Initial-Value Problems for Ordinary Differential Equations

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Outline

1. **Euler's Method**
   - Algorithm
   - Error analysis
2. **Higher-order Taylor methods**
   - Taylor methods

Obtain an approximation to the initial-value problem

$$\frac{dy}{dt} = f(t,y), \quad a \le t \le b, \quad y(a) = \alpha.$$

Subdivide $[a,b]$ into n subintervals of equal length $h = (b-a)/n$ with mesh points $\{t_0, t_1, \ldots, t_n\}$ where

$$t_i = a + ih, \ \forall\ i = 0,1,2,\ldots,n.$$

Recall the Taylor's Theorem

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + (t_{i+1}-t_i)y'(t_i) + \frac{(t_{i+1}-t_i)^2}{2} y''(\xi_i) \\ &= y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i) \\ &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i) \end{aligned} \tag{1}$$

for some $\xi_i \in (t_i, t_{i+1})$.

We have the formulation of Euler's method

$$\begin{aligned} t_{k+1} &= t_k + h, \\ y_{k+1} &= y_k + hf(t_k, y_k), \; y_0 = \alpha. \end{aligned}$$

Example

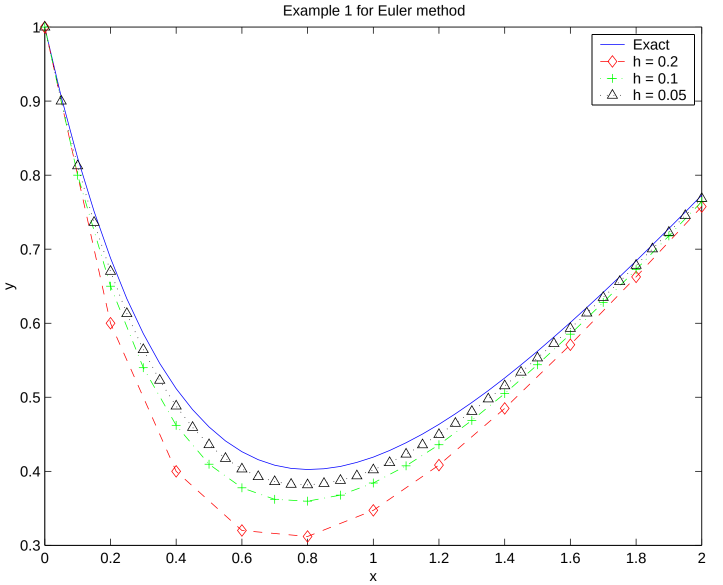
Use Euler's method to integrate

$$\frac{dy}{dx} = x - 2y, \quad y(0) = 1.$$

The exact solution is

$$y = \frac{1}{4}\left[2x - 1 + 5e^{-2x}\right].$$

Algorithm


Example 1 for Euler method
Exact
h = 0.2
h = 0.1
h = 0.05
y
x
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Lemma

$$0 \leq (1+x)^m \leq e^{mx}, \ \forall\, x \geq -1, \ m > 0.$$

Proof: Applying Taylor's Theorem,

$$e^x = 1 + x + \frac{1}{2}x^2 e^{\xi},$$

where ξ is between x and zero. Thus

$$\begin{aligned} & 0 \leq 1 + x \leq 1 + x + \frac{1}{2}x^2 e^{\xi} = e^x \\ \Rightarrow \quad & 0 \leq (1+x)^m \leq e^{mx} \end{aligned}$$

∎

Lemma

If $s, t \in \mathbb{R}^+$, $\{a_i\}_{i=0}^k$ *is a sequence satisfying* $a_0 \geq -t/s$, *and*

$$a_{i+1} \leq (1+s)a_i + t, \ \forall\, i = 0, 1, \ldots, k,$$

then

$$a_{i+1} \leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}.$$

Proof:

$$
\begin{aligned}
a_{i+1} &\leq (1+s)a_i + t \\
&\leq (1+s)\left[(1+s)a_{i-1} + t\right] + t \\
&\leq (1+s)\left\{(1+s)\left[(1+s)a_{i-2} + t\right] + t\right\} + t \\
&\vdots \\
&\leq (1+s)^{i+1}a_0 + \left[1 + (1+s) + (1+s)^2 + \cdots (1+s)^i\right] t \\
&= (1+s)^{i+1}a_0 + \frac{1-(1+s)^{i+1}}{1-(1+s)} t \\
&= (1+s)^{i+1}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s} \\
&\leq e^{(i+1)s}\left(a_0 + \frac{t}{s}\right) - \frac{t}{s}
\end{aligned}
$$

■

Theorem

Suppose $f \in C(D)$ and satisfies a Lipschitz condition with constant L on

$$D = \{(t, y) | a \leq t \leq b, -\infty < y < \infty\}$$

and $\exists$ M with

$$|y''(t)| \leq M, \; \forall \; t \in [a, b].$$

Let $y(t)$ denote the unique solution to (IVP)

$$y' = f(t, y), \; a \leq t \leq b, \; y(a) = \alpha,$$

and $y_0, y_1, \ldots, y_n$ be the approximations generated by Euler's method. Then

$$|y(t_i) - y_i| \leq \frac{hM}{2L} \left[e^{L(t_i - a)} - 1 \right], \; \forall \; i = 0, 1, \ldots, n.$$

Proof: Since $y(t_0) = y_0 = \alpha$, it is true for $i = 0$.
For $i = 0, 1, \ldots, n-1$,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} y''(\xi_i)$$

and

$$y_{i+1} = y_i + hf(t_i, y_i).$$

Consequently,

$$y(t_{i+1}) - y_{i+1} = y(t_i) - y_i + h\left[f(t_i, y(t_i)) - f(t_i, y_i)\right] + \frac{h^2}{2} y''(\xi_i)$$

and

$$\begin{aligned}
|y(t_{i+1}) - y_{i+1}| &\leq |y(t_i) - y_i| + h|f(t_i, y(t_i)) - f(t_i, y_i)| + \frac{h^2}{2}|y''(\xi_i)| \\
&\leq (1 + hL)|y(t_i) - y_i| + \frac{h^2 M}{2}
\end{aligned}$$

Referring to previous lemma and letting $s = hL$, $t = h^2M/2$ and $a_j = |y(t_j) - y_j|\ \forall\ j = 0, 1, \ldots, n$, we see that

$$\begin{aligned} |y(t_{i+1}) - y_{i+1}| &\le e^{(i+1)hL}\left(|y(t_0) - y_0| + \frac{h^2M}{2hL}\right) - \frac{h^2M}{2hL}. \\ &= \frac{hM}{2L}\left(e^{(i+1)hL} - 1\right) = \frac{hM}{2L}\left(e^{(t_{i+1}-a)L} - 1\right) \end{aligned}$$

since $(i+1)h = t_{i+1} - t_0 = t_{i+1} - a$. ∎

Definition (Local truncation error)

The difference method

$$\begin{aligned} y_0 &= \alpha, \\ y_{i+1} &= y_i + h\phi(t_i, y_i), \ \forall\ i = 0, 1, \ldots, n-1, \end{aligned}$$

has local truncation error

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y(t_{i+1}) - [y(t_i) + h\phi(t_i, y(t_i))]}{h} \\ &= \frac{y(t_{i+1}) - y(t_i)}{h} - \phi(t_i, y(t_i)), \end{aligned}$$

$\forall\ i = 0, 1, \ldots, n-1.$

For example, the local truncation error in Euler's method at *i*th step is

$$
\begin{aligned}
\tau_{i+1}(h) &= \frac{y(t_{i+1}) - y(t_i)}{h} - f(t_i, y(t_i)) \\
&= \frac{\left[y(t_i) + hy'(t_i) + h^2 y''(\xi_i)\right] - y(t_i)}{h} - f(t_i, y(t_i)) \\
&= \frac{h}{2} y''(\xi_i) \ \text{ for some } \ \xi_i \in (t_i, t_{i+1}).
\end{aligned}
$$

If $|y''(t)| \leq M \ \forall \ t \in [a, b]$, then

$$
|\tau_{i+1}(h)| \leq \frac{h}{2} M,
$$

so the local truncation error in Euler's method is $O(h)$.

To improve the convergence of difference methods, one way is selected difference-equations in such that their local truncation errors are $O(h^p)$ for as large a value of p as possible.

Suppose the solution y to (IVP) has $(n+1)$ continuous derivatives. Consider the nth Taylor polynomial of $y(t)$ at t_i,

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) \\ &\quad + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i) \end{aligned}$$

for some $\xi_i \in (t_i, t_{i+1})$. Since

$$\begin{aligned} y'(t) &= f(t,y), \\ y''(t) &= f'(t,y), \\ &\vdots \\ y^{(k)}(t) &= f^{(k-1)}(t,y), \end{aligned}$$

we get

$$\begin{aligned} y(t_{i+1}) &= y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2} f'(t_i, y(t_i)) + \cdots \qquad (2) \\ &+ \frac{h^n}{n!} f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)). \qquad (3) \end{aligned}$$

Taylor method of order n

$$\begin{aligned} y_0 &= \alpha, \\ y_{i+1} &= y_i + hT^{(n)}(t_i, y_i), \ \forall\, i = 0, 1, \ldots, n-1, \end{aligned}$$

where

$$T^{(n)}(t_i, y_i) = f(t_i, y_i) + \frac{h}{2} f'(t_i, y_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, y_i).$$

Example

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Consider Taylor's method of order two and four.

$$\begin{aligned}
f(t,y) &= y - t^2 + 1, \\
f'(t,y) &= \frac{d}{dt}\left(y - t^2 + 1\right) = y' - 2t = y - t^2 + 1 - 2t, \\
f''(t,y) &= \frac{d}{dt}\left(y - t^2 + 1 - 2t\right) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1, \\
f'''(t,y) &= \frac{d}{dt}\left(y - t^2 - 2t - 1\right) = y' - 2t - 2 \\
&= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1.
\end{aligned}$$

So

$$\begin{aligned} T^{(2)}(t_i, y_i) &= f(t_i, y_i) + \frac{h}{2} f'(t_i, y_i) \\ &= y_i - t_i^2 + 1 + \frac{h}{2}\left(y_i - t_i^2 - 2t_i + 1\right) \\ &= \left(1 + \frac{h}{2}\right)\left(y_i - t_i^2 + 1\right) - ht_i \end{aligned}$$

and

$$\begin{aligned} T^{(4)}(t_i, y_i) &= f(t_i, y_i) + \frac{h}{2} f'(t_i, y_i) + \frac{h^2}{6} f''(t_i, y_i) + \frac{h^3}{24} f'''(t_i, y_i) \\ &= y_i - t_1^2 + 1 + \frac{h}{2}\left(y_i - t_i^2 - 2t_i + 1\right) \\ &\quad + \frac{h^2}{6}\left(y_i - t_i^2 - 2t_i - 1\right) + \frac{h^3}{24}\left(y_i - t_i^2 - 2t_i - 1\right) \end{aligned}$$

That is

$$T^{(4)}(t_i, y_i) = \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(y_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right) h t_i$$
$$+1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}.$$

The Taylor methods of orders two and four are, consequently,

$$y_0 = 0.5,$$
$$y_{i+1} = y_i + h\left[\left(1 + \frac{h}{2}\right)\left(y_i - t_i^2 + 1\right) - h t_i\right]$$

and

$$y_0 = 0.5,$$
$$y_{i+1} = y_i + h\left[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(y_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right) h t_i\right.$$
$$\left. +1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right].$$

If $h = 0.2$, then $n = 10$ and $t_i = 0.2i \;\forall\; i = 1, 2, \ldots, 10$.

- The second-order method:

$$\begin{aligned} y_0 &= 0.5, \\ y_{i+1} &= y_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)\left(y_i - 0.04i^2 + 1\right) - 0.04i\right] \\ &= 1.22y_i - 0.0088i^2 - 0.008i + 0.22. \end{aligned}$$

- The fourth-order method:

$$\begin{aligned} y_{i+1} &= y_i + 0.2\left[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)\left(y_i - 0.04i^2\right)\right. \\ &\quad \left. - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\right] \\ &= 1.2214y_i - 0.008856i^2 - 0.00856i + 0.2186. \end{aligned}$$

- Exact solution $y(t) = (t+1)^2 - 0.5e^t$.

t_i	Exact $y(t_i)$	Taylor order 2 w_i	Error $\lvert y(t_i) - w_i \rvert$	Taylor order 4 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8300000	0.0007014	0.8293000	0.0000014
0.4	1.2140877	1.2158000	0.0017123	1.2140910	0.0000034
0.6	1.6489406	1.6520760	0.0031354	1.6489468	0.0000062
0.8	2.1272295	2.1323327	0.0051032	2.1272396	0.0000101
1.0	2.6408591	2.6486459	0.0077868	2.6408744	0.0000153
1.2	3.1799415	3.1913480	0.0114065	3.1799640	0.0000225
1.4	3.7324000	3.7486446	0.0162446	3.7324321	0.0000321
1.6	4.2834838	4.3061464	0.0226626	4.2835285	0.0000447
1.8	4.8151763	4.8462986	0.0311223	4.8152377	0.0000615
2.0	5.3054720	5.3476843	0.0422123	5.3055554	0.0000834

- The fourth-order results are vastly superior.

Theorem

If $y \in C^{n+1}[a, b]$, then the local truncation error of Taylor's method of order n is $O(h^n)$.

Proof: From Eq. (3), we have

$$\begin{aligned} y(t_{i+1}) &- y(t_i) - h\left[f(t_i, y(t_i)) + \frac{h}{2} f'(t_i, y(t_i)) + \cdots \right. \\ &+ \left. \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, y(t_i))\right] = \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)) \end{aligned}$$

for some ξ_i in (t_i, t_{i+1}). So the local truncation error is

$$\tau_{i+1}(h) = \frac{y(t_{i+1}) - y(t_i)}{h} - T^{(n)}(t_i, y(t_i)) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)).$$

Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i = O(h^n)$, $\forall\, i = 1, 2, \ldots, N$. ∎