Iterative Methods for Solving Systems of Linear Equations

NTNU

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1 – Classic Iterative Methods

1.1 – Basic Concept

First of all we give an example to illustrate the process of iterative methods for solving systems of linear equations.

Consider solving

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

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This system has the exact solution $x_1 = x_2 = 1$. Equivalently we can write the system as

$$\begin{cases} 3x_1 + 2x_2 = 5 \\ x_1 + 4x_2 = 5 \end{cases}$$

This implies that

$$\begin{cases} x_1 = \frac{1}{3}(5 - 2x_2) \\ x_2 = \frac{1}{4}(5 - x_1) \end{cases}$$

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that is, to use the iterative formulation

$$\begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \end{bmatrix} \right)$$

If we choose the initial guess $x_1^{(0)} = x_2^{(0)} = 0$, we would obtain

$$\begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1.6667 \\ 1.2500 \end{bmatrix}$$

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and

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By repeating the process, we have the following table

k	3	4	5	6	7
$x_1^{(k)}$	1.1111	0.9722	1.0185	0.9954	1.0031
$x_2^{(k)}$	1.0417	0.9722	1.0000	0.9954	1.0012

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This suggests an iterative process

$$x^{(k)} = (I - M^{-1}A)x^{(k-1)} + M^{-1}b \equiv Tx^{(k-1)} + c,$$

where T is usually called the iteration matrix. The initial vector $x^{(0)}$ can be arbitrary or be chosen according to certain conditions.

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- ☞ $x^{(k)}$ is easily computed. More precisely, the system $Mx^{(k)} = y$ is easy to solve;
- ☞ the sequence $\{x^{(k)}\}$ converges rapidly to the exact solution.

1.2 – Richard’s Method

When we choose $M = I$ such that $A = I - (I - A)$, we obtain the iteration procedure

$$x^{(k)} = (I - A)x^{(k-1)} + b = x^{(k-1)} - Ax^{(k-1)} + b \equiv x^{(k-1)} + r^{(k-1)}.$$

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Algorithm 1 (Richard's Method)

for $k = 1, 2, \ldots$ **do**

for $i = 1, 2, \ldots, n$ **do**

$$r_i^{(k-1)} = b_i - \sum_{j=1}^{n} a_{ij} x_j^{(k-1)}$$

$$x_i^{(k)} = x_i^{(k-1)} + r_i^{(k-1)}$$

end for

end for

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With this method, the iteration matrix $T = -D^{-1}(L+U)$ and $c = D^{-1}b$. Each component $x_i^{(k)}$ can be computed by

$$x_i^{(k)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Bigg/ a_{ii}.$$

$$
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a_{11}x_1^{(k)} + a_{12}x_2^{(k-1)} + a_{13}x_3^{(k-1)} + \cdots + a_{1n}x_n^{(k-1)} &= b_1 \\
a_{21}x_1^{(k-1)} + a_{22}x_2^{(k)} + a_{23}x_3^{(k-1)} + \cdots + a_{2n}x_n^{(k-1)} &= b_2 \\
&\vdots \\
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for $k = 1, 2, \ldots$ **do**

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Only the components of $x^{(k-1)}$ are used to compute $x^{(k)}$.
$\Rightarrow x_i^{(k)}, i = 1, \ldots, n,$ can be computed in parallel at each iteration k.

1.4 – Gauss-Seidel Method

When computing $x_i^{(k)}$ for $i > 1$, $x_1^{(k)}, \ldots, x_{i-1}^{(k)}$ have already been computed and are likely to be better approximations to the exact $x_1, \ldots, x_{i-1}$ than $x_1^{(k-1)}, \ldots, x_{i-1}^{(k-1)}$.

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The Gauss-Seidel method sets $M = D + L$ and defines the iteration as

$$x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b.$$

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Hence each component $x_i^{(k)}$ can be computed by

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Algorithm 3 (Gauss-Seidel Method)

for $k = 1, 2, \ldots$ **do**

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The question of choosing a good relaxation parameter ω is a very complex topic.

1.6 – Symmetric Successive Over Relaxation (SSOR) Method

In theory the symmetric successive over relaxation (SSOR) method chooses the splitting matrix $M = \frac{1}{\omega(2-\omega)}(D + \omega L)D^{-1}(D + \omega U)$ and iterates with the iteration matrix

$$T = (D + \omega U)^{-1} ((1 - \omega)D - \omega L) (D + \omega L)^{-1} ((1 - \omega)D - \omega U) .$$

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The idea is in fact to implement the SOR formulation twice, one forward and one backward, at each iteration. That is, SSOR method defines

$$\begin{aligned}(D + \omega L)x^{(k-\frac{1}{2})} &= \left((1-\omega)D - \omega U\right)x^{(k-1)} + \omega b \\ (D + \omega U)x^{(k)} &= \left((1-\omega)D - \omega L\right)x^{(k-\frac{1}{2})} + \omega b\end{aligned}$$

Each component $x_i^{(k)}$ is obtained by first computing

$$x_i^{(k-\frac{1}{2})} = \omega \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k-\frac{1}{2})} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right) \Big/ a_{ii} + (1-\omega) x_i^{(k)}$$

followed by

$$x_i^{(k)} = \omega \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-\frac{1}{2})} \right) \Big/ a_{ii} + (1-\omega) x_i^{(k-\frac{1}{2})}.$$

2 – Convergence Analysis

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Next we show that $(I - A)^{-1} = I + A + A^2 + \cdots .$

Iterative Methods for LS

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Lemma 3 *Suppose that* $A \in \mathbb{R}^{n\times n}$ *and* $\|\cdot\|$ *is a subordinate matrix norm. If* $\|A\| < 1$, *then* $I - A$ *is nonsingular and*

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$$(I-A)\left(\sum_{k=0}^{\infty} A^k\right) = (I-A)\left(\lim_{m\to\infty}\sum_{k=0}^{m} A^k\right) = I - \lim_{m\to\infty} A^{m+1} = I.$$

This shows that $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.

Finally, since $\|A\| < 1$,

$$\|(I - A)^{-1}\| = \left\| \sum_{k=0}^{\infty} A^k \right\| \leq \sum_{k=0}^{\infty} \|A^k\| \leq \sum_{k=0}^{\infty} \|A\|^k = \frac{1}{1 - \|A\|}.$$

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Theorem 2 *The following statements are equivalent.*

1. A *is a convergent matrix, i.e.,* $A^k \to 0$ *as* $k \to \infty$;
2. $\lim_{k\to\infty} \|A^k\| = 0$ *for some subordinate matrix norm;*
3. $\lim_{k\to\infty} \|A^k\| = 0$ *for all subordinate matrix norm;*
4. $\rho(A) < 1$;
5. $\lim_{k\to\infty} A^k x = 0$ *for any* x.

Iterative Methods for LS

Theorem 3 *For any* $x^{(0)} \in \mathbb{R}^n$*, the sequence produced by*

$$x^{(k)} = Tx^{(k-1)} + c, \quad k = 1, 2, \ldots,$$

converges to the unique solution of $x = Tx + c$ *if and only if*

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$$\begin{aligned}
x^{(1)} &= Tx^{(0)} + c \\
x^{(2)} &= Tx^{(1)} + c = T^2x^{(0)} + (T + I)c \\
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In general

$$x^{(k)} = T^k x^{(0)} + (T^{k-1} + T^{k-2} + \cdots T + I)c.$$

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It follows from theorem $\rho(T) < 1$. ■

Iterative Methods for LS

Corollary 1 *If* $\|T\| < 1$ *for some subordinate matrix norm, then the sequence produced by*

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Proof: Since $x = Tx + c$ and $x^{(k)} = Tx^{(k-1)} + c$,

$$\begin{aligned} x - x^{(k)} &= Tx + c - Tx^{(k-1)} - c \\ &= T(x - x^{(k-1)}) \\ &= T^2(x - x^{(k-2)}) = \cdots\cdots = T^k(x - x^{(0)}). \end{aligned}$$

This theorem implies that we can stop the iteration if $\|x^{(k)} - x^{(k-1)}\|$ is less than a small tolerance.

Theorem 5 *If* $\|T\| < 1$*, then the sequence* $x^{(k)}$ *converges to* x *for any initial* $x^{(0)}$ *and*

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The first statement can then be derived

$$\|x - x^{(k)}\| = \|T^k(x - x^{(0)})\| \le \|T\|^k \|x - x^{(0)}\|.$$

Iterative Methods for LS

For the second result, we first show that

$$\|x^{(n)} - x^{(n-1)}\| \leq \|T\|^{n-1} \|x^{(1)} - x^{(0)}\| \text{ for any } n \geq 1.$$

Iterative Methods for LS

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Since

$$\begin{aligned} x^{(n)} - x^{(n-1)} &= Tx^{(n-1)} + c - Tx^{(n-2)} - c \\ &= T\big(x^{(n-1)} - x^{(n-2)}\big) \\ &= T^2\big(x^{(n-2)} - x^{(n-3)}\big) = \cdots\cdots = T^{n-1}\big(x^{(1)} - x^{(0)}\big), \end{aligned}$$

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we have

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Let $m \geq k$,

$$\begin{aligned} & x^{(m)} - x^{(k)} \\ = & \left(x^{(m)} - x^{(m-1)}\right) + \left(x^{(m-1)} - x^{(m-2)}\right) + \cdots + \left(x^{(k+1)} - x^{(k)}\right) \\ = & T^{m-1}\left(x^{(1)} - x^{(0)}\right) + T^{m-2}\left(x^{(1)} - x^{(0)}\right) + \cdots + T^{k}\left(x^{(1)} - x^{(0)}\right) \\ = & \left(T^{m-1} + T^{m-2} + \cdots T^{k}\right)\left(x^{(1)} - x^{(0)}\right), \end{aligned}$$

hence

$$
\begin{aligned}
& \|x^{(m)} - x^{(k)}\| \\
\leq \quad & \left(\|T\|^{m-1} + \|T\|^{m-2} + \cdots + \|T\|^{k}\right) \|x^{(1)} - x^{(0)}\|
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Iterative Methods for LS

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= \quad & \|T\|^{k} \frac{1}{1 - \|T\|} \|x^{(1)} - x^{(0)}\|.
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This proves the second result. ■

Iterative Methods for LS

Theorem 6 *If A is strictly diagonal dominant, then both the Jacobi and Gauss-Seidel methods converges for any initial vector $x^{(0)}$.*

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For Jacobi method, the iteration matrix $T_J = -D^{-1}(L+U)$ has entries

$$[T_J]_{ij} = \begin{cases} -\frac{a_{ij}}{a_{ii}} & i \neq j \\ 0 & i = j \end{cases}$$

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Hence

$$\|T_J\|_\infty = \max_{1\le i\le n} \sum_{j=1, j\neq i}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| = \max_{1\le i\le n} \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1,$$

and this implies that the Jacobi method converges.

Iterative Methods for LS

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Hence for $i = 1, \ldots, n$,

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$$|\lambda||a_{ii}||y_i| \le |\lambda| \sum_{j=1}^{i-1} |a_{ij}||y_j| + \sum_{j=i+1}^{n} |a_{ij}||y_j|.$$

Choose the index k such that $|y_k| = 1 \geq |y_j|$ (this index can always be found since $\|y\|_\infty = 1$).

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Since λ is arbitrary, $\rho(T_{GS}) < 1$.

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Since λ is arbitrary, $\rho(T_{GS}) < 1$. This means the Gauss-Seidel method converges. ■

Theorem 7 *If A is positive definite and the relaxation parameter ω satisfying $0 < \omega < 2$, then the SOR iteration converges for any initial vector $x^{(0)}$.*

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Theorem 8 *If A is positive definite and tridiagonal, then $\rho(T_{GS}) = [\rho(T_J)]^2 < 1$ and the optimal choice of ω for the SOR iteration is*

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_J)]^2}}.$$

With this choice of ω, $\rho(T_{SOR}) = \omega - 1$.