

Project for large-scale eigenvalue problems

Tsung-Ming Huang

Department of Mathematics
National Taiwan Normal University, Taiwan

March 11, 2009



Outline

- 1 Model problem
- 2 Center difference discretization



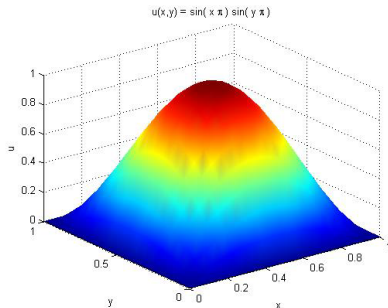
Consider the Dirichlet boundary-value problem:

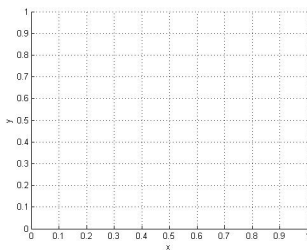
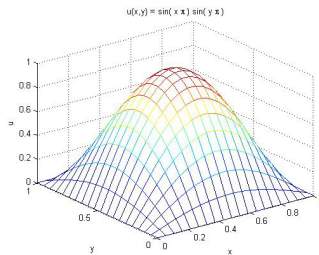
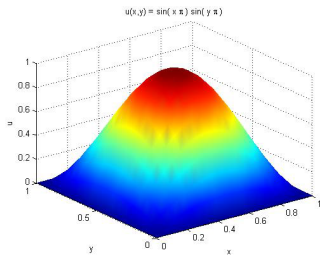
$$-\Delta u \equiv -u_{xx} - u_{yy} = 2\pi^2 \sin \pi x \sin \pi y, \text{ for } (x, y) \in \Omega, \quad (1)$$

$$u(x, y) = 0 \quad (x, y) \in \partial\Omega,$$

for $\Omega := \{x, y | 0 < x, y < 1\} \subseteq \mathbb{R}^2$ with boundary $\partial\Omega$, which has the exact solution

$$u(x, y) = \sin \pi x \sin \pi y.$$





To solve (1) by means of a difference methods, one replaces the differential operator by a difference operator. Let

$$\Omega_h := \{(x_i, y_i) | i, j = 1, \dots, n\},$$

$$\partial\Omega_h := \{(x_i, 0), (x_i, 1), (0, y_j), (1, y_j) | i, j = 0, 1, \dots, n + 1\},$$

where $x_i = ih$, $y_j = jh$, $i, j = 0, 1, \dots, n + 1$, $h := \frac{1}{n+1}$, $n \geq 1$, is an integer.



From the Taylor's theorem, we have

$$u(x_i + h) = u(x_i) + u'(x_i)h + \frac{h^2}{2}u''(x_i) + \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\xi_1)$$

$$u(x_i - h) = u(x_i) - u'(x_i)h + \frac{h^2}{2}u''(x_i) - \frac{h^3}{6}u'''(x_i) + \frac{h^4}{24}u^{(4)}(\xi_2),$$

where ξ_1 is between x_i and $x_i + h$ and ξ_2 is between x_i and $x_i - h$. Hence

$$\begin{aligned}u''(x_i) &= \frac{u(x_i + h) - 2u(x_i) + u(x_i - h)}{h^2} - \frac{h^2}{12}u^{(4)}(\xi) \\ &= \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1}))}{h^2} - \frac{h^2}{12}u^{(4)}(\xi),\end{aligned}$$

where ξ is between $x_i - h$ and $x_i + h$.



Similarly,

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j))}{h^2} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(\xi_i, y_j),$$

$$\frac{\partial^2 u}{\partial y^2}(x_i, y_j) = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1}))}{h^2} - \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(x_i, \eta_j).$$

where $\xi_i \in (x_{i-1}, x_{i+1})$ and $\eta_j \in (y_{j-1}, y_{j+1})$. It implies that

$$\begin{aligned} & \frac{\partial^2 u}{\partial x^2}(x_i, y_j) + \frac{\partial^2 u}{\partial y^2}(x_i, y_j) \\ = & \frac{u(x_i, y_{j-1}) + u(x_{i-1}, y_j) - 4u(x_i, y_j) + u(x_{i+1}, y_j) + u(x_i, y_{j+1}))}{h^2} \\ & - \frac{h^2}{12} \left[\frac{\partial^4 u}{\partial x^4}(\xi_i, y_j) + \frac{\partial^4 u}{\partial x^4}(x_i, \eta_j) \right]. \end{aligned}$$



Let u_{ij} denote an approximated value of function u at the grid point (x_i, y_j) for $i, j = 1, \dots, n + 1$. Then

$$\begin{aligned} & -u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) \\ \approx & \frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} \end{aligned}$$

with an error $O(h^2)$ and the equation

$$-u_{xx}(x_i, y_j) - u_{yy}(x_i, y_j) = 2\pi^2 \sin \pi x_i \sin \pi y_j \equiv f_{ij}$$

can be replaced by the following equation

$$\frac{-u_{i,j-1} - u_{i-1,j} + 4u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = f_{ij} \quad (2)$$

for $i, j = 1, \dots, n$.



For $j = 1$, we have

$$-u_{1,0} - u_{0,1} + 4u_{1,1} - u_{2,1} - u_{1,2} = h^2 f_{1,1}, \quad (3a)$$

$$-u_{2,0} - u_{1,1} + 4u_{2,1} - u_{3,1} - u_{2,2} = h^2 f_{2,1}, \quad (3b)$$

$$\vdots$$

$$-u_{n-1,0} - u_{n-2,1} + 4u_{n-1,1} - u_{n,1} - u_{n-1,2} = h^2 f_{n-1,1}, \quad (3c)$$

$$-u_{n,0} - u_{n-1,1} + 4u_{n,1} - u_{n+1,1} - u_{n,2} = h^2 f_{n,1}. \quad (3d)$$

By the boundary condition, it holds that

$$u_{1,0} = u_{2,0} = \cdots = u_{n,0} = 0, \quad (4a)$$

$$u_{0,1} = u_{n+1,1} = 0. \quad (4b)$$



Substituting (4) into (3), we get

$$4u_{1,1} - u_{2,1} - u_{1,2} = h^2 f_{1,1}, \quad (5a)$$

$$-u_{1,1} + 4u_{2,1} - u_{3,1} - u_{2,2} = h^2 f_{2,1}, \quad (5b)$$

$$\vdots$$

$$-u_{n-2,1} + 4u_{n-1,1} - u_{n,1} - u_{n-1,2} = h^2 f_{n-1,1}, \quad (5c)$$

$$-u_{n-1,1} + 4u_{n,1} - u_{n,2} = h^2 f_{n,1}. \quad (5d)$$

Let, for $j = 1, \dots, n$,

$$u_{:,j} = \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{n,j} \end{bmatrix}, f_{:,j} = \begin{bmatrix} f_{1,j} \\ f_{2,j} \\ \vdots \\ f_{n,j} \end{bmatrix}, A_1 = \begin{bmatrix} 4 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & & \\ & & & -1 & \\ & & & -1 & 4 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$



Then (5) can be rewritten as following matrix form:

$$\begin{bmatrix} A_1 & -I_n \end{bmatrix} \begin{bmatrix} u_{:,1} \\ u_{:,2} \end{bmatrix} = h^2 f_{:,1}.$$

For $j = 2, \dots, n-1$, using $u_{0,j} = u_{n+1,j} = 0$, we have

$$\begin{aligned} -u_{1,j-1} + 4u_{1,j} - u_{2,j} - u_{1,j+1} &= h^2 f_{1,j}, \\ -u_{2,j-1} - u_{1,j} + 4u_{2,j} - u_{3,j} - u_{2,j+1} &= h^2 f_{2,j}, \\ &\vdots \\ -u_{n-1,j-1} - u_{n-2,j} + 4u_{n-1,j} - u_{n,j} - u_{n-1,j+1} &= h^2 f_{n-1,j}, \\ -u_{n,j-1} - u_{n-1,j} + 4u_{n,j} - u_{n,j+1} &= h^2 f_{n,j}. \end{aligned}$$

Above equations can be represented as following matrix form:

$$\begin{bmatrix} -I_n & A_1 & -I_n \end{bmatrix} \begin{bmatrix} u_{:,j-1} \\ u_{:,j} \\ u_{:,j+1} \end{bmatrix} = h^2 f_{:,j}.$$



For $j = n$, using $u_{1,n+1} = u_{2,n+1} = u_{n,n+1} = 0$, we have

$$\begin{aligned}
 -u_{1,n-1} + 4u_{1,n} - u_{2,n} &= h^2 f_{1,n}, \\
 -u_{2,n-1} - u_{1,n} + 4u_{2,n} - u_{3,n} &= h^2 f_{2,n}, \\
 &\vdots \\
 -u_{n-1,n-1} - u_{n-2,n} + 4u_{n-1,n} - u_{n,n} &= h^2 f_{n-1,n}, \\
 -u_{n,n-1} - u_{n-1,n} + 4u_{n,n} &= h^2 f_{n,n}.
 \end{aligned}$$

Above equations can be represented as following matrix form:

$$\begin{bmatrix} -I_n & A_1 \end{bmatrix} \begin{bmatrix} u_{:,n-1} \\ u_{:,n} \end{bmatrix} = h^2 f_{:,n}.$$



Therefore, (2) with boundary conditions is equivalent to a linear system $Au = h^2 f$ with

$$A = \begin{bmatrix} A_1 & -I_n & & \\ -I_n & A_1 & \ddots & \\ & \ddots & \ddots & -I_n \\ & & -I_n & A_1 \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2}, \quad (6)$$

and

$$A_1 = \begin{bmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}, \quad u = \begin{bmatrix} u_{:,1} \\ u_{:,2} \\ \vdots \\ u_{:,n} \end{bmatrix}, \quad f = \begin{bmatrix} f_{:,1} \\ f_{:,2} \\ \vdots \\ f_{:,n} \end{bmatrix}.$$



Question

How to compute the eigenpair (λ, x) of the matrix A which λ is closest to a target value?

Exercise

Prove that the vector f is an eigenvector of

$$J = (4I - A)/4,$$

also an eigenvector of A . Furthermore,

$$Jf = \mu f \quad \text{and} \quad Af = 4(1 - \mu)f$$

with

$$\mu = \cos \pi h.$$

