The single-step method (SSM)

$$(D+L)x^{(i+1)} = -Ux^{(i)} + b$$

can be written in the form

$$x^{(i+1)} = x^{(i)} + \{-D^{-1}Lx^{(i+1)} - D^{-1}Ux^{(i)} + D^{-1}b - x^{(i)}\} := x^{(i)} + \nu^{(i)}. \tag{1}$$

Consider a general form of (1)

$$x^{(i+1)} = x^{(i)} + \omega\nu^{(i)} \tag{2}$$

with constant ω. (2) can be written as

$$Dx^{(i+1)} = Dx^{(i)} - \omega Lx^{(i+1)} - \omega Ux^{(i)} + \omega b - \omega Dx^{(i)}.$$

Then

$$x^{(i+1)} = (D+\omega L)^{-1}\left[(1-\omega)D - \omega U\right]x^{(i)} + \omega(D+\omega L)^{-1}b. \tag{3}$$

Hence the iteration matrix

$$T_\omega = (D+\omega L)^{-1}\left[(1-\omega)D - \omega U\right].$$

These methods is called for

$\omega < 1$: under relaxation,

$\omega = 1$: single-step method,

$\omega > 1$: over relaxation. (In general: relaxation methods.)

Let A be symmetric and $A = D + L + L^T$. The idea is in fact to implement the SOR formulation twice, one forward and one backward, at each iteration. That is, SSOR method defines

$$
\begin{aligned}
(D + \omega L)x^{(k-\frac{1}{2})} &= \left[(1-\omega)D - \omega L^T\right] x^{(k-1)} + \omega b \qquad (4)\\
(D + \omega L^T)x^{(k)} &= \left[(1-\omega)D - \omega L\right] x^{(k-\frac{1}{2})} + \omega b. \qquad (5)
\end{aligned}
$$

Define

$$
\begin{cases}
M_\omega : &= D + \omega L, \\
N_\omega : &= (1-\omega)D - \omega L^T.
\end{cases}
$$

Then from the iterations (4) and (5), it follows that

$$
\begin{aligned}
x_{i+1} &= \left(M_\omega^{-T} N_\omega^T M_\omega^{-1} N_\omega\right) x_i + \omega \left(M_\omega^{-T} N_\omega^T M_\omega^{-1} + M_\omega^{-T}\right) b \\
&\equiv T(\omega) x_i + M(\omega)^{-1} b.
\end{aligned}
$$

But

$$\begin{aligned}
&((1-\omega)D-\omega L)\,(D+\omega L)^{-1}+I \\
&=(-\omega L-D-\omega D+2D)(D+\omega L)^{-1}+I \\
&=-I+(2-\omega)D(D+\omega L)^{-1}+I \\
&=(2-\omega)D(D+\omega L)^{-1},
\end{aligned}$$

Thus

$$M(\omega)^{-1}=\omega\left(D+\omega L^T\right)^{-1}(2-\omega)D(D+\omega L)^{-1},$$

then the splitting matrix is

$$M(\omega)=\frac{1}{\omega(2-\omega)}(D+\omega L)D^{-1}\left(D+\omega L^T\right).$$

The iteration matrix is

$$T(\omega)=(D+\omega L^T)^{-1}\left[(1-\omega)D-\omega L\right](D+\omega L)^{-1}\left[(1-\omega)D-\omega L^T\right].$$