Boundary-Value Problems for Ordinary Differential Equations

NTNU

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The two-point boundary-value problems (BVP) considered in this chapter involve a second-order differential equation together with boundary condition in the following form:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha, \qquad y(b) = \beta \end{cases} \tag{1}$$

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The numerical procedures for finding approximate solutions to the initial-value problems can not be adapted to the solution of this problem since the specification of conditions involve two different points, $x = a$ and $x = b$. New techniques are introduced in this chapter for handling problems (1) in which the conditions imposed are of a boundary-value rather than an initial-value type.

1 – Mathematical Theories

Before considering numerical methods, a few mathematical theories about the two-point boundary-value problem (1), such as the existence and uniqueness of solution, shall be discussed in this section.

Theorem 1 *Suppose that f in (1) is continuous on the set*

$$D = \{(x, y, y')|a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\},$$

and that $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y'}$ are also continuous on D. If

1. $\dfrac{\partial f}{\partial y}(x, y, y') > 0$ *for all* $(x, y, y') \in D$, *and*
2. *a constant M exists, with* $\left|\dfrac{\partial f}{\partial y'}(x, y, y')\right| \le M, \forall\ (x, y, y') \in D$,

then (1) has a unique solution.

When the function $f(x,y,y')$ has the special form

$$f(x,y,y') = p(x)y' + q(x)y + r(x),$$

the differential equation become a so-called linear problem. The previous theorem can be simplified for this case.

Corollary 1 *If the linear two-point boundary-value problem*

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x) \\ y(a) = \alpha, \qquad y(b) = \beta \end{cases}$$

satisfies

1. $p(x), q(x)$, *and* $r(x)$ *are continuous on* $[a,b]$, *and*
2. $q(x) > 0$ *on* $[a,b]$,

then the problem has a unique solution.

Many theories and application models consider the boundary-value problem in a special form as follows.

$$\begin{cases} y'' = f(x,y) \\ y(0) = 0, \qquad y(1) = 0 \end{cases}$$

We will show that this simple form can be derived from the original problem by simple techniques such as changes of variables and linear transformation. To do this, we begin by changing the variable. Suppose that the original problem is

$$\begin{cases} y'' = f(x,y) \\ y(a) = \alpha, \qquad y(b) = \beta \end{cases} \tag{2}$$

where $y = y(x)$. Now let $\lambda = b - a$ and define a new variable

$$t = \frac{x-a}{b-a} = \frac{1}{\lambda}(x-a).$$

That is, $x = a + \lambda t$. Notice that $t = 0$ corresponds to $x = a$, and $t = 1$ corresponds to $x = b$. Then we define

$$z(t) = y(a + \lambda t) = y(x)$$

with $\lambda = b - a$. This gives

$$z'(t) = \frac{d}{dt}z(t) = \frac{d}{dt}y(a + \lambda t) = \left[\frac{d}{dx}y(x)\right]\left[\frac{d}{dt}(a + \lambda t)\right] = \lambda y'(x)$$

and, analogously,

$$z''(t) = \frac{d}{dt}z'(t) = \lambda^2 y''(x) = \lambda^2 f(x, y(x)) = \lambda^2 f(a + \lambda t, z(t)).$$

Likewise the boundary conditions are changed to

$$z(0) = y(a) = \alpha \quad \text{and} \quad z(1) = y(b) = \beta.$$

With all these together, the problem (2) is transformed into

$$\begin{cases} z''(t) = \lambda^2 f(a+\lambda t, z(t)) \\ z(0) = \alpha, \qquad z(1) = \beta \end{cases} \tag{3}$$

Thus, if $y(x)$ is a solution for (2), then $z(t) = y(a+\lambda t)$ is a solution for the boundary-value problem (3). Conversely, if $z(t)$ is a solution for (3), then $y(x) = z(\frac{1}{\lambda}(x-a))$ is a solution for (2).

Example 1 *Simplify the boundary conditions of the following equation by use of changing variables.*

$$\begin{cases} y'' = \sin(xy) + y^2 \\ y(1) = 3, \qquad y(4) = 7 \end{cases}$$

Solution: In this problem $a = 1, b = 4$, hence $\lambda = 3$. Now define the new variable $t = \frac{1}{3}(x-1)$, hence $x = 1+3t$, and let $z(t) = y(x) = y(1+3t)$. Then

$$\lambda^2 f(a+\lambda t, z) = 9\left[\sin(1+3t)z + z^2\right],$$

and the original equation is reduced to

$$\begin{cases} z''(t) = 9\sin((1+3t)z) + 9z^2 \\ z(0) = 3, \qquad z(1) = 7 \end{cases}$$

■

To reduce a two-point boundary-value problem

$$\begin{cases} z''(t) = g(t, z) \\ z(0) = \alpha, \qquad z(1) = \beta \end{cases}$$

to a homogeneous system, let

$$u(t) = z(t) - [\alpha + (\beta - \alpha)t]$$

then $u''(t) = z''(t)$, and

$$u(0) = z(0) - \alpha = 0 \quad \text{and} \quad u(1) = z(1) - \beta = 0$$

Moreover,

$$g(t,z) = g(t, u+\alpha+(\beta-\alpha)t) \equiv h(t,u).$$

The system is now transformed into

$$\begin{cases} u''(t) = h(t,u) \\ u(0) = 0, \qquad u(1) = 0 \end{cases}$$

Example 2 *Reduce the system*

$$\begin{cases} z'' = [5z - 10t + 35 + \sin(3z - 6t + 21)]e^t \\ z(0) = -7, \qquad z(1) = -5 \end{cases}$$

to a homogeneous problem by linear transformation technique.

Solution: Let

$$u(t) = z(t) - [-7 + (-5+7)t] = z(t) - 2t + 7.$$

Then $z(t) = u(t) + 2t - 7$, and

$$\begin{aligned} u'' = z'' = &\quad [5z - 10t + 35 + \sin(3z - 6t + 21)]e^t \\ = &\quad [5(u + 2t - 7) - 10t + 35 + \sin(3(u + 2t - 7) - 6t + 21)]e^t \\ = &\quad [5u + \sin(3u)]e^t \end{aligned}$$

The system is transformed to

$$\begin{cases} u''(t) = [5u + \sin(3u)]e^t \\ u(0) = u(1) = 0 \end{cases}$$

■

Example 3 *Reduce the following two-point boundary-value problem*

$$\begin{cases} y'' = y^2 + 3 - x^2 + xy \\ y(3) = 7, \qquad y(5) = 9 \end{cases}$$

to a homogeneous system.

Solution: In the original system, $a = 3, b = 5, \alpha = 7, \beta = 9$. Let $\lambda = b - a = 2$ and define a new variable

$$t = \frac{1}{2}(x-3) \quad \Longrightarrow \quad x = 2t + 3.$$

Let the function $z(t) = y(x) = y(2t+3)$. Then

$$\begin{aligned} z''(t) &= \lambda^2 y''(2t+3) = \lambda^2 f(2t+3, u) \\ &= 4[z^2 + 3 - (2t+3)^2 + (2t+3)z] \\ &= 4[z^2 + 3z + 2tz - 4t^2 - 12t - 6] \end{aligned}$$

The original problem is first transformed into

$$\begin{cases} z''(t) = 4[z^2 + 3z + 2tz - 4t^2 - 12t - 6] \\ z(0) = 7, \qquad z(1) = 9 \end{cases}$$

Next let

$$u(t) = z(t) - [7 + 2t], \quad \text{or equivalently,} \quad z(t) = u(t) + 2t + 7.$$

Then

$$u''(t) = 4[(z+2t+7)^2 + 3(u+2t+7) + 2t(u+2t+7) - 4t^2 - 12t - 6]$$
$$= 4[u^2 + 6tu + 17u + 4t^2 + 36t + 64].$$

The original problem is transformed into the following homogeneous system

$$\begin{cases} u''(t) = 4[u^2 + 6tu + 17u + 4t^2 + 36t + 64] \\ u(0) = u(1) = 0 \end{cases}$$

■

Theorem 2 *The boundary-value problem*

$$\begin{cases} y'' = f(x, y) \\ y(0) = 0, \qquad y(1) = 0 \end{cases}$$

has a unique solution if $\frac{\partial f}{\partial y}$ *is continuous, non-negative, and bounded in the strip* $0 \le x \le 1$ *and* $-\infty < y < \infty$.

Theorem 3 *If* f *is a continuous function of* (s, t) *in the domain* $0 \le s \le 1$ *and* $-\infty < t < \infty$ *such that*

$$|f(s,t_1) - f(s,t_2)| \le K|t_1 - t_2|, \qquad (K < 8).$$

Then the boundary-value problem

$$\begin{cases} y'' = f(x,y) \\ y(0) = 0, \qquad y(1) = 0 \end{cases}$$

has a unique solution in $C[0,1]$.

2 – Finite Difference Method For Linear Problems

We consider finite difference method for solving the linear two-point boundary-value problem of the form

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First, partition the interval $[a, b]$ into n equally-spaced subintervals by points $a = x_0 < x_1 < \ldots < x_n < x_n = b$. Each mesh point x_i can be computed by

$$x_i = a + i * h, \quad i = 0, 1, \ldots, n, \text{ with } h = \frac{b-a}{n}$$

where h is called the mesh size.

At the interior mesh points, x_i, for $i = 1, 2, \ldots, n-1$, the differential equation to be approximated satisfies

$$y''(x_i) = p(x_i)y'(x_i) + q(x_i)y(x_i) + r(x_i). \tag{5}$$

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The central finite difference formulae

$$y'(x_i) = \frac{y(x_{i+1}) - y(x_{i-1})}{2h} - \frac{h^2}{6}y^{(3)}(\eta_i), \tag{6}$$

for some η_i in the interval (x_{i-1}, x_{i+1}),

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Let u_i denote the approximate value of $y_i = y(x_i)$. If $y \in C^4[a, b]$, then a finite difference method with truncation error of order $O(h^2)$ can be obtained by using the approximations

$$y'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \quad \text{and} \quad y''(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}$$

for $y'(x_i)$ and $y''(x_i)$, respectively.

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The discrete version of equation (4) is then

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = p_i \frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i + r_i, \quad i = 1, 2, \ldots, n-1, \qquad (8)$$

together with boundary conditions $u_0 = \alpha$ and $u_n = \beta$.

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together with boundary conditions $u_0 = \alpha$ and $u_n = \beta$. Equation (8) can be written in the form

$$\left(1 + \frac{h}{2} p_i\right) u_{i-1} - \left(2 + h^2 q_i\right) u_i + \left(1 - \frac{h}{2} p_i\right) u_{i+1} = h^2 r_i, \tag{9}$$

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for $i = 1, 2, \ldots, n-1$. In (8), $u_1, u_2, \ldots, u_{n-1}$ are the unknown, and there are $n-1$ linear equations to be solved. The resulting system of linear equations can be expressed in the matrix form

$$Au = f, \tag{10}$$

where

$$A = \begin{bmatrix} -2-h^2q_1 & 1-\frac{h}{2}p_1 & & & \\ 1+\frac{h}{2}p_2 & -2-h^2q_2 & 1-\frac{h}{2}p_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & 1+\frac{h}{2}p_{n-2} & -2-h^2q_{n-2} & 1-\frac{h}{2}p_{n-2} \\ & & & 1+\frac{h}{2}p_{n-1} & -2-h^2q_{n-1} \end{bmatrix},$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix}, \quad \text{and} \quad f = \begin{bmatrix} h^2r_1 - \left(1+\frac{h}{2}p_1\right)\alpha \\ h^2r_2 \\ \vdots \\ h^2r_{n-2} \\ h^2r_{n-1} - \left(1-\frac{h}{2}p_{n-1}\right)\beta \end{bmatrix}$$

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Theorem 4 *Suppose that* $p(x), q(x)$*, and* $r(x)$ *in (4) are continuous on* $[a, b]$*, and* $q(x) > 0$ *on* $[a, b]$*. Then (10) has a unique solution provided that* $h < 2/L$*, where* $L = \max_{a \le x \le b} |p(x)|$.

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2.2 – Convergence Analysis

We shall analyze that when h converges to zero, the solution u_i of the discrete problem (8) converges to the solution y_i of the original continuous problem (5).

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for $i = 1, 2, \ldots, n-1$.

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for $i = 1, 2, \ldots, n-1$. Subtract (8) from (11) and let $e_i = y_i - u_i$, the result is

$$\frac{e_{i+1}-2e_i+e_{i-1}}{h^2} = p_i\frac{e_{i+1}-e_{i-1}}{2h} + q_i e_i + h^2 g_i, \qquad i = 1, 2, \ldots, n-1,$$

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and, hence

$$\begin{aligned} \left|2 + h^2 q_k\right| |e_k| &\leq \left|1 + \frac{h}{2} p_k\right| |e_{k-1}| + \left|1 - \frac{h}{2} p_k\right| |e_{k+1}| + h^4 |g_k| \\ &\leq \left|1 + \frac{h}{2} p_k\right| \|e\|_\infty + \left|1 - \frac{h}{2} p_k\right| \|e\|_\infty + h^4 \|g\|_\infty \end{aligned}$$

When $q(x) > 0, \forall x \in [a, b]$ and h is chosen small enough so that $|\frac{h}{2} p_i| < 1, \forall i$, then the the above inequality induces

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Hence $\frac{\|g\|_\infty}{\inf q(x)}$ is a bound independent of h. Thus we can conclude that $\|e\|_\infty$ is $O(h^2)$ as $h \to 0$.

3 – Shooting Methods

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In general

$$z_{k+1} = z_k - \phi(z_k)\frac{z_k - z_{k-1}}{\phi(z_k) - \phi(z_{k-1})}.$$

☞ Special BVP:

$$\begin{cases} y'' = u(x) + v(x)y + w(x)y' \\ y(a) = \alpha, \quad y(b) = \beta \end{cases} \tag{14}$$

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BVP of ODE

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for some parameter λ, we can show

$$y'' = u + vy + wy'$$

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In practice, we can solve the following two IVPs (in parallel)

$$\begin{cases} y'' = u(x) + v(x)y + w(x)y' \\ y(a) = \alpha, \quad y'(a) = 0 \end{cases}$$

and

$$\begin{cases} y'' = u(x) + v(x)y + w(x)y' \\ y(a) = \alpha, \quad y'(a) = 1 \end{cases}$$

i.e.,

$$\begin{cases} y_1' = y_3 \\ {y_3}' = {y_1}'' = u + vy_1 + w{y_1}' = u + vy_1 + wy_3 \\ y_1(a) = \alpha, \quad y_3(a) = {y_1}'(a) = 0 \end{cases}$$

and

$$\begin{cases} {y_2}' = y_4 \\ {y_4}' = u + vy_2 + wy_4 \\ y_2(a) = \alpha, \quad y_4(a) = 1 \end{cases}$$

to obtain approximate solutions y_1 and y_2, then compute λ and form the solution y.