Why we need high performance linear solver

Tsung-Ming Huang

Matrix Computation 2016

Maxwell’s Equations for electromagnetic waves

$$\nabla \times E = \imath\omega B, \quad \nabla \times H = -\imath\omega D$$

$$\nabla \cdot (\varepsilon E) = 0, \quad \nabla \cdot (\mu H) = 0$$

Dielectric materials

$$D = \varepsilon E, \quad B = \mu H$$

Metamaterials

$$D = \varepsilon E + \xi H,$$
$$B = \mu H + \zeta E$$

Dielectric materials (3D)

Math. model

$$\nabla \times \nabla \times E(\mathbf{r}) = \omega^2 \varepsilon(\mathbf{r}) E(\mathbf{r})$$

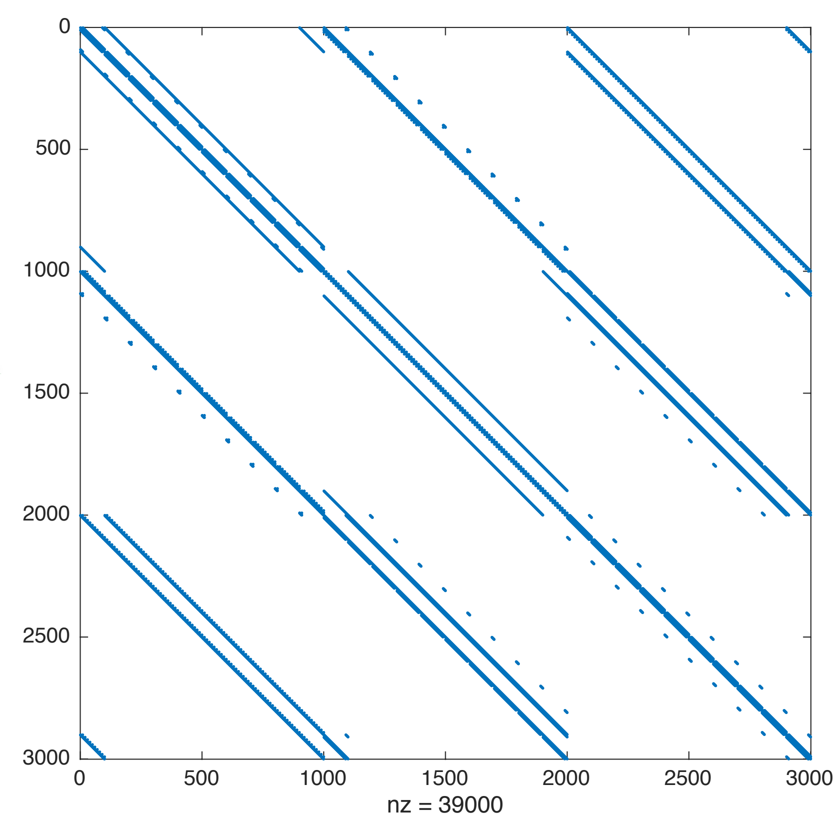
Eigenproblem

$$C^* C x = \lambda B x, \quad \lambda = \omega^2$$

Linear system

$$(C^* C - \sigma B) y = b$$

sparsity $C^* C$



Complex Media (3D)

Math. model

↓

Eigenproblem

↓

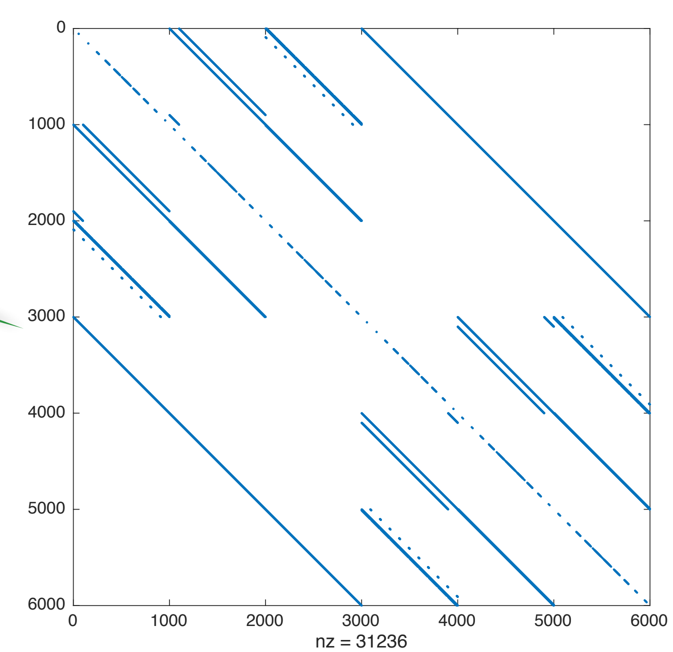
Linear system

$$\nabla \times E = \mathrm{i}\omega(\mu H + \zeta E),$$
$$\nabla \times H = -\mathrm{i}\omega(\varepsilon E + \xi H).$$

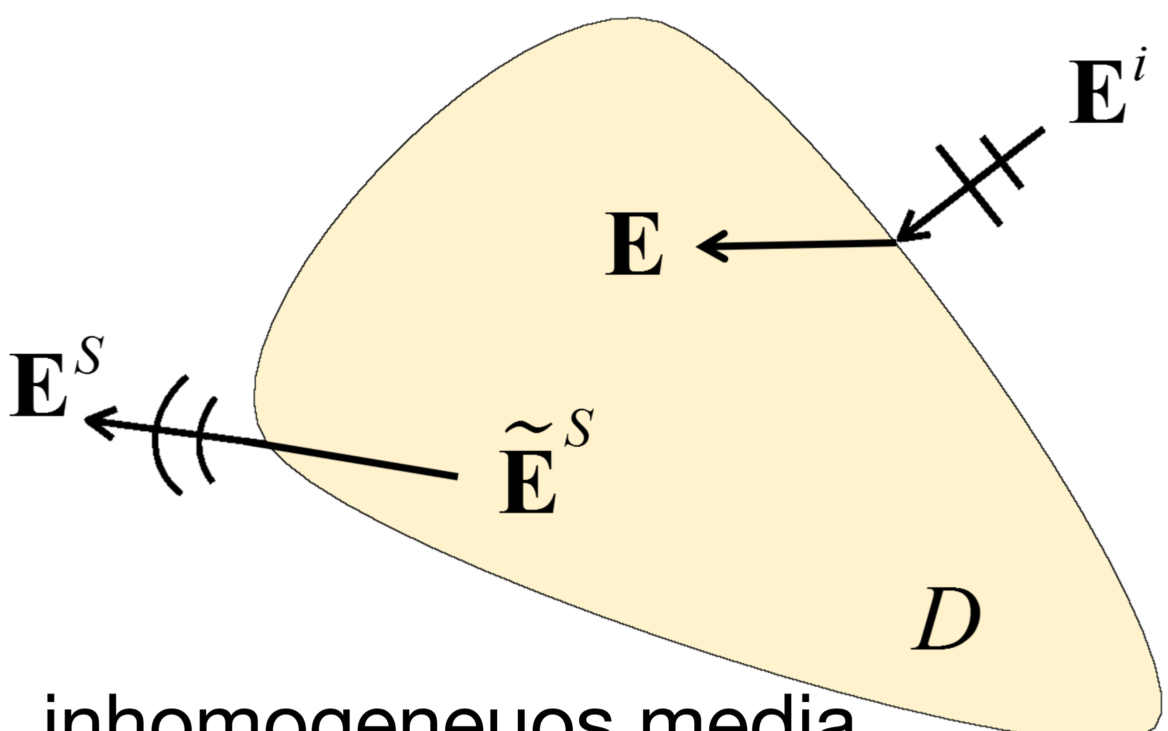
$$\begin{bmatrix} C & 0 \\ 0 & C^* \end{bmatrix}\begin{bmatrix} E \\ H \end{bmatrix} = \omega\left(\mathrm{i}\begin{bmatrix} \zeta_d & \mu_d \\ -\varepsilon_d & -\xi_d \end{bmatrix}\right)\begin{bmatrix} E \\ H \end{bmatrix}.$$

$$\left(\begin{bmatrix} C & 0 \\ 0 & C^* \end{bmatrix} - \sigma\begin{bmatrix} \zeta_d & \mu_d \\ -\varepsilon_d & -\xi_d \end{bmatrix}\right)x = b$$

sparsity



Transmission eigenvalue problem for Maxwell's eq.



inhomogeneuos media

$\mathbf{E}^i$: incident wave

$\mathbf{E}$: transmitted wave

$\mathbf{E}^s$: scattering wave

$\widetilde{\mathbf{E}}^s$: analytic extension of $\mathbf{E}^s$

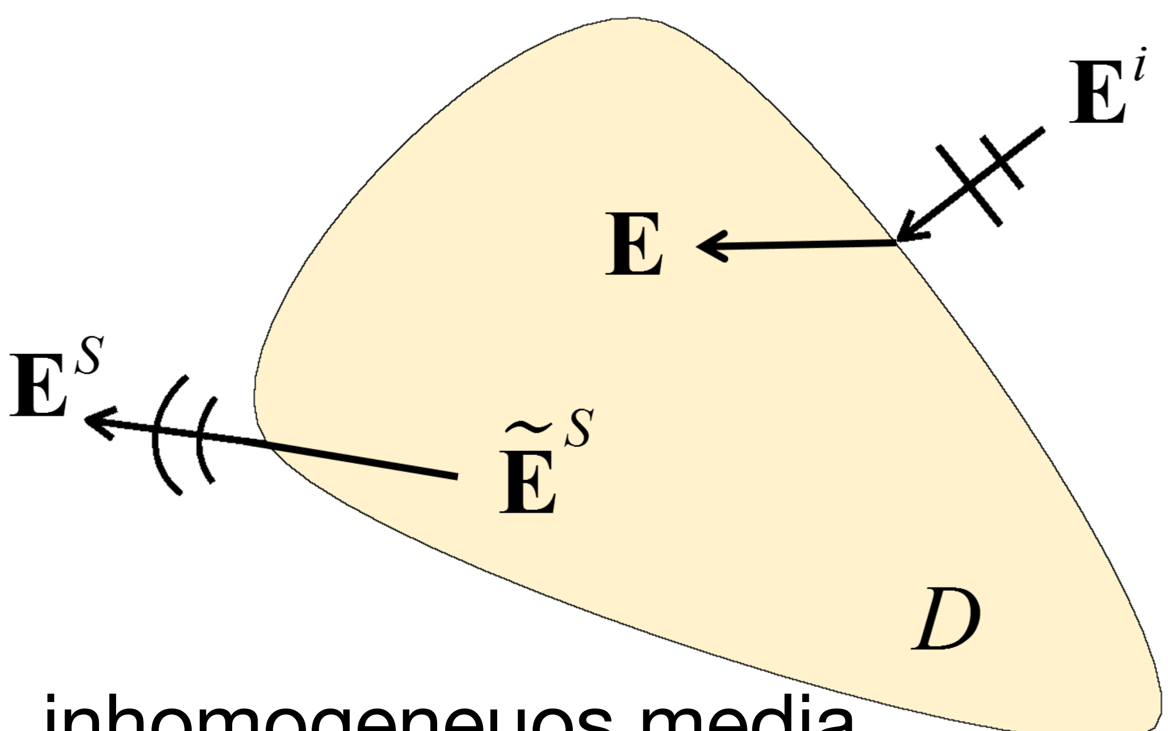
$$\mathbf{E}_0 := \widetilde{\mathbf{E}}^s + \mathbf{E}^i$$

Transmission eigenvalue problem for Maxwell's eq.

$$\nabla \times \nabla \times \mathbf{E} - \lambda N \mathbf{E} = 0 \quad \text{in } D,$$

$N = N(\mathbf{x})$: index of refraction

$N, (N-I)^{-1}$ (or $(I-N)^{-1}$): positive definite



inhomogeneuos media

$\mathbf{E}^i$: incident wave

$\mathbf{E}$: transmitted wave

$\mathbf{E}^s$: scattering wave

$\widetilde{\mathbf{E}}^s$: analytic extension of $\mathbf{E}^s$

$\mathbf{E}_0 := \widetilde{\mathbf{E}}^s + \mathbf{E}^i$

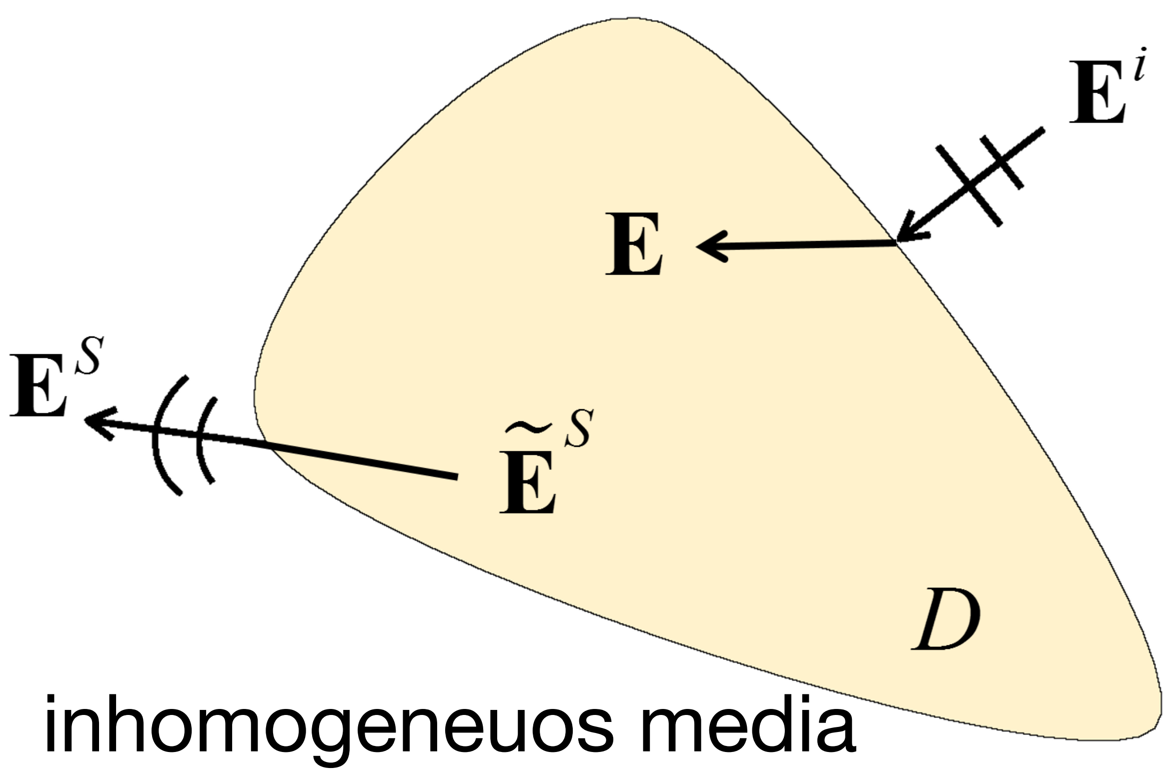
Transmission eigenvalue problem for Maxwell's eq.

$$\nabla \times \nabla \times \mathbf{E} - \lambda N \mathbf{E} = 0 \quad \text{in } D,$$

$N = N(\mathbf{x})$: index of refraction

$$\nabla \times \nabla \times \mathbf{E}_0 - \lambda \mathbf{E}_0 = 0 \quad \text{in } D,$$

$N, (N - I)^{-1} (\text{or } (I - N)^{-1})$: positive definite



$\mathbf{E}^i$: incident wave

$\mathbf{E}$: transmitted wave

$\mathbf{E}^s$: scattering wave

$\widetilde{\mathbf{E}}^s$: analytic extension of $\mathbf{E}^s$

$\mathbf{E}_0 := \widetilde{\mathbf{E}}^s + \mathbf{E}^i$

Transmission eigenvalue problem for Maxwell's eq.

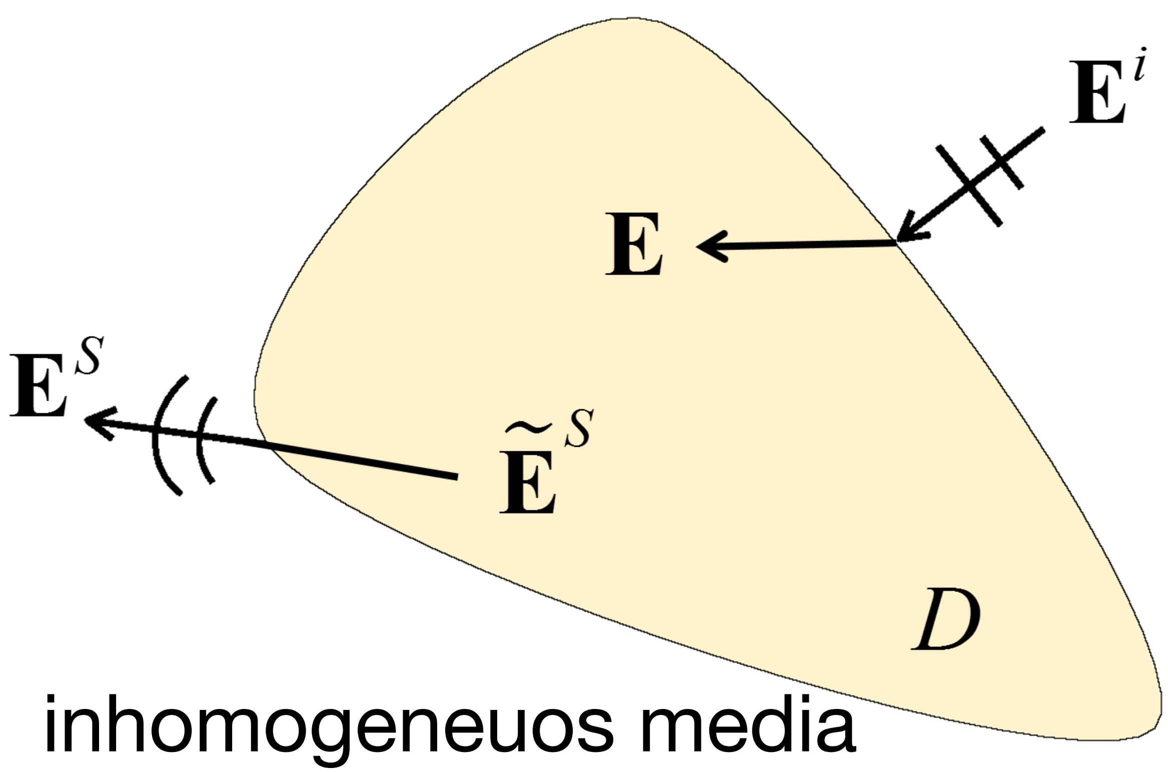
$$\nabla\times\nabla\times\mathbf{E}-\lambda N\mathbf{E}=0 \quad \text{in } D, \qquad N=N(\mathbf{x}):\ \text{index of refraction}$$

$$\nabla\times\nabla\times\mathbf{E}_0-\lambda\mathbf{E}_0=0 \quad \text{in } D,$$

$$\mathbf{E}\times\nu=\mathbf{E}_0\times\nu \ \text{ on } \partial D, \qquad \nu:\ \text{outer normal}$$

$$(\nabla\times\mathbf{E})\times\nu=(\nabla\times\mathbf{E}_0)\times\nu \ \text{ on } \partial D$$

$N, (N-I)^{-1}(\text{or } (I-N)^{-1})$: positive definite



$\mathbf{E}^i$: incident wave

$\mathbf{E}$: transmitted wave

$\mathbf{E}^s$: scattering wave

$\widetilde{\mathbf{E}}^s$: analytic extension of $\mathbf{E}^s$

$\mathbf{E}_0 := \widetilde{\mathbf{E}}^s + \mathbf{E}^i$

Transmission eigenvalue problem for Maxwell's eq.

$$\nabla\times\nabla\times\mathbf{E}-\lambda N\mathbf{E}=0 \quad \text{in } D,$$ $N=N(\mathbf{x})$: index of refraction

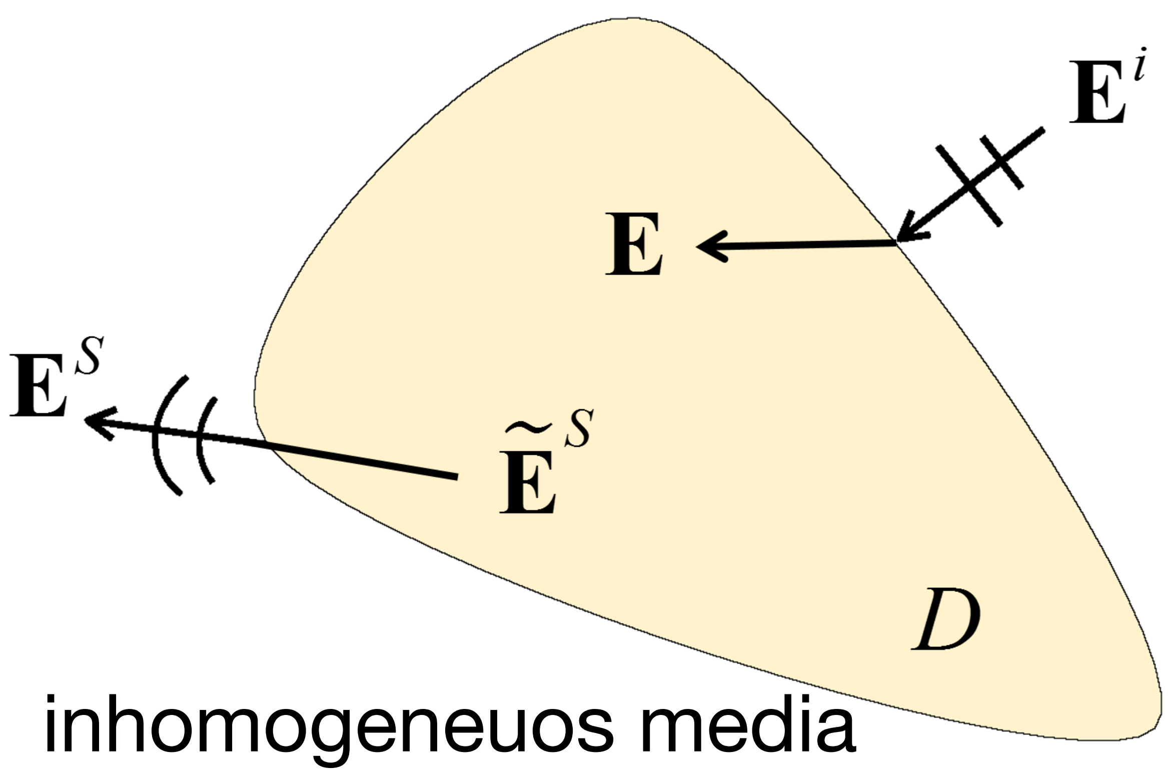
$$\nabla\times\nabla\times\mathbf{E}_0-\lambda\mathbf{E}_0=0 \quad \text{in } D,$$

$$\mathbf{E}\times\nu=\mathbf{E}_0\times\nu \ \text{ on } \partial D,$$ ν: outer normal

$$(\nabla\times\mathbf{E})\times\nu=(\nabla\times\mathbf{E}_0)\times\nu \ \text{ on } \partial D$$

Find smallest positive eigenvalues

$N,(N-I)^{-1}(\text{or } (I-N)^{-1})$: positive definite



$\mathbf{E}^i$: incident wave

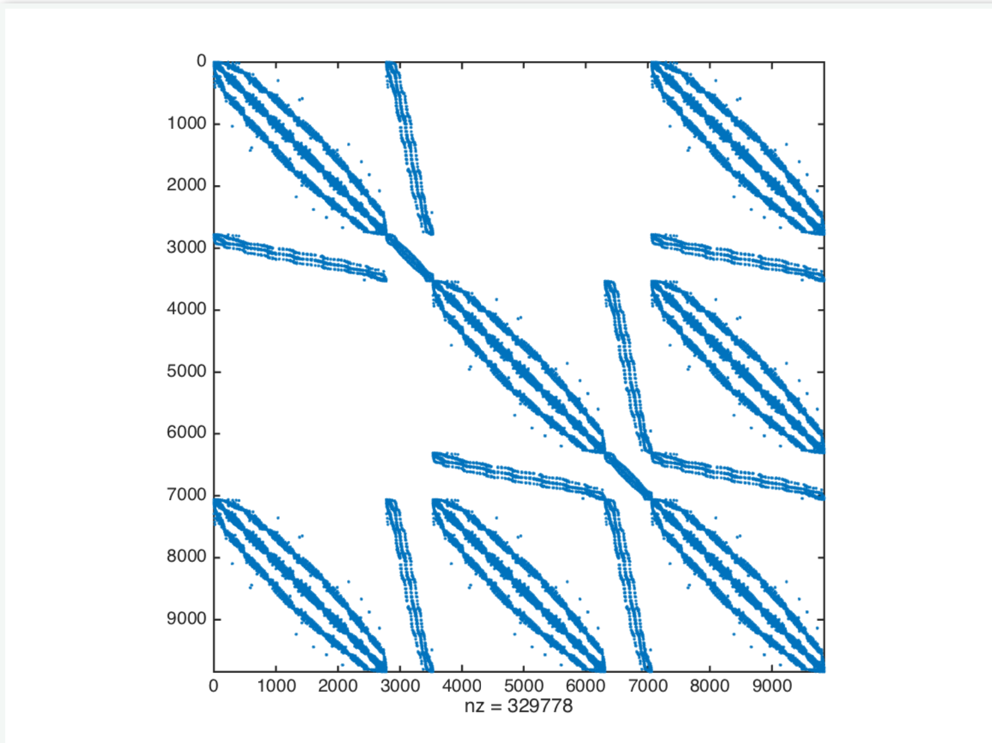
$\mathbf{E}$: transmitted wave

$\mathbf{E}^s$: scattering wave

$\widetilde{\mathbf{E}}^s$: analytic extension of $\mathbf{E}^s$

$\mathbf{E}_0 := \widetilde{\mathbf{E}}^s + \mathbf{E}^i$

Preconditioning linear system



Solve $\left(A_0 + \tau(A_1 + \mu_i^{(d)} A_2)\right)\mathbf{y} = \mathbf{r}_j$

$N(\mathbf{x})$: non-constant

$$\begin{bmatrix} \mathcal{M} & \mathbf{0} & -\mathcal{S}^\top \\ \mathbf{0} & \mathcal{M} & -\mathcal{T}_1^\top \\ \mathcal{S} - \tau\mathcal{T}_1 & \tau(\mu_i^{(d)}\mathcal{T}_1 - \mathcal{S}) & \tau(\mu_i^{(d)} M_1 - K) \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{u}} \\ \tilde{\mathbf{v}} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{r}_j \end{bmatrix}$$