# Solutions of equations in one variable

December 23, 2013

**Problem**: Solve the following nonlinear equation

$$f(x) \equiv \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) - x = 0, \quad x \in [0, 2\pi]. \tag{1}$$

- Bisection method: If $f(x) \in C[a,b]$ and $f(a)f(b) < 0$, then $\exists\, c \in (a,b)$ such that $f(c) = 0$.

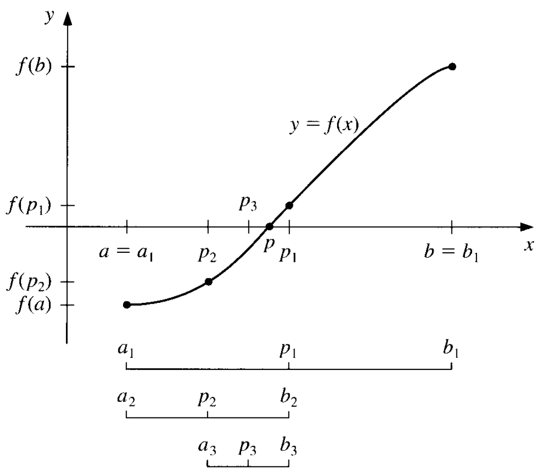


Figure 1: Bisection method

Given $f(x)$ defined on $(a,b)$, the maximal number of iterations $M$, and stop criteria $\delta$ and $\varepsilon$, this algorithm tries to locate one root of $f(x)$.

Compute $u = f(a)$, $v = f(b)$, and $e = b - a$
**If** $sign(u) = sign(v)$, **then** stop
**For** $k = 1, 2, \ldots, M$
  $e = e/2$, $c = a + e$, $w = f(c)$
  **If** $|e| < \delta$ or $|w| < \varepsilon$, **then** stop
  **If** $sign(w) \neq sign(u)$
    $b = c$, $v = w$
  **Else**
    $a = c$, $u = w$
  **End If**
**End For**

**Algorithm 1:** Bisection method

- Fixed-point iteration or functional iteration: Given a continuous function $g$, choose an initial point $x_0$ and generate $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = g(x_k), \quad k \geq 0.$$

Take $g(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right)$.

Given $x_0$, tolerance $TOL$, maximum number of iteration $M$.
Set $i = 1$ and $x = g(x_0)$.
While $i \leq M$ and $\frac{|x-x_0|}{|x|} \geq TOL$
Set $i = i + 1$, $x_0 = x$ and $x = g(x_0)$.
End While

**Algorithm 2:** Fixed point iteration

- Newton's method: Starts with an initial approximation $x_0$ and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Given $x_0$, tolerance $TOL$, maximum number of iteration $M$.
Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$.
While $i \leq M$ and $\frac{|x-x_0|}{|x|} \geq TOL$
Set $i = i + 1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$.
End While

**Algorithm 3:** Newton's method

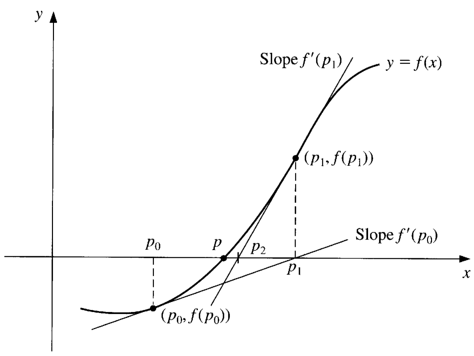


Figure 2: Newton's method

Consider the nonlinear eigenvalue problem

$$A(v)v \equiv \left[A_0 + \sin\left(\frac{v^\top B v}{v^\top v}\right) A_1\right] v = \lambda v,$$

where

$$A_0 = \frac{1}{10}\begin{bmatrix} 10 & 21 & 13 & 16 \\ 21 & -26 & 24 & 2 \\ 13 & 24 & -26 & 37 \\ 16 & 2 & 37 & -4 \end{bmatrix}, \quad A_1 = \frac{\beta}{10}\begin{bmatrix} 20 & 28 & 12 & 32 \\ 28 & 4 & 14 & 6 \\ 12 & 14 & 32 & 34 \\ 32 & 6 & 34 & 16 \end{bmatrix},$$

$$B = \frac{1}{10}\begin{bmatrix} -14 & 16 & -4 & 15 \\ 16 & 10 & 15 & -9 \\ -4 & 15 & 16 & 6 \\ 15 & -9 & 6 & -6 \end{bmatrix}.$$

The Jacobian matrix of $A(v)v$ is

$$J(v) = \frac{\partial}{\partial v}(A(v)v) = A(v) + 2\frac{\cos\left(\frac{v^\top Bv}{v^\top v}\right)}{\left(v^\top v\right)^2} A_1 v\left(\left(v^\top v\right)v^\top B - \left(v^\top Bv\right)v^\top\right)$$

Apply Newton method to solve the nonlinear eigenvalue problem and rewrite the nonlinear eigenvalue problem as

$$F\begin{pmatrix} v \\ \lambda \end{pmatrix} = \begin{pmatrix} A(v)v - \lambda v \\ \ell^T v - 1 \end{pmatrix} = 0, \tag{2}$$

where $\ell \in \mathbb{R}^n$ is a suitable fixed nonzero vector. The Jacobian matrix of (2) is

$$JF\begin{pmatrix} v_k \\ \lambda_k \end{pmatrix} = \begin{bmatrix} J(v_k) - \lambda_k I & -v_k \\ \ell^T & 0 \end{bmatrix}. \tag{3}$$

The Newton iteration is of the form

$$\begin{bmatrix} v_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} v_k \\ \lambda_k \end{bmatrix} - \left(JF\begin{pmatrix} v_k \\ \lambda_k \end{pmatrix}\right)^{-1}\begin{bmatrix} A(v_k)v_k - \lambda_k v_k \\ \ell^T v_k - 1 \end{bmatrix}. \tag{4}$$

- Secant method: Using the approximation

$$f'(x_{n-1}) \approx \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}.$$

Given $x_0, x_1$, tolerance $TOL$, maximum number of iteration $M$.
Set $i = 2$; $y_0 = f(x_0); y_1 = f(x_1)$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
While $i \le M$ and $\frac{|x - x_1|}{|x|} \ge TOL$
  Set $i = i + 1$; $x_0 = x_1; y_0 = y_1; x_1 = x; y_1 = f(x)$;
    $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
End While

**Algorithm 4:** Secant method

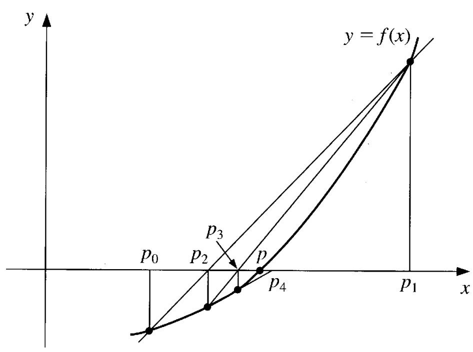


Figure 3: Secant method

**Home works**

1. Plot the figure of the function $f(x)$ on $[0, 2\pi]$.
2. Use Bisection method, fixed point iteration and Newton's method to solve (1). In each iteration, please output the approximation $x_1$ and the relative error $\frac{|x_1 - x_0|}{|x_1|}$.

Use MATLAB command function:

1. Build two functions, said "fun_f" and "fun_df", to compute the values $f(x)$ and $f'(x)$, respectively.
2. Rewrite your previous MATLAB code of Bisection method, fixed point iteration and Newton's method with using the functions "fun_f" and "fun_df".