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The convex and monotone functions associated with second-order cone

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Like the matrix-valued functions used in solutions methods for semidefinite programs (SDPs) and semidefinite complementarity problems (SDCPs), the vector-valued functions associated with second-order cones are defined analogously and also used in solutions methods for second-order-cone programs (SOCPs) and second-order-cone complementarity problems (SOCCPs). In this article, we study further about these vector-valued functions associated with second-order cones (SOCs). In particular, we define the so-called SOC-convex and SOC-monotone functions for any given function $f: \mathbb{R} \to \mathbb{R}$. We discuss the SOC-convexity and SOC-monotonicity for some simple functions, e.g., $f(t) = t^2$, t^3, $1/t$, $t^{1/2}$, $|t|$, and $[t]_+$. Some characterizations of SOC-convex and SOC-monotone functions are studied, and some conjectures about the relationship between SOC-convex and SOC-monotone functions are proposed.

Keywords: Second-order cone; Convex function; Monotone function; Complementarity; Spectral decomposition

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1. Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called the Lorentz cone, is defined by

$$\mathcal{K}^n = \left\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \leq x_1\right\}, \qquad (1)$$

where $\|\cdot\|$ denotes the Euclidean norm. If $n = 1$, let $\mathcal{K}^n$ denote the set of nonnegative reals $\mathbb{R}_+$. For any x, y in $\mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$; and write $x \succ_{\mathcal{K}^n} y$ if $x - y \in \text{int}(\mathcal{K}^n)$. In other words, we have $x \succeq_{\mathcal{K}^n} 0$ if and only if $x \in \mathcal{K}^n$, and $x \succ_{\mathcal{K}^n} 0$ if and only if $x \in \text{int}(\mathcal{K}^n)$. The relation $\succeq_{\mathcal{K}^n}$ is a partial ordering but not a linear ordering

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in $\mathcal{K}^n$, i.e., there exist $x, y \in \mathcal{K}^n$ such that neither $x \succeq_{\mathcal{K}^n} y$ nor $y \succeq_{\mathcal{K}^n} x$. To see this, for $n=2$, let $x=(1,1)$, $y=(1,0)$. Then, we have $x-y=(0,1) \notin \mathcal{K}^n$, $y-x=(0,-1) \notin \mathcal{K}^n$.

Recently, the SOC has received much attention in optimization, particularly in the context of applications and solutions methods for the second-order-cone program (SOCP) [14] and second-order-cone complementarity problem (SOCCP) [5–8]. For those solutions methods, *spectral decomposition* associated with SOC is required. The basic concepts are as follows. For any $x=(x_1,x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, x can be decomposed as

$$x = \lambda_1 u^{(1)} + \lambda_2 u^{(2)}, \tag{2}$$

where λ_1, λ_2 and $u^{(1)}$, $u^{(2)}$ are the spectral values and the associated spectral vectors of x given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|, \tag{3}$$

$$u^{(i)} = \begin{cases} \dfrac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\|x_2\|}\right), & \text{if } x_2 \neq 0, \\ \dfrac{1}{2}\left(1, (-1)^i w\right), & \text{if } x_2 = 0, \end{cases} \tag{4}$$

for $i=1$, 2 with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\|=1$. If $x_2 \neq 0$, the decomposition is unique.

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ ($n \geq 1$) was considered [8,10]:

$$f^{\text{soc}}(x) = f(\lambda_1)u^{(1)} + f(\lambda_2)u^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{5}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (5) is unambiguous whether $x_2 \neq 0$ or $x_2=0$. The cases of $f^{\text{soc}}(x)=x^{1/2}$, x^2, $\exp(x)$ are discussed in [9]. In fact, the equation (5) is analogous to one associated with the semidefinite cone $\mathcal{S}^n_+$ [19,21].

In this article, we further define the so-called SOC-convex and SOC-monotone functions (section 3), which are parallel to matrix-convex and matrix-monotone functions [2,11]. We study the SOC-convexity and SOC-monotinicity for some simple functions, e.g., $f(t)=t^2$, t^3, $1/t$, $t^{1/2}$, $|t|$, and $[t]_+$. Then, we explore the characterizations of SOC-convex and SOC-monotone functions. In addition, we state some conjectures about the relationship between SOC-convex and SOC-monotone functions. It is our intention to extend the existing properties of matrix-convex and matrix-monotone functions shown as in [2,11]. As will be seen in section 3, the vector-valued functions associated with SOC are accompanied by the Jordan product (will be defined in section 2). However, unlike matrix multiplication, the Jordan product associated with SOC is not associative, which is the main source of difficulty when we do the extension. Therefore, the ideas for proofs are usually quite different from those for matrix-valued functions. The vector-valued functions associated with SOC are heavily used in the solutions methods for SOCP and SOCCP. Therefore, further study on these functions will be helpful for developing and analyzing more solutions methods. That is one of the main motivations for this article.

In what follows and throughout the article, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product and $\|\cdot\|$ the Euclidean norm. The notation ":=" means "define". For any

$f: \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$, denotes the gradient of f and x. For any differentiable mapping $F = (F_1, F_2, \ldots, F_m)^T : \mathbb{R}^n \to \mathbb{R}^m$, $\nabla F(x) = [\nabla F_1(x) \cdots \nabla F_m(x)]$, is a $n \times m$ matrix denotes the transposed Jacobian of F at x. For any symmetric matrices $A, B \in \mathbb{R}^{n \times n}$, we write $A \succeq B$ (respectively, $A \succ B$) to mean $A - B$ is positive semidefinite (respectively, positive definite). We also use p.s.d. (respectively, p.d.) to represent the abbreviation of positive semidefinite (respectively, positive definite).

2. Jordan product and related properties

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (x^T y,\ y_1 x_2 + x_1 y_2). \tag{6}$$

We write x^2 to mean $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. Then $\circ$, $+$, together with $e = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$ have the following basic properties [9,10]: (1) $e \circ x = x$, for all $x \in \mathbb{R}^n$, (2) $x \circ y = y \circ x$, for all $x, y \in \mathbb{R}^n$, (3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, for all $x, y \in \mathbb{R}^n$ and (4) $(x + y) \circ z = x \circ z + y \circ z$, for all $x, y, z \in \mathbb{R}^n$. The Jordan product is not associative. For example, for $n = 3$, let $x = (1, -1, 1)$ and $y = z = (1, 0, 1)$, then we have $(x \circ y) \circ z = (4, -1, 4) \neq x \circ (y \circ z) = (4, -2, 4)$. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$. Thus, we may, without fear of ambiguity, write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. We define $x^0 = e$. Besides, $\mathcal{K}^n$ is not closed under Jordan product. For example, $x = (\sqrt{2}, 1, 1) \in \mathcal{K}^3$, $y = (\sqrt{2}, 1, -1) \in \mathcal{K}^3$, but $x \circ y = (2, 2\sqrt{2}, 0) \notin \mathcal{K}^3$.

For each $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the *determinant* and the *trace* of x are defined by

$$\det(x) = x_1^2 - \|x_2\|^2, \quad \operatorname{tr}(x) = 2x_1.$$

In general, $\det(x \circ y) \neq \det(x)\det(y)$ unless $x_2 = y_2$. A vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is said to be *invertible* if $\det(x) \neq 0$. If x is invertible, then there exists a unique $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ satisfying $x \circ y = y \circ x = e$. We call this y the inverse of x and denote it by x^{-1}. In fact, we have

$$x^{-1} = \frac{1}{x_1^2 - \|x_2\|^2}(x_1,\ -x_2) = \frac{1}{\det(x)}(\operatorname{tr}(x)e - x).$$

Therefore, $x \in \operatorname{int}(\mathcal{K}^n)$ if and only if $x^{-1} \in \operatorname{int}(\mathcal{K}^n)$. Moreover, if $x \in \operatorname{int}(\mathcal{K}^n)$, then $x^{-k} = (x^{-k})^{-1}$ is also well-defined. For any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{1/2}$ such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s}\right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 + \|x_2\|^2}\right)}.$$

In the preceding formula, the term x_2/s is defined to be the zero vector if $x_2 = 0$ and $s = 0$, i.e., $x = 0$.

For any $x \in \mathbb{R}^n$, we always have $x^2 \in \mathcal{K}^n$ (i.e., $x^2 \succeq_{\mathcal{K}^n} 0$). Hence, there exist a unique vector $(x^2)^{1/2} \in \mathcal{K}^n$ denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. It is also known that $|x| \succeq_{\mathcal{K}^n} x$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the nearest point (in Euclidean norm, since Jordan product does not induce a norm) projection of x

onto $\mathcal{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP: $[x]_+ = \arg\min\{\|x-y\| \,|y \in \mathcal{K}^n\}$. It is well-known that $[x]_+ = 1/2(x+|x|)$; see Property 2.2(f).

Next, for any $x=(x_1,x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$, we define a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ as

$$\begin{aligned} L_x : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ y &\longrightarrow L_x y := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} y. \end{aligned}$$

It can be easily verified that $x \circ y = L_x y$, $\forall y \in \mathbb{R}^n$, and L_x is positive definite (and hence invertible) if and only if $x \in \text{int}(\mathcal{K}^n)$. However, $L_x^{-1} y \neq x^{-1} \circ y$, for some $x \in \text{int}(\mathcal{K}^n)$ and $y \in \mathbb{R}^n$, i.e., $L_x^{-1} \neq L_{x^{-1}}$.

The spectral decomposition along with the Jordan algebra associated with SOC entail some basic properties as listed in the following text. We omit the proofs since they can be found in [9,10].

Property 2.1 For any $x=(x_1,x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$ with the spectral values λ_1, λ_2 and spectral vectors $u^{(1)}$, $u^{(2)}$ given as in equations (3)–(4), we have

(a) $u^{(1)}$ and $u^{(2)}$ are orthogonal under Jordan product and have length $1/\sqrt{2}$, i.e.,

$$u^{(1)} \circ u^{(2)} = 0, \quad \|u^{(1)}\| = \|u^{(2)}\| = \frac{1}{\sqrt{2}}.$$

(b) $u^{(1)}$ and $u^{(2)}$ are idempotent under Jordan product, i.e.,

$$u^{(i)} \circ u^{(i)} = u^{(i)}, \quad i = 1,2.$$

(c) λ_1, λ_2 are nonnegative (positive) if and only if $x \in \mathcal{K}^n$ ($x \in \text{int}(\mathcal{K}^n)$), i.e.,

$$\begin{aligned} \lambda_i \geq 0,\ \forall i = 1,2 &\Longleftrightarrow x \succeq_{\mathcal{K}^n} 0. \\ \lambda_i > 0,\ \forall i = 1,2 &\Longleftrightarrow x \succ_{\mathcal{K}^n} 0. \end{aligned}$$

(d) The determinant, the trace and the Euclidean norm of x can all be represented in terms of λ_1, λ_2:

$$\det(x) = \lambda_1\lambda_2, \quad \text{tr}(x) = \lambda_1 + \lambda_2, \quad \|x\|^2 = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right).$$

Property 2.2 For any $x=(x_1,x_2) \in \mathbb{R}\times\mathbb{R}^{n-1}$ with the spectral values λ_1, λ_2 and spectral vectors $u^{(1)}$, $u^{(2)}$ given as in equations (3)–(4), we have

(a) $x^2 = \lambda_1^2 u^{(1)} + \lambda_2^2 u^{(2)}$.
(b) If $x \in \mathcal{K}^n$, then $x^{1/2} = \sqrt{\lambda_1} u^{(1)} + \sqrt{\lambda_2} u^{(2)}$.
(c) $|x| = |\lambda_1| u^{(1)} + |\lambda_2| u^{(2)}$.
(d) $[x]_+ = [\lambda_1]_+ u^{(1)} + [\lambda_2]_+ u^{(2)}$, $[x]_- = [\lambda_1]_- u^{(1)} + [\lambda_2]_- u^{(2)}$.
(e) $|x| = [x]_+ + [-x]_+ = [x]_+ - [x]_-$.
(f) $[x]_+ = 1/2(x+|x|)$, $[x]_- = 1/2(x-|x|)$.

Property 2.3

(a) Any $x \in \mathbb{R}^n$ satisfies $|x| \succeq_{\mathcal{K}^n} x$.
(b) For any $x, y \succeq_{\mathcal{K}^n} 0$, if $x \succeq_{\mathcal{K}^n} y$, then $x^{1/2} \succeq_{\mathcal{K}^n} y^{1/2}$.

(c) For any $x, y \in \mathbb{R}^n$, if $x^2 \succeq_{\mathcal{K}^n} y^2$, then $|x| \succeq_{\mathcal{K}^n} |y|$.
(d) For any $x \in \mathbb{R}^n$, $x \succeq_{\mathcal{K}^n} 0 \Leftrightarrow \langle x, y\rangle \geq 0$, $\forall y \succeq_{\mathcal{K}^n} 0$.
(e) For any $x \succeq_{\mathcal{K}^n} 0$ and $y \in \mathbb{R}^n$, $x^2 \succeq_{\mathcal{K}^n} y^2 \Rightarrow x \succeq_{\mathcal{K}^n} y$.

In the following propositions, we study and explore more characterizations about spectral values, determinant, and trace of x as well as the partial order $\succeq_{\mathcal{K}^n}$. In fact, Propositions 2.1–2.4 are parallel results analogous to those associated with positive semidefinite cone [11]. Even though both $\mathcal{K}^n$ and $\mathcal{S}^n$ belong to self-dual cones and share similar properties, as we will see, the ideas for proving these results are quite different. One reason is that the Jordan product is not associative as mentioned earlier.

Proposition 2.1 *For any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *the following results hold.*

(a) *If* $x \succeq_{\mathcal{K}^n} y$, *then* $\det(x) \geq \det(y)$, $\mathrm{tr}(x) \geq \mathrm{tr}(y)$.
(b) *If* $x \succeq_{\mathcal{K}^n} y$, *then* $\lambda_i(x) \geq \lambda_i(y)$, $\forall i = 1, 2$.

Proof (a) From definition, we know that

$$\det(x) = x_1^2 - \|x_2\|^2, \quad \mathrm{tr}(x) = 2x_1,$$
$$\det(y) = y_1^2 - \|y_2\|^2, \quad \mathrm{tr}(y) = 2y_1.$$

Since $x - y = (x_1 - y_1, x_2 - y_2) \succeq_{\mathcal{K}^n} 0$, we have $\|x_2 - y_2\| \leq x_1 - y_1$. Thus, $x_1 \geq y_1$, and then $\mathrm{tr}(x) \geq \mathrm{tr}(y)$. Besides, the assumption on x and y gives

$$x_1 - y_1 \geq \|x_2 - y_2\| \geq \big|\|x_2\| - \|y_2\|\big|, \tag{7}$$

which is equivalent to $x_1 - \|x_2\| \geq y_1 - \|y_2\| > 0$ and $x_1 + \|x_2\| \geq y_1 + \|y_2\| > 0$. Hence,

$$\det(x) = x_1^2 - \|x_2\|^2 = (x_1 + \|x_2\|)(x_1 - \|x_2\|) \geq (y_1 + \|y_2\|)(y_1 - \|y_2\|) = \det(y).$$

(b) From definition of spectral values, we know that

$$\lambda_1(x) = x_1 - \|x_2\|, \quad \lambda_2(x) = x_1 + \|x_2\| \quad \text{and} \quad \lambda_1(y) = y_1 - \|y_2\|, \quad \lambda_2(y) = y_1 + \|y_2\|.$$

Then, by the inequality (7) in the proof of part (a), the results follow immediately. ■

Proposition 2.2 *For any* $x \succeq_{\mathcal{K}^n} 0$ *and* $y \succeq_{\mathcal{K}^n} 0$, *we have*

(a) $\det(x + y) \geq \det(x) + \det(y)$.
(b) $\det(x \circ y) \leq \det(x) \cdot \det(y)$.
(c) $\det(\alpha x + (1 - \alpha)y) \geq \alpha^2 \det(x) + (1 - \alpha)^2 \det(y)$, $\forall\, 0 < \alpha < 1$.
(d) $(\det(e + x))^{1/2} \geq 1 + \det(x)^{1/2}$, $\forall x \succeq_{\mathcal{K}^n} 0$.
(e) $\det(e + x + y) \leq \det(e + x) \cdot \det(e + y)$.

Proof

(a) For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, we know $\|x_2\| \leq x_1$ and $\|y_2\| \leq y_1$, which implies

$$|\langle x_2, y_2\rangle| \leq \|x_2\| \cdot \|y_2\| \leq x_1 y_1.$$

Hence, we obtain

$$\begin{aligned}\det(x + y) &= (x_1 + y_1)^2 - \|x_2 + y_2\|^2 \\ &= (x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) + 2(x_1 y_1 - \langle x_2, y_2\rangle) \\ &\geq (x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) \\ &= \det(x) + \det(y).\end{aligned}$$

(b) Applying the Cauchy inequality gives

$$\begin{aligned}\det(x \circ y) &= \langle x, y\rangle^2 - \|x_1 y_2 + y_1 x_2\|^2 \\ &= (x_1 y_1 + \langle x_2, y_2\rangle)^2 - \left(x_1^2\|y_2\|^2 + 2x_1 y_1\langle x_2, y_2\rangle + y_1^2\|x_2\|^2\right) \\ &= x_1^2 y_1^2 + \langle x_2, y_2\rangle^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2 \\ &\leq x_1^2 y_1^2 + \|x_2\|^2 \cdot \|y_2\|^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2 \\ &= (x_1^2 - \|x_2\|^2) \cdot (y_1^2 - \|y_2\|^2) \\ &= \det(x) \cdot \det(y).\end{aligned}$$

(c) For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, it is clear that $\alpha x \succ_{\mathcal{K}^n} 0$ and $(1-\alpha)y \succ_{\mathcal{K}^n} 0$ for every $0 < \alpha < 1$. In addition, we observe that $\det(\alpha x) = \alpha^2 \det(x)$, for all $\alpha > 0$. Hence,

$$\det(\alpha x + (1-\alpha)y) \geq \det(\alpha x) + \det((1-\alpha)y) = \alpha^2 \det(x) + (1-\alpha)^2 \det(y),$$

where the inequality is from part (a).

(d) For any $x \succeq_{\mathcal{K}^n} 0$, we know $\det(x) = \lambda_1\lambda_2 \geq 0$, where λ_i are the spectral values of x. Hence, $\det(e + x) = (1+\lambda_1)(1+\lambda_2) \geq (1 + \sqrt{\lambda_1\lambda_2})^2 = (1 + \det(x)^{1/2})^2$. Then, taking square root both sides yields the desired result.

(e) Again, For any $x \succeq_{\mathcal{K}^n} 0$ and $y \succeq_{\mathcal{K}^n} 0$, we have

$$\begin{cases} x_1 - \|x_2\| \geq 0, \\ y_1 - \|y_2\| \geq 0, \\ |\langle x_2, y_2\rangle| \leq \|x_2\| \cdot \|y_2\| \leq x_1 y_1. \end{cases} \tag{8}$$

Also, we know $\det(e + x + y) = (1 + x_1 + y_1)^2 - \|x_2 + y_2\|^2$, $\det(e + x) = (1 + x_1)^2 - \|x_2\|^2$ and $\det(e + y) = (1 + y_1)^2 - \|y_2\|^2$. Hence,

$$\begin{aligned}&\det(e + x) \cdot \det(e + y) - \det(e + x + y) \\ &\quad = \left((1 + x_1)^2 - \|x_2\|^2\right)\left((1 + y_1)^2 - \|y_2\|^2\right) \\ &\qquad - \left((1 + x_1 + y_1)^2 - \|x_2 + y_2\|^2\right) \\ &\quad = 2x_1 y_1 + 2\langle x_2, y_2\rangle + 2x_1 y_1^2 + 2x_1^2 y_1 - 2y_1\|x_2\|^2 - 2x_1\|y_2\|^2 \\ &\qquad + x_1^2 y_1^2 - y_1^2\|x_2\|^2 - x_1^2\|y_2\|^2 + \|x_2\|^2 \cdot \|y_2\|^2 \\ &\quad = 2(x_1 y_1 + \langle x_2, y_2\rangle) + 2x_1(y_1^2 - \|y_2\|^2) + 2y_1(x_1^2 - \|x_2\|^2) \\ &\qquad + (x_1^2 - \|x_2\|^2)(y_1^2 - \|y_2\|^2) \\ &\quad \geq 0,\end{aligned}$$

where we multiply out all the expansions to obtain the second equality and the last inequality holds by (8). ■

Proposition 2.3 *For any $x, y \in \mathbb{R}^n$, we have*

(a) $\mathrm{tr}(x + y) = \mathrm{tr}(x) + \mathrm{tr}(y)$.
(b) $\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) \leq \mathrm{tr}(x \circ y) \leq \lambda_1(x)\lambda_1(y) + \lambda_2(x)\lambda_2(y)$.
(c) $\mathrm{tr}(\alpha x + (1-\alpha)y) = \alpha \cdot \mathrm{tr}(x) + (1-\alpha) \cdot \mathrm{tr}(y)$, $\forall \alpha \in \mathbb{R}$.

Proof Parts (a) and (c) are trival. Thus, it remains to verify (b). Using the fact that $\mathrm{tr}(x \circ y) = 2\langle x, y\rangle$, we obtain

$$\begin{aligned}\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x) &= (x_1 - \|x_2\|)(y_1 + \|y_2\|_+(x_1 + \|x_2\|)(y_1 - \|y_2\|)\\ &= 2(x_1y_1 - \|x_2\|\|y_2\|)\\ &\leq 2(x_1y_1 + \langle x_2, y_2\rangle\\ &= 2\langle x, y\rangle = \mathrm{tr}(x \circ y)\\ &\leq 2(x_1y_1 + \|x_2\|\|y_2\|)\\ &= (x_1 - \|x_2\|)(y_1 - \|y_2\|) + (x_1 + \|x_2\|)(y_1 + \|y_2\|),\end{aligned}$$

which completes the proof. ■

The following two lemmas are well-known results in matrix analysis and are key to proving Proposition 2.4, which is an important extension about the function $\ln\det(\cdot)$ from positive semidefinite cone to SOC.

LEMMA 2.1 *For any nonzero vector $x \in \mathbb{R}^n$, the matrix xx^T is positive semidefinite* (*p.s.d.*) *with only one nonzero eigenvalue $\|x\|^2$.*

Proof The proof is routine, hence we omit it. ■

LEMMA 2.2 *Suppose that a symmetric matrix is partitioned as*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where A and C are square. Then this matrix is positive definite (*p.d.*) *if and only if A is positive definite and $C \succ B^TA^{-1}B$.*

Proof See Theorem 7.7.6 in [11]. ■

PROPOSITION 2.4 *For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, we have*

(a) *The real-valued function $f(x) = \ln(\det(x))$ is concave on* $\mathrm{int}(\mathcal{K}^n)$.
(b) $\det(\alpha x + (1-\alpha)y) \geq (\det(x))^\alpha(\det(y))^{1-\alpha}, \quad \forall 0 < \alpha < 1.$
(c) *The function real-valued $f(x) = \ln(\det(x^{-1})$ is convex on* $\mathrm{int}(\mathcal{K}^n)$.
(d) *The real-valued function $f(x) = \mathrm{tr}(x - 1)$ is convex on* $\mathrm{int}(\mathcal{K}^n)$.

Proof

(a) Since $\mathrm{int}(\mathcal{K}^n)$ is a convex set, it is enough to show that $\nabla^2 f(x)$ is negative semidefinite. From direct computation, we know

$$\nabla f(x) = \left(\frac{2x_1}{x_1^2 - \|x_2\|^2}, \frac{-2x_2}{x_1^2 - \|x_2\|^2}\right) = 2x^{-1},$$

and

$$\nabla^2 f(x) = \begin{bmatrix} \dfrac{-2x_1^2 - 2\|x_2\|^2}{(x_1^2 - \|x_2\|^2)^2} & \dfrac{4x_1 x_2^T}{(x_1^2 - \|x_2\|^2)^2} \\ \dfrac{4x_1 x_2}{(x_1^2 - \|x_2\|^2)^2} & \dfrac{-2(x_1^2 - \|x_2\|^2)I - 4x_2 x_2^T}{(x_1^2 - \|x_2\|^2)^2} \end{bmatrix}$$
$$= \frac{-2}{(x_1^2 - \|x_2\|^2)^2} \begin{bmatrix} (x_1^2 + \|x_2\|^2)^2 & -2x_1 x_2^T \\ -2x_1 x_2 & (x_1^2 - \|x_2\|^2)I + 2x_2 x_2^T \end{bmatrix}.$$

Let $\nabla^2 f(x)$ be denoted by the matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

given as in Lemma 2.2 (here A is a scalar). Then, we have

$$\begin{aligned} AC - B^T B &= (x_1^2 + \|x_2\|^2)\big((x_1^2 - \|x_2\|^2)I + 2x_2 x_2^T\big) - 4x_1^2 x_2 x_2^T \\ &= (x_1^4 - \|x_2\|^4)I - 2(x_1^2 - \|x_2\|^2)x_2 x_2^T \\ &= (x_1^2 - \|x_2\|^2)\big((x_1^2 + \|x_2\|^2)I - 2x_2 x_2^T\big) \\ &= (x_1^2 - \|x_2\|^2) \cdot M, \end{aligned}$$

were we denote the whole matrix in the big parenthesis of the last second equality by M. From Lemma 2.1, we know that $x_2 x_2^T$ is a p.s.d. with only one nonzero eigenvalue $\|x_2\|^2$. Hence, all the eigenvalues of the matrix M are $(x_1^2 + \|x_2\|^2) - 2\|x_2\|^2 = x_1^2 - \|x_2\|^2$ and $x_1^2 + \|x_2\|^2$ with multiplicity of $n-2$, which are all positive. Thus, M is positive definite which implies that $\nabla^2 f(x)$ is negative definite and hence negative semidefinite.

(b) From part (a), for all $0 < \alpha < 1$, we have

$$\begin{aligned} \ln(\det(\alpha x + (1-\alpha)y)) &\geq \alpha \ln(\det(x)) + (1-\alpha)\ln(\det(y)) \\ &= \ln(\det(x))^{\alpha} + \ln(\det(y))^{1-\alpha} \\ &= \ln(\det(x))^{\alpha}(\det(y))^{1-\alpha}. \end{aligned}$$

Since natural logarithm is an increasing function, the desired result follows.

(c) We observe that $\det(x^{-1}) = 1/\det(x)$, for all $x \in \mathrm{int}(\mathcal{K}^n)$. Therefore, $\ln\det(x^{-1}) = -\ln\det(x)$ is a convex function by part (a).

(d) The idea for proving this is the same as the one for part (a). Since $\mathrm{int}(\mathcal{K}^n)$ is a convex set, it is enough to show that $\nabla^2 f$ is positive semidefinite. Note that $f(x) = \mathrm{tr}(x^{-1}) = 2x_1/(x_1^2 - \|x_2\|^2)$. Thus, from direct computations, we have

$$\nabla^2 f(x) = \frac{2}{(x_1^2 - \|x_2\|^2)^3} \begin{bmatrix} 2x_1^3 + 6x_1\|x_2\|^2, & -(6x_1^2 + 2\|x_2\|^2)x_2^T \\ -(6x_1^2 + 2\|x_2\|^2)x_2, & 2x_1\big((x_1^2 - \|x_2\|^2)I + 4x_2 x_2^T\big) \end{bmatrix}.$$

Again, let $\nabla^2 f(x)$ be denoted by the matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

given as in Lemma 2.2 (here A is a scalar). Then, we have

$$\begin{aligned} AC - B^TB &= 2x_1\big(2x_1^3 + 6x_1\|x_2\|^2\big)\big((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\big) - \big(6x_1^2 + 2\|x_2\|^2\big)^2 x_2x_2^T \\ &= \big(4x_1^4 + 12x_1^2\|x_2\|^2\big)\big(x_1^2 - \|x_2\|^2\big)I - \big(20x_1^4 - 24x_1^2\|x_2\|^2 + 4\|x_2\|^4\big)x_2x_2^T \\ &= \big(4x_1^4 + 12x_1^2\|x_2\|^2\big)\big(x_1^2 - \|x_2\|^2\big)I - 4\big(5x_1^2 - \|x_2\|^2\big)\big(x_1^2 - \|x_2\|^2\big)x_2x_2^T \\ &= \big(x_1^2 - \|x_2\|^2\big)\big[\big(4x_1^4 + 12x_1^2\|x_2\|^2\big)I - 4\big(5x_1^2 - \|x_2\|^2\big)x_2x_2^T\big] \\ &= \big(x_1^2 - \|x_2\|^2\big) \cdot M, \end{aligned}$$

where we denote the whole matrix in the big parenthesis of the last second equality by M. From Lemma 2.1, we know that $x_2x_2^T$ is a p.s.d. with only one nonzero eigenvalue $\|x_2\|^2$. Hence, all the eigenvalues of the matrix M are $(4x_1^4 + 12x_1^2\|x\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4)$ and $4x_1^4 + 12x_1^2\|x_2\|^2$ with multiplicity of $n-2$, which are all positive since

$$\begin{aligned} 4x_1^4 + 12x_1^2\|x_2\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4 &= 4x_1^4 - 8x_1^2\|x_2\|^2 + 4\|x_2\|^4 \\ &= 4\big(x_1^2 - \|x_2\|^2\big) \\ &> 0. \end{aligned}$$

Thus, by Lemma 2.2, we obtain that $\nabla^2 f(x)$ is positive definite and hence is positive semidefinite. Therefore, f is convex on $\operatorname{int}(\mathcal{K}^n)$. ■

3. SOC-convex function and SOC-monotone function

In this section, we define the SOC-convexity and SOC-monotonicity and the study some examples of such functions.

Definition 3.1 Let $f : \mathbb{R} \to \mathbb{R}$. Then

(a) f is said to be SOC-monotone of order n if the corresponding vector-valued function $f^{\rm soc}$ satisfies the following:

$$x \succeq_{\mathcal{K}^n} y \Rightarrow f^{\rm soc}(x) \succeq_{\mathcal{K}^n} f^{\rm soc}(y).$$

(b) f is said to be SOC-convex of order n if the corresponding vector-valued function $f^{\rm soc}$ satisfies the following:

$$f^{\rm soc}((1-\lambda)x + \lambda y) \preceq_{\mathcal{K}^n} (1-\lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y), \tag{9}$$

for all $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$.

We say f is SOC-monotone (respectively, SOC-convex) if f is SOC-monotone of all order n (respectively, SOC-convex of all order n). If f is continuous, then the condition in equation (9) can be replaced by the more special condition:

$$f^{\rm soc}\Big(\frac{x+y}{2}\Big) \preceq_{\mathcal{K}^n} \frac{1}{2}\big(f^{\rm soc}(x) + f^{\rm soc}(y)\big). \tag{10}$$

It is clear that the set of SOC-monotone functions and the set of SOC-convex functions are both closed under positive linear combinations and under pointwise limits.

Proposition 3.1 *Let* $f:\mathbb{R}\to\mathbb{R}$ *be* $f(t)=\alpha+\beta t$, *then*

(a) f *is SOC-monotone on* $\mathbb{R}$ *for every* $\alpha\in\mathbb{R}$ *and* $\beta\geq 0$.
(b) f *is SOC-convex on* $\mathbb{R}$ *for all* $\alpha,\beta\in\mathbb{R}$.

Proof The proof is straightforward by checking that Definition 3.1 is satisfied. ■

Proposition 3.2

(a) *Let* $f:\mathbb{R}\to\mathbb{R}$ *be* $f(t)=t^2$, *then* f *is SOC-convex on* $\mathbb{R}$.
(b) *Hence, the function* $g(t)=\alpha+\beta t+\gamma t^2$ *is SOC-convex on* $\mathbb{R}$ *for all* $\alpha,\beta\in\mathbb{R}$ *and* $\gamma\geq 0$.

Proof

(a) For any $x,y\in\mathbb{R}^n$, we have

$$\frac{1}{2}(f(x)+f(x))-f\Big(\frac{x+y}{2}\Big)=\frac{x^2+y^2}{2}-\Big(\frac{x+y}{2}\Big)^2=\frac{1}{4}(x-y)^2\succeq_{\mathcal{K}^n}0.$$

Since f is continuous, the above implies that f is SOC-convex.
(b) This is an immediate consequence of (a). ■

Example 3.1 The function $f(t)=t^2$ is not SOC-monotone on $\mathbb{R}$.

To see this, let $x=(1,0)$, $y=(-2,0)$, then $x-y=(3,0)\succeq_{\mathcal{K}^n}0$. But, $x^2-y^2=(1,0)-(4,0)=(-3,0)\not\succeq_{\mathcal{K}^n}0$.

It is clear that $f(t)=t^2$ is also SOC-convex on the smaller interval $[0,\infty)$ by Proposition 3.2(a). We may ask a natural question: Is $f(t)=t^2$ SOC-monotone on the interval $[0,\infty)$? The answer is: it is true only for $n=2$ and is, false for general $n\geq 3$. We show this in the following example.

Example 3.2

(a) The function $f(t)=t^2$ is SOC-monotone on $[0,\infty)$ for $n=2$.
(b) However, $f(t)=t^2$ is not SOC-monotone on $[0,\infty)$ for $n\geq 3$.

(a) Let $x=(x_1,x_2)\succeq_{\mathcal{K}^2}y=(y_1,y_2)\succeq_{\mathcal{K}^2}0$. Then we have the following inequalities:

$$|x_2|\leq x_1,\quad |y_2|\leq y_1,\quad |x_2-y_2|\leq x_1-y_1,$$

which implies

$$\begin{cases} x_1-x_2\geq y_1-y_2\geq 0, \\ x_1+x_2\geq y_1+y_2\geq 0. \end{cases}\tag{11}$$

We want to prove that $f(x)-f(y)=(x_1^2+x_2^2-y_1^2-y_2^2,2x_1x_2-2y_1y_2)\succeq_{\mathcal{K}^2}0$, which is enough to verify that $x_1^2+x_2^2-y_1^2-y_2^2\geq|2x_1x_2-2y_1y_2|$. This can been seen by

$$
\begin{aligned}
&x_1^2+x_2^2-y_1^2-y_2^2-|2x_1x_2-2y_1y_2| \\
&\quad=\begin{cases} x_1^2+x_2^2-y_1^2-y_2^2-(2x_1x_2-2y_1y_2) & \text{if } x_1x_2-y_1y_2\geq 0 \\ x_1^2+x_2^2-y_1^2-y_2^2-(2y_1y_2-2x_1x_2) & \text{if } x_1x_2-y_1y_2\leq 0 \end{cases} \\
&\quad=\begin{cases} (x_1-x_2)^2-(y_1-y_2)^2 & \text{if } x_1x_2-y_1y_2\geq 0 \\ (x_1+x_2)^2-(y_1+y_2)^2 & \text{if } x_1x_2-y_2y_2\leq 0 \end{cases} \\
&\quad\geq 0,
\end{aligned}
$$

where the inequalities are true due to the inequalities (11).

(b) For $n\geq 3$, we give a counterexample to show that $f(t)=t^2$ is not SOC-monotone on the interval $[0,\infty)$. Let $x=(3,1,-2)\in\mathcal{K}^3$ and $y=(1,1,0)\in\mathcal{K}^3$. It is clear that $x-y=(2,0,-2)\succeq_{\mathcal{K}^3}0$. But $x^2-y^2=(14,6,-12)-(2,2,0)=(12,4,-12)\not\succeq_{\mathcal{K}^3}0$.

Now we look at the function $f(t)=t^3$. As expected, $f(t)=t^3$ is not SOC-convex. However, it is true that $f(t)=t^3$ is SOC-convex on $[0,\infty)$ for $n=2$, whereas it is false for $n\geq 3$. Besides, we will se $f(t)=t^3$ is neither SOC-monotone on $\mathbb{R}$ nor SOC-monotone on the interval $[0,\infty)$ in general. Nonetheless, it is true that it is SOC-monotone on the interval $[0,\infty)$ for $n=2$. The following two examples show what we have just said.

Example 3.3

(a) The function $f(t)=t^3$ is not SOC-convex on $\mathbb{R}$.
(b) Moreover, $f(t)=t^3$ is not SOC-convex on $[0,\infty)$ for $n\geq 3$.
(c) However, $f(t)=t^3$ is SOC-convex on $[0,\infty)$ for $n=2$.

To see (a), let $x=(0,-2)$, $y=(1,0)$. It can be verified that $1/2(f(x)+f(y))-f((x+y)/2)=(-9/8,\ -9/4)\not\succeq_{\mathcal{K}^2}0$, which says $f(t)=t^3$ is not SOC-convex on $\mathbb{R}$.

To see (b), let $x=(2,1,-1)$, $y=(1,1,0)\succeq_{\mathcal{K}^3}0$, then we have $1/2(f(x)+f(y))-f(x+y/2)=(3,1,-3)\not\succeq_{\mathcal{K}^3}0$, which implies $f(t)=t^3$ is not even SOC-convex on the interval $[0,\infty)$.

To see (c), it is enough to show that $f((x+y)/2)\preceq_{\mathcal{K}^2}1/2(f(x)+f(y))$ for any x,y in $\mathcal{K}^2$ 0. Let $x=(x_1,x_2)\succeq_{\mathcal{K}^2}0$ and $y=(y_1,y_2)\succeq_{\mathcal{K}^2}0$, then we have

$$
\begin{cases} x^3=(x_1^3+3x_1x_2^2,\,3x_1^2x_2+x_2^3), \\ y^3=(y_1^3+3y_1y_2^2,\,3y_1^2y_2+y_2^3), \end{cases}
$$

which yields

$$
\begin{cases} f\Big(\dfrac{x+y}{2}\Big)=\dfrac{1}{8}\big((x_1+y_2)^3+3(x_1+y_1)(x_2+y_2)^2,3(x_1+y_1)^2(x_2+y_2)+(x_2+y_2)^3\big), \\ \dfrac{1}{2}(f(x)+f(y))=\dfrac{1}{2}\big(x_1^3+y_1^3+3x_1x_2^2+3y_1y_2^2,x_2^3+y_2^3+3x_1^2x_2+3y_1^2y_2\big). \end{cases}
$$

After simplifications, we denote $1/2(f(x)+f(y))-f((x+y)/2):=1/8(\Xi_1,\Xi_2)$, where

$$\begin{cases} \Xi_1 = 4x_1^3+4y_1^3+12x_1x_2^2+12y_1y_2^2-(x_1+y_1)^3-3(x_1+y_1)(x_2+y_2)^2, \\ \Xi_2 = 4x_2^3+4y_2^3+12x_1^2x_2+12y_1^2y_2-(x_2+y_2)^3-3(x_1+y_1)^2(x_2+y_2). \end{cases}$$

To show that $\Xi_1 \geq |\Xi_2|$ we discuss two cases. First, if $\Xi_2 \geq 0$, then

$$\begin{aligned} \Xi_1-|\Xi_2| &= (4x_1^3+12x_1x_2^2-12x_1^2x_2-4x_2^3)+(4y_1^3+12y_1y_2^2-12y_1^2y_2-4y_2^3) \\ &\quad -\big((x_1+y_1)^3+3(x_1+y_1)(x_2+y_2)^2-3(x_1+y_1)^2(x_2+y_2)-(x_2+y_2)^3\big) \\ &= 4(x_1-x_2)^3+4(y_1-y_2)^3-((x_1+y_1)-(x_2+y_2))^3 \\ &= 4(x_1-x_2)^3+4(y_1-y_2)^3-((x_1-x_2)+(y_1-y_2))^3 \\ &= 3(x_1-x_2)^3+3(y_1-y_2)^3-3(x_1-x_2)^2(y_1-y_2)-3(x_1-x_2)(y_1-y_2)^2 \\ &= 3((x_1-x_2)+(y_1-y_2))\big((x_1-x_2)^2-(x_1-x_2)(y_1-y_2)+(y_1-y_2)^2\big) \\ &\quad -3(x_1-x_2)(y_1-y_2)((x_1-x_2)+(y_1-y_2)) \\ &= 3((x_1-x_2)+(y_1-y_2))((x_1-x_2)-(y_1-y_2))^2 \\ &\geq 0, \end{aligned}$$

where the inequality is true since $x,y\in\mathcal{K}^2$. Similarly, if $\Xi_2 \leq 0$, we also have,

$$\begin{aligned} \Xi_1-|\Xi_2| &= (4x_1^3+12x_1x_2^2+12x_1^2x_2+4x_2^3)+(4y_1^3+12y_1y_2^2+12y_1^2y_2+4y_2^3) \\ &\quad -\big((x_1+y_1)^3+3(x_1+y_1)(x_2+y_2)^2+3(x_1+y_1)^2(x_2+y_2)+(x_2+y_2)^3\big) \\ &= 4(x_1+x_2)^3+4(y_1+y_2)^3-((x_1+y_1)+(x_2+y_2))^3 \\ &= 4(x_1+x_2)^3+4(y_1+y_2)^3-((x_1+x_2)+(y_1+y_2))^3 \\ &= 3(x_1+x_2)^3+3(y_1+y_2)^3-3(x_1+x_2)^2(y_1+y_2)-3(x_1+x_2)(y_1+y_2)^2 \\ &= 3((x_1+x_2)+(y_1+y_2))\big((x_1+x_2)^2-(x_1+x_2)(y_1+y_2)+(y_1+y_2)^2\big) \\ &\quad -3(x_1+x_2)(y_1+y_2)((x_1+x_2)+(y_1+y_2)) \\ &= 3((x_1+x_2)+(y_1+y_2))((x_1+x_2)-(y_1+y_2))^2 \\ &\geq 0, \end{aligned}$$

where the inequality is true since $x,y\in\mathcal{K}^2$. Thus, we have verified that $f(t)=t^3$ is SOC-convex on $[0,\infty)$ for $n=2$.

Example 3.4

(a) The function $f(t)=t^3$ is not SOC-monotone on $\mathbb{R}$.
(b) Moreover, $f(t)=t^3$ is not SOC-monotone on $[0,\infty)$ for $n\geq 3$.
(c) However, $f(t)=t^3$ is SOC-monotone on $[0,\infty)$ for $n=2$.

To see (a) and (b), let $x=(2,1,-1)\succeq_{\mathcal{K}^3} 0$ and $y=(1,1,0)\succeq_{\mathcal{K}^3} 0$. It is clear that $x\succeq_{\mathcal{K}^3} y$. But, we have $f(x)=x^3=(20,14,-14)$ and $f(y)=y^3=(4,4,0)$, which gives $f(x) - f(y) = (16, 10, -14) \not\succeq_{\mathcal{K}^3} 0$. Thus, we show that $f(t) = t^3$ is not even SOC-monotone on the interval $[0, \infty)$.

To see (c), let $x = (x_1, x_2) \succeq_{\mathcal{K}^2} y = (y_1, y_2) \succeq_{\mathcal{K}^2} 0$. Again, we have the following inequalities:

$$|x_2| \leq x_1, \quad |y_2| \leq y_1, \quad |x_2 - y_2| \leq x_1 - y_1,$$

which leads to the inequalities (11). In addition, we know

$$\begin{aligned} f(x) = x^3 &= \left(x_1^3 + 3x_1x_2^2, \quad 3x_1^2x_2 + x_2^3\right), \\ f(y) = y^3 &= \left(y_1^3 + 3y_1y_2^2, \quad 3y_1^2y_2 + y_2^3\right). \end{aligned}$$

We denote $f(x) - f(y) := (\Xi_1, \Xi_2)$, where

$$\begin{cases} \Xi_1 = x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2, \\ \Xi_2 = x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2. \end{cases}$$

We wish to prove that $f(x) - f(y) = x^3 - y^3 \succeq_{\mathcal{K}^2} 0$, which is enough to show $\Xi_1 \geq |\Xi_2|$. This is true because

$$\begin{aligned} & x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - \left|x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2\right| \\ &= \begin{cases} x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 - (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2) & \text{if } \Xi_2 \geq 0 \\ x_1^3 - y_1^3 + 3x_1x_2^2 - 3y_1y_2^2 + (x_2^3 - y_2^3 + 3x_1^2x_2 - 3y_1^2y_2) & \text{if } \Xi_2 \leq 0 \end{cases} \\ &= \begin{cases} (x_1 - x_2)^3 - (y_1 - y_2)^3 & \text{if } \Xi_2 \geq 0 \\ (x_1 + x_2)^3 - (y_1 + y_2)^3 & \text{if } \Xi_2 \leq 0 \end{cases} \\ &\geq 0, \end{aligned}$$

where the inequalities are true by the inequalities (11).

Hence, we complete the verification.

Now, we move to another simple function $f(t) = 1/t$. We will prove that $-1/t$ is SOC-monotone on the interval $(0, \infty)$ and $1/t$ is SOC-convex on the interval $(0, \infty)$ as well. For the proof, we need the following technical lemmas.

Lemma 3.1 *For any* $a \geq b > 0$ *and* $c \geq d > 0$, *we always have*

$$\left(\frac{a}{b}\right) \cdot \left(\frac{c}{d}\right) \geq \frac{a+c}{b+d} \tag{12}$$

Proof The proof is followed by $ac(b+d) - bd(a+c) = ab(c-d) + cd(a-b) \geq 0$. ■

Lemma 3.2 *For any* $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathcal{K}^n$, *we have*

(a) $(x_1 + y_1)^2 - \|y_2\|^2 \geq 4x_1\sqrt{y_1^2 - \|y_2\|^2}$.
(b) $(x_1 + y_1 - \|y_2\|)^2 \geq 4x_1(y_1 - \|y_2\|)$.
(c) $(x_1 + y_1 + \|y_2\|)^2 \geq 4x_1(y_1 + \|y_2\|)$.
(d) $x_1y_1 - \langle x_2, y_2 \rangle \geq \sqrt{x_1^2 - \|x_2\|^2} \cdot \sqrt{y_1^2 - \|y_2\|^2}$.
(e) $(x_1 + y_1)^2 - \|x_2 + y_2\|^2 \geq 4\sqrt{x_1^2 - \|x_2\|^2} \cdot \sqrt{y_1^2 - \|y_2\|^2}$.

Proof

(a) The proof follows from

$$\begin{aligned}(x_1+y_1)^2-\|y_2\|^2 &= x_1^2+(y_1^2-\|y_2\|^2)+2x_1y_1\\ &\geq 2x_1\sqrt{y_1^2-\|y_2\|^2}+2x_1y_1\\ &\geq 2x_1\sqrt{y_1^2-\|y_2\|^2}+2x_1\sqrt{y_1^2-\|y_2\|^2}\\ &= 4x_1\sqrt{y_1^2-\|y_2\|^2},\end{aligned}$$

where the first inequality is true due to the fact that $a+b\geq 2\sqrt{ab}$ for any positive numbers a and b.

(b) The proof follows from

$$\begin{aligned}(x_1+y_1-\|y_2\|)^2-4x_1(y_1-\|y_2\|) &= x_1^2+y_1^2+\|y_2\|^2-2x_1y_1-2y_1\|y_2\|+2x_1\|y_2\|\\ &= (x_1-y_1+\|y_2\|)^2\geq 0.\end{aligned}$$

(c) Similarly, the proof follows from

$$\begin{aligned}(x_1+y_1+\|y_2\|)^2-4x_1(y_1+\|y_2\|) &= x_1^2+y_1^2+\|y_2\|^2-2x_1y_1+2y_1\|y_2\|-2x_1\|y_2\|\\ &= (x_1-y_1-\|y_2\|)^2\geq 0.\end{aligned}$$

(d) We know that $x_1y_1-\langle x_2,y_2\rangle\geq x_1y_1-\|x_2\|\cdot\|y_2\|\geq 0$, and

$$\begin{aligned}(x_1y_1-\|x_2\|\cdot\|y_2\|)^2-\left(x_1^2-\|x_2\|^2\right)\left(y_1^2-\|y_2\|^2\right) &= x_1^2\|y_2\|^2+y_1^2\|x_2\|^2-2x_1y_1\|x_2\|\cdot\|y_2\|\\ &= (x_1\|y_2\|-y_1\|x_2\|)^2\geq 0.\end{aligned}$$

Hence, we obtain $x_1y_1-\langle x_2,y_2\rangle\geq x_1y_1-\|x_2\|\cdot\|y_2\|\geq\sqrt{x_1^2-\|x_2\|^2}\cdot\sqrt{y_1^2-\|y_2\|^2}$.

(e) The proof follows from

$$\begin{aligned}(x_1+y_1)^2-\|x_2+y_2\|^2 &= \left(x_1^2-\|x_2\|^2\right)+\left(y_1^2-\|y_2\|^2\right)+2(x_1y_1-\langle x_2,y_2\rangle)\\ &\geq 2\sqrt{(x_1^2-\|x_2\|^2)(y_1^2-\|y_2\|^2)}+2(x_1y_1-\langle x_2,y_2\rangle)\\ &\geq 2\sqrt{(x_1^2-\|x_2\|^2)(y_1^2-\|y_2\|^2)}+2\sqrt{(x_1^2-\|x_2\|^2)(y_1^2-\|y_2\|^2)}\\ &= 4\sqrt{(x_1^2-\|x_2\|^2)(y_1^2-\|y_2\|^2)},\end{aligned}$$

where the first inequality is true since $a+b\geq 2\sqrt{ab}$ for all positive a, b and the second inequality is from part(d). ■

PROPOSITION 3.3 *Let* $f:(0,\infty)\to(0,\infty)$ *be* $f(t)=1/t$. *Then*

(a) $-f$ *is SOC-monotone on* $(0,\infty)$.

(b) f *is SOC-convex on* $(0,\infty)$.

Proof

(a) It suffices to show that $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}_n} 0$ implies $x^{-1} \preceq_{\mathcal{K}^n} y^{-1}$. For any $x, y \in \mathcal{K}^n$, we know that $y^{-1} = (1/\det(y))(y_1, -y_2)$, $x^{-1} = 1/\det(x)(x_1, -x_2)$. Thus,

$$\begin{aligned} y^{-1} - x^{-1} &= \left(\frac{y_1}{\det(y)} - \frac{x_1}{\det(x)}, \frac{x_2}{\det(x)} - \frac{y_2}{\det(y)}\right) \\ &= \frac{1}{\det(x)\det(y)}(\det(x)y_1 - \det(y)x_1, \ \det(y)x_2 - \det(x)y_2). \end{aligned}$$

To complete the proof, we need to verify two things.

(1) First, we have to show that $\det(x)y_1 - \det(y)x_1 \geq 0$. Applying Lemma 3.1 yields

$$\frac{\det(x)}{\det(y)} = \frac{x_1^2 - \|x_2\|^2}{y_1^2 - \|y_2\|^2} = \left(\frac{x_1 + \|x_2\|}{y_1 + \|y_2\|}\right)\left(\frac{x_1 - \|x_2\|}{y_1 - \|y_2\|}\right) \geq \frac{2x_1}{2y_1} = \frac{x_1}{y_1}.$$

Then cross multiplying gives $\det(x)y_1 \geq \det(y)x_1$, i.e., $\det(x)y_1 - \det(y)x_1 \geq 0$.

(2) Secondly, we show that $\|\det(y)x_2 - \det(x)y_2\| \leq \det(x)y_1 - \det(y)x_1$. This is true by

$$\begin{aligned} &(\det(x)y_1 - \det(y)x_1)^2 - \|\det(y)x_2 - \det(x)y_2\|^2 \\ &= (\det(x))^2 y_1^2 - 2\det(x)\det(y)x_1y_1 + (\det(y))^2 x_1^2 \\ &\quad - \big((\det(y))^2\|x_2\|^2 - 2\det(x)\det(y)\langle x_2, y_2\rangle + (\det(x))^2\|y_2\|^2\big) \\ &= (\det(x))^2(y_1^2 - \|y_2\|^2) + (\det(y))^2(x_1^2 - \|x_2\|^2) \\ &\quad - 2\det(x)\det(y)(x_1y_1 - \langle x_2, y_2\rangle) \\ &= (\det(x))^2\det(y) + (\det(y))^2\det(x) - 2\det(x)\det(y)(x_1y_1 - \langle x_2, y_2\rangle) \\ &= \det(x)\det(y)(\det(x) + \det(y) - 2x_1y_1 + 2\langle x_2, y_2\rangle) \\ &= \det(x)\det(y)\big((x_1^2 - \|x_2\|^2) + (y_1^2 - \|y_2\|^2) - 2x_1y_1 + 2\langle x_2, y_2\rangle\big) \\ &= \det(x)\det(y)\big((x_1 - y_1)^2 - (\|x_2\|^2 + \|y_2\|^2 - 2\langle x_2, y_2\rangle)\big) \\ &= \det(x)\det(y)\big((x_1 - y_1)^2 - (\|x_2 - y_2\|^2)\big) \\ &\geq 0, \end{aligned}$$

where the last step holds by the inequality (8) given as in the proof of Proposition 2.1(a). Thus, from all the above, we proved $y^{-1} - x^{-1} \in \mathcal{K}^n$, i.e., $y^{-1} \succeq_{\mathcal{K}^n} x^{-1}$.

(b) For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$ we have

$$\begin{cases} x_1 - \|x_2\| > 0 \\ y_1 - \|y_2\| > 0 \\ |\langle x_2, y_2\rangle| \leq \|x_2\| \cdot \|y_2\| \leq x_1y_1 \end{cases} \tag{13}$$

From $x^{-1} = 1/\det(x)(x_1, -x_2)$ and $y^{-1} = 1/\det(y)(y_1, -y_2)$, we also have

$$\frac{1}{2}(f(x) + f(y)) = \frac{1}{2}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}, -\frac{x_2}{\det(x)} - \frac{y_2}{\det(y)}\right),$$

and

$$f\left(\frac{x+y}{2}\right) = \left(\frac{x+y}{2}\right)^{-1} = \frac{2}{\det(x+y)}(x_1 + y_1, -(x_2 + y_2)).$$

We denote $1/2(f(x)+f(y))-f((x+y)/2) := 1/2(\Xi_1, \Xi_2)$, where $\Xi_1 \in \mathbb{R}$ and $\Xi_2 \in \mathbb{R}^{n-1}$ are given by

$$\begin{cases} \Xi_1 = \left(\dfrac{x_1}{\det(x)} + \dfrac{y_1}{\det(y)}\right) - \dfrac{4(x_1+y_1)}{\det(x+y)}, \\ \Xi_2 = \dfrac{4(x_2+y_2)}{\det(x+y)} - \left(\dfrac{x_2}{\det(x)} + \dfrac{y_2}{\det(y)}\right). \end{cases}$$

Again, to prove f is SOC-convex, it suffices to verify two things: $\Xi_1 \geq 0$ and $\|\Xi_2\| \leq \Xi_1$.

(1) First, we verify that $\Xi_1 \geq 0$. In fact, if we define the function

$$g(x) := \frac{x_1}{x_1^2 - \|x_2\|^2} = \frac{x_1}{\det(x)},$$

then we observe that

$$g\left(\frac{x+y}{2}\right) \leq \frac{1}{2}(g(x)+g(y)) \Longleftrightarrow \Xi_1 \geq 0.$$

Hence, to prove $\Xi_1 \geq 0$, it is equivalent to show g is convex on $\mathrm{int}(\mathcal{K}^n)$. Since $\mathrm{int}(\mathcal{K}^n)$ is a convex set, it is sufficient to show that $\nabla^2 g(x)$ is a positive semidefinite matrix. From direct computations, we have

$$\nabla^2 g(x) = \frac{1}{(x_1^2 - \|x_2\|^2)^3} \begin{bmatrix} 2x_1^3 + 6x_1\|x_2\|^2 & -(6x_1^2 + 2\|x_2\|^2)x_2^T \\ -(6x_1^2 + 2\|x_2\|^2)x_2 & 2x_1\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right) \end{bmatrix}.$$

Let $\nabla^2 g(x)$ be denoted by the matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

given as in Lemma 2.2 (here A is a scalar). Then, we have

$$\begin{aligned} AC - B^TB &= 2x_1\left(2x_1^3 + 6x_1\|x_2\|^2\right)\left((x_1^2 - \|x_2\|^2)I + 4x_2x_2^T\right) - \left(6x_1^2 + 2\|x_2\|^2\right)^2 x_2x_2^T \\ &= \left(4x_1^4 + 12x_1^2\|x_2\|^2\right)\left(x_1^2 - \|x_2\|^2\right)I - \left(20x_1^4 - 24x_1^2\|x_2\|^2 + 4\|x_2\|^4\right)x_2x_2^T \\ &= \left(4x_1^4 + 12x_1^2\|x_2\|^2\right)\left(x_1^2 - \|x_2\|^2\right)I - 4\left(5x_1^2 - \|x_2\|^2\right)\left(x_1^2 - \|x_2\|^2\right)x_2x_2^T \\ &= \left(x_1^2 - \|x_2\|^2\right)\left[\left(4x_1^4 + 12x_1^2\|x_2\|^2\right)I - 4\left(5x_1^2 - \|x_2\|^2\right)x_2x_2^T\right] \\ &= \left(x_1^2 - \|x_2\|^2\right) \cdot M, \end{aligned}$$

where we denote the whole matrix in the big parenthesis of the last second equality by M. From Lemma 2.1, we know that $x_2x_2^T$ is p.s.d. with only one nonzero eigenvalue $\|x_2\|^2$. Hence, all the eigenvalues of the matrix M are $(4x_1^4 + 12x_1^2\|x\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4)$ and $4x_1^4 + 12x_1^2\|x_2\|^2$ with multiplicity of $n-2$, which are all positive since

$$\begin{aligned} 4x_1^4 + 12x_1^2\|x_2\|^2 - 20x_1^2\|x_2\|^2 + 4\|x_2\|^4 &= 4x_1^4 - 8x_1^2\|x_2\|^2 + 4\|x_2\|^4 \\ &= 4\left(x_1^2 - \|x_2\|^2\right) \\ &> 0. \end{aligned}$$

Thus, by Lemma 2.2, we obtain that $\nabla^2 g(x)$ is positive definite and hence is positive semidefinite. It follows g is convex on $\mathrm{int}(\mathcal{K}^n)$, which says $\Xi_1 \geq 0$.
(2) It remains to show that $\Xi_1^2 - \|\Xi_2\|^2 \geq 0$:

$$\begin{aligned}
\Xi_1^2 - \|\Xi_2\|^2 &= \left[\left(\frac{x_1^2}{\det(x)^2} + \frac{2x_1y_1}{\det(x)\det(y)} + \frac{y_1^2}{\det(y)^2}\right) - \frac{8(x_1+y_1)}{\det(x+y)}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}\right)\right. \\
&\quad \left. + \frac{16}{\det(x+y)^2}(x_1^2 + 2x_1y_1 + y_1^2)\right] - \left\|\frac{4(x_2+y_2)}{\det(x+y)} - \left(\frac{x_2}{\det(x)} + \frac{y_2}{\det(y)}\right)\right\|^2 \\
&= \left[\left(\frac{x_1^2}{\det(x)^2} + \frac{2x_1y_1}{\det(x)\det(y)} + \frac{y_1^2}{\det(y)^2}\right) - \frac{8(x_1+y_1)}{\det(x+y)}\left(\frac{x_1}{\det(x)} + \frac{y_1}{\det(y)}\right)\right. \\
&\quad \left. + \frac{16}{\det(x+y)^2}(x_1^2 + 2x_1y_1 + y_1^2)\right] - \left[\frac{16}{\det(x+y)^2}(\|x_2\|^2 + 2\langle x_2, y_2\rangle + \|y_2\|^2)\right. \\
&\quad \left. - 8\left\langle \frac{x_2+y_2}{\det(x+y)}, \frac{x_2}{\det(x)} + \frac{y_2}{\det(y)}\right\rangle + \left(\frac{\|x_2\|^2}{\det(x)^2} + \frac{2\langle x_2, y_2\rangle}{\det(x)\det(y)} + \frac{\|y_2\|^2}{\det(y)^2}\right)\right] \\
&= \left[\frac{x_1^2 - \|x_2\|^2}{\det(x)^2} + \frac{2(x_1y_1 - \langle x_2, y_2\rangle)}{\det(x)\det(y)} + \frac{y_1^2 - \|y_2\|^2}{\det(y)^2}\right] \\
&\quad + \frac{16}{\det(x+y)^2}\left[(x_1^2 - \|x_2\|^2) + 2(x_1y_1 - \langle x_2, y_2\rangle) + (y_1^2 - \|y_2\|^2)\right] \\
&\quad - 8\left[\frac{x_1^2 - \|x_2\|^2}{\det(x+y)\det(x)} + \frac{x_1y_1 - \langle x_2, y_2\rangle}{\det(x+y)\det(x)} + \frac{x_1y_1 - \langle x_2, y_2\rangle}{\det(x+y)\det(y)} + \frac{y_1^2 + \|y_2\|^2}{\det(x+y)\det(y)}\right] \\
&= (x_1^2 - \|x_2\|^2)\left(\frac{1}{\det(x)^2} + \frac{16}{\det(x+y)^2} - \frac{8}{\det(x+y)\det(x)}\right) \\
&\quad + (y_1^2 - \|y_2\|^2)\left(\frac{1}{\det(y)^2} + \frac{16}{\det(x+y)^2} - \frac{8}{\det(x+y)\det(y)}\right) \\
&\quad + 2(x_1y_1 - \langle x_2, y_2\rangle)\left(\frac{1}{\det(x)\det(y)} + \frac{16}{\det(x+y)^2}\right. \\
&\quad \left. - \frac{4}{\det(x+y)\det(x)} - \frac{4}{\det(x+y)\det(y)}\right) \\
&= (x_1^2 - \|x_2\|^2)\left(\frac{\det(x+y) - 4\det(x)}{\det(x)\det(x+y)}\right)^2 + (y_1^2 - \|y_2\|^2)\left(\frac{\det(x+y) - 4\det(y)}{\det(y)\det(x+y)}\right)^2 \\
&\quad + 2(x_1y_1 - \langle x_2, y_2\rangle)\left(\frac{(\det(x+y) - 4\det(x))(\det(x+y) - 4\det(y))}{\det(x)\det(y)\det(x+y)^2}\right).
\end{aligned}$$

Now apply the fact that $\det(x) = x_1^2 - \|x_2\|^2$, $\det(y) = y_1^2 - \|y_2\|^2$, and $\det(x+y) - \det(x) - \det(y) = 2(x_1y_1 - \langle x_2, y_2\rangle)$, we can simplify the last equality (after a lot of algebra simplifications) and obtain

$$\Xi_1^2 - \|\Xi_2\|^2 = \frac{[\det(x+y) - 2\det(x) - 2\det(y)]^2}{\det(x)\det(y)\det(x+y)} \geq 0.$$

Hence, we proved that $f(x+y/2) \preceq_{\mathcal{K}^n} 1/2(f(x)+f(y))$, which says the function $f(t) = 1/t$ is SOC-convex on the interval $(0, \infty)$. ∎

PROPOSITION 3.4

(a) *The function* $f(t) = t/(1+t)$ *is SOC-monotone on* $(0, \infty)$.
(b) *For any* $\lambda > 0$, *the function* $f(t) = t/(\lambda + t)$ *is SOC-monotone on* $(0, \infty)$.

Proof

(a) Let $g(t) = -1/t$ and $h(t) = 1 + t$, then both functions are SOC-monotone from Propositions 3.2 and 3.3. Since $f(t) = 1 - 1/(1+t) = h(g(1+t))$, the result follows from that the composition of two SOC-monotone functions is also SOC-monotone.
(b) Similarly, let $g(t) = t/(1+t)$ and $h(t) = t/\lambda$, then both functions are SOC-monotone by part (a). Since $f(t) = g(h(t))$, the result is true by the same reason as in part (a). ■

PROPOSITION 3.5 *For any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *we have*

$$L_x \succeq_{\mathcal{K}^n} L_y \Longleftrightarrow L_y^{-1} \succeq_{\mathcal{K}^n} L_x^{-1} \Longleftrightarrow L_{y^{-1}} \succeq_{\mathcal{K}^n} L_{x^{-1}}$$

Proof By the property of L_x that $x \succeq_{\mathcal{K}^n} y \Longleftrightarrow L_x \succeq_{\mathcal{K}^n} L_y$, and Proposition 3.3(a), then proof follows. ■

Next, we examine another simple function $f(t) = \sqrt{t}$. We will see that it is SOC-monotone on the interval $[0, \infty)$, and $-\sqrt{t}$ is SOC-convex on $[0, \infty)$.

PROPOSITION 3.6 *Let* $f: [0, \infty) \to [0, \infty)$ *be* $f(t) = t^{1/2}$. *Then*

(a) f *is SOC-monotone on* $[0, \infty)$.
(b) $-f$ *is SOC-convex on* $[0, \infty)$.

Proof

(a) This is a consequence of Property 2.3(b).
(b) To show $-f$ is SOC-convex, it is enough to prove that $f((x+y)/2) \succeq_{\mathcal{K}^n} (f(x) + f(y)/2)$, which is equivalent to verify that $((x+y)/2)^{1/2} \succeq_{\mathcal{K}^n} (\sqrt{x} + \sqrt{y})/2$, $\forall x, y \in \mathcal{K}^n$. Since $x + y \succeq_{\mathcal{K}^n} 0$, by Property 2.3(e), it is sufficient to show that $((x+y)/2) \succeq_{\mathcal{K}^n} ((\sqrt{x} + \sqrt{y})/2)^2$. This can be seen by $((x+y)/2) - ((\sqrt{x} + \sqrt{y})/2)^2 = (\sqrt{x} - \sqrt{y})^2/4 \succeq_{\mathcal{K}^n} 0$. Thus, we complete the proof. ■

PROPOSITION 3.7 *Let* $f: [0, \infty) \to [0, \infty)$ *be* $f(t) = t^r$, $0 \le r \le 1$. *Then*

(a) f *is SOC-monotone on* $[0, \infty)$.
(b) $-f$ *is SOC-convex on* $[0, \infty)$.

Proof (a) Let r be a dyadic rational, i.e., a number of the form $r = m/2^n$, where n is any positive integer and $1 \le m \le 2^n$. It is enough to prove the assertion is true for such r since such r are dense in $[0, 1]$. We will show this by induction on n. Let $x, y \in \mathcal{K}^n$ with $x \succeq_{\mathcal{K}^n} y$, then by Property 2.3(b) we have $x^{1/2} \succeq_{\mathcal{K}^n} y^{1/2}$. Therefore, part (a) is true when $n = 1$. Suppose it is also true for all dyadic rational $m/2_j$, in which $1 \le j \le n-1$. Now let $r = m/2^n$ with $m \le 2^n$. By induction hypothesis, we know $x^{m/2^{n-1}} \succeq_{\mathcal{K}^n} y^{m/2^{n-1}}$. Then, by applying Property 2.3(b), we obtain $(x^{m/2^{n-1}})^{1/2} \succeq_{\mathcal{K}^n} (y^{m/2^n-1})^{1/2}$, which says $x^{m/2^n} \succeq_{\mathcal{K}^n} y^{m/2^n}$. Thus, we have shown that $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ implies $x^r \succeq_{\mathcal{K}^n} y^r$, for all dyadic rational r in $[0, 1]$. Such r are dense in $[0, 1]$, which says part (a) is true.

(b) The proof for part (b) is very similar to the preceding arguments. First, we observe that

$$\left(\frac{x+y}{2}\right) - \left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2 = \left(\frac{\sqrt{x}-\sqrt{y}}{2}\right)^2 \succeq_{\mathcal{K}^n} 0,$$

which implies $((x+y)/2)^{1/2} \succeq_{\mathcal{K}^n} 1/2(\sqrt{x}+\sqrt{y})$ by Property 2.3(b). Hence, we show that the assertion is true when $n=1$. By induction hypothesis, suppose $((x+y)/2)^{m/2^{n-1}} \succeq_{\mathcal{K}^n} ((x^{m/2^{n-1}} + y^{m/2^{n-1}})/2)$. Then we have

$$\begin{aligned}\left(\frac{x+y}{2}\right)^{m/2^{n-1}} - \left(\frac{x^{m/2^n}+y^{m/2^n}}{2}\right)^2 &\succeq_{\mathcal{K}^n} \left(\frac{x^{m/2^{n-1}}+y^{m/2^{n-1}}}{2}\right) - \left(\frac{x^{m/2^n}+y^{m/2^n}}{2}\right)^2 \\ &= \left(\frac{x^{m/2^n}-y^{m/2^n}}{2}\right)^2 \succeq_{\mathcal{K}^n} 0,\end{aligned}$$

which implies $((x+y)/2)^{m/2^n} \succeq_{\mathcal{K}^n} ((x^{m/2^n} + y^{m/2^n})/2)$ by Property 2.3(b). Following the same arguments about dyadic rational in part (a) yields the desired result. ■

From all the aforementioned examples, we know that f being monotone does not imply f^{soc} is SOC-monotone. Similarly, f being convex does not imply f^{soc} is SOC-convex. Now, we move onto some famous functions which are used very often for nonlinear complementarity problem (NCP), SDCP, and SOCCP. It would be interesting to know about the SOC-convexity and SOC-monotonicity of these functions. First, we will look at the Fischer–Burmeister function, $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, given by

$$\phi(x,y) = (x^2+y^2)^{1/2} - (x+y), \tag{14}$$

which is a well-known merit function for complementarity problem [13,18]. For SOCCP, it has been shown that squared norm of ϕ, i.e.,

$$\psi(x,y) = \|\phi(x,y)\|^2, \tag{15}$$

is continuously differentiable [7] whereas ψ is only shown differentiable for SDCP [21]. In addition, ϕ is proved to have semismoothness and Lipschitz continuity in recent paper [20] for both cases of SOCCP and SDCP. In NCP, ϕ is a convex function, so we may wish to have an analogy for SOCCP. Unfortunately, as shown in the following text, it is not a SOC-convex function.

Example 3.5 Let ϕ be defined as in (14) and ψ defined as in (15).

(a) The function $\rho(x,y) = (x^2+y^2)^{1/2}$ is not SOC-convex.
(b) The Fischer–Burmeister function ϕ is not SOC-convex.
(c) The function $\psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is not convex.

To see (a), a counter-example occurs when $x=(1,1)$ and $y=(1,0)$.

To see (b), suppose it is SOC-convex. Then we will have ρ is SOC-convex by $\rho(x,y) = \phi(x,y) + (x+y)$, which contradicts (a). Thus, ϕ is not SOC-convex.

To see (c), let $x=(1,-2)$, $y=(1,-1)$ and $u=(0,-1)$, $v=(1,-1)$. Then, we have

$$\phi(x,y)=\left(\frac{-3+\sqrt{13}}{2},\frac{7-\sqrt{13}}{2}\right)\Rightarrow\psi(x,y)=\|\phi(x,y)\|^2=21-5\sqrt{13}.$$
$$\phi(u,v)=\left(\frac{-1+\sqrt{5}}{2},\frac{5-\sqrt{5}}{2}\right)\Rightarrow\psi(u,v)=\|\phi(u,v)\|^2=9-3\sqrt{5}.$$

Thus, $1/2(\psi(x,y)+\psi(u,v))=1/2(30-5\sqrt{13}-3\sqrt{5})\approx 2.632$.

On the other hand, let $(a,b):=1/2(x,y)+1/2(u,v)$, that is, $a=(1/2,\ -3/2)$ and $b=(1,-1)$. Indeed, we have $a^2+b^2=(9/2,\ -7/2)$ and hence $(a^2+b^2)^{1/2}=(1+2\sqrt{2}/2,1-2\sqrt{2}/2)$, which implies $\psi(a,b)=\|\phi(a,b)\|^2=14-8\sqrt{2}\approx 2.686$. Therefore, we obtain

$$\psi\left(\frac{1}{2}(x,y)+\frac{1}{2}(u,v)\right)>\frac{1}{2}\psi(x,y)+\frac{1}{2}\psi(u,v),$$

which shows ψ is not convex.

Another function based on the Fischer–Burmeister function is $\psi_1:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, given by

$$\psi_1(x,y):=\|[\phi(x,y)]_+\|^2, \tag{16}$$

where ϕ is the Fischer–Burmeister function given as in equation (14). In the NCP case, it is known that ψ_1 is convex. It has been an open question whether this is still true for SDCP and SOCCP (see Q3 on p. 182 of [21]). In fact, Qi and Chen [16] gave the negative answer for the SDCP case. Here, we provide an answer to the question for SOCCP: ψ_1 is not convex in the SOCCP case.

Example 3.6 Let ϕ be defined as in (14) and ψ_1 defined as in (16).

(a) The function $[\phi(x,y)]_+=[(x^2+y^2)^{1/2}-(x+y)]_+$ is not SOC-convex.
(b) The function ψ_1 is not convex.

To see (a), let $x=(2,1,-1)$, $y=(1,1,0)$ and $u=(1,-2,5)$, $v=(-1,5,0)$. Also, we denote $\phi_1(x,y):=[\phi(x,y)]_+$. Then, by direct computations, we obtain

$$\frac{1}{2}\phi_1(x,y)+\frac{1}{2}\phi_1(u,v)-\phi_1\left(\frac{1}{2}(x,y)+\frac{1}{2}(u,v)\right)=\big(1.0794,0.4071,-1.0563\big)\not\succeq_{\mathcal{K}^3}0,$$

which says ϕ_1 is not SOC-convex.

To see (b), let $x=(17,5,16)$, $y=(20,-3,15)$ and $u=(2,3,3)$, $v=(9,-7,2)$. It can be easily verified that $1/2\psi_1(x,y)+1/2\psi_1(u,v)-\psi_2(1/2(x,y)+1/2(u,v))<0$, which implies ψ_1 is not convex.

Example 3.7

(a) The function $f(t)=|t|$ is not SOC-monotone.
(b) The function $f(t)=|t|$ is not SOC-convex.
(c) The function $f(t)=[t]_+$ is not SOC-monotone.
(d) The function $f(t)=[t]_+$ is not SOC-convex.

To see (a), let $x=(1,0)$, $y=(-2,0)$. It is clear that $x\succeq_{\mathcal{K}^2} y$. Besides, we have $x^2=(1,0)$, $y^2=(4,0)$, which yields $|x|=(1,0)$ and $|y|=(2,0)$. But, $|x|-|y|=(-1,0)\not\succeq_{\mathcal{K}^2} 0$.

To see (b), let $x=(1,1,1)$, $y=(-1,1,0)$. In fact, we have $|x|=(\sqrt{2},1/\sqrt{2},1/\sqrt{2})$, $|y|=(1,-1,0)$, and $|x+y|=(\sqrt{5},0,0)$. Therefore, $|x|+|y|-|x+y|=(\sqrt{2}+1-\sqrt{5},\ -1+1/\sqrt{2},1/\sqrt{2})\not\succeq_{\mathcal{K}^3} 0$. Thus, $f((x+y)/2)\not\preceq_{\mathcal{K}^3} 1/2(f(x)+f(x))$, which shows $f(t)=|t|$ is not SOC-convex.

To see (c) and (d), just follows (a) and (b) and the facts that $[t]_+=1/2(t+|t|)$, where $t\in\mathbb{R}$, and $[x]_+=1/2(x+|x|)$, where $x\in\mathbb{R}^n$.

To close this section, we check with one popular smoothing function. It is the function, $f(t)=1/2(\sqrt{t^2+4}+t)$, proposed by Chen and Harker [4], Kanzow [12], and Smale [17], and is called the CHKS function. The associated SOC-function is defined by

$$f(x)=\frac{1}{2}(\sqrt{x^2+4e}+x),$$

where $e=(1,0,\ldots,0)^T$. The function $f(t)$ is convex and monotone functions, so we may also wish to know whether the SOC-function is SOC-convex or SOC-monotone. Unfortunately, it is neither SOC-convex nor SOC-monotone for $n\geq 3$, although it is both SOC-convex and SOC-monotone for $n=2$. The following example shows what we have just said.

Example 3.8 Let $f:\mathbb{R}\to\mathbb{R}$ be $f(t)=(\sqrt{t^2+4}+t)/2$. Then

(a) f is not SOC-monotone for general $n\geq 3$.
(b) However, f is SOC-monotone for $n=2$.
(c) f is not SOC-convex for general $n\geq 3$.
(d) However, f is SOC-convex for $n=2$.

Again, let $x=(2,1,-1)$, $y=(1,1,0)$, then it is the counter-example for both (a) and (c). To see (b) and (d), it can be done by direct verification by using the same techniques as we have done in Example 3.2 and Example 3.3.

4. Characterization of SOC-convexity and SOC-monotonicity

Based on all the results in the previous section, one may expect some certain relation between SOC-convex function and SOC-monotone function. One may also like to know under what conditions a function is SOC-convex. The same question arises for SOC-monotone. In this section, we explore these relations. In fact, there already have some analogous results for matrix-functions (see Chapter V of [2]). However, not much for this kind of vector-valued SOC-functions, so further study on these topics are definitely necessary.

Proposition 4.1 *Let $f:[0,\infty)\to[0,\infty)$ be continuous. If f is SOC-concave, then f is SOC-monotone.*

Proof For any $0<\lambda<1$, we can write $\lambda x=\lambda y+(1-\lambda)\lambda/(1-\lambda)(x-y)$. Then the SOC-concavity of f yields that

$$f^{\mathrm{soc}}(\lambda x)\succeq_{\mathcal{K}^n}\lambda f^{\mathrm{soc}}(y)+(1-\lambda)f^{\mathrm{soc}}\left(\frac{\lambda}{1-\lambda}(x-y)\right)\succeq_{\mathcal{K}^n} 0,$$

where the second inequality is true since f is from $[0, \infty)$ into itself and $x - y \succeq_{\mathcal{K}^n} 0$. Letting $\lambda \to 1$, we obtain that $f^{\rm soc}(x) \succeq_{\mathcal{K}^n} f^{\rm soc}(y)$, which says that f is SOC-monotone. ■

We notice that if f is not a function from $[0, \infty)$ into itself, this proposition is false. For instance, $f(t) = -t^2$ is SOC-concave but not SOC-monotone.

Proposition 4.2 *A function $f : \mathbb{R} \to \mathbb{R}$ is SOC-convex if and only if the real-valued function $g(x) := \langle f^{\rm soc}(x), z \rangle$ is a convex function $\forall z \succeq_{\mathcal{K}^n} 0$.*

Proof Suppose f is SOC-convex and let $x, y \in \mathbb{R}^n$, $\lambda \in [0, 1]$. Then, we have

$$f^{\rm soc}((1-\lambda)x + \lambda y) \succeq_{\mathcal{K}^n} (1-\lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y).$$

Hence,

$$\begin{aligned} g((1-\lambda)x + \lambda y) &= \left\langle f^{\rm soc}((1-\lambda)x + \lambda y), z \right\rangle \\ &\leq \left\langle (1-\lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y), z \right\rangle \\ &= (1-\lambda)\langle f^{\rm soc}(x), z\rangle + \langle f^{\rm soc}(y), z\rangle \\ &= (1-\lambda) g(x) + \lambda g(y), \end{aligned}$$

where the inequality holds by Property 2.3(d). Thus, g is a convex function. For the other direction, from g is convex, we obtain

$$\left\langle f^{\rm soc}((1-\lambda)x + \lambda y), z \right\rangle \leq \left\langle (1-\lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y), z \right\rangle.$$

since $z \succeq_{\mathcal{K}^n} 0$, by Property 2.3(d) again, the preceding yields

$$f^{\rm soc}((1-\lambda)x + \lambda y) \succeq_{\mathcal{K}^n} (1-\lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y),$$

which says f is SOC-convex. ■

Proposition 4.3 *A differentiable function $f : \mathbb{R} \to \mathbb{R}$ is SOC-convex if and only if $f^{\rm soc}(y) \succeq_{\mathcal{K}^n} f^{\rm soc}(x) + \nabla f^{\rm soc}(x)(y - x)$ for all $x, y \in \mathbb{R}^n$.*

Proof By [8, Proposition 5.3], we know that f is differentiable if and only if $f^{\rm soc}$ is differentiable. Using the gradient formula given therein and following the arguments as in [1, Proposition B.3] or [3, Theorem 2.3.5], the proof can be done easily. We omit the details. ■

At last, we state two conjectures based on observing all the results and examples discussed in this article. The conjectures describe the relationship between SOC-convex and SOC-monotone functions. We are not able to complete the proof right now. Nonetheless, we notice that some interesting results related to the trace of x, for example [15, Proposition 6.2.9], might help towards proving our conjectures. Further study is certainly desirable. On the other hand, this article is just an initial start of research on SOC-convex and SOC-monotone functions. There are many more properties to be investigated and studied. For instance, there is a useful property [2, Theorem V3.6] for matrix-valued function that says every matrix-monotone function $f^{\rm mat}$ on an interval I is smooth. Similarly, we can ask whether the extension of this theorem to SOC functions is true or not. We leave for future research.

Conjecture 4.1 *If $f : (0, \infty) \to \mathbb{R}$ is continuous, convex, and nonincreasing, then*

(a) *$f^{\rm soc}$ is SOC-convex.*
(b) *$-f^{\rm soc}$ is SOC-monotone.*

CONJECTURE 4.2 *If* $f:[0,\infty)\to[0,\infty)$ *is continuous, then* $-f^{\mathrm{soc}}$ *is SOC-convex* $\Longleftrightarrow f^{\mathrm{soc}}$ *is SOC-monotone.*

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