Exercises

1 Basic Logic

1. 試分別在以下各小題，找出使之為 true statement 的所有可能整數 n (請不要只寫答案，盡量說明理由).

   (a) $n \geq 3$ and $n < 5$.

   (b) $n > 3$ or $n \leq 5$.

   (c) $n \geq 3$ and $n \leq 3$.

   (d) $n > 3$ or $n < 3$.

2. 在課堂上我們介紹了 $((P \wedge Q) \vee R) \sim ((P \vee R) \wedge (Q \vee R))$ 這稱為 distribution of disjunction over conjunction. 試利用 truth table 檢查以下的 distribution laws.

   (a) $((P \wedge Q) \wedge R) \sim ((P \wedge R) \wedge (Q \wedge R))$, distribution of conjunction over conjunction.

   (b) $((P \vee Q) \vee R) \sim ((P \vee R) \vee (Q \vee R))$, distribution of disjunction over disjunction.

   (c) $((P \vee Q) \wedge R) \sim ((P \wedge R) \vee (Q \wedge R))$, distribution of conjunction over disjunction.

3. **(Optional)** 在課堂上我們提及當一個 statement P 為 T, 我們可以定它的值 $p = 1$; 若為 F, 則定 $p = 0$. 假設 P, Q 為 statement 且其值分別為 p, q.

   (a) 試說明 $P \wedge Q$ 的值可表成 $p \times q$, 也可表成 $\min\{p, q\}$.

   (b) 試說明 $P \vee Q$ 的值可表成 $p + q - p \times q$, 也可表成 $\max\{p, q\}$.

   (c) 利用 truth table 以及 (a),(b) 的各種方法驗證 $(P \wedge P) \sim P$ 以及 $(P \vee P) \sim P$. 你覺得哪一種方法較好用？

   (d) 利用 $P \wedge Q$ 和 $P \vee Q$ 的值可分別表成 $p \times q$ 和 $p + q - p \times q$. 證明 $((P \wedge Q) \vee R) \sim ((P \vee R) \wedge (Q \vee R))$ 以及 $((P \vee Q) \wedge R) \sim ((P \wedge R) \vee (Q \wedge R))$.

   (e) 利用當 $a, b, c \geq 0$ 時 $\max\{a, b\} \times c = \max\{a \times c, b \times c\}$ (可嘗試證明看看), 證明 $((P \vee Q) \wedge R) \sim ((P \wedge R) \vee (Q \wedge R))$.

4. 試依照指定方式解決以下問題。

   (a) Negate $(P \Rightarrow Q) \Rightarrow P$（寫出否定句，不需化簡）。

   (b) 利用 $((P \wedge Q) \vee P) \sim P$ 說明 $\neg[(P \Rightarrow Q) \Rightarrow P]$ 和 $\neg P$ 是 logically equivalent.

   (c) 敘述 Proposition 1.2.2 且利用它說明 $\big((P \Rightarrow Q) \Rightarrow P\big) \Leftrightarrow P$ 是一個 tautology.

5. 將下列的 statement forms, 僅利用 $\neg$ and $\vee$ 寫下與之等價的（logically equivalent）statement forms.

   (a) $P \Leftrightarrow \neg Q$.

   (b) $(P \vee Q) \Rightarrow (P \wedge Q)$.

(c) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$.

6. 我們會將 $(\neg P) \wedge (\neg Q)$ 縮寫成 $P \triangle Q$. 僅利用 $\triangle$ 和 $\neg$ 寫下與 $P \Rightarrow Q$ 等價的 statement forms.

7. 證明 if "$\exists y < 0, \forall x > 0, P(x,y)$", then "$\forall x > 0, \exists y < 0, P(x,y)$".

8. Find an example for $P(x,y)$ so that the statement "$\forall x < 0, \exists y < 0, P(x,y)$" is true, but the statement "$\exists y < 0, \forall x < 0, P(x,y)$" is false.

9. Negate the statement: If $\forall x > 0, \exists y < 0, P(x) \wedge Q(y)$, then $\exists z \leq 0, R(x,z) \vee S(y,z)$.

10. Let P be the statement: If $x^2 - 3x + 2 < 0$, then $1.3 < x < 1.7$.

    (a) Write $\neg P$. （Hint: 注意在 if-then statement 中隱含的 "$\forall x$".）

    (b) 說明 P 或 $\neg P$ 哪一個是對的.

2 Methods of Proof

1. Using the indicated method to directly prove that if n is an integer, then $n^2 + 3n + 2$ is even.

    (a) Factorize $n^2 + 3n + 2$. (因式分解)

    (b) Divide n into 2 cases.

2. Suppose that a, b are integers. Let P be the statement: if $a \times b$ is even, then either a or b is even.

    (a) Write the contrapositive statement of P.

    (b) Prove P by using direct method (Hint: you can assume a is odd).

    (c) Prove P by using contrapositive method.

3. Let Q be the statement: if a, b are positive integers, then $a^2 - b^2 \neq 1$.

    (a) Negate Q.

    (b) Prove Q by using proof by cases (Hint: (1) $a < b$; (2) $a = b$; (3) $a > b$, and for (3) write $a = b + k$).

    (c) Prove Q by using contradiction method.

4. You have a five points in the plane, not all colinear. Prove that there is a line passing through just two of these points. (平面上有不共線的 5 個點。證明存在一條線僅通過其中兩個點)

5. Given $a, b \in \mathbb{N}$ with $b > 1$, prove that if there exist $h, r \in \mathbb{Z}$ with $0 \leq r < b$ such that $a = bh + r$, then h is unique and r is unique.

6. Use the pigeonhole principle to prove that no matter how 7 points are placed within an 8 by 9 rectangle, there will always be a pair whose distance is at most 5. （在 8×9 的長方形中任點 7 個點，一定有兩個點其距離不大於 5）(Hint: $3^2+4^2=5^2$)

7. Prove the pigeonhole principle by mathematical induction.

8. Using (strong) mathematical induction to prove the following:

   (a) If $a_1=1$ and $a_n=2a_{n-1}+1$, for $n\geq 2$, then $a_n=2^n-1$ for all $n\in\mathbb{N}$.

   (b) If $b_1=1$ and $b_n=n+\sum_{i=1}^{n-1}b_i$, for $n\geq 2$, then $b_n=2^n-1$ for all $n\in\mathbb{N}$.

3 Set

1. Let

$$A=\{n\in\mathbb{N}\mid n^2+5\},\quad B=\{n^2+5\mid n\in\mathbb{N}\},$$
$$C=\{n\in\mathbb{N}\mid n^2+5>35\},\quad D=\{n^2+5>35\mid n\in\mathbb{N}\}.$$

   (a) Two of the expressions above describe sets. Find Them. (以上哪兩個是正確集合表示？)

   (b) Write 3 elements for each of the two sets.

   (c) Is one of the set a subset of the other？Show your reasoning.

2. Let

$$S=\{n\in\mathbb{N}\mid 1\leq n\leq 5\},\quad T=\{n\in\mathbb{N}\mid 5\leq 2n-1\leq 17\},$$
$$U=\{2n-1\mid n\in S\},\quad V=\{2n\mid n=1,2,3,4\}.$$

   Calculate each of the following. (Hint: List all the elements of the set)

   (a) $S\cap U$.

   (b) $(S\cap T)\cup U$.

   (c) $S\cap(T\cup U)$.

   (d) $(S\cup T)\cap V$.

3. Let $S=\{x\in\mathbb{R}\mid 2<x<9\},\quad T=\{y\in\mathbb{R}\mid 5\leq y<14\}$.

   (a) Prove $S\cap T=\{z\in\mathbb{R}\mid 5\leq z<9\}$.

   (b) Write $S\cup T$ and prove your answer.

4. Let A,B,C,D be sets.

   (a) Prove that if $A\subseteq B$ and $C\subseteq B$, then $A\cup C\subseteq B$, by showing that every element of $A\cup C$ is an element of B.

   (b) Using B and D are both a subset of $B\cup D$ and (a) to prove that if $A\subseteq B$ and $C\subseteq D$, then $(A\cup C)\subseteq(B\cup D)$.

5. Let C, D be sets.

   (a) Which one of the following statements is equivalent to $C \not\subseteq D$.

      i. If $x \in C$, then $x \notin D$.

      ii. $x \in C$ and $x \notin D$.

      iii. If there is an $x \in C$, then $x \notin D$.

      iv. There is an $x \in C$ such that $x \notin D$.

   (b) Using your answer for (a) to explain why $\emptyset \not\subseteq D$ is wrong.

6. Let $A = \{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 0\}$ and $B = \{(x,y) \in \mathbb{R}^2 : x + y = 0\}$. Find $A \setminus B$.

7. Prove that $A = (A \cap B) \cup (A \setminus B)$.

8. Which of the following is true? (Please show your reasoning.)

$$(\mathbf{a})\ (C \setminus (B \setminus A)) \subseteq ((C \setminus B) \setminus A) \qquad (\mathbf{b})\ ((C \setminus B) \setminus A) \subseteq (C \setminus (B \setminus A)).$$

9. Suppose $B \setminus C \subseteq A$. Prove the following.

   (a) $B \setminus C \subseteq A \setminus C$.

   (b) If $A \cap C \subseteq A \cap B$ then $(A \setminus B) \cup (B \setminus C) \subseteq A \setminus C$

10. Let $I = \{i \in \mathbb{N} : i \geq 2\}$ and $\forall i \in I$, let $A_i = \{m/i : m \in \mathbb{Z}\}$.

   (a) Prove for any $k \in I$, $\displaystyle\bigcup_{i=k}^{\infty} A_i = \mathbb{Q}$.

   (b) Is it possible to find $j, k \in I$ with $j < k$ such that $\displaystyle\bigcap_{i=j}^{k} A_i = \mathbb{Q}$?

11. Let I be the closed interval $[1,2] = \{r \in \mathbb{R} : 1 \leq r \leq 2\}$. Suppose that for every $r \in I$, A_r is a set and $A_r \subseteq A_s$ for all $r, s \in I$ with $r > s$. We know that there exist $i, j \in I$ such that

$$\bigcap_{r \in I} A_r = A_i, \quad \bigcup_{r \in I} A_r = A_j.$$

   Find i and j. (Please show your reasoning.)

12. Let A_i, B_i are sets with index set I. Find an example to show the following are not always true.

$$(\bigcap_{i \in I} A_i) \cup (\bigcap_{i \in I} B_i) = \bigcap_{i \in I} (A_i \cup B_i),$$

$$(\bigcup_{i \in I} A_i) \cap (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (A_i \cap B_i).$$

   Please find correct statements for them with proof.

13. Let A, B, C, D be sets.

(a) Explain why $(C \setminus A) \times (D \setminus B) = ((C \setminus A) \cap C) \times (D \cap (D \setminus B))$.

(b) Use (a) to prove $(C \setminus A) \times (D \setminus B) = ((C \setminus A) \times D) \cap (C \times (D \setminus B))$.

(c) Use (b) to prove $(C \setminus A) \times (D \setminus B) = (C \times D) \setminus \big((A \times D) \cup (C \times B)\big)$.

(d) Explain why $(C \times D) \setminus (A \times B) = (C \times D) \setminus ((C \times D) \cap (A \times B))$.

(e) Use (d) to prove $(C \times D) \setminus (A \times B) = (C \times D) \setminus ((C \times B) \cap (A \times D))$.

(f) Use (e) to prove $(C \times D) \setminus (A \times B) = (C \times (D \setminus B)) \cup ((C \setminus A) \times D)$.

14. For a given set A, let $\mathfrak{T}_A = \{(a,S) \in A \times \mathscr{P}(A) : a \in S\}$.

(a) Please describe $\mathfrak{T}_\emptyset$, $\mathfrak{T}_{\{a\}}$ and $\mathfrak{T}_{\{a,b\}}$ using list mehthod.

(b) Let A be a finite set of n elements, find the number of elements of $\mathfrak{T}_A$. (you can use the fact that $\#(\mathscr{P}(A)) = 2^n$.)

(c) Let A, B be sets. Is it true that $\mathfrak{T}_{A\cap B} = \mathfrak{T}_A \cap \mathfrak{T}_B$? Explain your answer.

4 Relation and Order

1. Define a relation on $\mathbb{R}$ by setting $x \sim y$ if and only if $|x-y| < 1$. Is this relation an equivalence relation? Explain your answer.

2. Let $X = \{1,2,3\}$ and suppose $S \subseteq X \times X$ is an equivalence relation on X. Suppose further that $(1,2) \in S$ and $(2,3) \notin S$. Find all the elements of S. Let $X = \{1,2,3\}$. How many different equivalence relations can we find on X（在 X 中可以定義多少種 equivalence relation）?

(Hint: X 的 partition 和 X 的 equivalence relation 有著一對一的對應關係.)

3. Let $(\mathbb{C}, \prec)$ be a strict total ordered set with the following additional properties:

A: If $a \prec b$, then for every $c \in \mathbb{C}$, $a + c \prec b + c$

M: If $a \prec b$ and $0 \prec c$, then $ac \prec bc$

(a) Prove if $a \prec 0$, then $0 \prec -a$.

(b) Prove $0 \prec c^2$, $\forall c \in \mathbb{C}$.

4. Let A, B be nonempty subsets of a partial order set $(X, \preceq)$. Suppose that $A \subseteq B$.

(a) Prove that if there exists an upper bound of B, then there exists an upper bound of A.

(b) Suppose that a and b are the least upper bound of A and B, respectively. Prove that $a \preceq b$.

5. Define a relation $\prec$ on $\mathbb{Z}$ by the rule: $a \prec b$ if and only if either $(ab < 0) \wedge (a < b)$ or $(ab \geq 0) \wedge (|a| < |b|)$.

(a) Prove that $(\mathbb{Z}, \prec)$ is a strict total ordered set.

(b) Let $-\mathbb{N}$ be the set of negative integers. Show that 1 is the least upper bound of $-\mathbb{N}$.

(c) Prove that $(\mathbb{Z}, \prec)$ is a well ordered set (*i.e.* for any nonempty $S \subseteq \mathbb{Z}$, the least element of S exists).

5 Function

1. Let $f \subseteq \mathbb{R} \times \mathbb{R}$ be a relation from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$f = \{(a,b) : b - a = \begin{cases} 1, & \text{if } a \geq 0; \\ -1, & \text{if } a \leq 0. \end{cases} \}$$

Is f a function from $\mathbb{R}$ to $\mathbb{R}$? Show your reason.

2. Suppose that $f : X \to X$ is a function and consider f as a relation on X. Prove that if f is symmetric then $f \circ f = \mathrm{id}_X$.

3. Let $f : X \to Y$ and $g : Y \to Z$ be functions. Suppose $\forall x, x' \in X$, $y, y' \in Y$ and $z, z' \in Z$ we have $x \star x' \in X$, $y \diamond y' \in Y$ and $z * z' \in Z$. Moreover, suppose we have

$$f(x \star x') = f(x) \diamond f(x') \text{ and } g(y \diamond y') = g(y) * g(y').$$

Prove if $(g \circ f)(x) = z$ and $(g \circ f)(x') = z'$, then $(g \circ f)(x \star x') = z * z'$.

4. Consider the function $f : \{1,2,3\} \to \{1,2,3\}$ defined by $f(1) = f(3) = 2$ and $f(2) = 3$. List all subsets $A \subseteq \{1,2,3\}$ and find $f(A)$ and $f^{-1}(A)$.

5. Suppose that $f : X \to X$ is a function and $A \subseteq X$. Which of the following statements are true? Explain each case.

$$\begin{array}{ll} (a)\ f(A^c) \subseteq f(A)^c & (b)\ f(A)^c \subseteq f(A^c) \\ (c)\ f^{-1}(A^c) \subseteq f^{-1}(A)^c & (d)\ f^{-1}(A)^c \subseteq f^{-1}(A^c) \end{array}$$

6. Suppose that $g : X \to Y$ is a surjective function and $B \subseteq Y$.

   (a) Prove that there exists $A \subseteq X$ such that $g(A) = B$.

   (b) Prove $g(g^{-1}(B)) = B$.

   (c) Suppose $B' \subseteq Y$ and $B' \neq B$. Prove $g^{-1}(B') \neq g^{-1}(B)$.

7. Find functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that f is not onto and yet $g \circ f$ is onto.

8. Find functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that g is not injective but $g \circ f$ is injective.

9. Suppose that $f : X \to Y$ is a bijection and $g : Y \to Z$ is a function. Prove the following.

   (a) $g \circ f$ is surjective if and only if g is surjective.

(b) $g \circ f$ is injective if and only if g is injective.

10. Suppose $h: X \to Y$ is injective.

    (a) For $A \subseteq X$, prove $|A| = |h(A)|$.

    (b) For $B \subseteq Y$, prove $\left|h^{-1}(B)\right| \leq |B|$.

11. Suppose that A, B, C, D are nonempty sets with $|A| = |B|$ and $|C| = |D|$. Prove $|A \times C| = |B \times D|$.

12. Let X be a nonempty set and $Y, Z \subseteq X$.

    (a) Suppose Y is uncountable and Z is countable. Prove $Y \setminus Z$ is uncountable.

    (b) Using the fact that $\mathbb{R}$ is uncountable to show that the set of irrational numbers is uncountable.

    (c) Suppose that $Y \times Z$ is uncountable. Prove Y or Z is uncountable.

13. 給定 $n \in \mathbb{N}$, 試證明所有次數小於 n 的整係數多項式所成的集合為 countable. (Hint: 利用若 $S_1, \ldots, S_n$ 為 countable, 則 $S_1 \times \cdots \times S_n$ 為 countable)

14. 試證明所有整係數多項式所成的集合為 countable. (Hint: 利用若 S_i, $\forall i \in \mathbb{N}$ is countable, 則 $\bigcup_{i=1}^{\infty} S_i$ is countable)

15. 令 S 為所有整數部分為 0, 而小數點後各位數是 0 或 1 所組成的實數所成的集合. 假設 $f: \mathbb{N} \to S$ 是一個函數. 考慮 S 中的元素 $s = 0.a_1a_2a_3\ldots a_i\ldots$, 其中對任意 $i \in \mathbb{N}$, 我們令 s 的小數點後第 i 位的值 a_i 為

$$a_i = \begin{cases} 1, & \text{若 } f(i) \text{ 小數點後第 } i \text{ 位為 } 0; \\ 0, & \text{若 } f(i) \text{ 小數點後第 } i \text{ 位為 } 1. \end{cases}$$

    試證明 s 不在 $f: \mathbb{N} \to S$ 的 image 中, 並依此說明 S 是 uncountable.

16. For sets A, B, we define the set $A^B = \{f \mid f: B \to A \text{ is a function}\}$. Suppose that A_1, A_2, B are sets. Prove $\left|(A_1 \times A_2)^B\right| = \left|A_1^B \times A_2^B\right|$. (Hint: for $f_1 \in A_1^B, f_2 \in A_2^B$, define $(f_1 \times f_2) \in (A_1 \times A_2)^B$ by setting $(f_1 \times f_2)(b) = (f_1(b), f_2(b))$, $\forall b \in B$.)