Project for MATLAB Programming

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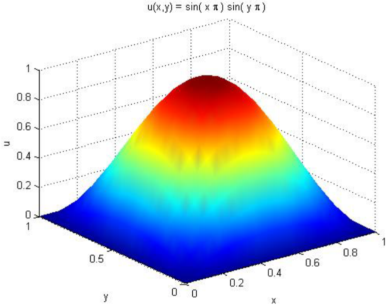
Outline

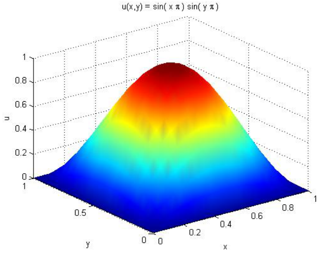
Consider the Dirichlet boundary-value problem:

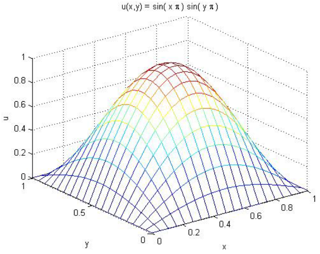
$$-\Delta u \equiv -u_{xx} - u_{yy} = 2\pi^2 \sin \pi x \sin \pi y, \text{ for } (x, y) \in \Omega, \quad (1)$$
$$u(x, y) = 0 \quad (x, y) \in \partial\Omega,$$

for $\Omega := \{x, y | 0 < x, y < 1\} \subseteq \mathbb{R}^2$ with boundary $\partial\Omega$, which has the exact solution

$$u(x, y) = \sin \pi x \sin \pi y.$$




u(x,y) = sin(x π) sin(y π)
u
1
0.8
0.6
0.4
0.2
0
1
0.5
0
y
0
0.2
0.4
0.6
0.8
1
x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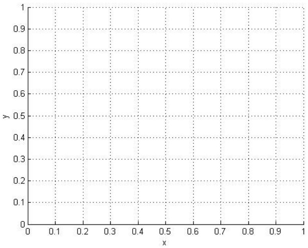
u
1
0.8
0.6
0.4
0.2
0
1
0.5
0
y
0
0.2
0.4
0.6
0.8
1
x


y
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

To solve (1) by means of a difference methods, one replaces the differential operator by a difference operator. Let

$$\begin{aligned}\Omega_h &:= \{(x_i, y_i) | i, j = 1, \ldots, n\}, \\ \partial\Omega_h &:= \{(x_i, 0), (x_i, 1), (0, y_j), (1, y_j) | i, j = 0, 1, \ldots, n+1\},\end{aligned}$$

where $x_i = ih,\ y_j = jh,\ i, j = 0, 1, \ldots, n+1,\ h := \frac{1}{n+1},\ n \geq 1,$ is an integer.

From the Taylor's theorem, we have

$$
\begin{aligned}
u(x_i+h) &= u(x_i) + u'(x_i)h + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\xi_1) \\
u(x_i-h) &= u(x_i) - u'(x_i)h + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\xi_2),
\end{aligned}
$$

where ξ_1 is between x_i and x_i+h and ξ_2 is between x_i and x_i-h. Hence

$$
\begin{aligned}
u''(x_i) &= \frac{u(x_i+h) - 2u(x_i) + u(x_i-h)}{h^2} - \frac{h^2}{12}u^{(4)}(\xi) \\
&= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} - \frac{h^2}{12}u^{(4)}(\xi),
\end{aligned}
$$

where ξ is between x_i-h and x_i+h.

Similarly,

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x_i, y_j) &= \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j), \\
\frac{\partial^2 u}{\partial y^2}(x_i, y_j) &= \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_i, \eta_j),
\end{aligned}$$

where $\xi_i \in (x_{i-1}, x_{i+1})$ and $\eta_j \in (y_{j-1}, y_{j+1})$. It implies that

$$\begin{aligned}
& \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j) \\
= \; & \frac{u(x_i, y_{j-1}) + u(x_{i-1}, y_j) - 4u(x_i, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j+1})}{h^2} \\
& - \frac{h^2}{12}\left[\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial x^4}(x_i, \eta_j)\right].
\end{aligned}$$

Let u_{ij} denote an approximated value of function u at the grid point (x_i, y_j) for $i, j = 1, \ldots, n+1$. Then

$$\begin{aligned} & -u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) \\ \approx \quad & \frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} \end{aligned}$$

with an error $O(h^2)$ and the equation

$$-u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) = 2\pi^2 \sin \pi x_i \sin \pi y_j \equiv f_{ij}$$

can be replaced by the following equation

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{ij} \tag{2}$$

for $i, j = 1, \ldots, n$.

For $j = 1$, we have

$$\begin{aligned}
-u_{1,0} - u_{0,1} + 4u_{1,1} - u_{2,1} - u_{1,2} &= h^2 f_{1,1}, && \text{(3a)}\\
-u_{2,0} - u_{1,1} + 4u_{2,1} - u_{3,1} - u_{2,2} &= h^2 f_{2,1}, && \text{(3b)}\\
&\vdots \\
-u_{n-1,0} - u_{n-2,1} + 4u_{n-1,1} - u_{n,1} - u_{n-1,2} &= h^2 f_{n-1,1}, && \text{(3c)}\\
-u_{n,0} - u_{n-1,1} + 4u_{n,1} - u_{n+1,1} - u_{n,2} &= h^2 f_{n,1}. && \text{(3d)}
\end{aligned}$$

By the boundary condition, it holds that

$$\begin{aligned}
u_{1,0} &= u_{2,0} = \cdots = u_{n,0} = 0, && \text{(4a)}\\
u_{0,1} &= u_{n+1,1} = 0. && \text{(4b)}
\end{aligned}$$

Substituting (4) into (3), we get

$$
\begin{aligned}
4u_{1,1} - u_{2,1} - u_{1,2} &= h^2 f_{1,1}, && \text{(5a)}\\
-u_{1,1} + 4u_{2,1} - u_{3,1} - u_{2,2} &= h^2 f_{2,1}, && \text{(5b)}\\
&\vdots \\
-u_{n-2,1} + 4u_{n-1,1} - u_{n,1} - u_{n-1,2} &= h^2 f_{n-1,1}, && \text{(5c)}\\
-u_{n-1,1} + 4u_{n,1} - u_{n,2} &= h^2 f_{n,1}. && \text{(5d)}
\end{aligned}
$$

Let, for $j = 1, \ldots, n$,

$$
u_{:,j} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix}, \; f_{:,j} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{n,j} \end{bmatrix}, \; A_1 = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n\times n}.
$$

Then (5) can be rewritten as following matrix form:

$$\begin{bmatrix} A_1 & -I_n \end{bmatrix} \begin{bmatrix} u_{:,1} \\ u_{:,2} \end{bmatrix} = h^2 f_{:,1}.$$

For $j = 2, \ldots, n-1$, using $u_{0,j} = u_{n+1,j} = 0$, we have

$$\begin{aligned}
-u_{1,j-1} + 4u_{1,j} - u_{2,j} - u_{1,j+1} &= h^2 f_{1,j}, \\
-u_{2,j-1} - u_{1,j} + 4u_{2,j} - u_{3,j} - u_{2,j+1} &= h^2 f_{2,j}, \\
&\vdots \\
-u_{n-1,j-1} - u_{n-2,j} + 4u_{n-1,j} - u_{n,j} - u_{n-1,j+1} &= h^2 f_{n-1,j}, \\
-u_{n,j-1} - u_{n-1,j} + 4u_{n,j} - u_{n,j+1} &= h^2 f_{n,j}.
\end{aligned}$$

Above equations can be represented as following matrix form:

$$\begin{bmatrix} -I_n & A_1 & -I_n \end{bmatrix} \begin{bmatrix} u_{:,j-1} \\ u_{:,j} \\ u_{:,j+1} \end{bmatrix} = h^2 f_{:,j}.$$

For $j = n$, using $u_{1,n+1} = u_{2,n+1} = u_{n,n+1} = 0$, we have

$$\begin{array}{rcl}
-u_{1,n-1} + 4u_{1,n} - u_{2,n} & = & h^2 f_{1,n}, \\
-u_{2,n-1} - u_{1,n} + 4u_{2,n} - u_{3,n} & = & h^2 f_{2,n}, \\
& \vdots & \\
-u_{n-1,n-1} - u_{n-2,n} + 4u_{n-1,n} - u_{n,n} & = & h^2 f_{n-1,n}, \\
-u_{n,n-1} - u_{n-1,n} + 4u_{n,n} & = & h^2 f_{n,n}.
\end{array}$$

Above equations can be represented as following matrix form:

$$\begin{bmatrix} -I_n & A_1 \end{bmatrix} \begin{bmatrix} u_{:,n-1} \\ u_{:,n} \end{bmatrix} = h^2 f_{:,n}.$$

Therefore, (2) with boundary conditions is equivalent to a linear system $Au = h^2 f$ with

$$A = \begin{bmatrix} A_1 & -I_n & & \\ -I_n & A_1 & \ddots & \\ & \ddots & \ddots & -I_n \\ & & -I_n & A_1 \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}, \tag{6}$$

and

$$A_1 = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}, \quad u = \begin{bmatrix} u_{:,1} \\ u_{:,2} \\ \vdots \\ u_{:,n} \end{bmatrix}, \quad f = \begin{bmatrix} f_{:,1} \\ f_{:,2} \\ \vdots \\ f_{:,n} \end{bmatrix}.$$

Question: How to solve such large sparse linear systems?

Exercise

Prove that

1. the vector f is an eigenvector of

$$J = (4I - A)/4,$$

   also an eigenvector of A. Furthermore,

$$Jf = \mu f \quad \text{with} \quad \mu = \cos \pi h;$$

2. the exact solution of $Au = h^2 f$ can be found

$$u = \frac{h^2}{4(1 - \cos \pi h)} f.$$