appeared and cited. From point of scientific research's view, we hereby provide correct proofs for them.

First, we recall what signomial function is. A function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ defined as

$$f(x) = cx_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where $c > 0$ and $\alpha_i \in \mathbb{R}$ for all i, is called a *monomial function* or simply a monomial, see [2]. Note that the exponents α_i of a monomial can be any real numbers, but the coefficient c must be nonnegative. A sum of monomials, namely, a function of the form

$$f(x) = \sum_{k=1}^{N} c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \cdots x_n^{\alpha_{nk}},$$

where $c_k > 0$ and $c_{ik} \in \mathbb{R}$, is called a *posynomial function* with N terms or simply a posynomial. A *signomial* is a linear combination of monomials of some positive variables $x_1, \ldots, x_n$. Generally speaking, signomials are more general than posynomials.

Next, we review some basic concepts and properties of symmetric matrices which will be used in subsequent analysis. These materials can be found in regular textbooks regarding matrix analysis and convex functions, e.g., [1, 3]. Let f be defined on an open convex set $D \subseteq \mathbb{R}^n$ and be twice differentiable, it is known that (i) f is convex on D if and only if the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite (p.s.d. for short) at each $x \in D$; (ii) if $\nabla^2 f(x)$ is positive definite (p.d. for short) at each $x \in D$, then f is strictly convex. The converse of (ii) is false, see the counterexample $f(x) = x^4$. Another important criterion for positive definiteness of a symmetric matrix A is via its leading principal minors as below. For convenience, we denote $\triangle_k$ as the leading principal minors of A.

2000 *Mathematics Subject Classification.* 90C25.

Key words and phrases. Convexity, positive definite, Hessian matrix, signomial function.

Lemma 1.1. *Let A be an $n \times n$ nonzero symmetric matrix.*

(a): *If A is positive semidefinite, then all its leading principal minors are nonnegative with not all of them being zero, i.e., $\triangle_k \geq 0$, $k = 1, 2, \ldots, n$ and not all $\triangle_k = 0$.*

(b): *A is positive definite if and only if all its leading principal minors are positive, i.e., $\triangle_k > 0$, for all $k = 1, 2, \ldots, n$.*

The positive definiteness of a symmetric matrix can be described not only by its leading principal minors, but also by all principal minors. More specifically, the positivity of any nested sequence of n principal minors of A (not just the leading principal minors) is necessary and sufficient for A to be positive definite (see [3, Theorem 7.2.5]). On the other hand, if all principal minors of A are nonnegative, then A is positive semidefinite (see [3, page 405]).

The converse of Lemma 1.1(a) is false. For example, let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, we have $\langle x, Ax \rangle = x_1^2 - x_3^2$ which is not always nonnegative for all $x \in \mathbb{R}^3$. But $\triangle_1 = 1 \geq 0$, $\triangle_2 = 0 \geq 0$, $\triangle_3 = 0 \geq 0$. In fact, the converse of Lemma 1.1(a) is true only for $n = 2$, see [1, page 112]. From the aforementioned discussion, we know that we can not tell the positive semidefiniteness of a symmetric matrix by its leading principal minors whereas we can do it for positive definiteness. Nonetheless, we still can reach the conclusion of the positive semidefiniteness of a symmetric matrix by the nonnegativeness of its eigenvalues. This can be seen as below.

Lemma 1.2. *Let A be an $n \times n$ nonzero symmetric matrix. Then, the followings hold.*

(a): *A is p.s.d. if and only if all of its eigenvalues are nonnegative with at least one eigenvalue being zero.*

(b): *A is p.d. if and only if all of its eigenvalues are positive.*

To close this section, we state another important relation between $\ln f(x)$ and $f(x)$ on their convexity that will be needed for proving our main results, i.e., suppose f is defined on a convex set $D \subseteq \mathbb{R}^n$ and $f(x) > 0$ for all $x \in D$, then the convexity of $\ln f(x)$ implies $f(x)$ being convex. Note that the converse is false, for instance, $f(x) = x^2$ is convex but $\ln f(x) = 2 \ln |x|$ is not convex.

2. Main results

Now we are ready to present our main results which show that the following two signomial functions are convex functions. As mentioned earlier, signomial functions play an important role in geometric programming. In particular, the convexity of such functions will help in designing solution methods for it which is the main motivation for this note.

Proposition 2.1. *Let $f_1 : \mathbb{R}^n_{++} \to \mathbb{R}$ be defined as $f_1(x) = c_1 \prod_{i=1}^{n} x_i^{\alpha_i}$, where $c_1 > 0$ and $\alpha_i \leq 0$ for all $i = 1, 2, \ldots, n$. Then f_1 is a convex function.*

Proof. Since $c_1 > 0$, it is enough to show that $\widetilde{f_1}(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ is convex.

Let $g(x)=\ln \widetilde{f_1}(x)=\sum_{i=1}^{n} \ln x_i^{\alpha_i}=\sum_{i=1}^{n} \alpha_i \ln x_i$. Then, we have

$$\nabla g(x) = \begin{bmatrix} \frac{\alpha_1}{x_1} & \frac{\alpha_2}{x_2} & \cdots & \frac{\alpha_n}{x_n} \end{bmatrix}^T \quad \text{and} \quad \nabla^2 g(x) = \begin{bmatrix} \frac{-\alpha_1}{x_1^2} & 0 & \cdots & 0 \\ 0 & \frac{-\alpha_2}{x_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{-\alpha_n}{x_n^2} \end{bmatrix}$$

Due to $\alpha_i \leq 0$ for all $i = 1, 2, \ldots, n$, we know that all eigenvalues of $\nabla^2 g(x)$ are nonnegative which implies (by Lemma 1.2(a)) that $\nabla^2 g(x)$ is positive semidefinite. Thus, $g(x)=\ln \widetilde{f}(x)$ is a convex function which yields $\widetilde{f_1}(x)$ being a convex function. $\square$

Proposition 2.2. *Let* $f_2 : \mathbb{R}^n_{++} \to \mathbb{R}$ *be defined as* $f_2(x) = c_2 \prod_{i=1}^{n} x_i^{\alpha_i}$, *where* $c_2 < 0$ *and* $\alpha_i > 0$ *for all* $i = 1, 2, \ldots, n$ *with* $1 - \sum_{i=1}^{n} \alpha_i \geq 0$. *Then* f_2 *is a convex function.*

Proof. It is not hard to compute that $[\nabla f_2(x)]_i = c_2 \alpha_i x_i^{\alpha_i - 1} \prod_{j=1, j\neq i}^{n} x_j^{\alpha_j}$. In other words,

$$\nabla f_2(x) = \begin{bmatrix} c_2\alpha_1 x_1^{\alpha_1-1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \\ c_2\alpha_2 x_1^{\alpha_1} x_2^{\alpha_2-1} \cdots x_n^{\alpha_n} \\ \vdots \\ c_2\alpha_n x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n-1} \end{bmatrix}.$$

In addition, it can be verified that

$$\left[\nabla^2 f_2(x)\right]_{ij} = \frac{\partial^2 f_2(x)}{\partial x_i \partial x_j} = \begin{cases} \frac{\alpha_i \alpha_j}{x_i x_j} f_2(x), & \text{if } i \neq j, \\ \frac{\alpha_i(\alpha_i - 1)}{x_i^2} f_2(x), & \text{if } i = j, \end{cases}$$

namely,

$$\nabla^2 f_2(x)$$

$$= \begin{bmatrix} c_2\alpha_1(\alpha_1-1)x_1^{-2}\prod_{i=1}^n x_i^{\alpha_i} & c_2\alpha_1\alpha_2 x_1^{-1}x_2^{-1}\prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2\alpha_1\alpha_n x_1^{-1}x_n^{-1}\prod_{i=1}^n x_i^{\alpha_i} \\ c_2\alpha_2\alpha_1 x_2^{-1}x_1^{-1}\prod_{i=1}^n x_i^{\alpha_i} & c_2\alpha_2(\alpha_2-1)x_2^{-2}\prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2\alpha_2\alpha_n x_2^{-1}x_n^{-1}\prod_{i=1}^n x_i^{\alpha_i} \\ \vdots & \vdots & \ddots & \vdots \\ c_2\alpha_n\alpha_1 x_n^{-1}x_1^{-1}\prod_{i=1}^n x_i^{\alpha_i} & c_2\alpha_n\alpha_2 x_n^{-1}x_2^{-1}\prod_{i=1}^n x_i^{\alpha_i} & \cdots & c_2\alpha_n(\alpha_n-1)x_n^{-2}\prod_{i=1}^n x_i^{\alpha_i} \end{bmatrix}$$

Moreover, the determinant of $\nabla^2 f_2(x)$ can be computed and be shown by induction as

$$\det\left[\nabla^2 f_2(x)\right] = (-c_2)^n\left(\prod_{i=1}^n \alpha_i x_i^{n\alpha_i-2}\right)\left(1-\sum_{i=1}^n \alpha_i\right). \tag{2.1}$$

Now, we will complete the proof by discussing the following two cases.

Case (i): If $1-\sum_{i=1}^n \alpha_i = 0$, we will show that $y^T\nabla^2 f_2(x)\, y \geq 0$ for any $y \in \mathbb{R}^n$ which says $\nabla^2 f_2(x)$ is a positive semidefinite matrix by definition, and hence $f_2(x)$ is a convex function under this case. To see this, we first write out the expression of $y^T\nabla^2 f_2(x)\, y$ as below

$$\begin{aligned} & y^T\nabla^2 f_2(x)\, y \\ = \; & c_2\prod_{i=1}^n x_i^{\alpha_i}\left\{\begin{array}{rl} & \alpha_1(\alpha_1-1)x_1^{-2}y_1^2+\alpha_1\alpha_2x_1^{-1}x_2^{-1}y_1y_2+\cdots+\alpha_1\alpha_nx_1^{-1}x_n^{-1}y_1y_n \\ + & \alpha_2\alpha_1x_2^{-1}x_1^{-1}y_1y_2+\alpha_2(\alpha_2-1)x_2^{-2}y_2^2+\cdots+\alpha_2\alpha_nx_2^{-1}x_n^{-1}y_2y_n \\ + & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + & \alpha_n\alpha_1x_n^{-1}x_1^{-1}y_1y_n+\alpha_n\alpha_2x_n^{-1}x_2^{-1}y_2y_n+\cdots+\alpha_n(\alpha_n-1)x_n^{-2}y_n^2 \end{array}\right\} \\ = \; & c_2\prod_{i=1}^n x_i^{\alpha_i}\left\{\begin{array}{rl} & \alpha_1x_1^{-1}y_1\left[(\alpha_1-1)x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n\right] \\ + & \alpha_2x_2^{-1}y_2\left[\alpha_1x_1^{-1}y_1+(\alpha_2-1)x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n\right] \\ + & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + & \alpha_nx_n^{-1}y_n\left[\alpha_1x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+(\alpha_n-1)x_n^{-1}y_n\right] \end{array}\right\} \\ = \; & c_2\prod_{i=1}^n x_i^{\alpha_i}\left\{\begin{array}{rl} & \alpha_1x_1^{-1}y_1\left[\alpha_1x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n-x_1^{-1}y_1\right] \\ + & \alpha_2x_2^{-1}y_2\left[\alpha_1x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n-x_2^{-1}y_2\right] \\ + & \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ + & \alpha_nx_n^{-1}y_n\left[\alpha_1x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n-x_n^{-1}y_n\right] \end{array}\right\} \\ = \; & c_2\prod_{i=1}^n x_i^{\alpha_i}\left\{\begin{array}{rl} & \left(\alpha_1x_1^{-1}y_1+\alpha_2x_2^{-1}y_2+\cdots+\alpha_nx_n^{-1}y_n\right)^2 \\ - & \left(\alpha_1x_1^{-2}y_1^2+\alpha_2x_2^{-2}y_2^2+\cdots+\alpha_nx_n^{-2}y_n^2\right) \end{array}\right\}. \end{aligned} \tag{2.2}$$

Next, we will argue that the whole thing inside the big parenthesis of (2.2) is non-positive by applying Cauchy-Schwarz inequality. In order to apply Cauchy-Schwarz inequality, we make the following arrangement:

$$\begin{aligned}
&\left[\left(\sqrt{\alpha_1}x_1^{-1}y_1\right)^2 + \left(\sqrt{\alpha_2}x_2^{-1}y_2\right)^2 + \cdots + \left(\sqrt{\alpha_n}x_n^{-1}y_n\right)^2\right] \\
&\left[\left(\sqrt{\alpha_1}\right)^2 + \left(\sqrt{\alpha_2}\right)^2 + \cdots + \left(\sqrt{\alpha_n}\right)^2\right] \\
&\geq \left[\alpha_1 x_1^{-1}y_1 + \alpha_2 x_2^{-1}y_2 + \cdots + \alpha_n x_n^{-1}y_n\right]^2 . \qquad (2.3)
\end{aligned}$$

Since $\left[\left(\sqrt{\alpha_1}\right)^2 + \left(\sqrt{\alpha_2}\right)^2 + \cdots + \left(\sqrt{\alpha_n}\right)^2\right] = 1$, inequality (2.3) is equivalent to

$$\left(\alpha_1 x_1^{-1}y_1 + \alpha_2 x_2^{-1}y_2 + \cdots + \alpha_n x_n^{-1}y_n\right)^2 - \left(\alpha_1 x_1^{-2}y_1^2 + \alpha_2 x_2^{-2}y_2^2 + \cdots + \alpha_n x_n^{-2}y_n^2\right) \leq 0.$$

This together with $c_2 < 0$ implies that $y^T\nabla^2 f_2(x)\ y \geq 0$ for any $y \in \mathbb{R}^n$. Thus, we complete the proof of case (i).

Case(ii): If $1 - \sum_{i=1}^{n} \alpha_i > 0$, then we know from (2.1) that

$$\triangle_i = (-c_2)^i \left(\prod_{j=1}^{i} \alpha_j\, x_j^{i\alpha_j - 2}\right)\left(1 - \sum_{j=1}^{i}\alpha_j\right), \qquad (2.4)$$

where $\triangle_i$ denotes the i-th leading principal minor of the Hessian matrix of $f_2(x)$. Note that $c_2 < 0$, $\alpha_i > 0$ for all $i = 1, 2, \cdots, n$, and $1 - \sum_{i=1}^{n}\alpha_i > 0$. Therefore, it can be seen that $\triangle_i > 0$ for all $i = 1, 2, \cdots, n$, which implies (by Lemma 1.1(b)) that $\nabla^2 f_2(x)$ is a positive definite matrix. This says that $f_2(x)$ is strictly convex under this case. $\square$

For Proposition 2.1, Tsai et al. claimed that (e.g. [4, Prop. 5(i)], [6, Prop. 1] and [9, Prop. 2]) all principal minors $\triangle_k \geq 0$ and concluded directly that f_1 is a convex function. As mentioned earlier, this property holds only for $n = 2$ and is not satisfied for general $n \geq 3$. For Proposition 2.2, Tsai et al. made the same mistakes again and did not notice that the case $1 - \sum_{i=1}^{n}\alpha_i = 0$ will cause the error therein (e.g. [4, Prop. 5(ii)], [6, Prop. 2] and [9, Prop. 3]).

We want to point out that our results also provide an alternative proof for the main result (Theorem 7) of [5]. Indeed, Maranas and Floudas in [5, Theorem 7] further discuss another condition as below

$$\exists j \ \text{ such that } \ \alpha_j \geq 1 - \sum_{i \neq j}^{n} \alpha_i, \ \text{ and } \ \alpha_i \leq 0,\ \forall i \neq j,\ i = 1, 2, \cdots n. \qquad (2.5)$$

to guarantee that f_1 defined as in Prop. 2.1 is a convex function. Our approach can be also employed to verify this fact. To see this, we arrange all powers α_i in decreasing order. In other words, without loss of generality, we assume

$$\alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n. \qquad (2.6)$$

Notice that condition (2.5) implies that α_1 is positive and all the other $\alpha_2, \cdots, \alpha_n$ are nonpositive with $\alpha_1 \geq 1 - \sum_{i=2}^{n}\alpha_i$. As mentioned in Prop. 2.1, we only need to show that the function $\widetilde{f}_1(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ is convex. By similar arguments as in the proof of Prop. 2.2, we know that

$$\widehat{\triangle}_i = (-1)^i \left(\prod_{j=1}^{i} \alpha_j \, x_j^{i\alpha_j - 2} \right) \left(1 - \sum_{j=1}^{i} \alpha_j \right),$$

where $\widehat{\triangle}_i$ denotes the i-th leading principal minor of the Hessian matrix of $\widetilde{f}_1(x)$. From conditions (2.5) and (2.6), it is easily verified that $\left(1 - \sum_{j=1}^{i} \alpha_j \right) < 0$ for each i. It is also not hard to observe that $\prod_{j=1}^{i} \alpha_j$ is positive if i is odd, and is negative if i is even. In other words, for each i there holds

$$(-1)^i \left(\prod_{j=1}^{i} \alpha_j \, x_j^{i\alpha_j - 2} \right) < 0.$$

In addition, we observe that $\widehat{\triangle}_n = 0$ when $\alpha_1 = 1 - \sum_{i=2}^{n} \alpha_i$. Thus, from all the above, we have either

$$\widehat{\triangle}_1 > 0, \cdots, \widehat{\triangle}_{n-1} > 0, \widehat{\triangle}_n > 0 \quad \text{if} \quad \alpha_1 > 1 - \sum_{i=2}^{n} \alpha_i \tag{2.7}$$

or

$$\widehat{\triangle}_1 > 0, \cdots, \widehat{\triangle}_{n-1} > 0, \widehat{\triangle}_n = 0 \quad \text{if} \quad \alpha_1 = 1 - \sum_{i=2}^{n} \alpha_i. \tag{2.8}$$

Then, Lemma 1.1(b) says that $\nabla^2 \widetilde{f}_1(x)$ is positive definite for case (2.7) whereas following the similar arguments as in Prop. 2.2 implies that $\nabla^2 \widetilde{f}_1(x)$ is positive semidefinite for case (2.8). Thus, we conclude that $\widetilde{f}_1$ is also a convex function under condition (2.5).

Acknowledgement

The authors would like to thank anonymous referees for their comments and suggestions which helps improve this paper a lot.

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Manuscript received March 24, 2008
revised January 12, 2009

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