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On unitary elements defined on Lorentz cone and their applications

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Abstract. In this paper, we illustrate a new concept regarding unitary elements defined on Lorentz cone, and establish some basic properties under the so-called unitary transformation associated with Lorentz cone. As an application of unitary transformation, we achieve a weaker version of the triangle inequality and several (weak) majorizations defined on Lorentz cone.

Keywords: Lorentz cone, Unitary element, Triangular inequality, Majorization.
MSC: 90C25, 15B99, 33E99

1 Introduction

A complex square matrix U is called *unitary* if its conjugate transpose U^* is also its inverse, that is,

$$U^*U = UU^* = I,$$

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where I is the identity matrix. In fact, unitary matrices play an important role on many algorithms in numerical matrix analysis and computing eigenvalues. They can be regarded as a bridge of "change of basis operators". No matter from computational or theoretical aspect, it is usually helpful and convenient to transform a given matrix by unitary congruence into another matrix with a special form. For example, let f be a real-valued function defined on an interval $I \subseteq \mathbb{R}$. If A is a Hermitian matrix whose eigenvalues are $\lambda_j \in I$, we may choose a unitary matrix U such that $A = UDU^*$, where D is a diagonal matrix. Then, one can define the associated matrix-valued function by $f(A) = Uf(D)U^*$, which is heavily used in matrix analysis and designing solutions methods. For more details about usages and properties of unitary matrices, please refers to [3, 4, 10, 14, 15].

It is known that both positive semidefinite cone and Lorentz cone are special cases of symmetric cones [8]. It is interesting to know whether there is a similar role like U in the setting of Lorentz cone. In other words, a natural question arises: what is the concept of *unitary* looks like in the setting of Lorentz cone? In this paper, we try to answer this question. More specifically, we try to extend the concept of unitary to the setting of Lorentz cone (also called second-order cone) by observing the role and properties of unitary matrices. With the observations, we illustrate how we define the unitary elements associated with Lorentz cone. Accordingly, the so-called unitary transformation is defined as well. Then, we establish some properties under the unitary transformation in Section 3. Moreover, using the unitary transformation, we derive a weak SOC triangular inequality, which is a parallel version to the matrix case given by Thompson in [14]. Several (weak) majorizations of the eigenvalues are deduced and some SOC inequalities are achieved as well.

2 Preliminary

In this section, we review some basic concepts and properties concerning Jordan algebras from the book [8] on symmetric cones and Lorentz cones (second-order cones) [5, 6, 7], which are needed in the subsequent analysis.

A *Euclidean Jordan algebra* is a finite dimensional inner product space $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;

(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$;

(iii) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathbb{V}$, where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of x and y. If a Jordan product only satisfies the conditions (i) and (ii) in the above definition, the algebra $\mathbb{V}$ is said to be a *Jordan algebra*. Moreover, if there is an (unique) element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *Jordan identity* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In a given Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* [8, Theorem III.2.1]. This means that $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \mathrm{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \to \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. The Lorentz cone, also called second-order cone, in $\mathbb{R}^n$ is an important example of symmetric cones, which is defined as follows:

$$\mathcal{K}^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \,|\, x_1 \geq \|x_2\| \right\}.$$

For $n = 1$, $\mathcal{K}^n$ denotes the set of nonnegative real number $\mathbb{R}_+$. Since $\mathcal{K}^n$ is a pointed closed convex cone, for any x, y in $\mathbb{R}^n$, we can define a partial order on it:

$$\begin{aligned} x \preceq_{\mathcal{K}^n} y &\iff y - x \in \mathcal{K}^n; \\ x \prec_{\mathcal{K}^n} y &\iff y - x \in \mathrm{int}(\mathcal{K}^n). \end{aligned}$$

Note that the relation $\preceq_{\mathcal{K}^n}$ (or $\prec_{\mathcal{K}^n}$) is only a partial ordering, not a linear ordering in $\mathcal{K}^n$. To see this, a counterexample occurs by taking $x = (1, 1)$ and $y = (1, 0)$ in $\mathbb{R}^2$. It is clear to see that $x - y = (0, 1) \notin \mathcal{K}^2$, $y - x = (0, -1) \notin \mathcal{K}^2$.

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define the *Jordan product* as

$$x \circ y = \left(x^T y,\ y_1 x_2 + x_1 y_2 \right).$$

We note that $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ acts as the Jordan identity. Besides, the Jordan product is *not associative* in general. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$. Without loss of ambiguity, we may denote x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for any positive integers m and n. Here, we set $x^0 = e$. In addition, $\mathcal{K}^n$ is *not closed* under Jordan product.

Given any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{\frac{1}{2}}$ such that $(x^{\frac{1}{2}})^2 = x^{\frac{1}{2}} \circ x^{\frac{1}{2}} = x$. Indeed,

$$x^{\frac{1}{2}} = \left(s, \frac{x_2}{2s} \right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2} \right)}.$$

In the above formula, the term x_2/s is defined to be the zero vector if $s = 0$, i.e., $x = 0$. For any $x \in \mathbb{R}^n$, we always have $x^2 \in \mathcal{K}^n$, i.e., $x^2 \succeq_{\mathcal{K}^n} 0$. Hence, there exists a unique vector $(x^2)^{\frac{1}{2}} \in \mathcal{K}^n$ denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. It is also known that $|x| \succeq_{\mathcal{K}^n} x$. For more details, please refer to [8, 9].

In a Euclidean Jordan algebra $\mathbb{V}$, an element $e^{(i)} \in \mathbb{V}$ is an *idempotent* if $(e^{(i)})^2 = e^{(i)}$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. The idempotents $e^{(i)}$ and $e^{(j)}$ are said to be *orthogonal* if $e^{(i)} \circ e^{(j)} = 0$. In addition, we say that a finite set $\{e^{(1)}, e^{(2)}, \cdots, e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e^{(i)} \circ e^{(j)} = 0 \ \text{ for } \ i \neq j, \ \text{ and } \ \sum_{i=1}^{r} e^{(i)} = e.$$

Note that $\langle e^{(i)}, e^{(j)} \rangle = \langle e^{(i)} \circ e^{(j)}, e \rangle = 0$ whenever $i \neq j$.

With the above, there have the spectral decomposition of an element x in $\mathbb{V}$.

Theorem 2.1. (The Spectral Decomposition Theorem) [8, Theorem III.1.2] *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \cdots, e^{(r)}\}$ and real numbers $\lambda_1(x), \cdots, \lambda_r(x)$ with*

$$x = \lambda_1(x) e^{(1)} + \cdots + \lambda_r(x) e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \cdots, r)$ are called the eigenvalues of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is called the spectral decomposition of x. Moreover, $\mathrm{tr}(x) := \sum_{i=1}^{r} \lambda_i(x)$ *is called the trace of x, and* $\det(x) := \lambda_1(x)\lambda_2(x)\cdots\lambda_r(x)$ *is called the determinant of x.*

In the setting of Lorentz cone in $\mathbb{R}^n$, the vector $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ can be decomposed as

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1}$$

where $\lambda_1(x), \lambda_2(x)$ and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \tag{2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\left(1, (-1)^i \frac{x_2}{\|x_2\|}\right) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \bar{v}\right) & \text{if} \quad x_2 = 0. \end{cases} \tag{3}$$

for $i = 1, 2$ with $\bar{v}$ being any vector in $\mathbb{R}^{n-1}$ satisfying $\|\bar{v}\| = 1$. If $x_2 \neq 0$, the decomposition is unique. Accordingly, the determinant, the trace, and the Euclidean norm of x can all be represented in terms of $\lambda_1(x)$ and $\lambda_2(x)$:

$$\begin{aligned} \det(x) &= \lambda_1(x)\lambda_2(x) = x_1^2 - \|x_2\|^2, \\ \mathrm{tr}(x) &= \lambda_1(x) + \lambda_2(x) = 2x_1, \\ \|x\|^2 &= \frac{1}{2}\left(\lambda_1(x)^2 + \lambda_2(x)^2\right). \end{aligned}$$

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ ($n \geq 1$) was considered in [5, 6]:

$$f^{\mathrm{soc}}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{4}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (4) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{\mathrm{soc}}(x) = x^{\frac{1}{2}}$, x^2, $\exp(x)$ are discussed in [8].

In a Euclidean Jordan algebras $\mathbb{V}$, for any $x \in \mathbb{V}$, the linear transformation $L(x) : \mathbb{V} \to \mathbb{V}$ is called *Lyapunov transformation*, which is defined as $L(x)y := x \circ y$ for all $y \in \mathbb{V}$. The so-called *quadratic representation* $P(x)$ is defined by

$$P(x) := 2L^2(x) - L(x^2). \tag{5}$$

For any $x \in \mathbb{V}$, the endomorphisms $L(x)$ and $P(x)$ are self-adjoint. For the quadratic representation $P(x)$, if x is invertible, then $P(x)$ is invertible with $P(x)^{-1} = P(x^{-1})$ and

$$P(x)\mathcal{K} = \mathcal{K} \quad \text{and} \quad P(x)\mathrm{int}(\mathcal{K}) = \mathrm{int}(\mathcal{K}).$$

For subsequent analysis, we list some properties of the trace and determinant concerning the mapping P whose proofs can be found in Proposition II.4.3 and Proposition III.4.2 of [8].

Lemma 2.2. *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra and $x, y, z \in \mathbb{V}$.*

(a) $\mathrm{tr}(x \circ (y \circ z)) = \mathrm{tr}((x \circ y) \circ z)$.

(b) $\det(P(x)y) = \det(x)^2 \det(y)$.

In addition, by the definition of P and Lemma 2.2(a), we also have

$$\begin{aligned} \mathrm{tr}(P(x)y) &= \mathrm{tr}(2x \circ (x \circ y) - x^2 \circ y) \\ &= 2\mathrm{tr}(x \circ (x \circ y)) - \mathrm{tr}(x^2 \circ y) \\ &= 2\mathrm{tr}((x \circ x) \circ y) - \mathrm{tr}(x^2 \circ y) \\ &= \mathrm{tr}(x^2 \circ y) \end{aligned}$$

which is often used in the following section. To close this section, we recall some basic properties as listed in the following text. We omit the proofs since they can be found in [5, 8, 9].

Lemma 2.3. *For any $x, y \in \mathbb{R}^n$ with spectral decomposition given as in (1)-(3), the following hold.*

(a) $x^{\frac{1}{2}} = \sqrt{\lambda_1(x)}u_x^{(1)} + \sqrt{\lambda_2(x)}u_x^{(2)}$ *whenever $x \in \mathcal{K}^n$.*

(b) $|x| = |\lambda_1(x)|u_x^{(1)} + |\lambda_2(x)|u_x^{(2)}$;

(c) *If $x \preceq_{\mathcal{K}^n} y$, then $\lambda_i(x) \leq \lambda_i(y)$ for all $i = 1, 2$.*

3 Unitary Elements

Note that the nonnegative orthant, the cone of positive semidefinite matrices, and the Lorentz cone are special cases of symmetric cones. In fact, the space $Sym(n;\mathbb{R})$ of $n \times n$ real symmetric matrices with the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ and the bilinear map $\mathrm{tr}(XY)$ forms a Euclidean Jordan algebra. In this case, its corresponding symmetric cone $\mathcal{K}$ is exactly the cone of positive definite matrices, and the quadratic representation is

$$P(X)Y = XYX, \quad \forall\ X, Y \in Sym(n;\mathbb{R}).$$

As an example of Euclidean Jordan algebra, we notice that

$$U^* \circ U = \frac{1}{2}(U^*U + UU^*) = I.$$

In addition, for any matrix $A \in \mathbb{R}^{n\times n}$, we shall write $A = UP$ for the polar decomposition of A, where U is a unitary and P is a positive semidefinite matrix. We note that $P = (A^*A)^{\frac{1}{2}}$ is the so-called absolute value of A and is denoted by $|A|$. Combining with the definition of absolute value associated with Lorentz cone in the previous section, it motivates us to define the unitary element w on Lorentz cone, even though there is no concept of conjugate associated with Lorentz cone yet.

Definition 3.1. *Let $\mathcal{K}^n$ be the Lorentz cone. An element w in $\mathbb{R}^n$ is called a unitary element defined on $\mathcal{K}^n$ if it satisfies*

$$w^2 = w \circ w = e.$$

After simple calculation, we obtain that for $w = (w_1, w_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$w^2 = (w_1^2 + \|w_2\|^2, 2w_1w_2) = (1, 0),$$

which says $w_1 = 0$ or $w_2 = 0$. Hence, we conclude that any unitary element w associated with Lorentz cone only has three types:

$$w = \begin{cases} e; \\ -e; \\ (0, \bar{w}) \text{ with } \|\bar{w}\| = 1. \end{cases} \tag{6}$$

We recall that transformation from one orthonormal basis to another one is accomplished by unitary matrix. The matrix of unitary transformation relative to an orthonormal basis is also a unitary matrix. In other words, the unitary matrix plays significant importance on the decomposition of matrix. In the setting of Lorentz cone, let $P(\cdot)$ be fined as in (5), we call $P(w)$ is the *unitary transformation* associated with Lorentz cone whenever w is a unitary element defined on Lorentz cone. With this definition, it is desired to see the role of the unitary transformation in the setting of Lorentz cone. All the propositions are devoted to answer this.

Proposition 3.2. *For any unitary element $w \in \mathbb{R}^n$ defined on Lorentz cone given as in* (6)*, there holds* $|\det(w)| = 1$.

Proof. According to the spectral decomposition (1)-(3) and the form of unitary w, it is obvious the spectral values of w is either 1 or -1, which implies $|\det(w)| = 1$. $\square$

We note that for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}, y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, if $x_2 = 0$ or $y_2 = 0$, then x, y may have the same Jordan frame via appropriately choosing in the spectral decomposition. Otherwise, x, y have the same Jordan frame if there exists $\alpha \in \mathbb{R}$ such that $x_2 = \alpha y_2$. Furthermore, x, y have the same ordered Jordan frame whenever $\alpha > 0$ and the reversely ordered Jordan frame whenever $\alpha < 0$.

In the sequel, given any $z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we shall denote $\bar{z} := \frac{z_2}{\|z_2\|}$ for the simplicity of notation.

Proposition 3.3. *For any $x, y \in \mathbb{R}^n$, there exists a unitary element $w \in \mathbb{R}^n$ defined on Lorentz cone such that $L(w)y$ and x have the same Jordan frame. Moreover, w can be chosen such that $L(w)y$ have the desired ordered Jordan frame.*

Proof. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}, y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, it is clear that the assertion holds by choosing $w = e$ whenever $x_2 = 0$ or $y_2 = 0$. Suppose that $x_2 \neq 0$ and $y_2 \neq 0$. For any $w = (0, \bar{w})$ with $\|\bar{w}\| = 1$, we have

$$L(w)y = w \circ y = \left(y_2^T \bar{w}, y_1 \bar{w}\right).$$

According to the spectral decomposition (1)-(3), we can choose a unitary element

$$w = \begin{cases} (0, \bar{x}), & \text{if } \; y_1 > 0 \\ (0, -\bar{x}), & \text{if } \; y_1 < 0 \\ (0, \bar{w}) \text{ with } \|\bar{w}\| = 1, & \text{if } \; y_1 = 0 \end{cases} \tag{7}$$

so that $L(w)y$ have the same ordered Jordan frame with respect to x. On the other hand, choosing $w' = -w$ leads $L(w')y$ to have the reversely ordered Jordan frame with respect to x. $\square$

Remark 3.4. In Proposition 3.3, the Lyapunov transformation $L(\cdot)$ could lead two elements into the same Jordan frame. However, it does not keep the spectral values after this transforming. Indeed, from (7), we notice that for $i = 1, 2$

$$\lambda_i(L(w)y) = y_2^T \bar{w} + (-1)^i \|y_1 \bar{w}\| = y_2^T \bar{w} + (-1)^i |y_1|$$

may not always coincide with $\lambda_i(y) = y_1 + (-1)^i \|y_2\|$, in general.

What kind of transformation will make any two elements $x, y \in \mathbb{R}^n$ to share the same Jordan frame and keep the spectral values? The question will be answered gradually from the following propositions.

Proposition 3.5. *Let $w \in \mathbb{R}^n$ be a unitary element defined on Lorentz cone given as in (6) and $P(w)$ be the unitary transformation. Then, for any $y \in \mathbb{R}^n$, the spectral values of $P(w)y$ coincide with the ones of y.*

Proof. From Lemma 2.2, we have

$$\begin{aligned} \operatorname{tr}(P(w)y) &= \operatorname{tr}(w^2 \circ y) = \operatorname{tr}(y), \\ \det(P(w)y) &= \det(w^2)\det(y) = \det(y), \end{aligned}$$

which imply that the spectral values of $P(w)y$ and y satisfy the same quadratic equation. Hence, $P(w)y$ and y have the same spectral values. $\square$

Proposition 3.6. *Let $w \in \mathbb{R}^n$ be a unitary element defined on Lorentz cone given as in (6) and $P(w)$ be the unitary transformation. Then, for any $y \in \mathbb{R}^n$, there holds*

$$\|P(w)y\| = \|y\|,$$

that is, the norm of y is invariant under the unitary transformation $P(w)$.

Proof. For any $z \in \mathbb{R}^n$, we note that $\|z\|^2 = \frac{1}{2}(\lambda_1(z)^2 + \lambda_2(z)^2)$. Applying Proposition 3.5, we know that the spectral values of $P(w)x$ coincide with the ones of y, and hence we obtain $\|P(w)y\| = \|y\|$. $\square$

In Proposition 3.5, we already know the spectral values is invariant under the transformation $P(w)$ with any unitary element w. It is natural to ask if $P(w)$ is able to change the Jordan frame to another one by suitable w. The answer is affirmative. To see this, we recall a theorem that Faraut and Korányi [8] established by the Peirce decomposition.

Theorem 3.7. [8, Theorem IV.2.5] *Let V be a simple Euclidean Jordan algebra. If $\{c^{(1)}, \dots, c^{(r)}\}$ and $\{d^{(1)}, \dots, d^{(r)}\}$ are two Jordan frames, then there exists an automorphism A such that*

$$Ac^{(j)} = d^{(j)} \ (1 \leq j \leq r),$$

where $A = P(w)$ for some w with $w^2 = e$.

In the setting of Lorentz cone, we offer another approach via the geometric view of $P(w)y$ and figure out the exact form of the suitable unitary element w. Given any

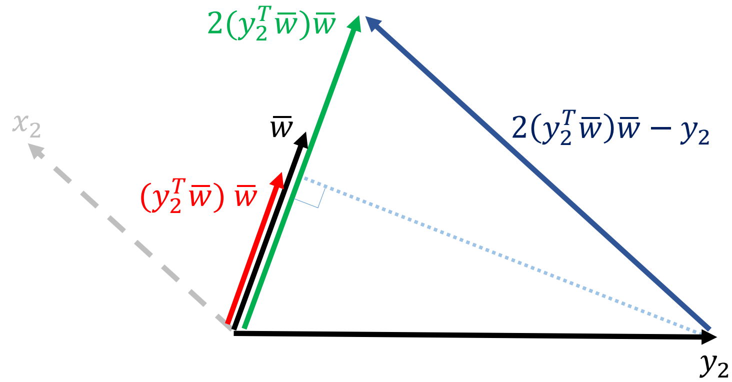

Figure 1: The geometric view of $P(w)y$.

$y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and unitary element $w = (0, \bar{w})$, we observe that

$$\begin{aligned} P(w)y &= 2w \circ (w \circ y) - w^2 \circ y \\ &= 2\,(0, \bar{w}) \circ \left(y_2^T\bar{w}, y_1\bar{w}\right) - e \circ (y_1, y_2) \\ &= 2\left(y_1, (y_2^T\bar{w})\bar{w}\right) - (y_1, y_2) \\ &= \left(y_1, 2(y_2^T\bar{w})\bar{w} - y_2\right). \end{aligned} \tag{8}$$

The geometric meaning of $P(w)y$ is depicted in Figure 1. In fact, it not only supports the conclusion of Proposition 3.6, but also tells us how to choose a suitable unitary element w defined on Lorentz cone.

Proposition 3.8. *For any $x, y \in \mathbb{R}^n$, there exists a unitary element $w \in \mathbb{R}^n$ defined on Lorentz cone such that $P(w)y$ and x have the same Jordan frame. Moreover, w can be chosen such that $P(w)y$ have the desired ordered Jordan frame.*

Proof. For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, if $x_2 = 0$ or $y_2 = 0$, then x, y may have the same Jordan frame with desired ordering via appropriately choosing in the spectral decomposition. Thus, we assume that $x_2 \neq 0$ and $y_2 \neq 0$.

(i) For x, y with the same Jordan frame, we may choose $w = e$ to keep the same ordered Jordan frame. On the other hand, if we choose any $w = (0, \bar{w})$ such that $y_2^T\bar{w} = 0$, then it follows from (8) that $P(w)y = (y_1, -y_2)$ and hence $P(w)y$ and y have the reversely ordered Jordan frame. In other words, $P(w)$ change the order of Jordan frame.

(ii) For x, y with the different Jordan frame, we choose $w = \left(0, \frac{\bar{x}+\bar{y}}{\|\bar{x}+\bar{y}\|}\right)$, and then applying

(8) again yields

$$\begin{aligned} P(w)y &= \left(y_1, 2\frac{y_2^T(\bar{x}+\bar{y})}{\|\bar{x}+\bar{y}\|^2}(\bar{x}+\bar{y}) - y_2\right) \\ &= \left(y_1, 2\frac{\|y_2\|(\bar{y}^T\bar{x}+1)}{2+2\bar{y}^T\bar{x}}(\bar{x}+\bar{y}) - y_2\right) \\ &= (y_1, \|y_2\|(\bar{x}+\bar{y}) - y_2) \\ &= (y_1, \|y_2\|\bar{x}). \end{aligned}$$

Hence, $P(w)y$ and x have the same ordered Jordan frame. On the other hand, while choosing $w = \left(0, \frac{-\bar{x}+\bar{y}}{\|-\bar{x}+\bar{y}\|}\right)$, it will lead $P(w)y$ and x to have the reversely ordered Jordan frame. $\Box$

Remark 3.9. Proposition 3.8 illustrates that we can change the Jordan frame to the desired one via $P(w)$ with suitable unitary element w. Conversely, for any Jordan frame $\{e^{(1)}, e^{(2)}\}$ and any unitary element w, the set $\{P(w)e^{(1)}, P(w)e^{(2)}\}$ still forms a Jordan frame. Indeed, we notice that for any $i = 1, 2$,

$$\begin{aligned} \operatorname{tr}(P(w)e^{(i)}) &= \operatorname{tr}(e^{(i)}) = 1, \\ \det(P(w)e^{(i)}) &= \det(e^{(i)}) = 0, \end{aligned}$$

which tell us that $P(w)e^{(i)}$ is of the form $\left(\frac{1}{2}, \frac{1}{2}\bar{v}^{(i)}\right)$ with $\|\bar{v}^{(i)}\| = 1$ since $P(w)e^{(i)} \in \mathcal{K}^n$. In addition,

$$P(w)e^{(1)} + P(w)e^{(2)} = P(w)e = w^2 = e,$$

which implies $\bar{v}^{(1)} + \bar{v}^{(2)} = 0$. Thus, the set $\{P(w)e^{(1)}, P(w)e^{(2)}\}$ actually forms a Jordan frame. Furthermore, for any pair of unitary elements w, w', there must have a unitary element $\hat{w}$ such that $P(\hat{w}) = P(w)P(w')$.

Proposition 3.10. *Let $w \in \mathbb{R}^n$ be a unitary element defined on Lorentz cone given as in* (6) *and $P(w)$ be the unitary transformation. Then, for any $x \in \mathbb{R}^n$, there holds*

$$|P(w)x| = P(w)|x|.$$

Proof. Denote $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$ by the spectral decomposition (1)-(3). Then, from Propositions 3.5–3.8 and Remark 3.9, there exists a Jordan frame $\{v^{(1)}, v^{(2)}\}$ such that

$$\begin{aligned} |P(w)x| &= \left|\lambda_1(x)v^{(1)} + \lambda_2(x)v^{(2)}\right| \\ &= |\lambda_1(x)|v^{(1)} + |\lambda_2(x)|v^{(2)} \\ &= P(w)|x|. \end{aligned}$$

Thus, the desired result is deduced. $\Box$

Proposition 3.11. *For any $x, y \in \mathcal{K}^n$, $P(x^{\frac{1}{2}})y$ and $P(y^{\frac{1}{2}})x$ have the same spectral values. Moreover, there exists a unitary element w defined on Lorentz cone such that*

$$P(x^{\frac{1}{2}})y = P(w)(P(y^{\frac{1}{2}})x).$$

Proof. By Lemma 2.2, we have

$$\begin{aligned} \mathrm{tr}(P(x^{\frac{1}{2}})y) &= \mathrm{tr}(x \circ y) = \mathrm{tr}(P(y^{\frac{1}{2}})x), \\ \det(P(x^{\frac{1}{2}})y) &= \det(x)\det(y) = \det(P(y^{\frac{1}{2}})x), \end{aligned}$$

which says that the spectral values of $P(x^{\frac{1}{2}})y$ and $P(y^{\frac{1}{2}})x$ satisfy the same quadratic equation, and hence they have the same spectral values. Moreover, the desired equality also holds by similar arguments as in Proposition 3.8. □

4 Applications

In this section, we provide two applications of unitary elements associated with Lorentz cone. First application is about an extended version of triangilar inequality. To this end, we begin with recalling the triangle inequality

$$|a + b| \leq |a| + |b|$$

for any real or complex numbers a, b. Moreover, in the Euclidean space $\mathbb{R}^n$ or normed linear spaces V, the triangle inequality is a property about distances, and it is written as

$$\|x + y\| \leq \|x\| + \|y\|$$

for any given $x, y \in \mathbb{R}^n$ (or V). However, the prospective triangle inequality for symmetric matrices X, Y

$$|X + Y| \leqslant |X| + |Y|$$

may not be true in general. The notation $X \leqslant Y$ means $Y - X$ is a positive semidefinite matrix. In fact, Thompson [14] established a weaker version of triangle inequality for two matrices: for any two matrices X and Y, there exist unitaries V and W such that

$$|X + Y| \leqslant V|X|V^* + W|Y|W^*. \tag{9}$$

Recently, Huang et. al [11] also discuss the triangle inequality on Lorentz cone and give a counterexample to illustrate the SOC triangular inequality

$$|x + y| \preceq_{\mathcal{K}^n} |x| + |y|$$

does not hold. Nevertheless, Huang et. al build up another SOC trace version of triangular inequality. We notice that the unitary matrix plays a crucial role in the proof of Thompson's theorem [14]. Here, we try to derive a parallel inequality in the setting of Lorentz cone by applying the concept of the unitary transformation discussed in Section 3.

Theorem 4.1. *For any $x, y \in \mathbb{R}^n$, there exist unitary elements $w, w' \in \mathbb{R}^n$ defined on Lorentz cone such that*

$$|x+y| \preceq_{\mathcal{K}^n} P(w)|x| + P(w')|y|. \tag{10}$$

Proof. Let $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. It is clear that inequality (10) holds if $x + y \in \mathcal{K}^n \cup (-\mathcal{K}^n)$ by choosing $w = w' = e$. Suppose that $x + y \notin \mathcal{K}^n \cup (-\mathcal{K}^n)$, it says that $\lambda_1(x+y) < 0 < \lambda_2(x+y)$ which implies

$$\begin{aligned}
|\lambda_1(x+y)| &= \|x_2 + y_2\| - x_1 - y_1 \le \|x_2\| + \|y_2\| - x_1 - y_1 = -\lambda_1(x) - \lambda_1(y), \\
|\lambda_2(x+y)| &= x_1 + y_1 + \|x_2 + y_2\| \le x_1 + y_1 + \|x_2\| + \|y_2\| = \lambda_2(x) + \lambda_2(y).
\end{aligned}$$

(i) For x, y with the same Jordan frame, we assume that $x = \lambda_1(x)e^{(1)} + \lambda_2(x)e^{(2)}$. Then, we discuss two subcases. If $y = \lambda_1(y)e^{(1)} + \lambda_2(y)e^{(2)}$, we have

$$\begin{aligned}
|x+y| &= \left|(\lambda_1(x) + \lambda_1(y))e^{(1)} + (\lambda_2(x) + \lambda_2(y))e^{(2)}\right| \\
&= \left|\lambda_1(x) + \lambda_1(y)\right|e^{(1)} + \left|\lambda_2(x) + \lambda_2(y)\right| e^{(2)} \\
&\preceq_{\mathcal{K}^n} (|\lambda_1(x)| + |\lambda_1(y)|)e^{(1)} + (|\lambda_2(x)| + |\lambda_2(y)|)e^{(2)} \\
&= |\lambda_1(x)|e^{(1)} + |\lambda_2(x)|e^{(2)} + |\lambda_1(y)|e^{(1)} + |\lambda_2(y)|e^{(2)} \\
&= |x| + |y|.
\end{aligned}$$

Similarly, if $y = \lambda_1(y)e^{(2)} + \lambda_2(y)e^{(1)}$, we can also derive $|x+y| \preceq_{\mathcal{K}^n} |x| + |y|$. Hence, we have the desired inequality by choosing $w = w' = e$.

(ii) For x, y with the different Jordan frame, we notice that $\bar{x} + \bar{y} \neq 0$, and

$$\begin{aligned}
|x+y| &= |\lambda_1(x+y)|u^{(1)}_{x+y} + |\lambda_2(x+y)|u^{(2)}_{x+y} \\
&\preceq_{\mathcal{K}^n} -\lambda_1(x)u^{(1)}_{x+y} - \lambda_1(y)u^{(1)}_{x+y} + \lambda_2(x)u^{(2)}_{x+y} + \lambda_2(y)u^{(2)}_{x+y} \\
&\preceq_{\mathcal{K}^n} |\lambda_1(x)|u^{(1)}_{x+y} + |\lambda_2(x)|u^{(2)}_{x+y} + |\lambda_1(y)|u^{(1)}_{x+y} + |\lambda_2(y)|u^{(2)}_{x+y}.
\end{aligned}$$

Applying Proposition 3.8, there exist unitary elements w, w' defined on Lorentz cone such that

$$\begin{aligned}
P(w)|x| &= |\lambda_1(x)|u^{(1)}_{x+y} + |\lambda_2(x)|u^{(2)}_{x+y}, \\
P(w')|y| &= |\lambda_1(y)|u^{(1)}_{x+y} + |\lambda_2(y)|u^{(2)}_{x+y}.
\end{aligned}$$

Thus, the desired inequality follows from all the above expressions. □

Another application is devoted to the majorizations of the eigenvalues. In particular, we derive several majorizations of the eigenvalues parallel to those for matrix case. In [1, 2], the authors give many matrix versions of inequalities for convex (or concave) function. In light of these concepts, we achieve their parallel inequalities in the setting of Lorentz cone. For convenience, we introduce some notations. For any $x \in \mathbb{R}^n$, we use $\lambda(x)$ to mean the vector $(\lambda_1(x), \lambda_2(x))$. For any $x, y \in \mathbb{R}^n$, we denote

(i) (spectral value inequalities)

$$\lambda(x) \leqslant \lambda(y) \Longleftrightarrow \lambda_i(x) \leq \lambda_i(y) \quad (i = 1, 2),$$

(ii) (weak majorization)

$$\begin{array}{rcl}\lambda(x) \prec_w \lambda(y) & \Longleftrightarrow & \lambda_2(x) \leq \lambda_2(y) \ \text{ and } \ \displaystyle\sum_{i=1}^{2} \lambda_i(x) \leq \sum_{i=1}^{2} \lambda_i(y), \\ \lambda(x) \prec_{w'} \lambda(y) & \Longleftrightarrow & \lambda_1(x) \leq \lambda_1(y) \ \text{ and } \ \displaystyle\sum_{i=1}^{2} \lambda_i(x) \leq \sum_{i=1}^{2} \lambda_i(y).\end{array}$$

We note that

$$x \preceq_{\mathcal{K}^n} y \Longrightarrow \lambda(x) \leqslant \lambda(y) \Longrightarrow \lambda(x) \prec_w \lambda(y) \ \text{ and } \ \lambda(x) \prec_{w'} \lambda(y).$$

Together with Proposition 3.8 and the proof of Theorem 4.1, we further have the following implication :

$$\lambda(x) \leqslant \lambda(y) \Longrightarrow x \preceq_{\mathcal{K}^n} P(w)y \ \text{ for some unitary element } w.$$

Lemma 4.2. *Suppose that $x \in \mathbb{R}^n$ has spectral values in $I \subseteq \mathbb{R}$. Let f be a convex function on I. Then, for every unit vector $v \in \mathbb{R}^n$ ($\|v\| = 1$), we have*

$$f(\langle x \circ v, v \rangle) \leq \langle f^{\mathrm{soc}}(x) \circ v, v \rangle.$$

In particular, for any vector $\tilde{v} = (\tilde{v}_1, \tilde{v}_2) \in \mathcal{K}^n$ with $\tilde{v}_1 = 1$, there holds

$$f(\langle x, \tilde{v} \rangle) \leq \langle f^{\mathrm{soc}}(x), \tilde{v} \rangle. \tag{11}$$

In addition, if $0 \in I$ and $f(0) \leq 0$, then for any arbitrary Jordan frame $\{e^{(1)}, e^{(2)}\}$ in $\mathcal{K}^n$ and for all $i = 1, 2$, there holds

$$f(\langle x, e^{(i)} \rangle) \leq \langle f^{\mathrm{soc}}(x), e^{(i)} \rangle.$$

Proof. First, we write $x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}$. Then, we have

$$\begin{aligned} f(\langle x \circ v, v \rangle) &= f\left(\langle (\lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}) \circ v, v \rangle \right) \\ &= f\left(\langle \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}, v^2 \rangle \right) \\ &= f\left(\lambda_1 \langle u_x^{(1)}, v^2 \rangle + \lambda_2 \langle u_x^{(2)}, v^2 \rangle \right), \end{aligned}$$

where the second equality holds by the condition (iii) of Euclidean Jordan algebra. We note that

$$\langle u_x^{(i)}, v^2 \rangle \geq 0 \quad \text{ for all } i = 1, 2$$

since $u_x^{(i)}, v^2 \in \mathcal{K}^n$ and

$$\langle u_x^{(2)}, v^2\rangle + \langle u_x^{(2)}, v^2\rangle = \langle e, v^2\rangle = \|v\|^2 = 1.$$

By the convexity of f, we have

$$\begin{array}{rcl} f(\langle x \circ v, v\rangle) & = & f\left(\lambda_1\langle u_x^{(1)}, v^2\rangle + \lambda_2\langle u_x^{(2)}, v^2\rangle\right) \\ & \leq & \langle u_x^{(1)}, v^2\rangle f\left(\lambda_1\right) + \langle u_x^{(2)}, v^2\rangle f\left(\lambda_2\right) \\ & = & \langle f\left(\lambda_1\right) u_x^{(1)}, v^2\rangle + \langle f\left(\lambda_2\right) u_x^{(2)}, v^2\rangle \\ & = & \langle f^{\mathrm{soc}}(x), v^2\rangle \\ & = & \langle f^{\mathrm{soc}}(x) \circ v, v\rangle. \end{array}$$

This proves the first assertion. In particular, for any vector $\tilde{v} \in \mathcal{K}^n$ with $\tilde{v}_1 = 1$, there has a vector $v \in \mathbb{R}^n$ such that $v^2 = \tilde{v}$. Moreover, the condition $\tilde{v}_1 = 1$ implies $\|v\| = 1$ by the definition of Jordan product. Thus, the second assertion is obtained.

If further $0 \in I$ and $f(0) \leq 0$, we obtain

$$f(t) = f\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2t\right) \leq \frac{1}{2} f\left(0\right) + \frac{1}{2} f\left(2t\right) \leq \frac{1}{2} f\left(2t\right),$$

which implies

$$\begin{array}{rcl} f(\langle x, e^{(i)}\rangle) & = & f\left(\frac{\lambda_1}{2}\langle u_x^{(1)}, 2e^{(i)}\rangle + \frac{\lambda_2}{2}\langle u_x^{(2)}, 2e^{(i)}\rangle\right) \\ & \leq & \langle u_x^{(1)}, 2e^{(i)}\rangle f\left(\frac{\lambda_1}{2}\right) + \langle u_x^{(2)}, 2e^{(i)}\rangle f\left(\frac{\lambda_2}{2}\right) \\ & = & \langle u_x^{(1)}, e^{(i)}\rangle 2f\left(\frac{\lambda_1}{2}\right) + \langle u_x^{(2)}, e^{(i)}\rangle 2f\left(\frac{\lambda_2}{2}\right) \\ & \leq & \langle u_x^{(1)}, e^{(i)}\rangle f\left(\lambda_1\right) + \langle u_x^{(2)}, e^{(i)}\rangle f\left(\lambda_2\right) \\ & = & \langle f\left(\lambda_1\right) u_x^{(1)}, e^{(i)}\rangle + \langle f\left(\lambda_2\right) u_x^{(2)}, e^{(i)}\rangle \\ & = & \langle f^{\mathrm{soc}}(x), e^{(i)}\rangle, \end{array}$$

for all $i = 1, 2$. Hence, we conclude the third assertion. $\square$

Lemma 4.3. *For any $x \in \mathbb{R}^n$ with spectral values $\lambda_1(x)$ and $\lambda_2(x)$. Then, we have*

$$\lambda_1(x) = \min\langle x \circ v, v\rangle \quad \text{and} \quad \lambda_2(x) = \max\langle x \circ v, v\rangle,$$

where the minimum and maximum are taken over all choices of unit vector v. Moreover, for any arbitrary Jordan frame $\{e^{(1)}, e^{(2)}\}$ in $\mathcal{K}^n$, there holds

$$\lambda_1(x) + \lambda_2(x) = \sum_{i=1}^{2}\langle x \circ \sqrt{2}e^{(i)}, \sqrt{2}e^{(i)}\rangle = \sum_{i=1}^{2}\langle x, 2e^{(i)}\rangle$$

Proof. Denote $x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}$ by the spectral decomposition (1)-(3). Then, for any unit vector $v \in \mathbb{R}^n$, we have

$$\begin{aligned}
\langle x \circ v, v \rangle &= \langle \big(\lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}\big) \circ v, v \rangle \\
&= \langle \lambda_1(x)u_x^{(1)}, v^2 \rangle + \langle \lambda_2(x)u_x^{(2)}, v^2 \rangle \\
&= \lambda_1(x)\langle u_x^{(1)}, v^2 \rangle + \lambda_2(x)\langle u_x^{(2)}, v^2 \rangle.
\end{aligned}$$

For $i = 1, 2$, we notice $\langle u_x^{(i)}, v^2 \rangle \geq 0$ since $u_x^{(i)}, v^2 \in \mathcal{K}^n$, and $\langle u_x^{(2)}, v^2 \rangle + \langle u_x^{(2)}, v^2 \rangle = 1$. This together the above yields

$$\lambda_1(x) \leq \langle x \circ v, v \rangle \leq \lambda_2(x),$$

and the minimum and maximum occur whenever $v = \pm\sqrt{2}u_x^{(1)}$ and $v = \pm\sqrt{2}u_x^{(2)}$, respectively.

In addition, for any arbitrary Jordan frame $\{e^{(1)}, e^{(2)}\}$ in $\mathcal{K}^n$, it can be verified that

$$\sum_{i=1}^{2} \langle x \circ \sqrt{2}e^{(i)}, \sqrt{2}e^{(i)} \rangle = \sum_{i=1}^{2} \langle x, 2e^{(i)} \rangle = \langle x, 2e \rangle = 2x_1 = \lambda_1(x) + \lambda_2(x).$$

Hence, the proof is complete. $\square$

We point out that Lemma 4.2–4.3 are the SOC versions of results as in [1, Lemma 2.1–2.2] (also see [4]). We use these two lammas to deduce a series of inequalities like the ones in [1] accordingly.

Theorem 4.4. *Let f be a convex real-valued function on $I \subseteq \mathbb{R}$. Then, for all x, y with spectral values in I and $0 \leq \alpha \leq 1$, there holds*

$$\lambda(f^{\rm soc}(\alpha x + (1-\alpha)y)) \prec_w \lambda(\alpha f^{\rm soc}(x) + (1-\alpha)f^{\rm soc}(y)).$$

If further $0 \in I$ and $f(0) \leq 0$, then

$$\lambda(f^{\rm soc}(P(s)x)) \prec_w \lambda(P(s)f^{\rm soc}(x))$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Proof. For any $x, y \in \mathbb{R}^n$ with $x \neq y$, we have

$$\begin{aligned}
\lambda_1(\alpha x + (1-\alpha)y) &= \alpha x_1 + (1-\alpha)y_1 - \|\alpha x_2 + (1-\alpha)y_2\| \geq \alpha\lambda_1(x) + (1-\alpha)\lambda_1(y), \\
\lambda_2(\alpha x + (1-\alpha)y) &= \alpha x_1 + (1-\alpha)y_1 + \|\alpha x_2 + (1-\alpha)y_2\| \leq \alpha\lambda_2(x) + (1-\alpha)\lambda_2(y),
\end{aligned}$$

which imply that

$$\alpha\lambda_1(x) + (1-\alpha)\lambda_1(y) \leq \lambda_1(\alpha x + (1-\alpha)y) \leq \lambda_2(\alpha x + (1-\alpha)y) \leq \alpha\lambda_2(x) + (1-\alpha)\lambda_2(y).$$

Hence, the spectral values of $\alpha x+(1-\alpha)y$ are also in I. For the simplicity of notation, we let $\hat{\lambda}_1$, $\hat{\lambda}_2$ be the spectral values of $\alpha x+(1-\alpha)y$ and $\{e^{(1)},e^{(2)}\}$ be the Jordan frame arranged such that $f(\hat{\lambda}_1)\le f(\hat{\lambda}_2)$. Then, we have

$$\begin{aligned}
\lambda_2(f^{\mathrm{soc}}(\alpha x+(1-\alpha)y)) &= f\left(\langle \alpha x+(1-\alpha)y, 2e^{(2)}\rangle\right)\\
&= f\left(\alpha\langle x, 2e^{(2)}\rangle+(1-\alpha)\langle y, 2e^{(2)}\rangle\right)\\
&\le \left[\alpha f\left(\langle x, 2e^{(2)}\rangle\right)+(1-\alpha)f\left(\langle y, 2e^{(2)}\rangle\right)\right]\\
&\le \left[\alpha\langle f^{\mathrm{soc}}(x), 2e^{(2)}\rangle+(1-\alpha)\langle f^{\mathrm{soc}}(y), 2e^{(2)}\rangle\right]\\
&= \langle \alpha f^{\mathrm{soc}}(x)+(1-\alpha)f^{\mathrm{soc}}(y), 2e^{(2)}\rangle\\
&\le \lambda_2(\alpha f^{\mathrm{soc}}(x)+(1-\alpha)f^{\mathrm{soc}}(y)),
\end{aligned}$$

where the three inequalities hold by the convexity of f, inequality (11) and Lemma 4.3, respectively. Similarly, we have

$$\begin{aligned}
\sum_{i=1}^{2}\lambda_i(f^{\mathrm{soc}}(\alpha x+(1-\alpha)y)) &= \sum_{i=1}^{2} f\left(\alpha\langle x, 2e^{(i)}\rangle+(1-\alpha)\langle y, 2e^{(i)}\rangle\right)\\
&\le \sum_{i=1}^{2}\left[\alpha f\left(\langle x, 2e^{(i)}\rangle\right)+(1-\alpha)f\left(\langle y, 2e^{(i)}\rangle\right)\right]\\
&\le \sum_{i=1}^{2}\left[\alpha\langle f^{\mathrm{soc}}(x), 2e^{(i)}\rangle+(1-\alpha)\langle f^{\mathrm{soc}}(y), 2e^{(i)}\rangle\right]\\
&= \sum_{i=1}^{2}\langle \alpha f^{\mathrm{soc}}(x)+(1-\alpha)f^{\mathrm{soc}}(y), 2e^{(i)}\rangle\\
&= \sum_{i=1}^{2}\lambda_i(f^{\mathrm{soc}}(x)+(1-\alpha)f^{\mathrm{soc}}(y)).
\end{aligned}$$

This proves the first assertion.

To prove the second assertion, let $\bar{\lambda}_1$, $\bar{\lambda}_2$ be the spectral values of $P(s)x$ and $\{d^{(1)},d^{(2)}\}$ be the Jordan frame arranged such that $f(\bar{\lambda}_1)\le f(\bar{\lambda}_2)$. Since $f(0)\le 0$, to prove the desired inequality we can assume that $P(s)d^{(i)}\neq 0$ for $i=1,2$. Moreover, we note that $s^2\preceq_{\mathcal{K}^n} e$ if and only if $P(s^2)\preccurlyeq P(e)$ by [13, Lemma 2.3], which gives $P(s)^2\preccurlyeq I$. Thus, we have

$$\|P(s)d^{(i)}\|^2={d^{(i)}}^T P(s)^T P(s)d^{(i)}\le {d^{(i)}}^T d^{(i)}=\frac{1}{2},$$

which says $\sqrt{2}\|P(s)d^{(i)}\|\le 1$. In addition, we note that $d^{(i)}$ is on the boundary of $\mathcal{K}^n$, which says $\det(d^{(i)})=0$. Furthermore, it also implies $\det(P(s)d^{(i)})=0$ by Lemma 2.2. Since the quadratic representation $P(s)$ is invariant on $\mathcal{K}^n$, we conclude $P(s)d^{(i)}$ is on the boundary of $\mathcal{K}^n$ as well. In particular, the first component of $P(s)d^{(i)}$ is $\frac{1}{\sqrt{2}}\|P(s)d^{(i)}\|$. On the other hand, the quadratic representation $P(s)$ can be expressed as

$$P(s)=2ss^T-\det(s)J,$$

where $J := \begin{bmatrix} 1 & 0^T \\ 0 & -I_{n-1} \end{bmatrix}$ with I_{n-1} being the identity matrix in $\mathbb{R}^{(n-1)\times(n-1)}$ (see [8, 12]), which says $P(s)$ is a symmetric transformation. Thus, for $i = 1, 2$, there holds

$$\lambda_i(f^{\mathrm{soc}}(P(s)x)) = f\left(\langle P(s)x, 2d^{(i)}\rangle\right) = f\left(\langle x, 2P(s)d^{(i)}\rangle\right).$$

Hence, we have

$$\begin{aligned}
& \lambda_2(f^{\mathrm{soc}}(P(s)x)) \\
= \;& f\left(\sqrt{2}\|P(s)d^{(2)}\| \cdot \left\langle x, \frac{\sqrt{2}P(s)d^{(2)}}{\|P(s)d^{(2)}\|}\right\rangle + (1-\sqrt{2}\|P(s)d^{(2)}\|)\cdot 0\right) \\
\le \;& \left[\sqrt{2}\|P(s)d^{(2)}\| \cdot f\left(\left\langle x, \frac{\sqrt{2}P(s)d^{(2)}}{\|P(s)d^{(2)}\|}\right\rangle\right) + (1-\sqrt{2}\|P(s)d^{(2)}\|)\cdot f(0)\right] \\
\le \;& \sqrt{2}\|P(s)d^{(2)}\| \cdot \left\langle f^{\mathrm{soc}}(x), \frac{\sqrt{2}P(s)d^{(2)}}{\|P(s)d^{(2)}\|}\right\rangle \\
= \;& \langle f^{\mathrm{soc}}(x), 2P(s)d^{(2)}\rangle \\
= \;& \langle P(s)f^{\mathrm{soc}}(x), 2d^{(2)}\rangle \\
\le \;& \lambda_2(P(s)f^{\mathrm{soc}}(x)),
\end{aligned}$$

by the convexity of f, the condition $f(0) \le 0$, inequality (11) and Lemma 4.3, respectively. Similarly, we can verify that

$$\begin{aligned}
& \sum_{i=1}^{2} \lambda_i(f^{\mathrm{soc}}(P(s)x)) \\
= \;& \sum_{i=1}^{2} f\left(\sqrt{2}\|P(s)d^{(i)}\| \cdot \left\langle x, \frac{\sqrt{2}P(s)d^{(i)}}{\|P(s)d^{(i)}\|}\right\rangle + \left(1-\sqrt{2}\|P(s)d^{(i)}\|\right)\cdot 0\right) \\
\le \;& \sum_{i=1}^{2} \left[\sqrt{2}\|P(s)d^{(i)}\| \cdot f\left(\left\langle x, \frac{\sqrt{2}P(s)d^{(i)}}{\|P(s)d^{(i)}\|}\right\rangle\right) + \left(1-\sqrt{2}\|P(s)d^{(i)}\|\right)\cdot f(0)\right] \\
\le \;& \sum_{i=1}^{2} \sqrt{2}\|P(s)d^{(i)}\| \cdot \left\langle f^{\mathrm{soc}}(x), \frac{\sqrt{2}P(s)d^{(i)}}{\|P(s)d^{(i)}\|}\right\rangle \\
= \;& \sum_{i=1}^{2} \langle P(s)f^{\mathrm{soc}}(x), 2d^{(i)}\rangle \\
= \;& \sum_{i=1}^{2} \lambda_i(P(s)f^{\mathrm{soc}}(x)).
\end{aligned}$$

This completes the proof. $\square$

Remark 4.5. According to the argument of Theorem 4.4, we similarly have the following weak majorization for concave real-valued function g defined on $I \subset \mathbb{R}$:

$$\lambda(\alpha g^{\text{soc}}(x) + (1-\alpha) g^{\text{soc}}(y)) \prec_{w'} \lambda(g^{\text{soc}}(\alpha x + (1-\alpha)y))$$

for all x, y with spectral values in I and $0 \leq \alpha \leq 1$. If further $0 \in I$ and $g(0) \geq 0$, then

$$\lambda(P(s) g^{\text{soc}}(x)) \prec_{w'} \lambda(g^{\text{soc}}(P(s)x))$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Corollary 4.6. *Let f be a nonnegative decreasing convex function on $[0, \infty)$. Then, for any $x, y \in \mathcal{K}^n$, there holds*

$$\lambda\left(f^{\text{soc}}(x+y)\right) \prec_w \lambda\left(f^{\text{soc}}(x) + f^{\text{soc}}(y)\right).$$

Proof. Note that every nonnegative decreasing function f on $[0, \infty)$ satisfies

$$f(2t) \leq 2f(t), \quad \forall t \in [0, \infty).$$

This further implies that $f^{\text{soc}}(2x) \leq 2f^{\text{soc}}(x)$ and $f^{\text{soc}}(2y) \leq 2f^{\text{soc}}(y)$. Using this fact and applying Theorem 4.4, we have

$$\begin{aligned} \lambda\left(f^{\text{soc}}(x+y)\right) &\prec_w \lambda\left(\frac{f^{\text{soc}}(2x) + f^{\text{soc}}(2y)}{2}\right) \\ &\prec_w \lambda\left(f^{\text{soc}}(x) + f^{\text{soc}}(y)\right), \end{aligned}$$

which says the desired result. □

Corollary 4.7. *Let f be a nonnegative convex function on $I \subseteq \mathbb{R}$. Then,*

$$\|f^{\text{soc}}(\alpha x + (1-\alpha)y)\| \leq \|\alpha f^{\text{soc}}(x) + (1-\alpha) f^{\text{soc}}(y)\|$$

for any x, y with spectral values in I and $0 \leq \alpha \leq 1$. If further $0 \in I$ and $f(0) = 0$, then

$$\|f^{\text{soc}}(P(s)x)\| \leq \|P(s) f^{\text{soc}}(x)\|$$

for any $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Proof. We define a function $\Phi : \mathbb{R}^2 \to [0, \infty)$ by

$$\Phi(a, b) = \left[\frac{1}{2}(a^2 + b^2)\right]^{\frac{1}{2}}.$$

For any $z \in \mathbb{R}^n$, we notice that $\|z\| = \Phi(\lambda_1(z), \lambda_2(z))$. Then, the results follow from Theorem 4.4 and Problem II.5.12(iv) [4, page 53]. □

Theorem 4.8. *Let f be a monotone convex function on $I \subseteq \mathbb{R}$. Then, for any x, y with spectral values in I and $0 \le \alpha \le 1$, there holds*

$$\lambda\left(f^{\rm soc}(\alpha x + (1-\alpha)y)\right) \leqslant \lambda\left(\alpha f^{\rm soc}(x) + (1-\alpha) f^{\rm soc}(y)\right).$$

If further $0 \in I$ and $f(0) \le 0$, then

$$\lambda\left(f^{\rm soc}(P(s)x)\right) \leqslant \lambda\left(P(s) f^{\rm soc}(x)\right)$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Proof. According to the proof of Theorem 4.4, it remains to show that

$$\lambda_1(f^{\rm soc}(\alpha x + (1-\alpha)y)) \le \lambda_1(\alpha f^{\rm soc}(x) + (1-\alpha) f^{\rm soc}(y)).$$

If f is increasing (or decreasing), then

$$\begin{array}{rcl} \lambda_1(f^{\rm soc}(z)) & = & f(\lambda_1(z)) = f\left(\min\{\langle z \circ v, v\rangle \mid \|v\| = 1\}\right) \\ & (\text{or} & f(\lambda_2(z)) = f\left(\max\{\langle z \circ v, v\rangle \mid \|v\| = 1\}\right)) \\ & = & \min\{f(\langle z \circ v, v\rangle) \mid \|v\| = 1\}. \end{array}$$

In light of the convexity of f and Lemma 4.2, we obtain

$$\begin{array}{rcl} f\left(\langle(\alpha x + (1-\alpha)y) \circ v, v\rangle\right) & = & f\left(\alpha\langle x \circ v, v\rangle + (1-\alpha)\langle y \circ v, v\rangle\right) \\ & \le & \alpha f\left(\langle x \circ v, v\rangle\right) + (1-\alpha) f\left(\langle y \circ v, v\rangle\right) \\ & \le & \langle(\alpha f^{\rm soc}(x) + (1-\alpha) f^{\rm soc}(y)) \circ v, v\rangle, \quad (\|v^2\| = 1) \end{array}$$

Then, applying Lemma 4.3 yields the desired inequality after taking the minimum over all choices of unit vector v. This completes the first assertion. The arguments for the second assertion are similar and are omitted. $\square$

Remark 4.9. Suppose that $C : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction and composite of invertible quadratic representations. According to the proof of Theorem 4.4, we note the contraction of C implies $\|C^T d^{(i)}\|^2 \le \frac{1}{2}$. Applying Lemma 2.2(b), we obtain $\det(C^T d^{(i)}) = 0$ since C is composite of invertible quadratic representations, and hence the first component of $C^T d^{(i)}$ is $\frac{1}{\sqrt{2}}\|C^T d^{(i)}\|$. Therefore, for a convex real-valued function f on I with $0 \in I$ and $f(0) \le 0$, we can conclude

$$\lambda\left(f^{\rm soc}(Cx)\right) \prec_w \lambda\left(C f^{\rm soc}(x)\right)$$

by following the same arguments of Theorem 4.4. If further f is monotone, then

$$\lambda\left(f^{\rm soc}(Cx)\right) \leqslant \lambda\left(C f^{\rm soc}(x)\right).$$

Remark 4.10. Following the argument of Theorem 4.8, we can obtain the majorization for monotone concave real-valued function g defined on $I \subseteq \mathbb{R}$:

$$\lambda(\alpha g^{\rm soc}(x) + (1-\alpha) g^{\rm soc}(y)) \preccurlyeq \lambda(g^{\rm soc}(\alpha x + (1-\alpha)y))$$

for all x, y with spectral values in I and $0 \le \alpha \le 1$. If further $0 \in I$ and $g(0) \ge 0$, then

$$\lambda(P(s) g^{\rm soc}(x)) \preccurlyeq \lambda(g^{\rm soc}(P(s)x))$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Theorem 4.11. *Let f be a monotone convex function on $I \subseteq \mathbb{R}$. Then, for any x, y with spectral values in I and $0 \le \alpha \le 1$, there exists a unitary element w such that*

$$f^{\rm soc}\left(\alpha x + (1-\alpha)y)\right) \preceq_{\mathcal{K}^n} P(w)\left(\alpha f^{\rm soc}(x) + (1-\alpha) f^{\rm soc}(y)\right).$$

If further $0 \in I$ and $f(0) \le 0$, then

$$f^{\rm soc}(P(s)x) \preceq_{\mathcal{K}^n} P(w)\left(P(s) f^{\rm soc}(x)\right)$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Proof. The inequalities hold by Proposition 3.8 and Theorem 4.8. $\Box$

Remark 4.12. For monotone concave real-valued function g defined on $I \subseteq \mathbb{R}$, we also get the SOC inequality as below:

$$P(w)\left(\alpha g^{\rm soc}(x) + (1-\alpha) g^{\rm soc}(y)\right) \preceq_{\mathcal{K}^n} g^{\rm soc}\left(\alpha x + (1-\alpha)y)\right)$$

for all x, y with spectral values in I and $0 \le \alpha \le 1$. If further $0 \in I$ and $g(0) \ge 0$, then

$$P(w)\left(P(s) g^{\rm soc}(x)\right) \preceq_{\mathcal{K}^n} g^{\rm soc}(P(s)x)$$

for all $x \in \mathbb{R}^n$ with spectral values in I and $0 \prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$.

Theorem 4.13. *Let f be a nonnegative increasing convex function on $[0, \infty)$ with $f(0) = 0$ and $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$. Then, there exist unitary elements w, w' such that*

$$P(w) f^{\rm soc}(x) + P(w') f^{\rm soc}(y) \preceq_{\mathcal{K}^n} f^{\rm soc}(x+y).$$

Proof. Suppose that $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$. Since $x \preceq_{\mathcal{K}^n} x+y$, we have $x^{\frac{1}{2}} \preceq_{\mathcal{K}^n} (x+y)^{\frac{1}{2}}$ by [5, Property 2.3(b)], and hence $P(x^{\frac{1}{2}}) \leqslant P((x+y)^{\frac{1}{2}})$ by [13, Lemma 2.3]. This together with [3, Lemma V.1.7] implies $\|P(x^{\frac{1}{2}})P((x+y)^{-\frac{1}{2}})\| \le 1$, which says $P(x^{\frac{1}{2}})P((x+y)^{-\frac{1}{2}})$ is a contraction. We note that all x, y, and $x+y$ are invertible. Then, we have

$$\begin{aligned}
& x = P(x^{\frac{1}{2}})P((x+y)^{-\frac{1}{2}})(x+y) \\
\Longrightarrow\quad & f^{\rm soc}(x) = f^{\rm soc}\left(P(x^{\frac{1}{2}})P((x+y)^{-\frac{1}{2}})(x+y)\right) \\
\Longrightarrow\quad & f^{\rm soc}(x) \preceq_{\mathcal{K}^n} P(\tilde{w})P(x^{\frac{1}{2}})P((x+y)^{-\frac{1}{2}})f^{\rm soc}(x+y) \\
\Longleftrightarrow\quad & P(\tilde{w})f^{\rm soc}(x) \preceq_{\mathcal{K}^n} P(x^{\frac{1}{2}})((x+y)^{-1} \circ f^{\rm soc}(x+y)) \\
\Longrightarrow\quad & P(\tilde{w})f^{\rm soc}(x) \preceq_{\mathcal{K}^n} P(\hat{w})P\left(((x+y)^{-1} \circ f^{\rm soc}(x+y))^{\frac{1}{2}}\right)x \\
\Longleftrightarrow\quad & P(\hat{w})P(\tilde{w})f^{\rm soc}(x) \preceq_{\mathcal{K}^n} P\left(((x+y)^{-1} \circ f^{\rm soc}(x+y))^{\frac{1}{2}}\right)x
\end{aligned}$$

by Remark 4.9 and Proposition 3.11 with some unitary elements $\hat{w}, \tilde{w}$. Hence by Remark 3.9, there exists a unitary element w such that

$$P(w)f^{\rm soc}(x) \preceq_{\mathcal{K}^n} P\left(((x+y)^{-1} \circ f^{\rm soc}(x+y))^{\frac{1}{2}}\right)x. \tag{12}$$

Similarly, there exists a unitary element w' such that

$$P(w')f^{\rm soc}(y) \preceq_{\mathcal{K}^n} P\left(((x+y)^{-1} \circ f^{\rm soc}(x+y))^{\frac{1}{2}}\right)y. \tag{13}$$

Adding inequalities (12) and (13) yields

$$\begin{aligned}
P(w)f^{\rm soc}(x) + P(w')f^{\rm soc}(y) \quad & \preceq_{\mathcal{K}^n} \quad P\left(((x+y)^{-1} \circ f^{\rm soc}(x+y))^{\frac{1}{2}}\right)(x+y) \\
& = \quad (x+y)^{-1} \circ f^{\rm soc}(x+y) \circ (x+y) \\
& = \quad f^{\rm soc}(x+y)
\end{aligned}$$

since $x+y$, $(x+y)^{-1}$, and $f^{\rm soc}(x+y)$ share the same Jordan frame. $\square$

Similarly, if f is a nonnegative increasing concave function on $[0,\infty)$ with $f(0) \ge 0$ and $x, y \in \mathcal{K}^n$, then there exist unitary elements w, w' such that

$$f^{\rm soc}(x+y) \preceq_{\mathcal{K}^n} P(w)f^{\rm soc}(x) + P(w')f^{\rm soc}(y).$$

This inequality can be viewed as a generalization of (10) in Theorem 4.1. Furthermore, we may obtain the following norm inequality.

Theorem 4.14. *Let f be a nonnegative increasing concave function on $[0,\infty)$ with $f(0) \ge 0$ and $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$. Then, there holds*

$$\|f^{\rm soc}(x+y)\| \le \|f^{\rm soc}(x)\| + \|f^{\rm soc}(y)\|.$$

Proof. Following the arguments of Corollary 4.7, we have

$$\begin{aligned}\|f^{\mathrm{soc}}(x+y)\| &\leq \|P(w)f^{\mathrm{soc}}(x)+P(w')f^{\mathrm{soc}}(y)\| \\ &\leq \|P(w)f^{\mathrm{soc}}(x)\|+\|P(w')f^{\mathrm{soc}}(y)\| \\ &= \|f^{\mathrm{soc}}(x)\|+\|f^{\mathrm{soc}}(y)\|,\end{aligned}$$

where the equality holds by Proposition 3.6. $\square$

5 Concluding Remarks

In this paper, we discuss the concept of unitary elements associated with Lorentz cone, and derive several properties under the unitary transformation $P(w)$ for suitable unitary element w. We believe that these new concepts will be helpful for designing algorithm on second-order cone programming, in which it always needs the inequalities or majorizations to analyze the convergence. Our new discovery just steps out and there are some other directions to be clarified in the future. We raise two of them as below.

Recall that a function $f:\mathbb{R}\to\mathbb{R}$ is said to be SOC-monotone if

$$x \preceq_{\mathcal{K}^n} y \implies f^{\mathrm{soc}}(x) \preceq_{\mathcal{K}^n} f^{\mathrm{soc}}(y),$$

and is said to be SOC-convex if

$$f^{\mathrm{soc}}((1-\lambda)x+\lambda y) \preceq_{\mathcal{K}^n} (1-\lambda)f^{\mathrm{soc}}(x)+\lambda f^{\mathrm{soc}}(y)$$

for any given x,y and $0\leq\lambda\leq 1$. Thinking about the convexity of f being replaced by SOC-monotonicity or SOC-convexity, it raises the following two questions.

Question 5.1. *If the function $f:[0,\infty)\to\mathbb{R}$ is SOC-monotone and $x\in\mathcal{K}^n$ and $0\prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$, then*

$$P(s)f^{\mathrm{soc}}(x) \preceq_{\mathcal{K}^n} f^{\mathrm{soc}}(P(s)x)?$$

Question 5.2. *The function $f:[0,\infty)\to\mathbb{R}$ is SOC-convex and $f(0)\leq 0$ if and only if for any $x\in\mathcal{K}^n$ and $0\prec_{\mathcal{K}^n} s^2 \preceq_{\mathcal{K}^n} e$*

$$f^{\mathrm{soc}}(P(s)x) \preceq_{\mathcal{K}^n} P(s)f^{\mathrm{soc}}(x)?$$

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