Solutions of Non-linear Equations in One Variable

NTNU

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$$|x_{n+1} - x^*| \le c|x_n - x^*|, \quad \forall\, n \ge N;$$

2. *superlinear if* $\exists$ $\{c_n\}$, $c_n \to 0$ *as* $n \to \infty$, *and an integer* $N > 0$ *such that*

$$|x_{n+1} - x^*| \le c_n|x_n - x^*|, \quad \forall\, n \ge N,$$

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3. *quadratic if* $\exists$ *a constant* $c > 0$ *(not necessarily less than 1) and an integer* $N > 0$ *such that*

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^2, \quad \forall\, n \geq N.$$

In general, if there are positive constants c and α and an integer $N > 0$ such that

$$|x_{n+1} - x^*| \leq c|x_n - x^*|^{\alpha}, \quad \forall\, n \geq N,$$

then we say the rate of convergence is of order α.

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Example 1 *Compare the convergence behavior of $\{x_n\}$ and $\{y_n\}$, where*

$$x_n = \frac{n+1}{n^2}, \quad \text{and} \quad y_n = \frac{n+3}{n^3}.$$

Solution: Note that both

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Let $\alpha_n = \frac{1}{n}$ and $\beta_n = \frac{1}{n^2}$. Then

$$\begin{aligned} |x_n - 0| &= \frac{n+1}{n^2} \le \frac{n+n}{n^2} = \frac{2}{n} = 2\alpha_n, \\ |y_n - 0| &= \frac{n+3}{n^3} \le \frac{n+3n}{n^3} = \frac{4}{n^2} = 4\beta_n. \end{aligned}$$

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This shows that $\{y_n\}$ converges to 0 much faster than $\{x_n\}$. ■

2 – Bisection Method

Idea: if $f(x) \in C[a, b]$ and $f(a)f(b) < 0$, then $\exists\, c \in (a, b)$ such that $f(c) = 0$.

Algorithm 1 (Bisection Method) *Given $f(x)$ defined on (a, b), the maximal number of iterations M, and stop criteria δ and ε, this algorithm tries to locate one root of $f(x)$.*

compute $u = f(a)$, $v = f(b)$, and $e = b - a$.
if $sign(u) = sign(v)$, **then** stop
for $k = 1, 2, \ldots, M$ **do**
  $e = e/2$, $c = a + e$, $w = f(c)$.
  if $|e| < \delta$ or $|w| < \varepsilon$, **then** stop
  if $sign(w) \neq sign(u)$ **then**
    $b = c$, $v = w$.
  else
    $a = c$, $u = w$
  end if
end for

Sol. Non-linear Fun.

Let $\{c_n\}$ be the sequence of numbers produced. The algorithm should stop if one of the following conditions is satisfied.

1. the iteration number $k > M$,
2. $|c_k - c_{k-1}| < \delta$, or
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$\Rightarrow \lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.

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Since

$$
\begin{aligned}
b_1 - a_1 &= \frac{1}{2}(b_0 - a_0) \\
b_2 - a_2 &= \frac{1}{2}(b_1 - a_1) = \frac{1}{4}(b_0 - a_0) \\
&\vdots \\
b_n - a_n &= \frac{1}{2^n}(b_0 - a_0)
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Therefore

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Since f is a continuous function

$$\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(z) \quad \text{and} \quad \lim_{n\to\infty} f(b_n) = f(\lim_{n\to\infty} b_n) = f(z).$$

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Let $c_n = \frac{1}{2}(a_n + b_n)$. Then

$$\begin{aligned}
|z - c_n| &= \left|\lim_{n\to\infty} a_n - \frac{1}{2}(a_n + b_n)\right| \\
&= \left|\frac{1}{2}\left[\lim_{n\to\infty} a_n - b_n\right] + \frac{1}{2}\left[\lim_{n\to\infty} a_n - a_n\right]\right| \\
&\leq \frac{1}{2}\max\left\{\left|\lim_{n\to\infty} a_n - b_n\right|, \left|\lim_{n\to\infty} a_n - a_n\right|\right\} \\
&\leq \frac{1}{2}|b_n - a_n| = \frac{1}{2^{n+1}}|b_0 - a_0|.
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&\leq \frac{1}{2}\max\left\{\left|\lim_{n\to\infty} a_n - b_n\right|, \left|\lim_{n\to\infty} a_n - a_n\right|\right\} \\
&\leq \frac{1}{2}|b_n - a_n| = \frac{1}{2^{n+1}}|b_0 - a_0|.
\end{aligned}$$

This proves the following theorem.

Theorem 1 *Let* $\{[a_n, b_n]\}$ *denote the intervals produced by the bisection algorithm. Then* $\lim_{n\to\infty} a_n$ *and* $\lim_{n\to\infty} b_n$ *exist, are equal, and represent a zero of* $f(x)$. *If*

$$z = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n \quad \text{and} \quad c_n = \frac{1}{2}(a_n + b_n),$$

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Example 2 *If bisection method starts with interval* $[50, 75]$, *then how many steps should be taken to compute a root with relative error that is less than* 10^{-12}?

Solution: Seek an n such that

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That is, we solve

$$2^{-(n+1)}(75 - 50) \leq 50 \times 10^{-12}$$

for n, which gives $n \geq 38$. ■

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$$\begin{aligned} 0 = f(x^*) = f(x+h) &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots \\ &= f(x) + f'(x)h + O(h^2). \end{aligned}$$

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Since h is small, $O(h^2)$ is negligible. It is reasonable to drop $O(h^2)$ terms. This implies

$$f(x) + f'(x)h \approx 0 \quad \text{and} \quad h \approx -\frac{f(x)}{f'(x)}, \quad \text{if} \quad f'(x) \neq 0.$$

Hence

$$x + h = x - \frac{f(x)}{f'(x)}$$

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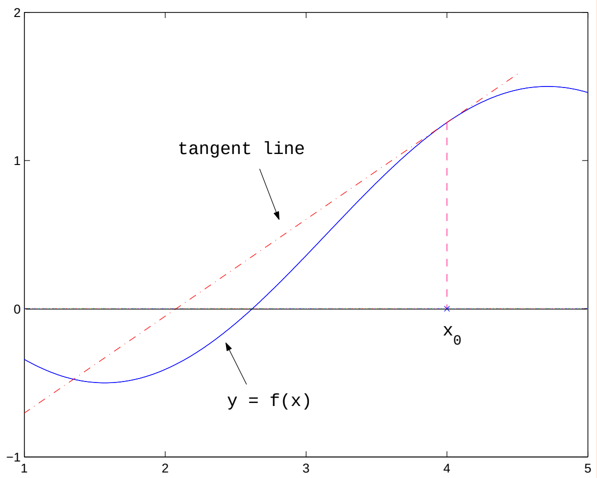

Figure 1: Newton-Rapbson method

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$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(x_k)(x - x_k)^2 + \cdots .$$

At x_k, one uses the tangent line

$$y = \ell(x) = f(x_k) + f'(x_k)(x - x_k)$$

to approximate the curve of $f(x)$ and uses the zero of the tangent line to approximate the zero of $f(x)$.

Algorithm 2 (Newton's Method) *Given a function $f : \mathbb{R} \to \mathbb{R}$, an initial guess x_0 to the zero of f, and stop criteria M, δ, and ε, this algorithm performs the Newton's iteration to approximate one root of f.*

$u = f(x_0)$
$v = f'(x_0)$
$x_1 = x_0 - \frac{u}{v}$
$k = 1$
$u = f(x_k)$
while $(k < M)$ and $(|x_k - x_{k-1}| \geq \delta)$ and $(|f(x_k)| \geq \varepsilon$ **do**
    $v = f'(x_k)$
    $x_{k+1} = x_k - \frac{u}{v}$
    $k = k + 1$
    $u = f(x_k)$
end while

3.2 – Convergence Analysis

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$$D = \{x; |x - x^*| \leq \delta\}$$

and

$$\gamma = \frac{1}{2} \cdot \frac{\max_{x \in D} |f''(x)|}{\min_{x \in D} |f'(x)|}.$$

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Choose δ such that $\rho = \delta\gamma < 1$.

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Suppose that f'' is continuous and x^* is a simple zero of f, i.e., $f(x^*) = 0$ but $f'(x^*) \neq 0$. Choose $\delta > 0$ and let

$$D = \{x; |x - x^*| \leq \delta\}$$

and

$$\gamma = \frac{1}{2} \cdot \frac{\max_{x \in D} |f''(x)|}{\min_{x \in D} |f'(x)|}.$$

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In general,

$$|e_k| \le \rho |e_{k-1}| \le \cdots \le \rho^k |e_0|.$$

Since $\rho < 1$, $|e_k| \to 0$ as $k \to 0$, that is, $x_k \to x^*$.

Sol. Non-linear Fun.

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To investigate the convergence rate, we start with

$$e_{k+1} = x_{k+1} - x^* = x_k - \frac{f(x_k)}{f'(x_k)} - x^* = \frac{f'(x_k)e_k - f(x_k)}{f'(x_k)}.$$

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$$f'(x_k)e_k - f(x_k) = \frac{1}{2}f''(\xi_k)e_k^2.$$

Hence

$$|e_{k+1}| = \frac{|f''(\xi_k)|}{2|f'(x_k)|}|e_k|^2 \approx \frac{|f''(x^*)|}{2|f'(x^*)|}|e_k|^2 \equiv C|e_k|^2.$$

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Theorem 2 *Assume* $f(x^*) = 0,\ f'(x^*) \neq 0$ *and* $f(x)$, $f'(x)$ *and* $f''(x)$ *are continuous on* $N_\varepsilon(x^*)$. *Then if* x_0 *is chosen sufficiently close to* x^*, *then*

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Moreover,

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Definition 3 (Lipschitz Continuous) *A function* $f(x)$ *is Lipschitz continuous with Lipschitz constant* γ *in a set* X, *written* $f \in \mathcal{L}ip_\gamma(X)$, *if*

$$|f(x) - f(y)| \leq \gamma|x - y|, \quad \text{for all } x, y \in X.$$

Lemma 1 *Suppose* $f : \Omega \to \mathbb{R}$ *for some open interval* $\Omega \subseteq \mathbb{R}$ *and* $f' \in \mathcal{L}ip_\gamma(\Omega)$. *Then for all* $x, y \in \Omega$,

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By changing variable $u = x + t(y - x)$, $du = (y - x)\, dt$ and $f' \in \mathcal{L}ip_\gamma(\Omega)$, we have

$$\begin{aligned} |f(y) - f(x) - f'(x)(y - x)| &= \left| \int_0^1 [f'(x + t(y - x)) - f'(x)]\, (y - x)\, dt \right| \\ &\leq |y - x| \int_0^1 \gamma |t(y - x)|\, dt \\ &= \frac{\gamma}{2} |y - x|^2. \end{aligned}$$

■

Theorem 3 *Let* $f : \Omega \to \mathbb{R}$ *for some open interval* $\Omega \subseteq \mathbb{R}$*. Assume*

1. $\exists\, x^* \in \Omega$ *such that* $f(x^*) = 0$;
2. $f' \in \mathcal{L}ip_\gamma(\Omega)$;
3. $\exists\, \rho > 0$ *such that* $|f'(x)| \geq \rho \; \forall\, x \in \Omega$, *that is,* $f'(x) \neq 0 \; \forall\, x \in \Omega$.

Then $\exists\, \eta > 0$ *such that if* $|x_0 - x^*| < \eta$, *then*

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, 2, \ldots,$$

converges to x^**. Furthermore,*

$$|x_{k+1} - x^*| \leq \frac{\gamma}{2\rho} |x_k - x^*|^2$$

or, equivalently,

$$|e_{k+1}| \leq \frac{\gamma}{2\rho} |e_k|^2.$$

Proof: With the result of previous Lemma,

$$\begin{aligned}
|x_{k+1} - x^*| &= \left| x_k - \frac{f(x_k)}{f'(x_k)} - x^* \right| \\
&= \left| x_k - x^* - \frac{f(x_k) - f(x^*)}{f'(x_k)} \right| \\
&= \frac{1}{|f'(x_k)|} \left| f(x^*) - f(x_k) - f'(x_k)(x_k - x^*) \right| \\
&\le \frac{\gamma}{2|f'(x_k)|} |x_k - x^*|^2 \\
&\le \frac{\gamma}{2\rho} |x_k - x^*|^2.
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(ii) If $f'(x^*) = 0$, i.e., x^* is a multiple root of f, then Newton's method converges only linearly.

(iii) When f is a linear function, Newton's method will converge in one step.

3.3 – Examples

Example 3 *The following table shows the convergence behavior of Newton's method applied to solving* $f(x) = x^2 - 1 = 0$*. Observe the quadratic convergence rate.*

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k	x_k	$\|e_k\|$
0	2.0	1
1	1.25	0.25
2	1.025	2.5e-2
3	1.0003048780488	3.048780488e-4
4	1.0000000464611	4.64611e-8
5	1.0	0

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Example 4 *Newton's method will fail (the sequence* $\{x_k\}$ *does not converge) on solving* $f(x) = x^2 - 4x + 5 = 0$ *since* f *does not have any real root.*

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Example 5 *When Newton's method applied to* $f(x) = \cos x$ *with starting point* $x_0 = 3$*, which is close to the root* $\frac{\pi}{2}$ *of* f*, it produces* $x_1 = -4.01525, x_2 = -4.8526, \cdots$ *, which converges to another root* $-\frac{3\pi}{2}$*.*

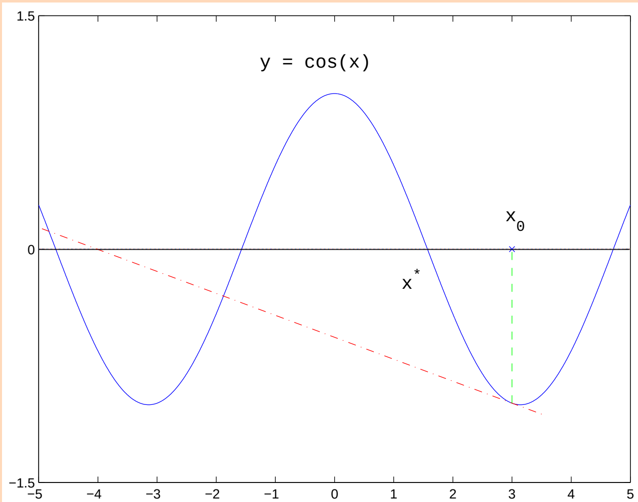

Figure 2: one step of Newton method

Example 6 *When Newton's method applied to* $f(x) = x^3 - x - 3$ *with starting point* $x_0 = 0$*, it produces*

$x_1 = -3$	$x_2 = -1.961538$
$x_3 = -1.147176$	$x_4 = -0.00679$
$x_5 = -3.000389$	$x_6 = -1.961818$
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$$x_1 = 1.727272,$$

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The sequence converges to the root 1.671699881*. This example illustrates that the starting point* x_0 *must be close enough to the zero of* f*.*

4 – Quasi-Newton's Method (Secant Method)

4.1 – The Secant Method

Newton's iteration:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

4 – Quasi-Newton's Method (Secant Method)

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Since

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

$\Rightarrow$

$$f'(x_k) \approx \frac{f(x_k + h_k) - f(x_k)}{h_k} \quad \text{for some small} \quad h_k.$$

$\Rightarrow$ The finite-difference Newton's iteration

$$x_{k+1} = x_k - f(x_k)\frac{h_k}{f(x_k + h_k) - f(x_k)}$$

⇒ The finite-difference Newton's iteration

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From geometric point of view, we use a secant line through x_k and a near-by point $x_k + h_k$ instead of the tangent line to approximate the function at the point x_k.

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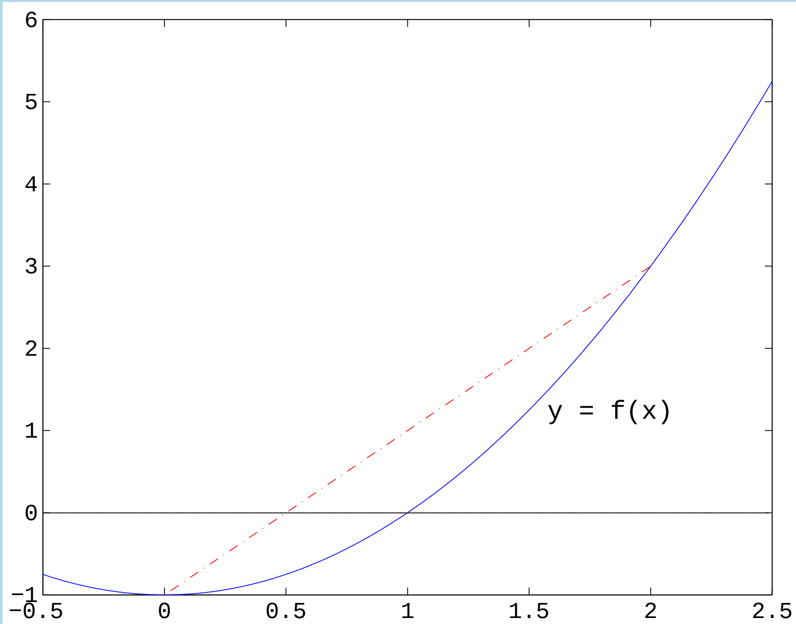

Figure 3: Secant method

From geometric point of view, we use a secant line through x_k and a near-by point $x_k + h_k$ instead of the tangent line to approximate the function at the point x_k.

Sol. Non-linear Fun.

The slope of the secant line is

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The zero of the secant line

$$x = x_k - \frac{f(x_k)}{s_k} = x_k - f(x_k)\frac{h_k}{f(x_k + h_k) - f(x_k)}$$

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Set

$$h_k = x_k - x_{k-1}.$$

This leads to the so-called secant method or quasi-Newton method.

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}.$$

Sol. Non-linear Fun.

Note that the secant method requires two initial guesses x_0 and x_{-1}.

Example 7 *The following table shows the convergence history for the finite-difference Newton's method with $h_k = 10^{-7} \times x_k$ and secant method for solving $f(x) = x^2 - 1 = 0$.*

	finite-difference Newton	*secant method*
x_0	*2*	*2*
x_1	*1.2500000266453*	*1.2500000266453*
x_2	*1.0250000179057*	*1.0769230844910*
x_3	*1.0003048001120*	*1.0082644643823*
x_4	*1.0000000464701*	*1.0003048781354*
x_5	*1*	*1.0000012544523*
x_6		*1.0000000001912*
x_7		*1*

4.2 – Error Analysis of Secant Method

Let x^* denote the exact solution of $f(x) = 0$, $e_k = x_k - x^*$ be the errors at the k-th step. Then

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Sol. Non-linear Fun.

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Since $x_k - x_{k-1} = e_k - e_{k-1}$ and

$$\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \longrightarrow \frac{1}{f'(x^*)},$$

we have

$$\begin{aligned} e_{k+1} &\approx e_k e_{k-1} \left(\frac{\frac{1}{2}(e_k - e_{k-1}) f''(x^*)}{e_k - e_{k-1}} \cdot \frac{1}{f'(x^*)} \right) = \frac{1}{2} \frac{f''(x^*)}{f'(x^*)} e_k e_{k-1} \\ &\equiv C e_k e_{k-1} \end{aligned} \tag{1}$$

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Remark 2 *Two steps of secant method would require a little more work than one step of Newton's method.*

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Two questions arise: "When does a function have a fixed point?" and "How to find a fixed point?".

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Fixed-point iteration or **functional iteration**: Given a continuous function f, choose an initial point x_0 and generate $\{x_k\}_{k\geq 0}$ by

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Note that Newton's method for solving $g(x) = 0$

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}$$

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Definition 5 *A function (mapping)* f *is said to be contractive if there exists a constant* $0 \leq \lambda < 1$ *such that*

$$|f(x) - f(y)| \leq \lambda |x - y|$$

for all x, y *in the domain of* f*.*

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Theorem 4 (Contractive Mapping Theorem) *Suppose $f : D \rightarrow D$, where $D \subseteq \mathbb{R}$ is a closed set, is a contractive mapping. Then f has a unique fixed point in D. Moreover, this fixed point is the limit of every sequence obtained by*

$$x_{k+1} = f(x_k)$$

with any initial point x_0.

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$\{x_k\}_{k\geq 0}$ converges if and only if $\sum_{i=1}^{\infty} (x_i - x_{i-1})$ converges and it is sufficient to show $\sum_{i=1}^{\infty} |x_i - x_{i-1}|$ converges.

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Since f is contractive, we have

$$\begin{aligned} |x_i - x_{i-1}| &= |f(x_{i-1}) - f(x_{i-2})| \\ &\leq \lambda|x_{i-1} - x_{i-2}| \\ &\leq \lambda^2|x_{i-2} - x_{i-3}| \\ &\vdots \\ &\leq \lambda^{i-1}|x_1 - x_0|. \end{aligned}$$

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Sol. Non-linear Fun.

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Theorem 5 *If $f \in C[a, b]$ such that $a \leq f(x) \leq b$ for all $x \in [a, b]$, then f has a fixed point in $[a, b]$. Suppose, in addition, that $f'(x)$ exists in (a, b) and there exists a positive constant $M < 1$ such that $|f'(x)| \leq M < 1$ for all $x \in (a, b)$. Then the fixed point is unique.*

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5.2 – Convergence Analysis

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Let $e_k = x_k - x^*$. Then by the Mean-Value theorem

$$e_{k+1} = x_{k+1} - x^* = f(x_k) - f(x^*) = f'(\xi_k)(x_k - x^*) = f'(\xi_k)e_k, \quad (2)$$

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Sol. Non-linear Fun.

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Therefore we have

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Therefore we have

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This shows that the convergence rate of the functional iteration is m.