Solutions of equations in one variable

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Problem: Solve the following nonlinear equation

$$f(x) \equiv \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) - x = 0, \quad x \in [0, 2\pi]. \tag{1}$$

- Bisection method: If $f(x) \in C[a,b]$ and $f(a)f(b) < 0$, then $\exists\ c \in (a,b)$ such that $f(c) = 0$.

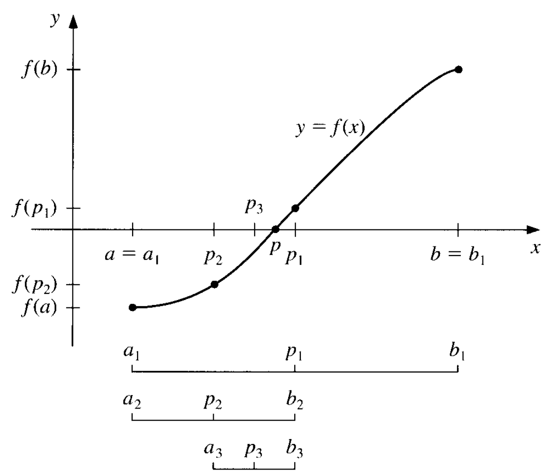

Figure 1: Bisection method

- Fixed-point iteration or functional iteration: Given a continuous function g, choose an initial point x_0 and generate $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = g(x_k), \quad k \geq 0.$$

  Take $g(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right)$.

- Newton's method: Starts with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Given $f(x)$ defined on (a, b), the maximal number of iterations M, and stop criteria δ and ε, this algorithm tries to locate one root of $f(x)$.

Compute $u = f(a)$, $v = f(b)$, and $e = b - a$
If $sign(u) = sign(v)$, **then** stop
For $k = 1, 2, \ldots, M$
  $e = e/2$, $c = a + e$, $w = f(c)$
  If $|e| < \delta$ or $|w| < \varepsilon$, **then** stop
  If $sign(w) \neq sign(u)$
    $b = c$, $v = w$
  Else
    $a = c$, $u = w$
  End If
End For

Algorithm 1: Bisection method

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = g(x_0)$.
While $i \leq M$ and $\frac{|x - x_0|}{|x|} \geq TOL$
  Set $i = i + 1$, $x_0 = x$ and $x = g(x_0)$.
End While

Algorithm 2: Fixed point iteration

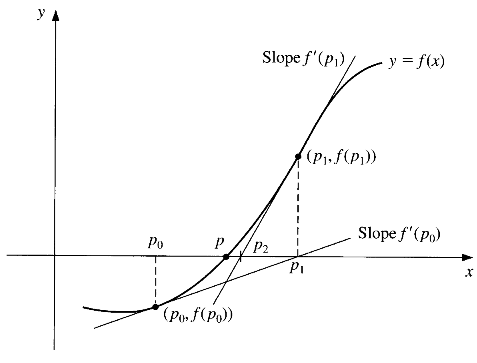

Figure 2: Newton's method

Given x_0, tolerance TOL, maximum number of iteration M. Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$. While $i \leq M$ and $\frac{\lvert x-x_0\rvert}{\lvert x\rvert} \geq TOL$ Set $i = i+1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$. End While

Algorithm 3: Newton's method

- Secant method: Using the approximation

$$f'(x_{n-1}) \approx \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}.$$

Home works

1. Plot the figure of the function $f(x)$ on $[0, 2\pi]$.
2. Use bisection method, fixed point iteration, Newton's method and Secant method to solve (1). In each iteration, please output the approximation x_1 and the relative error $\frac{\lvert x_1 - x_0\rvert}{\lvert x_1\rvert}$.
3. Let

Given x_0, x_1, tolerance TOL, maximum number of iteration M.
Set $i = 2$; $y_0 = f(x_0); y_1 = f(x_1)$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
While $i \leq M$ and $\frac{|x-x_1|}{|x|} \geq TOL$
    Set $i = i + 1$; $x_0 = x_1; y_0 = y_1; x_1 = x; y_1 = f(x)$;
        $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
End While

Algorithm 4: Secant method

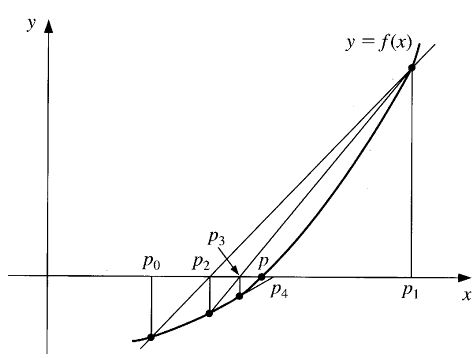

Figure 3: Secant method

$$A\sin\alpha\cos\alpha + B\sin^2\alpha - C\cos\alpha - E\sin\alpha = 0,$$

where

$$A = \ell\sin\beta_1, \quad B = \ell\cos\beta_1, \quad C = (h+0.5D)\sin\beta_1 - 0.5D\tan\beta_1,$$
$$E = (h+0.5D)\cos\beta_1 - 0.5D.$$

(a) It is stated that when $\ell = 89$ in., $h = 49$ in., $D = 55$ in., and $\beta_1 = 11.5°$, angle α is approximately $33°$. Verify this result.

(b) Find α for the situation when ℓ, h, and β_1 are the same as in part (a) but $D = 30$ in..