2003 程式設計第三次作業

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Problem: Solving the eigenvalue problem $Ax = \lambda x$, where

$$A = \begin{bmatrix} 2 & -1 & & -1 \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{m\times m}.$$

Lemma 1

$$\sum_{j=2}^{k} \sin\left(\frac{2\pi(j-1)\ell}{k}\right) = 0, \quad \textit{for} \ \ k \geq 2 \ \ \textit{and all integers} \ \ \ell,$$

$$\sum_{j=1}^{k} \cos\left(\frac{2\pi(j-1)\ell}{k}\right) = \begin{cases} k, & \textit{if} \ \ \ell \ \ \textit{is a multiple of} \ \ k, \\ 0, & \textit{otherwise.} \end{cases}$$

Use the formulas in Lemma 1 to prove the following results.

Theorem 2 *Assume that m is an even number. Let*

$$W = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 1 & \cos(\theta) & \cos(2\theta) & \cdots & \cos(\frac{m}{2}\theta) & \sin(\theta) & \sin(2\theta) & \cdots & \sin(\frac{m}{2}-1)\theta \\ 1 & \cos(2\theta) & \cos(4\theta) & \cdots & \cos(m\theta) & \sin(2\theta) & \sin(4\theta) & \cdots & \sin 2(\frac{m}{2}-1)\theta \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & \cos(m-1)\theta & \cos 2(m-1)\theta & \cdots & \cos(m-1)\frac{m}{2}\theta & \sin(m-1)\theta & \sin 2(m-1)\theta & \cdots & \sin(m-1)(\frac{m}{2}-1)\theta \end{bmatrix}$$

and

$$D = \mathit{diag}\left(\sqrt{\frac{1}{m}}, \sqrt{\frac{2}{m}}, \cdots, \sqrt{\frac{2}{m}}, \sqrt{\frac{1}{m}}, \sqrt{\frac{2}{m}}, \cdots, \sqrt{\frac{2}{m}}\right).$$

Then

$$(WD)^T(WD) = I_m$$

and

$$(WD)^T A (WD) = \mathit{diag}(\lambda_1, \ \cdots, \ \lambda_m) \equiv \Lambda,$$

where

$$\lambda_i = \begin{cases} 4\sin^2[(i-1)\pi/m], & \textit{for} \ \ 1 \leq i \leq m/2+1, \\ 4\sin^2[(i-m/2-1)\pi/m], & \textit{for} \ \ m/2+2 \leq i \leq m, \end{cases}$$

[**Numerical method:**] Let

$$\mathbf{F}(x,\lambda) = \begin{bmatrix} \lambda x - Ax \\ x^T x - 1 \end{bmatrix}.$$

Then

$$D\mathbf{F}(x,\lambda) = \begin{bmatrix} \lambda I - A & x \\ 2x^T & 0 \end{bmatrix}.$$

Find the eigenvalue $\lambda = 4$ and the associated eigenvector of the eigenvalue problem $\mathbf{F}(x,\lambda) = 0$ by using the following Newton's method with stopping tolerance $\| \mathbf{F}(x,\lambda) \|_2 < 10^{-14}$.

Algorithm 3 (Newton's Method) *Given $F : \mathbb{R}^n \to \mathbb{R}^n$, an initial guess x_0 to the zero of F, and stop criteria M, δ, and ε, this algorithm performs the Newton's iteration to approximate one root of F.*

> $x_1 = x_0 - DF(x_0)^{-1}F(x_0)$
> $k = 1$
> **while** $(k < M)$ or $(\| x_k - x_{k-1} \|_2 \geq \delta)$ or $(\| F(x_k) \|_2 \geq \varepsilon)$ **do**
> $x_{k+1} = x_k - DF(x_k)^{-1}F(x_k)$
> $k = k + 1$
> **end while**

[**Requirement:**]

(a) m is an even number.

(b) Use the subroutine Gaussian elimination in program 2 to solve linear system $DF(x_k)y = F(x_k)$.

(c) Use double precision for all floating points.

(d) Take $M = 100$, $\delta = \varepsilon = 10^{-14}$, $\lambda_0 = 4 + \sigma$ and $x_0 = [1 + \sigma, -1 + \sigma, \ldots, 1 + \sigma, -1 + \sigma]$ in Newton's algorithm where σ is a constant.