Linear System

May 29, 2013

Let $n \geq 1$ be an integer and $h = \frac{1}{n+1}$. Define

$$f_{ij} \equiv 2h^2\pi^2 \sin \pi x_i \sin \pi y_j$$

with $x_i = ih,\ y_j = jh,\ i, j = 0, 1, \ldots, n+1$. Let

$$f = \begin{bmatrix} f_{:,1} \\ f_{:,2} \\ \vdots \\ f_{:,n} \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & -I_n & & \\ -I_n & A_1 & \ddots & \\ & \ddots & \ddots & -I_n \\ & & -I_n & A_1 \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}$$

where, for $j = 1, \ldots, n$,

$$f_{:,j} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{n,j} \end{bmatrix}, A_1 = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Problem: Solve the linear system $Ax = f$.

Requirements:

- Use *function* command to construct the coefficient matrix A and right hand side vector b.
- Call *lu*, i.e., [L,U,P] = *lu*(A), which is the MATLAB built in function to compute LU factorization of A:

  $$PA = LU,$$

  where L and U are lower and upper triangular, respectively, and P is a permutation matrix.
- Compute the solution of the linear system by $x = U^{-1}L^{-1}P^T f$.

– **Forward substitution**: When a linear system $Lx = b$ is lower triangular of the form

$$\begin{bmatrix} \ell_{11} & 0 & \cdots & 0 \\ \ell_{21} & \ell_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{n1} & \ell_{n2} & \cdots & \ell_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

where all diagonals $\ell_{ii} \neq 0$, x_i can be obtained by the following procedure

$$\begin{aligned} x_1 &= b_1/\ell_{11}, \\ x_2 &= (b_2 - \ell_{21}x_1)/\ell_{22}, \\ x_3 &= (b_3 - \ell_{31}x_1 - \ell_{32}x_2)/\ell_{33}, \\ &\vdots \\ x_n &= (b_n - \ell_{n1}x_1 - \ell_{n2}x_2 - \cdots - \ell_{n,n-1}x_{n-1})/\ell_{nn}. \end{aligned}$$

The general formulation for computing x_i is

$$x_i = \left(b_i - \sum_{j=1}^{i-1} \ell_{ij}x_j \right) \Big/ \ell_{ii}, \qquad i = 1, 2, \ldots, n.$$

Algorithm 1 (Forward Substitution) *Suppose that $L \in \mathbb{R}^{n\times n}$ is nonsingular lower triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Lx = b$.*

```
For i = 1, ..., n
    tmp = 0
    For j = 1, ..., i − 1
        tmp = tmp + L(i, j) ∗ x(j)
    End for
    x(i) = (b(i) − tmp)/L(i, i)
End for
```

– **Backward substitution**: Solve the linear system $Ux = b$:

$$\begin{aligned} u_{11}x_1 + u_{12}x_2 + \cdots + u_{1n}x_n &= b_1, \\ u_{22}x_2 + \cdots + u_{2n}x_n &= b_2, \\ &\vdots \\ u_{nn}x_n &= b_n \end{aligned}$$

* Solving the nth equation for x_n gives

$$x_n = \frac{b_n}{u_{nn}}.$$

* Solving the $(n-1)$th equation for x_{n-1} and using the value for x_n yields

$$x_{n-1} = \frac{b_{n-1} - u_{n-1,n}x_n}{u_{n-1,n-1}}.$$

* In general,

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} u_{ij}x_j}{u_{ii}}, \ \forall\ i = n-1, n-2, \ldots, 1.$$

Algorithm 2 (Backward Substitution) *Suppose that $U \in \mathbb{R}^{n\times n}$ is nonsingular upper triangular and $b \in \mathbb{R}^n$. This algorithm computes the solution of $Ux = b$.*

```
For i = n, ..., 1
    tmp = 0
    For j = i + 1, ..., n
        tmp = tmp + U(i, j) * x(j)
    End for
    x(i) = (b(i) - tmp)/U(i, i)
End for
```