Solutions of equations in one variable

October 21, 2012

Problem: Solve the following nonlinear equation

$$f(x) \equiv \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) - x = 0, \quad x \in [0, 2\pi]. \tag{1}$$

- Fixed-point iteration or functional iteration: Given a continuous function g, choose an initial point x_0 and generate $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = g(x_k), \quad k \geq 0.$$

Take $g(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right)$.

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = g(x_0)$.
While $i \leq M$ and $\frac{|x-x_0|}{|x|} \geq TOL$
  Set $i = i + 1$, $x_0 = x$ and $x = g(x_0)$.
End While

Algorithm 1: Fixed point iteration

- Newton's method: Starts with an initial approximation x_0 and generates the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Given x_0, tolerance TOL, maximum number of iteration M.
Set $i = 1$ and $x = x_0 - f(x_0)/f'(x_0)$.
While $i \leq M$ and $\frac{|x-x_0|}{|x|} \geq TOL$
  Set $i = i + 1$, $x_0 = x$ and $x = x_0 - f(x_0)/f'(x_0)$.
End While

Algorithm 2: Newton's method

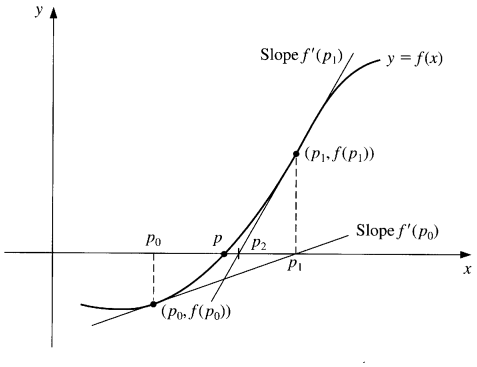

Figure 1: Newton's method

- Secant method: Using the approximation

$$f'(x_{n-1}) \approx \frac{f(x_{n-1}) - f(x_{n-2})}{x_{n-1} - x_{n-2}}.$$

for $f'(x_{n-1})$ in Newton's formula gives

$$x_n = x_{n-1} - \frac{f(x_{n-1})(x_{n-1} - x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}.$$

Given x_0, x_1, tolerance TOL, maximum number of iteration M.
Set $i = 2$; $y_0 = f(x_0)$; $y_1 = f(x_1)$; $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
While $i \leq M$ and $\frac{|x - x_1|}{|x|} \geq TOL$
  Set $i = i + 1$; $x_0 = x_1$; $y_0 = y_1$; $x_1 = x$; $y_1 = f(x)$;
    $x = x_1 - y_1(x_1 - x_0)/(y_1 - y_0)$.
End While

Algorithm 3: Secant method

Home works

1. Plot the figure of the function $f(x)$ on $[0, 2\pi]$.
2. Use fixed point iteration, Newton's method and Secant method to solve (1). In each iteration, please output the approximation x_1 and the relative error $\frac{|x_1 - x_0|}{|x_1|}$.
   (a) Fixed point iteration: 40040112S, 498711079, 40040126S

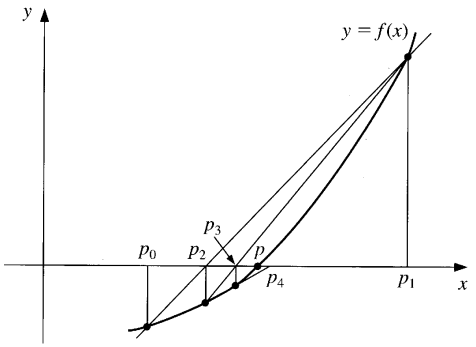

Figure 2: Secant method

(b) Newton's method: 40040114S, 498402331, 40040316S, 40140229S

(c) Secant method: 40040118S, 498401052, 40040326S