Exercise

1 Vector Spaces

In this set of exercises, V is always a vector space over a field F.

1. Let U be a subspace of V. For $\mathbf{v} \in V$ define $\mathbf{v} + U = \{\mathbf{v} + \mathbf{u} \mid \mathbf{u} \in U\}$.
   (a) Prove that $\mathbf{v} + U$ is a subspace of V if and only if $\mathbf{v} \in U$.
   (b) For $\mathbf{v}, \mathbf{w} \in V$, prove the following are equivalent:
      i. $(\mathbf{v} + U) \cap (\mathbf{w} + U) \neq \emptyset$.
      ii. $\mathbf{v} - \mathbf{w} \in U$.
      iii. $\mathbf{v} + U = \mathbf{w} + U$.

2. Let S, U, W be subspaces of V.
   (a) Show that $S \cap (U + W) \supseteq (S \cap U) + (S \cap W)$.
   (b) Find an example that $S \cap (U + W) = (S \cap U) + (S \cap W)$ is not true.
   (c) Show that if $W \subseteq S$, then $S \cap (U + W) = (S \cap U) + (S \cap W)$.
   (d) Prove $S \cap (U + (S \cap W)) = (S \cap U) + (S \cap W)$.

3. Let $P_n(F) = \{a_n x^n + \cdots + a_1 x + a_0 \mid a_i \in F\}$ and let $F[x]$ be the set of polynomials with coefficients in F.
   (a) Let $f_0(x) \neq 0, f_1(x), \ldots, f_n(x) \in P_n(F)$ with $\deg(f_i(x)) = i$, for $i = 0, \ldots, n$. Prove that $\{f_0(x), f_1(x), \ldots, f_n(x)\}$ is a basis of $P_n(F)$.
   (b) Show that $F[x]$ is a vector space over F, but is not a finite dimensional vector space over F.

4. Let V be a finite dimensional vector space over F and let U, W be subspaces of V.
   (a) Show that $\max\{\dim(U), \dim(W)\} \leq \dim(U + W) \leq \dim(U) + \dim(W)$.
   (b) Prove that $\dim(U + W) + \dim(U \cap W) = \dim(U) + \dim(W)$.
   (c) Suppose $\dim(V) = 8$, $\dim(U) = 7$ and $\dim(W) = 5$. Suppose further that $W \nsubseteq U$. Show that $\dim(U \cap W) = 4$.

5. Let F' be a subfield of F.
   (a) Prove that F is a vector space over F'.
   (b) Suppose that F is finite dimensional over F' and V is a finite dimensional vector space over F. Prove that V is a finite dimensional vector space over F' and $\dim_{F'}(V) = \dim_F(V) \dim_{F'}(F)$.

6. Let W be a subspace of V and consider the quotient space V/W with a subspace $\widetilde{U}$.
   (a) Let $U = \{\mathbf{u} \in V \mid \overline{\mathbf{u}} \in \widetilde{U}\}$. Show that U is a subspace of V and $W \subseteq U$.
   (b) Prove $U/W = \widetilde{U}$.

2 Linear Transformation

1. Let $T : V \to W$ be a linear transformation. Suppose that $\dim(V) \geq 2$ and $\dim(W) \geq 2$.

   (a) Prove the following are equivalent:

      i. T is one-to-one.

      ii. $T^{-1}(T(\{\mathbf{O}_V\})) = \{\mathbf{O}_V\}$.

      iii. For every nontrivial subspace V' of V, $T^{-1}(T(V')) = V'$.

   (b) Prove the following are equivalent:

      i. T is onto.

      ii. $T(T^{-1}(W)) = W$.

      iii. For every nontrivial subspace W' of W, $T(T^{-1}(W')) = W'$.

2. Let $T_1 : V \to W$ and $T_2 : W \to U$ be linear transformations. Consider the composition $T_2 \circ T_1 : V \to U$.

   (a) Show that $\mathrm{Ker}(T_2 \circ T_1) = T_1^{-1}(\mathrm{Ker}(T_2))$.

   (b) Prove that $T_2 \circ T_1$ is one-to-one if and only if T_1 is one-to-one and
   $$\mathrm{Ker}(T_2) \cap \mathrm{Im}(T_1) = \{\mathbf{O}_W\}.$$

   (c) Show that $\mathrm{Im}(T_2 \circ T_1) = T_2(\mathrm{Im}(T_1))$.

   (d) Prove that $T_2 \circ T_1$ is onto if and only if T_2 is onto and
   $$\mathrm{Ker}(T_2) + \mathrm{Im}(T_1) = W.$$

   (e) Suppose that W is finite dimensional. Prove
   $$\dim(\mathrm{Im}(T_1)) + \dim(\mathrm{Im}(T_2)) - \dim(W) \leq \dim(\mathrm{Im}(T_2 \circ T_1)) \leq \min\{\dim(\mathrm{Im}(T_1)), \dim(\mathrm{Im}(T_2))\}.$$
   (Hint: Consider the restriction map $T_2|_{\mathrm{Im}(T_1)} : \mathrm{Im}(T_1) \to U$ for the first inequality.)

3. Suppose that V_1, V_2 are vector spaces and U_1, U_2 are subspaces of V_1, V_2 respectively.

   (a) Prove that $U_1 \oplus U_2$ is a subspace of $V_1 \oplus V_2$.

   (b) Show that
   $$(V_1 \oplus V_2)/(U_1 \oplus U_2) \simeq (V_1/U_1) \oplus (V_2/U_2).$$

4. Let V, W be finite dimensional vector space such that $\dim(V) = n$, $\dim(W) = m$ and let β, β' be an order basis of V, W, respectively. Suppose that $T : V \to W$ is a linear transformation and let ${}_{\beta'}[T]_{\beta}$ be the representative matrix of T with respect to β, β'.

   (a) Show that $C({}_{\beta'}[T]_{\beta})$ (the column space of ${}_{\beta'}[T]_{\beta}$) is isomorphic to $\mathrm{Im}(T)$ and prove the following are equivalent:

      i. T is onto

      ii. There exists a linear transformation $T' : W \to V$ such that $T \circ T'$ is the identity map of W.

iii. There exists an $n \times m$ matrix A such that ${}_{\beta'}[T]_\beta \cdot A = I_m$ (where I_m is the $m \times m$ identity matrix).

iv. The rank of ${}_{\beta'}[T]_\beta$ is m.

(b) Show that $N({}_{\beta'}[T]_\beta)$ (the null space of ${}_{\beta'}[T]_\beta$) is isomorphic to $\mathrm{Ker}(T)$ and prove the following are equivalent:

i. T is one-to-one

ii. There exists a linear transformation $T'' : W \to V$ such that $T'' \circ T$ is the identity map of V.

iii. There exists an $n \times m$ matrix B such that $B \cdot {}_{\beta'}[T]_\beta = I_n$.

iv. The rank of ${}_{\beta'}[T]_\beta$ is n.

5. For a vector space V over F, let $V^* = \mathcal{L}(V, F)$ be the set of linear transformations from V to F (called the *dual space* of V). Let $\beta = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be an ordered basis of V and $\epsilon = (1)$ be the standard basis of F. For every $\mathbf{v}_i$, $i = 1, \ldots, n$, consider $\mathbf{v}_i^* \in V^*$, the unique linear transformation satisfying $\mathbf{v}_i^*(\mathbf{v}_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$ For $\mathbf{v} \in V$, write $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, with $c_i \in F$. Let $\mathbf{v}^* = c_1\mathbf{v}_1^* + \cdots + c_n\mathbf{v}_n^*$.

(a) For $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, find the representative matrix of $\mathbf{v}^* \in V^*$ with respect to β, ϵ.

(b) Prove that $* : V \to V^*$, defined by $*(\mathbf{v}) = \mathbf{v}^*, \forall\, \mathbf{v} \in V$ is a linear transformation. Furthermore, prove that $* : V \to V^*$ is an isomorphism.

(c) Show that $\{\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*\}$ is a basis of V^* (this is called a *dual basis*). Consider $\beta^* = (\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*)$ as an ordered basis of V^*. Find the representative matrix of $* : V \to V^*$ with respect to β, β^*

6. Continuing Exercise 5, let W be a vector space over F with an ordered basis $\gamma = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ and let γ^* be the ordered dual basis $(\mathbf{w}_1^*, \ldots, \mathbf{w}_m^*)$ of W^*. For $\mathbf{w} = c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m$, with $c_i \in F$, let $\mathbf{w}^* = c_1\mathbf{w}_1^* + \cdots + c_m\mathbf{w}_m^*$. Consider a linear transformation $T : V \to W$ and let ${}_\gamma[T]_\beta$ be the representative matrix of T with respective to β, γ.

(a) Consider the map $T' : V^* \to W^*$ defined by $T'(\mathbf{v}^*) = T(\mathbf{v})^*, \forall\, \mathbf{v}^* \in V^*$. Prove that T' is a linear transformation.

(b) Find ${}_{\gamma^*}[T']_{\beta^*}$ (the representative matrix of T' with respective to β^*, γ^*) by using ${}_\gamma[T]_\beta$.

(c) Consider the map $T^* : W^* \to V^*$ defined by $T^*(f) = f \circ T, \forall\, f \in W^*$. Prove that T^* is a linear transformation.

(d) Let ${}_\epsilon[\mathbf{w}^*]_\gamma$ be the representative matrix of $\mathbf{w}^* \in W^*$ with respect to γ, ϵ. Find the representative matrix of $T^*(\mathbf{w}^*) \in V^*$ with respective to β, ϵ by using ${}_\gamma[T]_\beta$ and ${}_\epsilon[\mathbf{w}^*]_\gamma$.

(e) Find ${}_{\beta^*}[T^*]_{\gamma^*}$ (the representative matrix of T^* with respective to γ^*, β^*), by using ${}_\gamma[T]_\beta$.

7. * (This exercise is more challenging) Continuing Exercise 6, consider the linear transformation $T : V \to W$ and its dual $T^* : W^* \to V^*$, defined by $T^*(f) = f \circ T, \forall\, f \in W^*$. Let $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ be a basis of $\mathrm{Im}(T)$. Extending S to an order basis $\gamma = (\mathbf{w}_1, \ldots, \mathbf{w}_r, \ldots, \mathbf{w}_m)$ and let $\gamma^* = (\mathbf{w}_1^*, \ldots, \mathbf{w}_r^*, \ldots, \mathbf{w}_m^*)$ the

ordered dual basis of W^*. Let β be an ordered basis of V and β^* the ordered dual basis of V^*.

(a) Show that $\mathrm{Ker}(T^*) = \{f \in W^* \mid \mathrm{Im}(T) \subseteq \mathrm{Ker}(f)\}$.

(b) Prove that $\{\mathbf{w}_{r+1}^*, \ldots, \mathbf{w}_m^*\}$ is a basis of $\mathrm{Ker}(T^*)$ and show that

$$m = \dim(\mathrm{Im}(T)) + \dim(\mathrm{Ker}(T^*)).$$

(c) Prove that $\dim(\mathrm{Im}(T)) = \dim(\mathrm{Im}(T^*))$ and show that the rank of ${}_\gamma[T]_\beta$ is equal to the rank of ${}_{\beta^*}[T^*]_{\gamma^*}$.

(d) Using the result of Exercise 6(e), show that for any matrix A, the rank of A is equal to the rank of its transpose A^T (this is equivalent to the dimension of the column space of A is equal to the dimension of the row space of A).

(e) Use results in Exercise 4 to show that T is onto if and only if T^* is one-to-one and show that T is one-to-one if and only if T^* is onto.

3 Linear Operator

In this set of exercises, we let V be a finite dimensional vector space, $\mathcal{L}(V)$ be the vector space of F-linear operator and $M_n(F)$ be the vector space of $n \times n$ matrices over F.

For an ordered basis β of V and a linear operator $T : V \to V$, let $[T]_\beta$ be the representative matrix of T with respect to β and $\chi_T(x), \mu_T(x)$ be the characteristic and minimal polynomials of T, respectively.

1. For $T_1, T_2 \in \mathcal{L}(V)$, define the "multiplication" of T_1, T_2 by $T_1 \circ T_2$.

   (a) Prove that under this multiplication and the original addition, $\mathcal{L}(V)$ is a ring.

   (b) For an ordered basis β of V, let $\Phi : \mathcal{L}(V) \to M_n(F)$ be the linear transformation defined by $\Phi(T) = [T]_\beta$. Prove that Φ is a ring Isomorphism.

2. Determinant the characteristic and minimal polynomials of each of the following matrices:
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

3. Suppose that $T \in \mathcal{L}(V)$ and $p(x)$ is an irreducible polynomial in $F[x]$ such that $p(T)$ is not one-to-one. Prove that $p(x) \mid \chi_T(x)$ and $p(x) \mid \mu_T(x)$.

4. Suppose that $T \in \mathcal{L}(V)$ and $p(x), q(x) \in F[x]$ are relatively prime.

   (a) Prove that $\mathrm{Im}(p(T)) + \mathrm{Im}(q(T)) = V$.

   (b) Prove that $\mathrm{Ker}(p(T)) \cap \mathrm{Ker}(q(T)) = \{\mathbf{O}\}$.

   (c) Suppose that $\mu_T(x) = p(x)q(x)$. Prove that $\mathrm{Ker}(p(T)) = \mathrm{Im}(q(T))$ and hence show
$$V = \mathrm{Ker}(p(T)) \oplus \mathrm{Im}(p(T)) \text{ and } V = \mathrm{Im}(p(T)) \oplus \mathrm{Im}(q(T)).$$

5. For each of the following matrix A, (using its minimal polynomial found in 2) find an invertible matrix P so that $P^{-1} \cdot A \cdot P$ is a block diagonal matrix.
$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

6. Suppose that $T \in \mathcal{L}(V)$ and $\chi_T(x) = p_1(x)^{c_1} \cdots p_k(x)^{c_k}, \mu_T(x) = p_1(x)^{m_1} \cdots p_k(x)^{m_k}$ where $c_i, m_i \in \mathbb{N}$ and $p_1(x), \ldots, p_k(x)$ are distinct monic irreducible polynomials.

   (a) Show that $\dim(\mathrm{Ker}(p_i(T)^{\circ m_i})) = c_i \deg(p_i(x)), \forall\, i = 1, \ldots, k$.

   (b) Prove that $\mathrm{Ker}(p_1(T)^{\circ m_1}) = \mathrm{Im}(p_2(T)^{\circ m_2} \circ \cdots \circ p_k(T)^{\circ m_k})$.

   (c) Prove that $\mathrm{Ker}(p_i(T)^{\circ m_i}) = \ker(p_i(T)^{\circ m}), \forall\, m > m_i$.

7. Suppose that $T \in \mathcal{L}(V)$ and $V = U \oplus W$, where U, W are T-invariant. Consider a map $\pi_U : V \to V$ defined by $\pi_U(\mathbf{v}) = \mathbf{u}$, if $\mathbf{v} = \mathbf{u} + \mathbf{w}$ with $\mathbf{u} \in U, \mathbf{w} \in W$.

   (a) Show that π_U is a linear transformation and find $\mathrm{Im}(\pi_U), \mathrm{Ker}(\pi_U)$.

   (b) Prove that $\pi_U \circ T = T \circ \pi_U$.

   (c) Suppose that $\mu_T(x) = f(x)g(x)$ with $f(x), g(x) \in F[x]$ relatively prime. Suppose further that $\mathrm{Ker}(f(T)) = U$ and $\mathrm{Ker}(g(T)) = W$. Prove that there exists $h(x) \in F[x]$ such that $\pi_U = h(T)$.

4 Form Reduction

In this set of exercises, for a given square matrix A, $\chi_A(x)$ is the characteristic polynomial of A and $\mu_A(x)$ is the minimal polynomials of A. For a given F-linear operator $T : V \to V$ and for $\mathbf{v} \in V$, $C_{\mathbf{v}}$ is the T-cyclic space spanned by $\mathbf{v}$.

1. Let $\theta \in \mathbb{R}$ and consider $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ as a matrix over $\mathbb{C}$. Find the eigenvalues of A in $\mathbb{C}$ and find its corresponding eigenspace.

2. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F)$ with $b \neq 0$.

   (a) Suppose that $\lambda \in F$ is an eigenvalue of A. Prove that $\begin{pmatrix} b \\ \lambda \end{pmatrix}$ is an eigenvector of A.

   (b) Suppose that λ_1, λ_2 are distinct eigenvalues of A. Let $P = \begin{pmatrix} b & b \\ \lambda_1 & \lambda_2 \end{pmatrix}$. Show that $P^{-1} \cdot A \cdot P$ is a diagonal matrix.

3. Let $A \in M_n(F)$ and $\lambda \in F$ is an eigenvalue of A.

   (a) Suppose that A is invertible. Prove that λ^{-1} is an eigenvalue of A.

   (b) Let $f(x) \in F[x]$. Prove that $f(\lambda)$ is an eigenvalue of $f(A)$.

4. Suppose that $A \in M_n(\mathbb{R})$ is diagonalizable.

   (a) Suppose that A is invertible. Prove that A^{-1} is also diagonalizable.

   (b) Prove that there exists $B \in M_n(\mathbb{R})$ such that $B^3 = A$.

5. Let T_1, T_2 be linear operators of $P_2(\mathbb{R})$ where

$$T_1(ax^2 + bx + c) = (-3a + b - c)x^2 + (-7a + 5b - c)x + (-6a + 6b - 2c),$$

$$T_2(ax^2 + bx + c) = (a - 3b + 3c)x^2 + (3a - 5b + 3c)x + (6a - 6b + 4c).$$

   Which operator is diagonalizable and find an ordered basis β of $P_2(\mathbb{R})$ so that its representative matrix with respect to β is a diagonal matrix.

6. Let $A = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix} \in M_4(F)$. Suppose that r, s, t are nonzero. Find an invertible matrix $P \in M_4(F)$ such that $P^{-1} \cdot A \cdot P = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ r & \lambda & 0 & 0 \\ 0 & s & \lambda & 0 \\ 0 & 0 & t & \lambda \end{pmatrix}$.

7. Find the Jordan form J of the following matrix A and find an invertible P such that $P^{-1} \cdot A \cdot P = J$.

$$A = \begin{pmatrix} 1 & 8 & 6 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & -5 & -4 & -3 & -2 \end{pmatrix}.$$

8. Suppose that A, B are square matrices over F. For the following given characteristic polynomials, find all the possible minimal polynomials of A and B and find all the possible Jordan forms for the corresponding minimal polynomial.

   (a) $\chi_A(x) = (x-\lambda)^5$.

   (b) $\chi_B(x) = (x-\lambda_1)^2(x-\lambda_2)^3$, where $\lambda_1 \neq \lambda_2$.

9. Let A, B be square matrices over F with $\chi_A(x) = \chi_B(x) = (x-\lambda_1)^2(x-\lambda_2)^3$ where $\lambda_1 \neq \lambda_2$. Suppose further that $\mu_A(x) = \mu_B(x)$. Prove A and B are similar.

10. Consider the following nilpotent matrices

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

    Show that A and B are not similar.

11. Let $\mathbf{u}, \mathbf{w} \in V$ suppose that the T-annihilators $\mu_{\mathbf{u}}(x)$ and $\mu_{\mathbf{w}}(x)$ are relatively prime. Let $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

    (a) Show that $C_{\mathbf{u}} \cap C_{\mathbf{w}} = \{\mathbf{O}_V\}$.

    (b) Prove that $\mu_{\mathbf{v}}(x) = \mu_{\mathbf{u}}(x)\mu_{\mathbf{w}}(x)$.

    (c) Show that $C_{\mathbf{v}} = C_{\mathbf{u}} \oplus C_{\mathbf{w}}$.

12. Let $\mathbf{u}, \mathbf{w} \in V$ and suppose that $C_{\mathbf{u}} \cap C_{\mathbf{w}} = \{\mathbf{O}_V\}$. Show that $C_{\mathbf{u}} \oplus C_{\mathbf{w}}$ is a T-cyclic space if and only if $\mu_{\mathbf{u}}(x)$ and $\mu_{\mathbf{w}}(x)$ are relatively prime.

13. Prove that there exists a cyclic decomposition $V = C_{\mathbf{v}_1} \oplus C_{\mathbf{v}_2} \oplus \cdots \oplus C_{\mathbf{v}_k}$ such that $\mu_{\mathbf{v}_{i+1}}(x) \mid \mu_{\mathbf{v}_i}(x)$ for all $i \in \{1, \ldots, k-1\}$. (Definition : $(\mu_{\mathbf{v}_1}(x), \mu_{\mathbf{v}_2}(x), \ldots, \mu_{\mathbf{v}_k}(x))$ is called the *invariant factors* of T.)

14. Prove the invariant factors of T is unique.

15. Let $(\mu_{\mathbf{v}_1}(x), \mu_{\mathbf{v}_2}(x), \ldots, \mu_{\mathbf{v}_k}(x))$ be the invariant factors of T. Prove that

$$\chi_T(x) = \mu_{\mathbf{v}_1}(x)\mu_{\mathbf{v}_2}(x)\cdots\mu_{\mathbf{v}_k}(x) \quad \text{and} \quad \mu_T(x) = \mu_{\mathbf{v}_1}(x).$$

16. For two F-linear operators $T : V \to V$ and $T' : V \to V$, prove that T and T' have the same rational form (or classical form) if and only if they have the same invariant factors.

17. Suppose that $\tilde{F}$ is a field extension of F. For $A \in M_n(F)$, we can consider A as a matrix in $M_n(\tilde{F})$.

    (a) Suppose that $\mathbf{v} \in F^n$ with $\mu_{\mathbf{v}}(x) = p(x)^m$ where $p(x)$ is a monic irreducible polynomial in $F[x]$. Suppose that $p(x) = q_1(x)^{n_1} \cdots q_l(x)^{n_l}$, where $q_i(x)$ are distinct monic irreducible polynomial in $\tilde{F}[x]$. Prove that there exist $\mathbf{v}_1, \ldots, \mathbf{v}_l \in \tilde{F}^n$ such that $C_{\mathbf{v}} = C_{\mathbf{v}_1} \oplus \cdots \oplus C_{\mathbf{v}_l}$ as a vector space over $\tilde{F}$ and $\mu_{\mathbf{v}_i}(x) = q_i(x)^{mn_i}$.

(b) Let $(p_1(x)^{m_1}, \ldots, p_t(x)^{m_t})$ be the elementary divisors of A as matrix in $M_n(F)$ ($p_i(x)$ not necessary distinct). Suppose that $p_i(x) = q_{i,1}(x)^{n_{i,1}} \cdots q_{i,l_i}(x)^{n_{i,l_i}}$, where $q_{i,j}(x)$ are distinct monic irreducible polynomial in $\tilde{F}[x]$. Prove that

$$(p_{1,1}(x)^{m_1 n_{1,1}}, \ldots, p_{1,l_1}(x)^{m_1 n_{1,l_1}}, \ldots, p_{t,1}(x)^{m_t n_{t,1}}, \ldots, p_{t,l_t}(x)^{m_t n_{t,l_t}})$$

is the elementary divisors of A as matrix in $M_n(\tilde{F})$.

(c) Show that A has the same invariant factors no matter considering A as a matrix in $M_n(F)$ or considering A as a matrix in $M_n(\tilde{F})$.

(d) Prove that if $A, B \in M_n(F)$ and $A \sim B$ in $M_n(\tilde{F})$, then $A \sim B$ in $M_n(F)$.

5 Operators on Inner Product Space

1. Let V be an inner product space over $\mathbb{C}$ and let W be a subspace of V. Prove that $\mathbf{v} \in W^{\perp}$ if and only if $\langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle \leq \langle \mathbf{w}, \mathbf{w} \rangle$, $\forall\, \mathbf{w} \in W$.

2. Let W be a subspace of a finite dimensional inner product space V. Prove that for every $\mathbf{v} \in V$, there exists a unique $\mathbf{u} \in W^{\perp}$ such that $\overline{\mathbf{u}} = \overline{\mathbf{v}}$ in V/W.

3. Let V be a vector space (not necessary being finite dimensional) and let U, W be subspaces of V such that $V = U \oplus W$.

   (a) Prove that $(V/W)^*$ is isomorphic to W^0.

   (b) Prove that V^*/W^0 is isomorphic to W^*.

   (c) Prove that $V^* = \mathrm{Im}(\pi^{\mathrm{t}}_{W,U}) \oplus \mathrm{Im}(\pi^{\mathrm{t}}_{U,W})$. (Where $\pi^{\mathrm{t}}_{W,U} : V^* \to V^*$ is the transpose of $\pi_{W,U} : V \to V$.)

4. Let V, W be finite dimensional vector spaces over the same field F and let $\mathcal{L}(V, W)$ be the vector space of F-linear transformations from V to W.

   (a) Prove that the mapping $\psi : \mathcal{L}(V, W) \to \mathcal{L}(W^*, V^*)$ given by $\psi(T) = T^{\mathrm{t}}$ is an isomorphism.

   (b) Suppose that V, W are inner product spaces. Prove that the mapping $\varphi : \mathcal{L}(V, W) \to \mathcal{L}(W, V)$ given by $\varphi(T) = T^*$ is a conjugate isomorphism.

5. For a given field F, let $M_{m\times n}(F)$ be the vector space of $m \times n$ matrices over F. For $A \in M_{n\times n}(F)$, let $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{i,i}$, where $a_{i,i}$ is the (i, i)-th entry of A.

   (a) For the case $F = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, consider $\langle A, B \rangle = \mathrm{tr}(B^*A)$, $\forall\, A, B \in M_{m\times n}(F)$. Prove $\langle\, ,\rangle$ is an inner product on $M_{m\times n}(F)$.

   (b) For an arbitrary field F, given $B \in M_{m\times n}(F)$, consider $\phi_B : M_{m\times n} \to F$ defined by $\phi_B(A) = \mathrm{tr}(B^*A)$, $\forall\, A \in M_{m\times n}(F)$. Prove that $\phi_B \in (M_{m\times n}(F))^*$ (i.e. ϕ_B is a linear functional on $M_{m\times n}(F)$). Prove also that for every linear functional $f \in (M_{m\times n}(F))^*$, there exists a unique $B \in M_{m\times n}(F)$ such that $f = \phi_B$.

   (c) For the case $F = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, consider the inner product on $M_{m\times n}(F)$ as in (a). For every $C \in M_{m\times l}(F)$, let $T_C : M_{l\times n}(F) \to M_{m\times n}(F)$ be the linear transformation given by $T_C(A) = CA$, $\forall\, A \in M_{l\times n}(F)$. Find the adjoint T_C^* of T_C.

6. Let V be a finite dimensional inner product space over $\mathbb{C}$. For all $\mathbf{u}, \mathbf{w} \in V$ with $\mathbf{u} \neq \mathbf{O}_V$ and $\mathbf{w} \neq \mathbf{O}_V$, let $T_{\mathbf{u},\mathbf{w}} : V \to V$ be given by $T_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{w} \rangle \mathbf{u}$, $\forall\, \mathbf{v} \in V$.

   (a) Prove that $T_{\mathbf{u},\mathbf{w}}$ is a linear operator and $T_{\mathbf{u},\mathbf{v}} \circ T_{\mathbf{v},\mathbf{w}} = \|\mathbf{v}\|^2 T_{\mathbf{u},\mathbf{w}}$, $\forall\, \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.

   (b) Prove that $T^*_{\mathbf{u},\mathbf{w}} = T_{\mathbf{w},\mathbf{u}}$.

   (c) Prove that $T_{\mathbf{u},\mathbf{w}}$ is a normal operator if and only if there exists $\lambda \in \mathbb{C}$ such that $\mathbf{u} = \lambda \mathbf{w}$.

   (d) Prove that $T_{\mathbf{u},\mathbf{w}}$ is a self-adjoint operator if and only if there exists $\lambda \in \mathbb{R}$ such that $\mathbf{u} = \lambda \mathbf{w}$.

7. Let V be a finite dimensional inner product space over F and let $T : V \to V$ be a linear operator.

(a) Suppose $F = \mathbb{C}$. Prove that T is self-adjoint if and only if $\langle T(\mathbf{v}), \mathbf{v} \rangle \in \mathbb{R}$, $\forall\, \mathbf{v} \in V$.

(b) Suppose that $F = \mathbb{R}$ and T is self-adjoint. Prove that $T = \mathbf{O}$ if and only if $\langle T(\mathbf{v}), \mathbf{v} \rangle = 0$, $\forall\, \mathbf{v} \in V$.

(c) Suppose $F = \mathbb{R}$. Prove that T is skew-adjoint if and only if $\langle T(\mathbf{v}), \mathbf{v} \rangle = 0$, $\forall\, \mathbf{v} \in V$.

(d) Suppose $F = \mathbb{R}$. Prove that there is a unique self-adjoint operator $T_1 : V \to V$ such that $\langle T(\mathbf{v}), \mathbf{v} \rangle = \langle T_1(\mathbf{v}), \mathbf{v} \rangle$, $\forall\, \mathbf{v} \in V$.

8. Let V be a finite dimensional inner product space over F and let $T : V \to V$ be a self-adjoint operator.

   (a) Prove the following are equivalent.

   i. $\langle T(\mathbf{v}), \mathbf{v} \rangle \geq 0$, $\forall\, \mathbf{v} \in V$.

   ii. Every eigenvalue of T is greater than or equal to 0.

   iii. There is a self-adjoint operator $T_1 : V \to V$ such that $T_1^{\circ 2} = T$.

   iv. There is a linear operator $T_2 : V \to V$ such that $T_2^* \circ T_2 = T$.

   (Definition: A self-adjoint operator $T : V \to V$ is said to be *positive* if it satisfies $\langle T(\mathbf{v}), \mathbf{v} \rangle \geq 0$, $\forall\, \mathbf{v} \in V$.)

   (b) Prove the following are equivalent.

   i. $\langle T(\mathbf{v}), \mathbf{v} \rangle > 0$, $\forall\, \mathbf{v} \in V \setminus \{\mathbf{O}_V\}$.

   ii. Every eigenvalue of T is greater than 0.

   iii. There is a self-adjoint isomorphism $T_1 : V \to V$ such that $T_1^{\circ 2} = T$.

   iv. There is an isomorphism $T_2 : V \to V$ such that $T_2^* \circ T_2 = T$.

   (Definition: A self-adjoint operator $T : V \to V$ is said to be *positive definite* if it satisfies $\langle T(\mathbf{v}), \mathbf{v} \rangle > 0$, $\forall\, \mathbf{v} \in V \setminus \{\mathbf{O}_V\}$.)

9. Let V be a finite dimensional vector space over F with a given inner product $\langle\, , \rangle$.

   (a) Prove that if $\langle\langle\, , \rangle\rangle$ is another inner product on V, then there is a unique positive definite $T : V \to V$ such that $\langle\langle \mathbf{v}, \mathbf{w} \rangle\rangle = \langle T(\mathbf{v}), \mathbf{w} \rangle$, $\forall\, \mathbf{v}, \mathbf{w} \in V$.

   (b) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of V. Prove there is a unique self-adjoint $n \times n$ matrix A such that if $\mathbf{v} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$ and $\mathbf{w} = y_1\mathbf{v}_1 + \cdots + y_n\mathbf{v}_n$, then

   $$\langle \mathbf{v}, \mathbf{w} \rangle = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} A \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix}.$$

   Prove also that A is a positive definite matrix (i.e., every eigenvalue of A is positive.)

10. Let V be a finite dimensional inner product space and let $T : V \to V$ be a normal operator.

   (a) Prove that if T is nilpotent, then $T = \mathbf{O}$.

   (b) Prove that if $T^{\circ n+1} = T^{\circ n}$, for some $n \in \mathbb{N}$, then T is a self-adjoint operator.

   (c) Prove that T is an orthogonal projection if and only if T is a projection.