Newton's method

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Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f \in C^2[a, b]$, i.e., f'' exists and is continuous. If $f(x^*) = 0$ and $x^* = x + h$ where h is small, then by Taylor's theorem

$$\begin{aligned} 0 = f(x^*) &= f(x+h) \\ &= f(x) + f'(x)h + \frac{1}{2}f''(x)h^2 + \frac{1}{3!}f'''(x)h^3 + \cdots \\ &= f(x) + f'(x)h + O(h^2). \end{aligned}$$

Since h is small, $O(h^2)$ is negligible. It is reasonable to drop $O(h^2)$ terms. This implies

$$f(x) + f'(x)h \approx 0 \quad \text{and} \quad h \approx -\frac{f(x)}{f'(x)}, \quad \text{if} \quad f'(x) \neq 0.$$

Hence

$$x + h = x - \frac{f(x)}{f'(x)}$$

is a better approximation to x^*.

This sets the stage for the Newton-Rapbson's method, which starts with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Since the Taylor's expansion of $f(x)$ at x_k is given by

$$f(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{1}{2}f''(x_k)(x - x_k)^2 + \cdots .$$

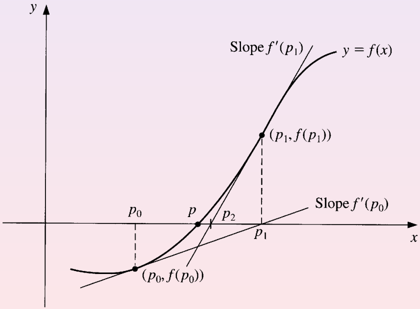
At x_k, one uses the tangent line

$$y = \ell(x) = f(x_k) + f'(x_k)(x - x_k)$$

to approximate the curve of $f(x)$ and uses the zero of the tangent line to approximate the zero of $f(x)$.

Newton's Method

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$.
While $i \leq M$ and $|x - x_0| \geq TOL$
  Set $i = i + 1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$.
End While



Problem

The equation $f(x) \equiv x^2 - 10\cos x = 0$ has two solutions ± 1.3793646. Use Newton's method to approximate the solutions with initial values ± 25.

Requirements

1. Write two MATLAB functions, said fun_f and fun_df, to compute the values of f and f', respectively.
2. Implement the Newton's algorithm as a MATLAB function:
    - Input arguments: fun_f, fun_df, initial value
    - Output arguments: approximated solution of the equation